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JACOBI-MAUPERTUIS METRIC OF LIÉNARD TYPE EQUATIONS AND JACOBI LAST MULTIPLIER

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Abstract. We present a construction of the Jacobi-Maupertuis (JM) principle for an equation of the Liénard type,

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0,$$

using Jacobi's last multiplier. The JM metric allows us to reformulate the Newtonian equation of motion for a variable mass as a geodesic equation for a Riemannian metric. We illustrate the procedure with examples of Painlevé-Gambier XXI, the Jacobi equation and the Henon-Heiles system.

1. Introduction

Nonlinear differential equations of the Liénard type occupy a special place in the study of dynamical systems as they serve to model various physical, chemical and biological processes. The standard Liénard equation involves a dissipative term depending linearly on the velocity. However there are practical problems in which higher order dependance on velocities are appropriate. Such equations have the generic form $\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$. It is interesting to note that equations of this type naturally arise in Newtonian dynamics when the mass, instead of being a constant, is allowed to vary with the position coordinate – the so called position dependent mass (PDM) scenario. Recently it has been shown [11, 1] that an inhomogeneous compactification of the extra dimension of a five-dimensional Kaluza-Klein metric has been shown to generate a PDM in the corresponding four-dimensional system. But this method yields a very special class of PDM Hamiltonian which directly related to the solution of the Lioville equation. In a related project Cariñena et al [2] formulated a method that starts with the study of the existence of Killing vector fields for the PDM geodesic motion and the construction of the associated Noether momenta.

There is also an alternative mechanism in which this dependance on a mass function manifests itself in the context of differential systems, namely through Jacobi's last multiplier (JLM). The JLM originally arose in the problem of reducing a system of first-order ordinary differential equations to quadrature and has a long and chequered history. In recent years its role in the context of the inverse problem of

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dynamical systems has led to a revival of interest in the JLM. In this brief note we examine the connection between the JLM and the principle of least action within the framework of a Liénard type differential equation with a quadratic dependance on the velocity.

It is known that the Liénard type equation is connected to the Painlevé-Gambier equations [8, 9]. So it is natural for us to ask whether we can reformulate the subclass of the Painlevé-Gambier family as geodesic equations for a Riemannian metric using the Jacobi-Maupertuis principle. There are several choices for a Riemannian manifold and metric tensor: a space-time configuration manifold and the Eisenhart metric (for example, [4, 5, 6, 14], a configuration manifold and the Jacobi-Maupertuis metric [12, 13]. In this paper we choose a configuration space of an analyzed system for a Riemannian manifold. The crux of the matter is that the Hamiltonian or energy function provided by the JLM should remain constant for these equations. In [3], geometrical theory for the mechanics of a position-dependent mass particle is developed using proper generalization of Euler-Maupertuis' theory and generalized Jacobi's principle. In this paper we generalize this to Liénard type equation and show that these are equivalent to the geodesic equations for the JM metric. We illustrate our construction with some interesting examples.

Main Result. Let $\mathcal{V}$ be a Hamiltonian vector field of the Liénard type equation $\ddot{x}+f(x)\dot{x}^2+g(x)=0$ in $\mathbb{R}^2$ with Hamiltonian $H=\frac{1}{2}M(x)\dot{x}^2+U(x)$, where $M(x)=exp(2\int^x f(s)ds)$ and $U(x)=\int^x M(s)g(s)ds$. Then by Maupertuis principle, $\mathcal{V}$ coincides with the trajectories of the modified vector field $\mathcal{V}'$ on the fixed isoenergy level $H(x,\dot{x})=E$ for the Hamiltonian $\tilde{H}=\frac{1}{2(E-U(x))}M(x)\dot{x}^2$. This defines a geodesic flow of some Riemannian metric given by Jacobi. In other words, solutions to the Liénard type equation with energy E are, after reparametrization, geodesics for the Jacobi-Maupertuis metric.

A corollary of the main result shows that we can reformulate the Newtonian equation of motion for a variable mass, Painlevé-Gambier XXI equation, the Jacobi equation and Henon-Heiles system in terms of geodesic flows of the Jacobi-Maupertuis metric.

The outline of this article is as follows: in section 2 we introduce the Jacobi Last Multiplier and point out its connection to the Lagrangian of a second-order ODE. Thereafter we explicitly derive the Lagrangian and the Hamiltonian functions for a Liénard equation of the second kind, i.e., with a quadratic dependance on the velocity and highlight the role of the position dependant mass term. In section 3 we express the equation in terms of geodesic flows of the Jacobi-Maupertuis metric and some observations regarding the geometric consequences of the PDM are outlined. Explicit examples from the Painlevé-Gambier family of equations are considered along with the two-dimensional Henon-Heiles system.

2. Lagrangians and the Jacobi last multiplier

Let $M=M(x^1,\dots,x^n)$ be a non-negative C^1 function non-identically vanishing on any open subset of $\mathbb{R}^n$, then M is a Jacobi multiplier of the vector field $\mathbb{X}=W^i\frac{\partial}{\partial x^i}$ if

$$\int_D M(x^1,\dots,x^n)dx^1\dots dx^n=\int_{\phi_t(D)} M(x^1,\dots,x^n)dx^1\dots dx^n \tag{2.1}$$

where D is any open subset of $\mathbb{R}^n$ and $\phi_t(\cdot)$ is the flow generated by the solution $\mathbf{x} = \mathbf{x}(t)$ of the system

$$\frac{dx^i}{dt} = W^i(x^1, \dots, x^n) \quad i = 1, \dots, n. \tag{2.2}$$

Thus the Jacobi multiplier can be viewed as the density associated with the invariant measure $\int_D M dx$. The divergence free condition is

$$\frac{dM}{dt} + \frac{\partial W^i}{\partial x^i} M = 0. \tag{2.3}$$

The appellation 'last' is a historical legacy. If a Jacobi multiplier is known together with $(n-2)$ first integrals, we can reduce locally the n dimensional system to a two-dimensional vector field on the intersection of the $(n-2)$ level sets formed by the first integrals. The existence of a Jacobi Last Multiplier [7] then implies the existence of an extra first integral and the system may therefore be reduced to quadrature.

For the second-order ODE

$$\ddot{x} = F(x, \dot{x}, t) \quad \Rightarrow \quad \dot{x} = y, \ \dot{y} = F(x, y, t) \tag{2.4}$$

we have

$$\frac{dM}{dt} + \frac{\partial F}{\partial y} M = 0. \tag{2.5}$$

On the other hand by expanding the Euler-Lagrange equation of motion

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}}\Big) = 0, \tag{2.6}$$

we have

$$\frac{\partial L}{\partial x} = \dot{y}\Big(\frac{\partial^2 L}{\partial \dot{x}^2}\Big) + \dot{x}\frac{\partial}{\partial x}\Big(\frac{\partial L}{\partial \dot{x}}\Big) = \dot{y}\Big(\frac{\partial^2 L}{\partial \dot{x}^2}\Big) + y\frac{\partial}{\partial \dot{x}}\Big(\frac{\partial L}{\partial x}\Big).$$

Differentiating it with respect to, $\dot{x} = y$, gives

$$\frac{\partial}{\partial \dot{x}}\Big(\frac{\partial L}{\partial x}\Big) = \frac{\partial \dot{y}}{\partial y}\Big(\frac{\partial^2 L}{\partial \dot{x}^2}\Big) + \dot{y}\Big(\frac{\partial^3 L}{\partial \dot{x}^3}\Big) + \frac{\partial}{\partial \dot{x}}\Big(\frac{\partial L}{\partial x}\Big) + y\frac{\partial^2}{\partial \dot{x}^2}\Big(\frac{\partial L}{\partial x}\Big),$$

implies

$$\frac{\partial F}{\partial y}\Big(\frac{\partial^2 L}{\partial \dot{x}^2}\Big) + \Big[\dot{y}\frac{\partial}{\partial \dot{x}}\Big(\frac{\partial^2 L}{\partial \dot{x}^2}\Big) + y\frac{\partial}{\partial x}\Big(\frac{\partial^2 L}{\partial \dot{x}^2}\Big)\Big] = 0.$$

Therefore

$$\frac{d}{dt}\Big(\frac{\partial^2 L}{\partial \dot{x}^2}\Big) + \Big(\frac{\partial F}{\partial y}\Big)\Big(\frac{\partial^2 L}{\partial \dot{x}^2}\Big) = 0. \tag{2.7}$$

Thus, by comparing (2.7) to (2.5), we may identify the JLM as follows:

$$M = \frac{\partial^2 L}{\partial \dot{x}^2}. \tag{2.8}$$

Given a JLM we can easily integrate (2.8) twice to obtain

$$L(x, \dot{x}, t) = \int^{\dot{x}} \Big(\int^{y} M dz\Big) dy + R(x,t)\dot{x} + S(x,t). \tag{2.9}$$

where R and S are functions arising from integration. To determine these functions we substitute the Lagrangian of (2.9) into the Euler-Lagrange equation of motion (2.6) and compare the resulting equation with the given ODE (2.4).

Consider now a Liénard equation of the second kind,

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0, \tag{2.10}$$

where f and g are defined in a neighbourhood of $0 \in \mathbb{R}$. We assume that $g(0) = 0$, which says that O is a critical point, and $xg(x) > 0$ in a punctured neighbourhood of $0 \in \mathbb{R}$, which ensures that the origin is a centre.

Proposition 2.1. *A Liénard equation of the second kind, $\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$, admits a Hamiltonian of the form $H = 1/2M(x)\dot{x}^2 + U(x)$ which is a constant of motion where $M(x)$ is the Jacobi last multiplier and $U(x)$ is a potential function.*

Proof. From the definition (2.5) of the last multiplier it follows that for the equation under consideration

$$M(x) = \exp(2F(x)) \quad \text{where } F(x) = \int^x f(s)ds. \tag{2.11}$$

Consequently by (2.9), we have

$$L = \frac{1}{2}M(x)\dot{x}^2 + R(x,t)\dot{x} + S(x,t). \tag{2.12}$$

From the Euler-Lagrange equation one finds that the functions R and S must satisfy

$$S_x - R_t = -M(x)g(x)$$

This gives us the freedom to set $S = G_t - U(x)$ and $R = G_x$ for some gauge function G, so that there exists a potential function $U(x)$ given by

$$U(x) = \int^x M(s)g(s)ds. \tag{2.13}$$

The Lagrangian then has the form

$$L = \frac{1}{2}M(x)\dot{x}^2 - U(x) + \frac{dG}{dt}. \tag{2.14}$$

Clearly the total derivative term may be ignored and by means of the standard Legendre transformation the Hamiltonian is given by

$$H = \frac{1}{2}M(x)\dot{x}^2 + U(x). \tag{2.15}$$

It is now straight forward to verify that $dH/dt = 0$ so that $H = E(\text{say})$ is a constant of motion. This completes the proof. $\square$

From (2.15) it is evident that the JLM, $M(x)$, plays the role of a variable mass term. We can reduce the differential system to a unit mass problem by defining a transformation $x \longrightarrow X = \int_0^x \sqrt{M(s)}ds = \psi(x)$ whence

$$\frac{1}{2}\dot{X}^2 + \int_0^{\psi^{-1}(X)} M(s)g(s)ds = E. \tag{2.16}$$

In terms of X the equation of motion is given by

$$\ddot{X} + e^{F(\psi^{-1}(X))}g(\psi^{-1}(X)) = 0. \tag{2.17}$$

We now proceed to cover some fundamentals regarding the Jacobi metric, and deduce it for the Liénard equation. We mainly follow the Nair et al formalism of Jacobi-Maupertuis principle [12] and elaborate on it in the next section.

3. Jacobi-Maupertuis metric and Liénard type equation

When the Hamiltonian is not explicitly time dependent, i.e., $H = E_0$, a constant, then the solutions may be restricted to the energy surface $E = E_0$. Suppose Q is a manifold with local coordinates $x = \{x^i\}, i = 1, \dots, n$ and $x(\tau) \in Q \subseteq \mathbb{R}^n$ be a curve with $\tau \in [0, T]$. Let T_xQ and T_x^*Q be the tangent and cotangent spaces with velocity $\dot{x}(\tau) \in T_xQ \subseteq \mathbb{R}^n$ and momenta $p(\tau) \in T_x^*Q \subseteq \mathbb{R}^n$. Denote by γ a curve in the manifold Q parametrized by $t \in [a, b]$ with $\gamma(a) = x_0$ and $\gamma(b) = x_N$. The according to the Maupertuis principle among all the curves $x(t)$ connecting the two points x_0 and x_n parametrized such that $H(x, p) = E_0$ the trajectory of the Hamiltons equation of motion is an extremal of the integral of action

$$\int_\gamma p dx = \int_\gamma p\dot{x} dt = \int_\gamma \frac{\partial L(t)}{\partial \dot{x}} \dot{x}(t) dt. \tag{3.1}$$

Here L is assumed to be a regular Lagrangian $L : TQ \to \mathbb{R}$ where $L = K - U$ and the kinetic energy $K : TQ \to \mathbb{R}$.

Proposition 3.1. *Let the Hamiltonian $H = K + U$ be a constant of motion i.e., $H = E$ (say) with the kinetic energy K being a homogeneous quadratic function of the velocities and $U(x)$ is some potential function such that $U(x) < E$: then there exists a Riemannian metric defined by $d\widetilde{s} = \sqrt{2(E - U(x))} ds$ with $K = 1/2(ds/dt)^2$ such that the trajectories are the geodesic equations corresponding to the Jacobi-Maupertuis principle of least action.*

Proof. Let ds^2 be a Riemannian metric on the configuration space with kinetic energy

$$K = \frac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j = \frac{1}{2} \left(\frac{ds}{dt} \right)^2. \tag{3.2}$$

As the total energy is a constant E with potential $U(x) < E$ the Hamiltonian satisfies $H = K + U = E$. Because K is a homogeneous quadratic function hence Euler theorem implies $2K = \dot{x}^i \partial L / \partial \dot{x}^i = (ds/dt)^2$. Therefore from (3.1) we have

$$\begin{aligned} \int_\gamma \frac{\partial L(t)}{\partial \dot{x}} \dot{x}(t) dt &= \int_\gamma 2K dt = \int_\gamma 2K \frac{ds}{\sqrt{2K}} = \int_\gamma \sqrt{2K} ds \\ &= \int_\gamma \sqrt{2(E - U(x))} ds = \int_\gamma d\widetilde{s}, \end{aligned}$$

where the Riemannian metric $\widetilde{s}$ is defined by $d\widetilde{s} = \sqrt{2(E - U(x))} ds$. This shows that it is possible to derive a metric which is given by the kinetic energy itself [4] and the trajectories are geodesics in the metric $d\widetilde{s}$. From (3.2) one finds $ds = \sqrt{q_{ij} dx^i dx^j}$ and the Maupertuis principle involves solving for the stationary points of the action $\int \sqrt{2K} ds$, i.e.,

$$\delta \int \sqrt{2K} ds = 0 \quad \text{or} \quad \delta \int \sqrt{2(E - U(x)) g_{ij} dx^i dx^j} = 0, \tag{3.3}$$

with the integral being over the generalized coordinates $\{x^i\}$ along all paths connecting $\gamma(a)$ and $\gamma(b)$.

It is evident from $d\widetilde{s} = \sqrt{2(E - U(x)) g_{ij} dx^i dx^j}$ that

$$d\widetilde{s}^2 = \widetilde{g}_{ij} dx^i dx^j \qquad \text{where } \widetilde{g}_{ij}(x) = 2(E - U(x)) g_{ij}(x). \tag{3.4}$$

The geodesic equation corresponding to the least action $\delta\int_{s_1}^{s_2} dt\sqrt{\widetilde{g}_{ij}\dot{x}^i\dot{x}^j} = 0$ is given by

$$\frac{d^2x^i}{d\widetilde{s}^2} + \Gamma^i_{jk}\frac{dx^j}{d\widetilde{s}}\frac{dx^j}{d\widetilde{s}} = 0, \quad \text{where } \Gamma^i_{jk} = \frac{1}{2}\widetilde{g}^{il}\Big(\frac{\partial\widetilde{g}_{jl}}{\partial x^k} + \frac{\partial\widetilde{g}_{kl}}{\partial x^j} - \frac{\partial\widetilde{g}_{jk}}{\partial x^l}\Big). \tag{3.5}$$

This complete the proof. □

For an equation of the Liénard type (2.10), from Proposition (2.1) we have

$$K = \frac{1}{2}M(x)\dot{x}^2 \quad \text{where } M(x) = \exp(2F(x))$$

so that $g_{11}(x) = M(x)$ while from the Jacobi-Maupertuis (JM) metric (3.4) we observe that $\widetilde{g}_{11} = 2(E-U(x))M(x)$. The geodesic equation (3.5) therefore reduces to

$$\frac{d^2x}{d\widetilde{s}^2} + \Gamma^1_{11}\Big(\frac{dx}{d\widetilde{s}}\Big)^2 = 0 \quad \text{with } \Gamma^1_{11} = \frac{M'(x)}{2M(x)} - \frac{U'(x)}{2(E-U(x))},$$

or in explicit terms

$$\frac{d^2x}{d\widetilde{s}^2} + \Big(\frac{M'(x)}{2M(x)} - \frac{U'(x)}{2(E-U(x))}\Big)\Big(\frac{dx}{d\widetilde{s}}\Big)^2 = 0. \tag{3.6}$$

Equation (3.6) gives the geodesic for the JM-metric of a Liénard equation of the type (2.10).

Proposition 3.2. *The geodesic equation* (3.6) *and* (2.10) *are equivalent.*

Proof. From $K = E - U(x) = 1/2M(x)\dot{x}^2$ we have

$$\dot{x}^2 = 2(E-U(x))/M(x) \tag{3.7}$$

and as $d\widetilde{s}^2 = \widetilde{g}_{11}dx^2 = 2((E-U(x))M(x)dx^2$, it follows that

$$\frac{d\widetilde{s}}{dt} = 2(E-U(x)) \;\Rightarrow\; \frac{dx}{dt} = 2(E-U(x))\frac{dx}{d\widetilde{s}}. \tag{3.8}$$

This enables us to obtain

$$\begin{aligned} \frac{d^2x}{d\widetilde{s}^2} &= \frac{1}{2(E-U(x))}\frac{d}{dt}\Big\{\frac{1}{2(E-U(x))}\frac{dx}{dt}\Big\} \\ &= \frac{1}{4(E-U(x))^2}\Big[\frac{d^2x}{dt^2} + \frac{U'(x)}{(E-U(x))}\dot{x}^2\Big] \end{aligned} \tag{3.9}$$

Consequently (3.6), taking (3.7) into account, assumes the form

$$\frac{1}{4(E-U(x))^2}\Big[\frac{d^2x}{dt^2} + \frac{U'(x)}{(E-U(x))}\dot{x}^2\Big] = \Big[\frac{U'(x)}{2(E-U(x))} - \frac{M'(x)}{2M(x)}\Big]\frac{1}{4(E-U(x))^2}\dot{x}^2;$$

in other words we have

$$\frac{d^2x}{dt^2} + \frac{M'(x)}{2M(x)}\dot{x}^2 + \frac{U'(x)}{2(E-U(x))}\dot{x}^2 = 0. \tag{3.10}$$

However as $\dot{x}^2 = 2(E-U(x))/M(x)$ the last term of the above equation can be expressed as $U'(x)/M(x)$ and as a result the equation has the appearance

$$\frac{d^2x}{dt^2} + \frac{M'(x)}{2M(x)}\dot{x}^2 + \frac{U'(x)}{M(x)} = 0. \tag{3.11}$$

This equation reduces to (2.10) upon making the identifications $M(x) = \exp(2F(x))$ which implies $M'(x)/2M(x) = f(x)$ and $U(x) = \int^x M(y)g(y)dy$ which implies

$U'(x)/M(x) = g(x)$ where $g(x)$ refers to the forcing term of the Liénard equation (2.10). □

Remark 3.3. Finally it is interesting to note how (2.10) or equivalently (3.11) may be viewed geometrically. To this end we write (3.11) as

$$\frac{d^2x}{dt^2} + \frac{M'(x)}{2M(x)}\dot{x}^2 = -\frac{U'(x)}{M(x)} \tag{3.12}$$

and look upon the right hand side as an external force function. Restricting ourselves to the left hand side we consider a 1+1 dimensional line element of the form $ds^2 = c^2dt^2 - M(x)dx^2 = c^2d\tau^2$ which yields the following geodesic equations for a free particle moving in this spacetime, namely

$$\frac{d^2x}{d\tau^2} + \frac{M'(x)}{2M(x)}\Big(\frac{dx}{d\tau}\Big)^2 = 0, \quad \frac{d^2t}{d\tau^2} = 0.$$

These equations imply upon elimination of the proper time τ the left hand side of (3.12)and the latter may be recast as

$$\frac{d}{dt}\left(M(x)\dot{x}\right) = \frac{M'(x)}{2}\dot{x}^2.$$

Thus from a Newtonian perspective we see that the position dependent mass function $M(x)$ changes the geometry of spacetime in a manner such that the particle experiences an additional geometric force $f_G = M'(x)\dot{x}^2/2$. However unlike the case when the PDM is also a function of time [15] the curvature of spacetime is flat because as a result of the transformation $dX = \sqrt{M(x)}dx$ one has $ds^2 = c^2dt^2 - dX^2$ and the resulting geodesic equation of a free particle in this transformed spacetime is just $\frac{d^2X}{dt^2} = 0$ or

$$\frac{d}{dt}\Big(\sqrt{M(x)}\frac{dx}{dt}\Big) = 0, \quad \text{or} \quad \frac{1}{2}M(x)\dot{x}^2 = const.$$

which implies the conservation of the kinetic energy.

To complete this article we illustrate our results with a few examples.

Example 3.4. We consider the Painléve-Gambier XXI

$$\ddot{x} - \frac{3}{4x}\dot{x}^2 - 3x^2 = 0,$$

for which we have $F(x) = -3/4\int dx/x = -3/4\log|x|$ so that $M(x) = |x|^{-3/2}$; and as $2K = M(x)\dot{x}^2 = g_{11}(x)\dot{x}^2$ we have $g_{11}(x) = M(x) = |x|^{-3/2}$ while $U(x) = \int^x M(z)g(z)dz = \mp 2x^{3/2}$ depending on whether $x > 0$ or $x < 0$. As a result we find $\widetilde{g}_{11} = 2(E \pm 2x^{3/2})|x|^{-3/2}$.

Example 3.5. We consider the Jacobi equation

$$\ddot{x} + \frac{1}{2}\phi_x\dot{x}^2 + \phi_t\dot{x} + B(t,x) = 0,$$

for which we have $M(x,t) = \exp(\phi(x,t)) = g_{11}$ and the Lagrangian

$$L = \frac{1}{2}e^{\phi}\dot{x}^2 - U(x,t), \quad \text{where} \quad U(x,t) = \int^x e^{\phi(y,t)}B(y,t)dy\,.$$

It canb be verified that the Hamiltonian is a constant of motion and $\widetilde{g}_{11} = 2(E - U(x,t))\exp(\phi(x,t))$. The geodesic equation is

$$\frac{d^2x}{d\widetilde{s}^2} + \Gamma^1_{11}\Big(\frac{dx}{d\widetilde{s}}\Big)^2 = 0, \quad \text{with } \Gamma^1_{11} = \frac{\phi_x}{2} - \frac{U_x}{2(E-U(x,t))}.$$

Example 3.6. We consider the Henon-Heiles system

$$\ddot{x} = -(Ax + 2\alpha xy),$$
$$\ddot{y} = -(By + \alpha x^2 - \beta y^2)$$

which has has the Lagrangian

$$L(x,y,\dot{x},\dot{y}) = \frac{1}{2}(\dot{x}^2+\dot{y}^2) - \Big(\frac{A}{2}x^2 + \frac{B}{2}y^2 + \alpha x^2 y - \frac{\beta}{3}y^3\Big).$$

It is therefore easily seen that $M_{xx} = M_{yy} = 1$ and it admits the first integral

$$I = \frac{1}{2}(\dot{x}^2+\dot{y}^2) + \Big(\frac{A}{2}x^2 + \frac{B}{2}y^2 + \alpha x^2 y - \frac{\beta}{3}y^3\Big),$$

which is just the Hamiltonian of the system. Consequently we have $g_{11} = M_{xx} = 1$ and $g_{22} = M_{yy} = 1$ while

$$\widetilde{g}_{11} = 2(E - U(x,y)) = \widetilde{g}_{22},$$

where

$$U(x,y) = \frac{1}{2}(\dot{x}^2+\dot{y}^2) - \Big(\frac{A}{2}x^2 + \frac{B}{2}y^2 + \alpha x^2 y - \frac{\beta}{3}y^3\Big)$$

The geodesic equations have the form

$$\frac{d^2x}{d\widetilde{s}^2} - \frac{1}{2(E-U(x,y))}\Big(U_x\Big(\frac{dx}{d\widetilde{s}}\Big)^2 + 2U_y\Big(\frac{dx}{d\widetilde{s}}\Big)\Big(\frac{dy}{d\widetilde{s}}\Big) + U_x\Big(\frac{dy}{d\widetilde{s}}\Big)^2\Big) = 0,$$
$$\frac{d^2y}{d\widetilde{s}^2} - \frac{1}{2(E-U(x,y))}\Big(U_y\Big(\frac{dx}{d\widetilde{s}}\Big)^2 + 2U_x\Big(\frac{dx}{d\widetilde{s}}\Big)\Big(\frac{dy}{d\widetilde{s}}\Big) + U_y\Big(\frac{dy}{d\widetilde{s}}\Big)^2\Big) = 0$$

Conclusion. In this article, we studied the so called Liénard type equations, such equations naturally appear in physical system such as position dependent mass particles, and show these are equivalent to the geodesic equations for the Jacobi-Maupertuis (JM) metric. We have illustrated our construction with some explicit examples including The Painlevé-Gambier XXI, The Jacobi and the Henon-Heiles system of equations.

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