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RENORMALIZED SOLUTIONS FOR NONLINEAR PARABOLIC EQUATIONS WITH GENERAL MEASURE DATA

MOHAMMED ABDELLAOUI, ELHOUSSINE AZROUL

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Abstract. We prove the existence of parabolic initial boundary value problems of the type

$$\begin{gathered} u_t - \operatorname{div}(a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)) = \mu_\epsilon \quad \text{in } Q := (0,T)\times\Omega, \\ u_\epsilon = 0 \quad \text{on } (0,T)\times\partial\Omega, \quad u_\epsilon(0) = u_{0,\epsilon} \quad \text{in } \Omega, \end{gathered}$$

with respect to suitable convergence of the nonlinear operators a_ϵ and of the measure data μ_ϵ. As a consequence, we obtain the existence of a renormalized solution for a general class of nonlinear parabolic equations with right-hand side measure.

1. Introduction

In this article we consider the parabolic problem

$$\begin{gathered} u_t - \operatorname{div}(a(t,x,u,\nabla u)) = \mu \quad \text{in } Q := (0,T)\times\Omega, \\ u = 0 \quad \text{on } (0,T)\times\partial\Omega, \\ u(0) = u_0 \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, $T > 0$ and Q is the cylinder $(0,T)\times\Omega$, $(0,T)\times\partial\Omega$ being its lateral surface, the operator of Leray-Lions $u \mapsto -\operatorname{div}(a(t,x,u,\nabla u))$ is pseudo-monotone defined on the space $L^p(0,T;W_0^{1,p}(\Omega))$ with values in its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$, $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. We assume that $u_0 \in L^2(\Omega)$ and the data μ is a Radon measure with bounded variation on Q.

Under some assumptions on a, If $\mu \in L^{p'}(Q)$ the existence and unicity of a weak solution u of (1.1) belonging to suitable energy space and to $C([0,T;L^2(\Omega)])$ was proved in [18]. In the case of linear operators the difficulty can be overcome by defining the solution through the adjoint operator, this method is used in [27] and yields a formulation having a unique solution. For nonlinear operators, the authors in [4] and [21] extends the results in two different directions, assuming that $\mu \in L^1(Q)$ and $u_0 \in L^1(\Omega)$, they prove existence of a renormalized solution, and of entropy solution, the same notions of solutions are used to ensure existence and uniqueness of equations with bounded Radon measures on Q that does not charge

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the sets of zero parabolic p-capacity (See [4, 15, 24]), the authors show in [14] that these two notions of solution actually coincide.

Here we use the notion of renormalized solution, introduced in [12, 20, 23]. Roughly speaking, a renormalized solution to (1.1) is a measurable function with all the truncations in the space $L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(0,T;L^1(\Omega))$ and such that for every $S\in W^{2,\infty}(\mathbb{R})(S(0)=0)$ with S' has compact support on $\mathbb{R}$, we have

$$\begin{aligned} &-\int_\Omega S(u_0)\varphi(0)\,dx-\int_0^T\langle\varphi_t,S(u-g)\rangle\,dt\\ &+\int_Q S'(u-g)a(t,x,u,\nabla u)\cdot\nabla\varphi\,dx\,dt\\ &+\int_Q S''(u-g)a(t,x,u,\nabla u)\cdot\nabla(u-g)\varphi\,dx\,dt\\ &=\int_Q S'(u-g)\varphi\,d\tilde{\mu}_0, \end{aligned}\tag{1.2}$$

for every function $\varphi\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$, $\varphi_t\in L^{p'}(0,T;W^{-1,p'}(\Omega))$, with $\varphi(T,x)=0$, such that $S'(u-g)\varphi\in L^p(0,T;W_0^{1,p}(\Omega))$, g_t is the time derivative part of μ_0 and $\hat{\mu}_0=\mu-g_t-\mu_s=f-\operatorname{div}(G)$. Moreover, for every $\psi\in C(\overline{Q})$ we have

$$\begin{gathered} \lim_{n\to+\infty}\frac{1}{n}\int_{\{n\le v<2n\}}a(t,x,u,\nabla u)\cdot\nabla v\psi\,dx\,dt=\int_Q\psi d\mu_s^+,\\ \lim_{n\to+\infty}\frac{1}{n}\int_{\{-2n<v\le -n\}}a(t,x,u,\nabla u)\cdot\nabla v\psi\,dx\,dt=\int_Q\psi d\mu_s^-, \end{gathered}$$

where μ_s^+ and μ_s^- are respectively the positive and the negative part of the singular part of the measure μ w.r.t. the p-capacity.

In the proof of [23, Theorem 2], they used the fact that the approximating sequences μ_ϵ having a splitting converging to μ, the estimate concerning u_ϵ and $u_\epsilon-g_\epsilon^t$, next they prove the strong convergence of $T_k(u_\epsilon-g_\epsilon)$ in $L^p(0,T;W_0^{1,p}(\Omega))$. To obtain this result, they use the same technique as in [12] adapted to the parabolic case.

In the present paper we generalize this existence result to renormalized solutions of problems depending on u and ∇u

$$\begin{gathered} (u_\epsilon)_t-\operatorname{div}(a(t,x,u_\epsilon,\nabla u_\epsilon))=\mu_\epsilon\quad\text{in } Q:=(0,T)\times\Omega,\\ u_\epsilon=0\quad\text{on } (0,T)\times\partial\Omega,\\ u_\epsilon(0)=u_0\quad\text{in } \Omega, \end{gathered}\tag{1.3}$$

where (μ_ϵ) is a sequences of measures with splitting converging to μ, and

$$\lim_{\epsilon\to 0}a_\epsilon(t,x,s_\epsilon,\zeta_\epsilon)=a_0(t,x,s,\zeta),$$

for every sequence $(s_\epsilon,\zeta_\epsilon)\in\mathbb{R}\times\mathbb{R}^N$ converging to (s,ζ) and for a.e. $(t,x)\in Q$.

The main point which allows to go further the previous works, is the proof of the almost everywhere convergence of gradients in Proposition 5.2 using the technique developed in [24, 25]. To underline the importance of this tool, we have chosen to plan the paper in the following way: in Sect. 2, we recall some basic notations and we investigate the link between measures in Q and the notion of parabolic capacity, this notion can be obtained from the result of the "elliptic capacity" contained in

[8], which can be slightly adapted to this context of parabolic spaces, and we show the decomposition method for more general measures with bounded total variation in order to find a sense of solution to Cauchy-Dirichlet problems.

In Sect. 3, we introduce and study a special type of approximating sequences of measures obtained via convolution arguments. In Sect. 4 we show the interest of cut-off functions and intermediary lemmas. In the last two sections, we establish the fundamental a priori estimates and we use the proof of strong convergence of truncates to obtain our main result.

2. Preliminaries

2.1. **Assumptions on the operator.** Throughout this paper Ω will be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, p and p' will be real numbers, with $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. In what follows, $|\zeta|$ and $\zeta \cdot \zeta'$ will denote respectively the Euclidean norm of a vector $\zeta \in \mathbb{R}^N$ and the scalar product between ζ and $\zeta' \in \mathbb{R}^N$.

Fixed three positive constants c_0, c_1, c_2, and a non-negative function $b_0 = b(t,x) \in L^{p'}(Q)$, we say that a function $a : (0,T)\times\Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ satisfies the assumptions $H(c_0, c_1, c_2, b_0)$ if a is a Carathéodory function (that is, $a(\cdot,\cdot,s,\zeta)$ is measurable on Q for every (s,ζ) in $\mathbb{R}\times\mathbb{R}^N$, and $a(t,x,\cdot)$ is continuous on $\mathbb{R}\times\mathbb{R}^N$ for almost every (t,x) in Q) such that, for every $s \in \mathbb{R}$, $\zeta, \zeta' \in \mathbb{R}^N$ with $\zeta \neq \zeta'$, satisfying the following properties.

$$a(t,x,s,\zeta)\cdot\zeta \geq c_0|\zeta|^p, \tag{2.1}$$

$$|a(t,x,s,\zeta)| \leq b_0(t,x) + c_1|s|^{p-1} + c_2|\zeta|^{p-1}, \tag{2.2}$$

$$(a(t,s,s,\zeta) - a(t,x,s,\zeta'))\cdot(\zeta - \zeta') > 0. \tag{2.3}$$

Notice that, as a consequence of (2.1) and of the continuity of a with respect to ζ, we have that $a(t,x,s,0) = 0$ for a.e. (t,x) in Q and for every $s \in \mathbb{R}$. Thanks to assumptions $H(c_0,c_1,c_2,b_0)$, the map $u \mapsto -\operatorname{div}(a(t,x,u,\nabla u))$ is a coercive, continuous, bounded and monotone operator defined on $L^p(0,T;W_0^{1,p}(\Omega))$ with values into its dual space $L^{p'}(0,T;W^{-1,p'}(\Omega))$; hence by the standard theory of monotone operators (see, e.g.,[18]), for every F in $L^{p'}(Q)$ and $u_0 \in L^2(\Omega)$ there exists a variational solution u of the problem

$$\begin{gathered} u_t - \operatorname{div}(a(t,x,v,\nabla v)) = F \quad \text{in } Q := (0,T)\times\Omega, \\ v = 0 \quad \text{on } (0,T)\times\partial\Omega, \\ v(0) = u_0 \quad \text{in } \Omega, \end{gathered}$$

in the sense that v belongs to $W\cap C(0,T;L^2(\Omega))$ (where $W = \{u \in L^p(0,T;V), u_t \in L^{p'}(0,T;V')\}$ with $V = W_0^{1,p}(\Omega)\cap L^2(\Omega)$), and

$$\begin{aligned} &-\int_\Omega u_0\varphi(0)\,dx - \int_0^T \langle\varphi_t, v\rangle\,dt + \int_Q a(t,x,v,\nabla v)\cdot\nabla\varphi\,dx\,dt \\ &= \int_0^T \langle F,\varphi\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}dt, \end{aligned} \tag{2.4}$$

for all $\varphi \in W$ such that $\varphi(T) = 0$. (Here and in the following $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$).

2.2. **Capacity and measures.** For every set $B \subseteq Q$, its p-capacity $\text{cap}_p(B,Q)$ with respect to Q is defined by

$$\inf\{\|u\|_W\}$$

where the infimum is taken over all the functions $u \in W$ such that $u \geq 1$ almost everywhere in a neighborhood of B.

We say that a property $\mathcal{P}(t,x)$ holds cap_p-quasi everywhere if $\mathcal{P}(t,x)$ holds for every (x,t) outside a subset of Q of zero p-capacity. A function u defined on Q is said to be cap_p-quasi continuous if for every $\epsilon > 0$ there exists $B \subseteq Q$ with $\text{cap}_p(B,Q) < \epsilon$ such that the restriction of u to $Q\backslash B$ is continuous. It is well known that every function in W has a unique, up to sets of p-capacity zero, cap_p-quasi continuous representative, whose values are defined cap_p-quasi everywhere in Q. In what follows we always identify a function $u \in W$ with its cap_p-quasi continuous representative.

We define $\mathcal{M}_b(Q)$ as the space of all Radon measures on Q with bounded total variation, and $C_b(Q)$ as the space of all bounded, continuous functions on Q, so that $\int_Q \varphi d\mu$ is defined for $\varphi \in C_b(Q)$ and μ in $\mathcal{M}_b(Q)$. The positive part, the negative part, and the total variation of a measure μ in $\mathcal{M}_b(Q)$ are denoted by μ^+, μ^-, and $|\mu|$, respectively.

We recall that for a measure μ in $\mathcal{M}_b(Q)$, and a Borel set $E \subseteq Q$, the measure $\mu \perp E$ is defined by $(\mu \perp E)(Q) = \mu(E \cap B)$ for any Borel set $B \subseteq Q$.

In the sequel we suppose that p satisfies $p > 2 - \frac{1}{N+1}$. Then the embedding $W_0^{1,p}(\Omega) \subset L^2(\Omega)$ is valid, i.e.,

$$X = L^p((0,T); W_0^{1,p}(\Omega)), \quad X' = L^{p'}((0,T); W^{-1,p'}(\Omega)).$$

We say that a sequence (μ_n) of measures in $\mathcal{M}_b(Q)$ converges in the narrow topology to a measure μ in $\mathcal{M}_b(Q)$ if

$$\lim_{n\to+\infty} \int_Q \varphi d\mu_n = \int_Q \varphi d\mu \tag{2.5}$$

for every $\varphi \in C(\overline{Q})$. If (2.5) holds only for all the continuous functions φ with compact support in Q, then we have the usual weak_* convergence in $\mathcal{M}_b(Q)$.

We define $\mathcal{M}_0(Q)$ as the set of all measures μ in $\mathcal{M}_b(Q)$ which satisfy $\mu(B) = 0$ for every Borel set $B \subseteq Q$ such that $\text{cap}_p(B,Q) = 0$, while $\mathcal{M}_s(Q)$ will be the set of all measures μ in $\mathcal{M}_b(Q)$ for which there exists a Borel set $B \subset Q$, with $\text{cap}_p(B,Q) = 0$, such that $\mu = \mu \perp E$. For every $\mu \in \mathcal{M}_b(Q)$ there exist a unique pair (μ_0, μ_s) such that $\mu = \mu_0 + \mu_s$, $\mu_0 \in \mathcal{M}_0(Q)$, $\mu_s \in \mathcal{M}_s(Q)$ (see [17, Lemma 2.1]). In addition, a measure μ belongs to $\mathcal{M}_0(Q)$ if and only if μ belongs to $L^1(Q) + L^{p'}(0,T; W^{-1,p'}(\Omega)) + L^p(0,T;V)$ (see [15, Theorem 1.1]). Hence a measure $\mu \in \mathcal{M}_b(Q)$ can be decomposed (not in a unique way) as

$$\mu = f + F + g_t + \mu_s^+ - \mu_s^- \tag{2.6}$$

with $f \in L^1(Q)$, $F \in L^{p'}(0,T; W^{-1,p'}(\Omega))$, $g_t \in L^p(0,T;V)$ and $\mu_s \perp p$-capacity.

2.3. **Definition of renormalized solution.** For any $k > 0$, we define the truncation function $T_k : \mathbb{R} \to \mathbb{R}$ by

$$T_k(t) = \max(-k, \min(k,t)), \quad t \in \mathbb{R}$$

Let us consider the space of all measurable functions, finite a.e. in Q such that $T_k(u)$ belongs to $L^p(0,T; W_0^{1,p}(\Omega))$ for every $k > 0$.

We can see that every function u in this space has a cap_p-quasi continuous representative, that will always be identified with u. Moreover, there exists a measurable function $v : Q \to \mathbb{R}^N$, which is unique up to almost everywhere equivalence, such that $\nabla T_k(u) = v\chi_{\{|u|<k\}}$ a.e. in Q, for every $k > 0$, (see [7, Lemma 2.1]). Hence it is possible to define a generalized gradient ∇u of u, setting $\nabla u = v$. If $u \in L^1(0, T; W_0^{1,1}(\Omega))$, this gradient coincide with the usual gradient in distributional sense.

Let $T_k(t)$ be the Lipschitz continuous function $T_k : \mathbb{R} \to \mathbb{R}$, so that we can define the auxiliary functions

$$\Theta_n(s) = T_1(s - T_n(s)), \quad h_n(s) = 1 - (\Theta_n(s)), \quad S_n(s) = \int_0^s h_n(r)dr, \ \forall s \in \mathbb{R}.$$

We are now in a position to introduce (following [23]) the notion of renormalized solution. To simplify the notation, let us define $v = u - g$, where u is the solution and g is the time-derivative part of μ_0, and $\hat{\mu}_0 = \mu - g_t - \mu_s = f - \mathrm{div}(G)$.

Definition 2.1. Let $u_0 \in L^1(\Omega)$, $\mu \in \mathcal{M}_b(Q)$. A measurable function u is a renormalized solution of problem (1.1) if there exists a decomposition (f, G, g) of μ_0 such that

$$\begin{gathered} v = u - g \in L^q(0, T; W_0^{1,q}(\Omega)) \cap L^\infty(0, T; L^1(\Omega)) \quad \forall q < p - \frac{N}{N+1}, \\ T_k(v) \in X \quad \forall k > 0, \end{gathered} \tag{2.7}$$

and, for every $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on $\mathbb{R}$, and $S(0) = 0$,

$$\begin{aligned} &- \int_\Omega S(u_0)\varphi(0)\,dx - \int_0^T \langle \varphi_t, S(v)\rangle\,dt + \int_Q S'(v)a(t, x, u, \nabla u) \cdot \nabla\varphi\,dx\,dt \\ &+ \int_Q S''(v)a(t, x, u, \nabla u) \cdot \nabla v\varphi\,dx\,dt = \int_Q S'(v)\varphi\,d\tilde{\mu}_0, \end{aligned} \tag{2.8}$$

for any $\varphi \in X \cap L^\infty(Q)$ such that $\varphi_t \in X' + L^1(Q)$ and $\varphi(\cdot, T) = 0$; for any $\psi \in C(\overline{Q})$

$$\begin{aligned} \lim_{n\to+\infty} \frac{1}{n} \int_{\{n\le v<2n\}} a(t, x, u, \nabla u) \cdot \nabla v\psi\,dx\,dt &= \int_Q \psi d\mu_s^+, \\ \lim_{n\to+\infty} \frac{1}{n} \int_{\{-2n<v\le -n\}} a(t, x, u, \nabla u) \cdot \nabla v\psi\,dx\,dt &= \int_Q \psi d\mu_s^-, \end{aligned} \tag{2.9}$$

Remark 2.2. Notice that, if u is a renormalized solution of (1.1), then

$$\begin{aligned} &(S(u - g))_t - \mathrm{div}(a(t, x, u, \nabla u)S'(u - g)) + S''(u - g)a(t, x, u, \nabla u) \cdot \nabla(u - g) \\ &= S'(u - g)f + S''(u - g)G \cdot \nabla(u - g) - \mathrm{div}(GS'(u - g)) \end{aligned} \tag{2.10}$$

is satisfied in the sense of distributions. Hence we can put as test functions not only functions in $C_0^\infty(Q)$ but also in $L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$.

3. Statement of results

In what follows the variable ϵ will belong to a sequence of positive numbers converging to zero. Let $a_\epsilon : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a sequence of functions satisfying the hypothesis $H(c_0, c_1, c_2, b_0)$. Assume that there exists a function $a_0 : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying the hypothesis $H(c_0, c_1, c_2, b_0)$, and such that

$$\lim_{\epsilon\to 0} a_\epsilon(t, x, s_\epsilon, \zeta_\epsilon) = a_0(t, x, s, \zeta), \tag{3.1}$$

for every sequence $(s_\epsilon, \zeta_\epsilon) \in \mathbb{R} \times \mathbb{R}^N$ which converges to (s, ζ) and for almost $(t, x) \in Q$. Fixed $\mu \in \mathcal{M}_b(Q)$, we consider a special type of approximating sequence μ_ϵ, defined as follows.

Definition 3.1. Let $\mu \in \mathcal{M}_b(Q)$ be decomposed as $\mu = f + F + g_t + \mu_s^+ - \mu_s^-$, with $f \in L^1(Q)$, and $F = -\operatorname{div}(G)$, $G \in (L^{p'}(Q))^N$, $g_t \in L^{p'}(0, T; W^{-1,p'}(\Omega))$. Let (μ_ϵ) be a sequence of measures in $\mathcal{M}_b(Q)$, we say that (μ_ϵ) has a splitting $(f_\epsilon, F_\epsilon, g^t_\epsilon, \lambda^\oplus_\epsilon, \lambda^\ominus_\epsilon)$ converging to μ. If for every ϵ the measure μ_ϵ can be decomposed as

$$\mu_\epsilon = f_\epsilon + F_\epsilon + g^t_\epsilon + \lambda^\oplus_\epsilon - \lambda^\ominus_\epsilon, \tag{3.2}$$

and the following holds

(i) (f_ϵ) is a sequence of $C_c^\infty(Q)$ functions converging to f weakly in $L^1(Q)$;
(ii) (G_ϵ) is a sequence of functions in $(C_c^\infty(Q))^N$ that converges to g strongly in $(L^{p'}(Q))^N$;
(iii) (g^t_ϵ) is a sequence of functions in $(C_c^\infty(Q))^N$ that converges to g_t in $L^p(0, T; V)$;
(iv) $(\lambda^\oplus_\epsilon)$ is a sequence of non-negative measures in $\mathcal{M}_b(Q)$ such that $\lambda^\oplus_\epsilon = \lambda^{1,\oplus}_{\epsilon,0} - \operatorname{div}(\lambda^{2,\oplus}_{\epsilon,0}) + \lambda^\oplus_{\epsilon,s}$ with $(\lambda^{1,\oplus}_{\epsilon,0} \in L^1(Q)$, $\lambda^{2,\oplus}_{\epsilon,0} \in (L^{p'}(Q))^N$ and $\lambda^\oplus_{\epsilon,s} \in \mathcal{M}^+_s(Q))$ that converges to μ_s^+ in the narrow topology of measures;
(v) $(\lambda^\ominus_\epsilon)$ is a sequence of non-negative measures in $\mathcal{M}_b(Q)$ such that $\lambda^\ominus_\epsilon = \lambda^{1,\ominus}_{\epsilon,0} - \operatorname{div}(\lambda^{2,\ominus}_{\epsilon,0}) + \lambda^\ominus_{\epsilon,s}$ with $(\lambda^{1,\ominus}_{\epsilon,0} \in L^1(Q)$, $\lambda^{2,\ominus}_{\epsilon,0} \in (L^{p'}(Q))^N$ and $\lambda^\ominus_{\epsilon,s} \in \mathcal{M}^+_s(Q))$ that converges to μ_s^- in the narrow topology of measures.

Moreover, let $u_0^\epsilon \in C_0^\infty(\Omega)$ that approaches u_0 in $L^1(\Omega)$, notice that this approximation can be easily obtained via a standard convolution arguments and we can also assume

$$\|\mu_\epsilon\|_{L^1(Q)} \le C|\mu|; \quad \|u_{0,\epsilon}\|_{L^1(\Omega)} \le C\|u_0\|_{L^1(\Omega)}.$$

Remark 3.2. Let us introduce the following function that we will often use in the following

$$H_n(r) = \chi_{[-n,n]}(r) + \frac{2n - |s|}{n}\chi_{\{n<|s|\le 2n\}}(r), \quad \overline{H}_n(r) = \int_0^r H_n(\tau)d\tau,$$

and another auxiliary function introduced in terms of $H_n(s)$

$$B_n(s) = 1 - H_n(s).$$

Proposition 3.3. *Let $v = u - g$ be a renormalized solution of problem* (1.1)*. Then, for every, $k > 0$, we have*

$$\int_Q |\nabla T_k(v)|^p dx\, dt \le C(k + 1),$$

where C is a positive constant not depending on k.

For a proof of the above proposition see [23, Proposition 2].

Remark 3.4. If we decompose the measures, μ_ϵ, $\lambda^\oplus_\epsilon$, $\lambda^\ominus_\epsilon$ respectively as $\mu_\epsilon = \mu_{\epsilon,0} + \mu_{\epsilon,s}$, $\lambda^\oplus_\epsilon = \lambda^\oplus_{\epsilon,0} + \lambda^\oplus_{\epsilon,s}$ $(\lambda^\oplus_{\epsilon,0} = \lambda^{1,\oplus}_{\epsilon,0} - \operatorname{div}(\lambda^{2,\oplus}_{\epsilon,0}))$, $\lambda^\ominus_\epsilon = \lambda^\ominus_{\epsilon,0} + \lambda^\ominus_{\epsilon,s}$ $(\lambda^\ominus_{\epsilon,0} = \lambda^{1,\ominus}_{\epsilon,0} - \operatorname{div}(\lambda^{2,\ominus}_{\epsilon,0}))$, with $\mu_{\epsilon,0}, \lambda^\oplus_{\epsilon,0}, \lambda^\ominus_{\epsilon,0}$ in $\mathcal{M}_0(Q)$, and $\mu_{\epsilon,s}$, $\lambda^\oplus_{\epsilon,s}$, $\lambda^\ominus_{\epsilon,s}$ in $\mathcal{M}_s(Q)$, then clearly $\lambda^{\oplus}_{\epsilon,0}$, $\lambda^{\ominus}_{\epsilon,0}$, $\lambda^{\oplus}_{\epsilon,s}$, $\lambda^{\ominus}_{\epsilon,s}$ are non-negative, $\mu_{\epsilon,0} = f_\epsilon + F_\epsilon + g_\epsilon + \lambda^{\oplus}_{\epsilon,0} - \lambda^{\ominus}_{\epsilon,0}$ and $\mu_{\epsilon,s} = \lambda^{\oplus}_{\epsilon,s} - \lambda^{\ominus}_{\epsilon,s}$. In particular we have

$$0 \le \mu^+_{\epsilon,s} \le \lambda^{\oplus}_{\epsilon,s}, \quad 0 \le \mu^-_{\epsilon,s} \le \lambda^{\ominus}_{\epsilon,s}. \tag{3.3}$$

We are interested in the asymptotic behaviour of a sequence of renormalized solutions (u_ϵ) to the problem

$$\begin{gathered} (u_\epsilon)_t - \operatorname{div}(a(t,x,u_\epsilon,\nabla u_\epsilon)) = \mu_\epsilon \quad \text{in } Q := (0,T)\times\Omega, \\ u_\epsilon = 0 \quad \text{on } (0,T)\times\partial\Omega, \\ u_\epsilon(0) = u_0 \quad \text{in } \Omega, \end{gathered} \tag{3.4}$$

in the sense of Definition 2.1. Our main result reads as follows.

Theorem 3.5. *Let $(a_\epsilon), a_0$ be functions satisfying $H(c_0,c_1,c_2,b_0)$ and (3.1). Let $\mu \in \mathcal{M}_b(Q)$ be decomposed as $f + F + g_t + \mu^+_s - \mu^-_s$, and let (μ_ϵ) a sequence of measures in $\mathcal{M}_b(Q)$ which have a splitting $(f_\epsilon, F_\epsilon, g_\epsilon, \lambda^{\oplus}_\epsilon, \lambda^{\ominus}_\epsilon)$ converging to μ. Assume that u_ϵ is a renormalized solution of (3.4). Then there exists a subsequence, still denoted by (u_ϵ), and a renormalized solution u to the problem*

$$\begin{gathered} u_t - \operatorname{div}(a_0(t,x,u,\nabla u)) = \mu \quad \text{in } Q := (0,T)\times\Omega, \\ u = 0 \quad \text{on } (0,T)\times\partial\Omega, \\ u(0) = u_0 \quad \text{in } \Omega, \end{gathered} \tag{3.5}$$

such that (u_ϵ) converges to u a.e. in Q, and $(v_\epsilon) = (u_\epsilon - g_\epsilon)$ converges to $v = u - g$ a.e. in Q.

Remark 3.6. The convergence of u_ϵ to u is not merely pointwise. The kind of converges obtained are listed in Proposition 5.2, where the existence of the limit function u is obtained.

Remark 3.7. Let z_ν be a sequence of functions such that

$$\begin{gathered} z_\nu \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega), \quad \|z_\nu\|_{L^\infty(\Omega)} \le k, \\ z_\nu \to T_k(u_0) \text{ a.e. in } \Omega \text{ as } \nu \text{ tends to infinity}, \\ \frac{1}{\nu}\|z_\nu\|^p_{W^{1,p}_0(\Omega)} \to 0 \text{ as } \nu \text{ tends to infinity}. \end{gathered}$$

Then, for fixed $k > 0$, and $\nu > 0$, we denote by $(T_k(v))_\nu$ (Landes-time regularization of the truncate function $T_k(v)$ introduced in [19] and used in several articles (see [2, 5, 13]) the unique solution of the problem

$$\begin{gathered} \frac{dT_k(v)_\nu}{dt} = \nu(T_k(v) - T_k(v)_\nu) \quad \text{in the sense of distributions}, \\ T_k(v)_\nu = z_\nu \quad \text{in } \Omega, \end{gathered}$$

therefore, $T_k(v)_\nu \in L^p(0,T;W^{1,p}_0(\Omega)\cap L^\infty(Q))$ and $\frac{dT_k(v)}{dt} \in L^p(0,T;W^{1,p}_0(\Omega))$, and it can be proved that, up to a subsequences, as ν diverges

$$\begin{gathered} T_k(v)_\nu \to T_k(v) \quad \text{strongly in } L^p(0,T;W^{1,p}_0(\Omega)) \text{ and a.e. in } Q, \\ \|T_k(v)_\nu\|_{L^\infty(Q)} \le k \quad \forall \nu > 0 \end{gathered}$$

Then choosing this approximation in parabolic case with fact that (μ_ϵ) approximates μ in the sense of Definition 3.1. Hence we obtain, as consequence of the strong convergence of truncates the existence of renormalized solution of (3.5) obtained as stated in the following theorem.

Theorem 3.8. *Let a_0 be a function satisfying $H(c_0, c_1, c_2, b_0)$ and $u_0 \in L^1(\Omega)$, $\mu \in \mathcal{M}_b(Q)$. Then there exists a renormalized solution u to problem*

$$u_t - \operatorname{div}(a_0(t, x, u, \nabla u)) = \mu \quad \text{in } Q := (0, T) \times \Omega,$$
$$u = 0 \quad \text{on } (0, T) \times \partial\Omega,$$
$$u(0) = u_0 \quad \text{in } \Omega.$$

4. Some remarks on measures

We recall that a sequence (μ_ϵ) of non-negative measures converges to μ in the narrow topology if and only if $(\mu_\epsilon(Q))$ converges to $\mu(Q)$ and (2.5) holds for every $\varphi \in C_c^\infty(Q)$. In particular a sequence (μ_ϵ) of non-negative measures converges to μ in the narrow topology if and only if (2.5) holds for every $\varphi \in C_c(\overline{Q})$. The following lemma states a consequence result of the Dunford-pettis theorem.

Lemma 4.1. *Let (ρ_ϵ) be a sequence in $L^1(Q)$ converging to ρ weakly in $L^1(Q)$ and (σ_ϵ) a bounded sequence in $L^\infty(Q)$ converging to σ a.e. in Q. Then*

$$\lim_{\epsilon \to 0} \int_Q \rho_\epsilon \sigma_\epsilon dx\, dt = \int_Q \rho\sigma\, dx\, dt$$

Next we need to localize some integrals near the support of $\mu_s \in \mathcal{M}_s(Q)$ (singular measure with respect to p-capacity). This will be done in terms of the following cut-off functions (see [23, Lemma 5]).

Lemma 4.2. *Let μ_s be a measure in $\mathcal{M}_s(Q)$, and let μ_s^+, μ_s^- be respectively the positive and the negative part of μ_s. Then for every $\delta > 0$, there exists two functions $\psi_\delta^+, \psi_\delta^-$ in $C_0^1(Q)$, such that the following hold*

(i) $0 \le \psi_\delta^+ \le 1$ *and* $0 \le \psi_\delta^- \le 1$ *on* Q;
(ii) $\lim_{\delta\to0} \psi_\delta^+ = \lim_{\delta\to0} \psi_\delta^- = 0$ *strongly in* $L^p(0, T; W_0^{1,p}(\Omega))$ *and weakly-* in* $L^\infty(Q)$;
(iii) $\lim_{\delta\to0}(\psi_\delta^+)_t = \lim_{\delta\to0}(\psi_\delta^-)_t = 0$ *strongly in* $L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q)$;
(iv) $\int_Q \psi_\delta^- d\mu_s^+ \le \delta$ *and* $\int_Q \psi_\delta^+ d\mu_s^- \le \delta$;
(v) $\int_Q (1 - \psi_\delta^+ \psi_\eta^+) d\mu_s^+ \le \delta + \eta$ *and* $\int_Q (1 - \psi_\delta^- \psi_\eta^-) d\mu_s^- \le \delta + \eta$ *for all* $\eta > 0$.

Lemma 4.3. *Let μ_s be a measure in $\mathcal{M}_s(\Omega)$, decomposed as $\mu_s = \mu_s^+ - \mu_s^-$, with μ_s^+ and μ_s^- concentrated on two disjoint subsets E^+ and E^- of zero p-capacity. Then, for every $\delta > 0$, there exists two compact sets $K_\delta^+ \subseteq E^+$ and $K_\delta^- \subseteq E^-$ such that*

$$\mu_s^+(E^+ \backslash K_\delta^+) \le \delta, \quad \mu_s^-(E^- \backslash K_\delta^-) \le \delta, \tag{4.1}$$

and there exists ψ_δ^+, $\psi_\delta^- \in C_0^1(Q)$, such that

$$\psi_\delta^+, \psi_\delta^- \equiv 1 \quad \text{respectively on } K_\delta^+, K_\delta^-, \tag{4.2}$$

$$0 \le \psi_\delta^+, \psi_\delta^- \le 1, \tag{4.3}$$

$$\operatorname{supp}(\psi_\delta^+) \cap \operatorname{supp}(\psi_\delta^-) \equiv \emptyset. \tag{4.4}$$

Moreover

$$\|\psi_\delta^+\|_S \le \delta, \quad \|\psi_\delta^-\|_S \le \delta, \tag{4.5}$$

and, in particular, there exists a decomposition of $(\psi_\delta^+)_t$ and a decomposition of $(\psi_\delta^-)_t$ such that

$$\|(\psi_\delta^+)_t^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \le \frac{\delta}{3}, \quad \|(\psi_\delta^+)_t^2\|_{L^1(Q)} \le \frac{\delta}{3}, \tag{4.6}$$

$$\|(\psi_\delta^-)_t^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \le \frac{\delta}{3}, \quad \|(\psi_\delta^-)_t^2\|_{L^1(Q)} \le \frac{\delta}{3}, \tag{4.7}$$

*and both ψ_δ^+ and ψ_δ^- converges to zero *_weakly in $L^\infty(Q)$, in $L^1(Q)$, and up to subsequences, almost everywhere as δ vanishes.*

Moreover, if $\lambda_\epsilon^\oplus$ and $\lambda_\epsilon^\ominus$ are as in (3.2) *we have*

$$\int_Q \psi_\delta^- d\lambda_\epsilon^\oplus = \omega(\epsilon,\delta), \quad \int_Q \psi_\delta^- d\mu_s^+ \le \delta, \tag{4.8}$$

$$\int_Q \psi_\delta^+ d\lambda_\epsilon^\ominus = \omega(\epsilon,\delta), \quad \int_Q \psi_\delta^+ d\mu_s^- \le \delta, \tag{4.9}$$

$$\int_Q (1-\psi_\delta^+\psi_\eta^+) d\lambda_\epsilon^\oplus = \omega(\epsilon,\delta,\eta), \quad \int_Q (1-\psi_\delta^+\psi_\eta^+) d\mu_s^+ \le \delta+\eta, \tag{4.10}$$

$$\int_Q (1-\psi_\delta^-\psi_\eta^-) d\lambda_\epsilon^\ominus = \omega(\epsilon,\delta,\eta), \quad \int_Q (1-\psi_\delta^-\psi_\eta^-) d\mu_s^- \le \delta+\eta. \tag{4.11}$$

For a proof of the above lemma see [23, Lemma 5].

Remark 4.4. If $\lambda_\epsilon^\oplus$ and $\lambda_\epsilon^\ominus$ satisfy (iii) and (iv) of Definition 3.1, respectively, and ψ_δ^- and ψ_δ^+ are the functions defined in Lemma 4.2, as an easy consequence of the narrow convergence we obtain

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_Q \psi_\delta^- d\lambda_\epsilon^\oplus = 0, \quad \lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_Q \psi_\delta^+ d\lambda_\epsilon^\ominus = 0, \tag{4.12}$$

$$\lim_{\eta\to 0}\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_Q (1-\psi_\delta^+\psi_\eta^+) d\lambda_\epsilon^\oplus = 0, \quad \lim_{\eta\to 0}\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_Q (1-\psi_\delta^-\psi_\eta^-) d\lambda_\epsilon^\ominus = 0. \tag{4.13}$$

5. Existence of a limit function

The following lemma is the main tool in order to establish the fundamental a priori estimates for the sequence (u_ϵ).

Lemma 5.1. *Let u, v as defined before, and assume that there exists $C > 0$ such that*

$$\begin{gathered} \|u\|_{L^\infty(0,T;L^1(\Omega))} \le C; \quad \|v\|_{L^\infty(0,T;L^1(\Omega))} \le C, \\ \int_Q |\nabla T_k(u)|^p dx\,dt \le Ck; \quad \int_Q |\nabla T_k(v)|^p dx\,dt \le C(k+1), \end{gathered} \tag{5.1}$$

for every $k > 0$. Then there exists $C = C(N,M,p) > 0$ such that

(i) $\operatorname{meas}\{|u| \ge k\} \le Ck^{-(p-1+\frac{p}{n})}$, $\operatorname{meas}\{|v| \ge k\} \le Ck^{-(p-1+\frac{p}{n})}$,
(ii) $\operatorname{meas}\{|\nabla u| \ge k\} \le Ck^{-(p-\frac{N}{N+1})}$, $\operatorname{meas}\{|\nabla v| \ge k\} \le Ck^{-(p-\frac{N}{N+1})}$.

Proof. (i) We can improve this kind of estimate by using a suitable Gagliardo-Niremberg typeinequality (see [11, Proposition 3.1]) which asserts that is $w \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, with $q \ge 1$, $\sigma \ge 1$. Then $w \in L^\sigma(Q)$ with $\sigma = q\frac{N+\rho}{N}$ and

$$\int_Q |w|^\sigma dx\,dt \le C\|w\|_{L^\infty(0,T;L^\rho(\Omega))}^{\frac{\rho q}{N}} \int_Q |\nabla w|^q dx\,dt.$$

Indeed, in this way we obtain

$$\int_Q |T_k(u)|^{p+\frac{p}{N}} dx\,dt \le Ck,$$

and so, we can write

$$K^{p+\frac{p}{N}} meas\{|u| \geq k\} \leq \int_{\{|u|\geq k\}} |T_k(u)|^{p+\frac{p}{N}}\,dx\,dt \leq \int_Q |T_k(u)|^{p+\frac{p}{N}}\,dx\,dt \leq Ck,$$

Then,

$$\operatorname{meas}\{|u| \geq k\} \leq \frac{C}{k^{p-1+\frac{p}{N}}}.$$

(ii) We are interested about a similar estimate on the gradients of functions u; let us emphasize that these estimates hold true. First of all, observe that

$$\operatorname{meas}\{|\nabla u| \neq \lambda\} \leq \operatorname{meas}\{|\nabla u| \neq \lambda; |u| \leq k\} + \operatorname{meas}\{|\nabla u| \neq \lambda; |u| > k\}$$

with regard to the first term in the right hand side, we have

$$\begin{aligned} \operatorname{meas}\{|\nabla u| \neq \lambda; |u| \leq k\} &\leq \frac{1}{\lambda^p} \int_{\{|\nabla u|\geq\lambda; |u|\leq k\}} |\nabla u|^p dx \\ \frac{1}{\lambda^p} \int_{\{|u|\leq k\}} |\nabla u|^p dx &= \frac{1}{\lambda^p} \int_Q |\nabla T_k(u)|^p dx \leq \frac{Ck}{\lambda^p}; \end{aligned} \tag{5.2}$$

while for the last term, thanks to (i), we can write

$$\operatorname{meas}\{|\nabla u| \geq \lambda; |u| > k\} \leq \operatorname{meas}\{|u| \geq k\} \leq \frac{\overline{C}}{K^\sigma},$$

with $\sigma = p - 1 + \frac{p}{N}$. So, finally, we obtain

$$\operatorname{meas}\{|\nabla u| \geq \lambda\} \leq \frac{\overline{C}}{k^\sigma} + \frac{Ck}{\lambda^p},$$

and we obtain a better estimate by taking the minimum over k of the right-hand side; the minimum is achieved for the value

$$k_0 = \big(\frac{\sigma C}{\overline{C}}\big)^{\frac{1}{\sigma+1}} \lambda^{\frac{p}{\sigma+1}}$$

and so we obtain the desired estimate

$$\operatorname{meas}\{|\nabla u| \geq \lambda\} \leq C\lambda^{-\gamma}$$

with $\gamma = p(\frac{\sigma}{\sigma+1}) = \frac{Np+p-N}{N+1} = p - \frac{N}{N+1}$. Then, we found that u (resp v) is uniformally bounded in the Marcinkiewicz space $\mathcal{M}^{p-1+\frac{p}{N}}(Q)$ and ∇u (resp ∇v) is equibounded in $\mathcal{M}^\gamma(Q)$, with $\gamma = p - \frac{N}{N+1}$. $\square$

From now we always assume that (a_ϵ), a_0 are functions satisfying $H(c_0, c_1, c_2, b_0)$ and (3.1), that $\mu \in \mathcal{M}_b(Q)$ is decomposed as $f + F + g_t + \mu_s$, $f \in L^1(Q)$, $F \in L^{p'}(0, T; W^{-1,p'}(\Omega))$, $g_t \in L^p(0, T; V)$, $\mu_s \in \mathcal{M}_s(Q)$, and that (μ_s) is a sequence of measure in $\mathcal{M}_b(Q)$, which have a splitting $(f_\epsilon, F_\epsilon, g_\epsilon, \lambda_\epsilon^\oplus, \lambda_\epsilon^\ominus)$ converging to μ. We shall denotes by u_ϵ a renormalized solution of (3.4) with μ_ϵ as datum. Hence it satisfies:

$$\begin{aligned} &\int_0^T \langle (v_\epsilon)_t, \varphi\rangle\,dt + \int_Q a(t, x, u_\epsilon, \nabla u_\epsilon) \cdot \nabla\varphi\,dx\,dt \\ &= \int_Q f_\epsilon \varphi\,dx\,dt + \int_0^T \langle F_\epsilon, \varphi\rangle\,dx\,dt + \int_Q \varphi\,d(\lambda_\epsilon^\oplus - \lambda_\epsilon^\ominus), \end{aligned} \tag{5.3}$$

for all $\varphi \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, $\varphi_t \in L^{p'}(0, T; W^{-1,p'}(\Omega))$, with $\varphi(T, 0) = 0$.

As a first step, we find a function $u \in L^\infty(0,T;L^1(\Omega))$ such that $T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega))$ which is the limit, up to a subsequence, of (u_ϵ) in suitable topologies.

Proposition 5.2. *Let $\mu_\epsilon \in \mathcal{M}_b(Q)$, $(u_{0,\epsilon}) \in L^1(\Omega)$, with $sup_\epsilon|\mu_\epsilon(Q)| < \infty$ and $\|u_{0,\epsilon}\|_{1,\Omega} < \infty$. Let (u_ϵ) be a sequence of renormalized solutions of (3.4), and let $v_\epsilon = u_\epsilon - g_\epsilon$. Then there exists $C > 0$ such that*

$$\begin{gathered}
\|u_\epsilon\|_{L^\infty(0,T;L^1(\Omega))} \le C, \quad \int_Q |\nabla T_k(u_\epsilon)|^p dx\,dt \le Ck, \\
\|v_\epsilon\|_{L^\infty(0,T;L^1(\Omega))} \le C, \quad \int_Q |\nabla T_k(v_\epsilon)|^p dx\,dt \le C(k+1),
\end{gathered} \tag{5.4}$$

for every ϵ and for every $k > 0$. Moreover there exists a subsequence, still denoted by u_ϵ (resp v_ϵ) and a measurable function u (resp v) such that the following convergence hold.

(i) *u_ϵ (resp (v_ϵ)) converges to u (resp v) a.e. in Q;*
(ii) *u (resp v) belongs to $L^\infty(0,T;L^1(\Omega))$ and for every $k > 0$, the sequence $(T_k(u_\epsilon))$ (resp $T_k(v_\epsilon)$) converges to $T_k(u)$ (resp $T_k(v)$) $\in L^p(0,T;W_0^{1,p}(\Omega))$ in the weak topology of $L^p(0,T;W_0^{1,p}(\Omega))$;*
(iii) *∇u_ϵ (resp (∇v_ϵ)) converges to ∇u (resp ∇v) a.e. in Q;*
(iv) *$a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)$ converges to $a_0(t,x,u,\nabla u)$ in the strong topology of the space $L^q(0,T;W_0^{1,q}(\Omega))$ for every $q < p - \frac{N}{N+1}$, while $a_\epsilon(t,x,u,\nabla T_k(u_\epsilon))$ converges to $a_0(t,x,u,\nabla T_k(u))$ in the weak topology of $(L^{p'}(Q))^N$ for every $k > 0$.*

Proof. **Step 1.** a priori estimates. Let us choose $T_k(u_\epsilon)$ as test function in (5.3) and we integrate in $]0,t[$ to obtain

$$\begin{aligned}
&\int_\Omega \Theta_k(u_\epsilon(t))\,dx + \int_0^t \int_\Omega a(t,x,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon)\,dx\,dt \\
&= \int_0^t \int_\Omega T_k(u_\epsilon) d\mu_\epsilon + \int_\Omega \Theta_k(u_{0,\epsilon})\,dx
\end{aligned} \tag{5.5}$$

using (3.1) and the fact that $\|u_{0,\epsilon}\|_{L^1(\Omega)}$ and $\|\mu_\epsilon\|_{L^1(Q)}$ are bounded:

$$\int_\Omega \Theta_k(u_\epsilon)(t)\,dx + \int_0^t \int_\Omega |\nabla T_k(u_\epsilon)|^p dx\,dt \le Ck$$

Since $\Theta_k(s) \ge 0$ and $|\Theta_1(s)| \ge |s| - 1$, we obtain

$$\int_\Omega |u_\epsilon(t)|\,dx + \int_0^t \int_\Omega |\nabla T_k(u_\epsilon)|^p dx\,dt \le C(k+1), \quad \forall k > 0, \forall t \in [0,T].$$

Taking the supremum on $(0,T)$. As a consequence we obtain the estimate of u_ϵ in $L^\infty(0,T;L^1(\Omega))$

$$\|u_\epsilon\|_{L^\infty(0,T;L^1(\Omega))} \le C,$$

We repeat here the same argument to get the estimate on v_ϵ: let us choose $T_k(v_\epsilon)$ as test function in (5.3). by integration by parts (recall that g_ϵ has compact support in Q, so that $(v_\epsilon(0) = u_\epsilon(0) = u_{0,\epsilon})$) and using (3.1)

$$\int_\Omega \Theta(v_\epsilon)(t)\,dx + \alpha \int_0^t \int_\Omega |\nabla u_\epsilon|^p \chi_{\{|v_\epsilon \le k|\}}\,dx\,ds$$

$$
\begin{aligned}
&\le \int_\Omega \Theta_k(u_{0,\epsilon})\,dx + \int_Q f_\epsilon T_k(v_\epsilon)\,dx\,dt + \int_0^t\int_\Omega G_\epsilon\cdot\nabla u_\epsilon \chi_{\{|v_\epsilon\le k|\}}dxds\\
&\quad - \int_0^t\int_\Omega G_\epsilon\cdot\nabla g_\epsilon\chi_{\{|v_\epsilon\le k|\}}dxds + \int_0^t\int_\Omega a(s,x,u_\epsilon,\nabla u_\epsilon)\nabla g_\epsilon\chi_{\{|v_\epsilon|\le k\}}dsds\\
&\quad + \int_Q T_k(v_\epsilon)d\lambda_\epsilon^\oplus - \int_Q T_k(v_\epsilon)d\lambda_\epsilon^\ominus.
\end{aligned}
$$

thanks to (3.2) and young's inequality,

$$
\begin{aligned}
&\int_\Omega \Theta(v_\epsilon)(t)\,dx + \frac{\alpha}{2}\int_0^t\int_\Omega |\nabla u_\epsilon|^p\chi_{\{|v_\epsilon\le k|\}}\,dx\,ds\\
&\le \int_Q |f_\epsilon|dx\,dt + C\int_Q |G_\epsilon|^{p'}dx\,dt + C\int_Q|\nabla g_\epsilon|^p dx\,dt\\
&\quad + C\int_Q |b(t,x)|^{p'}dx\,dt + k\int_\Omega |u_{0,\epsilon}|dx + k\int_Q d\lambda_\epsilon^\oplus + k\int_Q d\lambda_\epsilon^\ominus.
\end{aligned}
$$

Using that G_ϵ is bounded in $L^{p'}(Q)$, g_ϵ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, f_ϵ, $\lambda_\epsilon^\oplus$ and $\lambda_\epsilon^\ominus$ are bounded in $L^1(Q)$ and $u_{0,\epsilon}$ is bounded in $L^1(\Omega)$, we have

$$
\int_\Omega \Theta_1(v_\epsilon)\,dx \le C \quad \forall t\in[0,T],
$$

In this way the same estimate of u_ϵ follows for v_ϵ in $L^\infty(0,T;L^1(\Omega))$:

$$
\begin{gathered}
\|v_\epsilon\|_{L^\infty(0,T;L^1(\Omega))}\le C,\\
\int_Q |\nabla u_\epsilon|^p\chi_{\{|v_\epsilon|\le k\}}dx\,dt \le C(k+1),
\end{gathered}
$$

which yields that $T_k(v_\epsilon)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ for any $k>0$ (recall that g_ϵ itself is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$). Then

$$
\int_Q |\nabla T_k(v_\epsilon)|^p dx\,dt \le C(k+1).
$$

Step 2. Up to a subsequence, u_ϵ is a Cauchy sequence in measure. We are going to prove now that, up to subsequences, u_ϵ converges almost everywhere in Q towards a measurable function u. Lemma 5.1 gives the usual estimates for parabolic equation with measure data, that is to say u_ϵ is bounded in $L^q(0,T;W_0^{1,q}(\Omega))$ for every $q<p-\frac{N}{N+1}$ and in $L^\infty(0,T;L^1(\Omega))$, for which we can deduce that

$$
\lim_{k\to+\infty}\operatorname{meas}\{(x,t)\in Q: |u_\epsilon|>k\}=0 \quad \text{uniformly with respect to } u.
$$

From (5.4) we have that $T_k(u_\epsilon)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ for every $k>0$. Now, if we multiply the approximating equation by $\mathcal{T}_k'(v_\epsilon)$, where $\mathcal{T}_k(s)$ is a $C^2(\mathbb{R})$, nondecreasing function such that $\mathcal{T}_k(s)=s$ for $|s|\le\frac{k}{2}$ and $\mathcal{T}_k(s)=k$ for $|s|>k$, we obtain

$$
\begin{aligned}
&(\mathcal{T}_k(v_\epsilon))_t - \operatorname{div}(a(t,x,u_\epsilon,\nabla u_\epsilon)\mathcal{T}_k'(v_\epsilon)) + a(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla v_\epsilon\mathcal{T}_k''(v_\epsilon)\\
&= \mathcal{T}_k'(v_\epsilon)f_\epsilon + \mathcal{T}_k''(v_\epsilon)G_\epsilon\cdot\nabla v_\epsilon - \operatorname{div}(G_\epsilon\mathcal{T}_k'(v_\epsilon)) + (\lambda_\epsilon^\oplus-\lambda_\epsilon^\ominus)\mathcal{T}_k'(v_\epsilon).
\end{aligned}
$$

in the sense of distributions. This implies, thanks to the last equality and to the fact that $\mathcal{T}_k'$ has compact support, that $\mathcal{T}_k(v_\epsilon)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ while its time derivative $(\mathcal{T}_k(v_\epsilon))_t$ is bounded in $L^p(0,T;W^{-1,p'}(\Omega))+L^1(Q)$, hence a classical compactness result (see [26]) allows us to conclude that $\mathcal{T}_k(v_\epsilon)$ is compact in $L^2(Q)$. Thus for a subsequence, it also converges in measure, and almost everywhere in Q. Since we have, for $\sigma > 0$,

$$\begin{aligned}
&\operatorname{meas}\{(x,t) : |v_n - v_m| > \sigma\} \\
&\leq \operatorname{meas}\{(x,t) : |v_n| > \frac{k}{2}\} + \operatorname{meas}\{(x,t) : |v_n| > \frac{k}{2}\} \\
&\quad + \operatorname{meas}\{(x,t) : |\mathcal{T}_k(v_n) - \mathcal{T}_k(v_m)| > \sigma, \}
\end{aligned}$$

by (5.4) for every fixed $\epsilon > 0$ we can choose $\overline{k}$ large enough to have

$$\operatorname{meas}\{(x,t) : |v_n - v_m| > \sigma\} \leq \operatorname{meas}\{(x,t) : |\mathcal{T}_k(v_n) - \mathcal{T}_{\overline{k}}(v_m)| > \sigma\} + \epsilon, \tag{5.6}$$

for all $n, m \in \mathbb{N}$. The fact that $\mathcal{T}_k(v_\epsilon)$ converges in measure for every $k > 0$ implies, using (2.8), that, up to subsequences, v_ϵ also converges in measure and almost everywhere in Q. In particular, we have found out that there exists a measurable function v in $L^\infty(0,T;L^1(\Omega)) \cap L^q(0,T;W_0^{1,q}(\Omega))$ for every $q < p - \frac{N}{N+1}$ such that $T_k(v)$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))$ for every $k > 0$, and for a subsequences, not relabeled,

$$T_k(v_\epsilon) \to T_k(v) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \text{ strongly in } L^p(Q) \text{ and a.e. in } Q.$$

We deduce that

$$v_\epsilon \to v \text{ a.e. in } Q,$$

and since g_ϵ strongly converges to g in $L^p(0,T;W_0^{1,p}(\Omega))$, there exists a measurable function u such that

$$u_\epsilon \to u \text{ a.e. in } Q,$$

The estimate (5.4) also imply that $u \in L^\infty(0,T;L^1(\Omega))$. Indeed, using Fatou's Lemma on the first term of the left-hand of

$$\int_\Omega |u_\epsilon(t)|\,dx + \int_0^t \int_\Omega |\nabla T_k(u_\epsilon)|^p dx\,dt \leq C(k+1), \quad \forall k > 0, \forall t \in [0,T].$$

where

$$T_k(u_\epsilon) \rightharpoonup T_k(u) \quad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega))$$

and in addition

$$\int_Q |\nabla T_k(u)|^p dx\,dt \leq Ck, \quad \int_Q |\nabla T_k(v)|^p dx\,dt \leq C(k+1), \tag{5.7}$$

that is property (ii) holds.

Step 3. ∇u_ϵ is a Cauchy sequence in measure. Let us show that ∇u_ϵ is a Cauchy sequence in measure, which will yields $\nabla u_\epsilon \to \nabla u$ almost everywhere, for a convenient subsequence. Given $\delta > 0$ for every $\eta > 0$ and $k > 0$ one has

$$\begin{aligned}
&\{(t,x), |\nabla u_n - \nabla u_m| \geq \delta\} \\
&\subseteq \{(t,x), |u_n| > k\} \cup \{(t,x), |u_m| > k\} \\
&\quad \cup \{(t,x), |\nabla u_n| > k\} \cup \{(t,x), |\nabla u_m| > k\} \cup \{(t,x), |u_n - u_m| > \eta\} \\
&\quad \cup \big\{(t,x), |\nabla u_n - \nabla u_m| \geq \delta, |u_n \leq k|, |\nabla u_n| \leq k, \\
&\qquad |u_n| \leq k, |\nabla u_m| \leq k, |u_n - u_m| \leq \eta\big\}.
\end{aligned} \tag{5.8}$$

We will denote A_1 to A_6 the six sets of the right hand side. One could remark, in the sequel of the proof, that only the upper bound of the measure of A_6 uses the equation of which u_n and u_m are solutions. The other bounds use the boundedness of (u_n) and (∇u_n).

Let us bound $\operatorname{meas}(A_1)$ and $\operatorname{meas}(A_2)$, we have

$$k \operatorname{meas}(A_1) \le \int_{A_1} |\nabla u_n| dx\, dt \le \int_0^T \int_\Omega |\nabla u_n|\, dx\, dt$$

hence

$$\operatorname{meas}(A_1) \le \frac{1}{k} \int_0^T \int_\Omega |\nabla u_n| dx\, dt \le \frac{C}{k} \le \varepsilon,$$

for k large enough, because (∇u_n) is bounded in $L^q((0,T)\times\Omega)$ for $q < p - \frac{N}{N+1}$ and hence in $L^1((0,T)\times\Omega)$. Let us fix k such that

$$\operatorname{meas}(A_1) \le \varepsilon \quad \text{and} \quad \operatorname{meas}(A_2) \le \varepsilon \quad \text{for all } n, m \in \mathbb{N},$$

Now let us bound $\operatorname{meas}(A_3)$, we have (u_n) is a Cauchy sequence in $L^1((0,T)\times\Omega)$ hence for a given n, there exist n_0 such that for $n, m \ge n_0$ one has

$$\operatorname{meas}(A_3) \le \varepsilon$$

it is now sufficient to bound $\operatorname{meas}(A_4)$, and to choose η. Thanks to the monotonicity of A, we have $[a(t,x,s,\zeta_1) - a(t,x,s,\zeta_2)](\zeta_1 - \zeta_2) > 0$ for $\zeta_1 - \zeta_2 \ne 0$. Since the set of (ζ_1, ζ_2) such that: $\{(t,x), |s| \le k, |\zeta_1| \le k, |\zeta_2| \le k \text{ and } |\zeta_1 - \zeta_2| \ge \delta\}$ is compact and a is continuous with respect to ζ for almost all t and x, $[a(t,x,s,\zeta_1) - (a(t,x,s,\zeta_2)](\zeta_1 - \zeta_2)$ reaches on this compact its minimum that we will denotes $\gamma(t,x)$, and that verifies $\gamma(t,x) > 0$ a.e. Since $\gamma(t,x) > 0$ a.e., there exists $\epsilon' > 0$ such that, for all measurable set $A \subset (0,T)\times\Omega$,

$$\int_A \gamma \le \varepsilon' \Longrightarrow \operatorname{meas}(A) \le \varepsilon$$

hence, to obtain $\operatorname{meas}(A_4) \le \varepsilon$, it is sufficient to show that

$$\int_{A_4} \gamma \le \varepsilon' \tag{5.9}$$

By definition of γ and A_4, we have

$$\int_{A_4} \gamma \le \int_{A_4} (a(t,x,u_n,\nabla u_m) - a(t,x,u_m,\nabla u_m))(\nabla u_n - \nabla u_m)\chi_{\{|u_n - u_m| \le \eta\}}.$$

Moreover the term to be integrated is non negative and $\nabla T_\eta(u_n - u_m) = (\nabla u_n - \nabla u_m)\chi_{\{|u_n - u_m| \le \eta\}}$, hence we have

$$\int_{A_4} \gamma \le \int_0^T (a(t,x,u_n,\nabla u_n) - a(t,x,u_m,\nabla u_m)) \cdot \nabla T_\eta(u_n - u_m),$$

if one chooses $\varphi = T_\eta(u_n - u_m) \in L^p(0,T;W^{1,p}(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$, which satisfies $T_\eta(u_n - u_m)_t \in L^{p'}(]0,T[;W^{-1,p'}(\Omega))$, in equation in the sense of distributions written successively with u_n and u_m one gets

$$\begin{aligned}
&\int_0^T \langle (u_n - u_m)_t, T_\eta(u_n - u_m)\rangle \\
&+ \int_0^T \int_\Omega (a(t,x,u_n,\nabla u_n) - a(t,x,u_m,\nabla u_m))\nabla T_\eta(u_n - u_m) \\
&= \int_0^T \int_\Omega (\mu_n - \mu_m) T_\eta(u_n - u_m).
\end{aligned}$$

that is (using Θ_η the primitive of T_η)

$$
\begin{aligned}
&\int_\Omega \Theta_\eta(u_n-u_m)(T)-\int_\Omega \Theta_\eta(u_n-u_m)(0)\\
&+\int_0^T\int_\Omega (a(t,x,u_n,\nabla u_n)-a(t,x,u_m,\nabla u_m))\nabla T_\eta(u_n-u_m)\\
&=\int_0^T\int_\Omega (\mu_n-\mu_m)T_\eta(u_n-u_m)
\end{aligned}
$$

Since the first term is non-negative ($\Theta_\eta(x)\geq 0$), and $\Theta_\eta(x)\leq \eta|x|$ one has

$$
\begin{aligned}
&\int_0^T\int_\Omega (a(t,x,u_n,\nabla u_n)-a(t,x,u_m,\nabla u_m))\cdot\nabla T_\eta(u_n-u_m)\\
&\leq \eta\int_0^T\int_\Omega |\mu_n-\mu_m|+\eta\int_\Omega |u_0^n-u_0^m|\\
&\leq 2\eta(\|\mu(Q)\|+\|u_0\|_{1,\Omega})\,.
\end{aligned}
$$

Then for η small enough, one has $\int_{A_4}\gamma\leq\varepsilon'$ and thus $\operatorname{meas}(A_4)\leq\varepsilon$ and therefore for all $n,m\geq n_0$ we have

$$
\operatorname{meas}(\{|(\nabla u_n-\nabla u_m)(x)|\geq\delta\})\leq 4\varepsilon,
$$

thus, we obtain that ∇u_ϵ is a Cauchy sequence in measure. Passing to a subsequence, we assume that

$$
\nabla u_\epsilon\to\nabla u \quad \text{almost everywhere in } Q.
$$

Similarly, we obtain the convergence a.e of v_ϵ, this gives

$$
\nabla v_\epsilon\to\nabla v \text{ almost everywhere in } Q.
$$

that is property (iii) holds.

It remains to prove (iv). By (5.5), Lemma 5.1, and (2.2), $a(t,x,u_\epsilon,\nabla u_\epsilon)$ is bounded in $L^q(0,T;W_0^{1,q}(\Omega))$ for every $q<p-\frac{N}{N+1}$. Moreover, by (3.1), (i) and (iii), $a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)$ converges to $a_0(t,x,u,\nabla u)$ a.e. in Q. Hence by Vitali's Theorem, we have that $a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)$ converges to $a_0(t,x,u,\nabla u)$ in the strong topology of $L^q(0,T;W_0^{1,q}(\Omega))$, $1\leq q<p-\frac{N}{N+1}$. Finally, by (ii) and (2.2), the sequence $(a_\epsilon(t,x,u_\epsilon,\nabla T_k(u_\epsilon))$ is bounded in $L^{p'}(Q)$, which easily implies that it converges to $a_0(t,x,u,\nabla T_k(u))$ in the weak topology of $L^{p'}(Q)$. □

6. Proof of Theorem 3.5

At this point we have a subsequence (u_ϵ) of renormalized solutions to (3.4) and a measurable function u with $T_k(u)\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(0,T;L^1(\Omega))$ such that all the convergences stated in Proposition 5.2 hold. We have to prove that the function u is a renormalized solution to (3.5). By Proposition 5.2 (ii) condition (a) of Definition 2.1 is satisfied, while by (5.7) and Lemma 5.1, we obtain that u satisfies condition (2.7) of Definition 2.1. Hence, it is enough to prove (2.8). Let $S\in W^{2,\infty}(\mathbb{R})$, and let $\varphi\in C_0^1([0,T]\times\Omega)$. We choose $S'(v_\epsilon)\varphi$ as test function in the equation solved by u_ϵ, obtaining

$$\begin{aligned} &-\int_\Omega S(u_{0,\epsilon})\varphi(0)\,dx - \int_0^T \langle \varphi_t, S(v_\epsilon)\rangle + \int_Q S'(v_\epsilon)a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla\varphi\,dx\,dt \\ &+ \int_Q S''(v_\epsilon)a_\epsilon(t,x,u_\epsilon,\nabla v_\epsilon)\cdot\nabla v_\epsilon\varphi\,dx\,dt \\ &= \int_Q S'(v_\epsilon)\varphi d\hat{\mu}_\epsilon + \int_Q S'(v_\epsilon)\varphi d\lambda_\epsilon^{\oplus} - \int_Q S'(v_\epsilon)\varphi d\lambda_\epsilon^{\ominus}. \end{aligned} \tag{6.1}$$

As $supp(S') \subset [-M,M]$, we have

$$\int_Q a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla v_\epsilon S''(v_\epsilon)\varphi\,dx\,dt = \int_Q a_\epsilon(x,t,u_\epsilon,\nabla T_M(v_\epsilon)\varphi)\,dx\,dt$$

To pass to the limit in this term, we need the following improvement of Proposition 5.2 (ii).

Proposition 6.1. *Let $(a_\epsilon), a_0$ be functions satisfying $H(c_0,c_1,c_2,b_0)$ and (3.1). Let $\mu \in \mathcal{M}_b(Q)$ be fixed, and $\mu = f + F + g_t + \mu_s$, $f \in L^1(Q)$, $F \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, $\mu_s \in \mathcal{M}_s(Q)$. Assume that (μ_ϵ) is a sequence of measures in $\mathcal{M}_b(Q)$ having a splitting $(f_\epsilon, F_\epsilon, g_{t,\epsilon}, \lambda_\epsilon^{\oplus}, \lambda_\epsilon^{\ominus})$ which converges to μ. Let (u_ϵ) a sequence of renormalized solutions of (3.4), and let u be its limit in the sense of Proposition 5.2. Then for every $k > 0$ the sequence $(T_k(u_\epsilon))$ converges strongly in $L^p(0,T;W_0^{1,p}(\Omega))$ to $T_k(u)$ as ϵ goes to 0.*

Proof. It is sufficient to follow the lines of the long and not easy proof of the same result, for a fixed operator independent of u, for the elliptic case in [12, sections 5–8], for the parabolic case in [23, section 7]. The assumptions on a_ϵ allow to obtain some estimates for varying operators explicitly depending on u.

For any $\delta, \eta > 0$, let $\psi_\delta^+, \psi_\eta^+, \psi_\delta^-$ and ψ_η^- as in Lemma 4.3 and let E^+ and E^- be the sets where, respectively, μ_s^+, μ_s^- are concentrated; setting

$$\Phi_{\delta,\eta} = \psi_\delta^+\psi_\eta^+ + \psi_\delta^-\psi_\eta^-.$$

Suppose that, the estimate near E,

$$I_1 = \int_{\{|v_\epsilon|\le k\}} \Phi_{\delta,\eta} a(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(v_\epsilon - T_k(v)_\nu) \le \omega(\epsilon,\nu,\delta,\eta), \tag{6.2}$$

and far from E,

$$I_2 = \int_{\{|v_\epsilon|\le k\}} (1-\Phi_{\delta,\eta}) a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(v_\epsilon - T_k(v)_\nu) \le \omega(\epsilon,\nu,\delta,\eta). \tag{6.3}$$

Putting these statements together we obtain

$$\limsup_{\nu\to 0,\epsilon\to 0} \int_{\{|v_\epsilon|\le k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(v_\epsilon - T_k(v)_\nu) \le 0, \tag{6.4}$$

so that using the convergence of $(T_k(v)_\nu)$ to $T_k(v)$ in X we deduce

$$\limsup_{\epsilon\to 0} \int_{\{|v_\epsilon|\le k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(v_\epsilon - T_k(v)) \le 0, \tag{6.5}$$

since by the weak convergence of $T_k(v_\epsilon)$ to $T_k(v)$ in X, Proposition 5.2 implies that

$$\int_{\{|v_\epsilon|\le k\}} a(t,x,u,\nabla(T_k(v)+g_\epsilon))\cdot\nabla(T_k(v_\epsilon) - T_k(v)) = \omega(\epsilon). \tag{6.6}$$

then we obtain

$$\int_{\{|v_\epsilon|\le k\}} (a(t,x,u_\epsilon,\nabla u_\epsilon) - a(t,x,u,\nabla(T_k(v)+g_\epsilon)))\cdot\nabla(u_\epsilon-(T_k(v)+g_\epsilon)) = \omega(\epsilon).$$

we also have, using the convergence of ∇u_ϵ to ∇u a.e. in Q

$$(a(t,x,u_\epsilon,\nabla u_\epsilon)) \rightharpoonup a(t,x,u,\nabla u) \quad \text{in } (L^{p'}(Q))^N, \tag{6.7}$$

then we obtain

$$\limsup_{\epsilon\to 0}\int_Q a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla T_k(v_\epsilon) \le \int_Q a(t,x,u,\nabla u)\cdot\nabla T_k(v).$$

so that by Proposition 5.2, since $(a(t,x,u_\epsilon,\nabla(T_k(v_\epsilon+g_\epsilon))$ converges weakly in $(L^{p'}(Q))^N$ to some F_k, it follows that $F_k = a(t,x,u,\nabla(T_k(u)+g))$. We get

$$\begin{aligned}
&\limsup_{\epsilon\to 0}\int_Q a(t,x,u_\epsilon,\nabla(T_k(v_\epsilon)+g_\epsilon))\cdot\nabla(T_k(v_\epsilon)+g_\epsilon)\\
&\le \limsup_{\epsilon\to 0}\int_Q a(t,x,u_\epsilon,\nabla v_\epsilon)\cdot\nabla T_k(v_\epsilon) + \limsup_{\epsilon\to 0}\int_Q a(t,x,\nabla(T_k(v_\epsilon)+g_\epsilon))\cdot\nabla g_\epsilon\\
&\le \int_Q a(t,x,u,\nabla(T_k(v)+h))\cdot\nabla(T_k(v)+g).
\end{aligned}$$

We finally deduce

$$(T_k(v_\epsilon)) \quad \text{converges to } T_k(v) \text{ strongly in } X \text{ for all } k>0. \tag{6.8}$$

□

The next Lemma is devoted to establish the preliminary essential estimate.

Lemma 6.2. *Near E we have the estimate*

$$I_1 = \int_{\{|v_\epsilon|\le k\}} \Phi_{\delta,\eta} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(v_\epsilon - T_k(v)_\nu) \le \omega(\epsilon,\nu,\delta,\eta).$$

Proof. We have

$$I_1 = \int_Q \Phi_{\delta,\eta} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla T_k(v_\epsilon) - \int_{\{|v_\epsilon|\le k\}} \Phi_{\delta,\eta} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(T_k(v))_\nu.$$

so that, from Proposition 5.2 (iv) and since $a(t,x,u_\epsilon,\nabla T_k(v_\epsilon)+g_\epsilon)\nabla T_k(v)_\nu$ converges weakly in $L^1(Q)$ to $F_k\nabla(T_k(v))_\nu$, $\chi_{\{|v_\epsilon|\le k\}}$ converges to $\chi_{\{|v|\le k\}}$ a.e in Q, $\Phi_{\delta,\eta}$ converges to 0 a.e. in Q as $\delta\to 0$ and $\Phi_{\delta,\eta}$ takes its values in $[0,1]$, using Lemma 4.1, we have the first integral

$$\begin{aligned}
&\int_{\{|v_\epsilon|\le k\}} \Phi_{\delta,\eta} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(T_k(v))_\nu\\
&= \int_Q \chi_{\{|v_\epsilon|\le k\}}\Phi_{\delta,\eta} a(t,x,u_\epsilon,\nabla(T_k(v_\epsilon)+g_\epsilon))\cdot\nabla(T_k(v))_\nu\\
&= \int_Q \chi_{\{|v|\le k\}}\Phi_{\delta,\eta} F_k\cdot\nabla(T_k(v))_\nu + \omega(\epsilon)\\
&= \omega(\epsilon,\nu,\delta).
\end{aligned}$$

To obtain the second integral, we set, for any $n>k>0$, and any $s\in\mathbb{R}$

$$\hat{S}_{n,k}(s) = \int_0^s (k-T_k(r))H_n(r)dr$$

where H_n is defined at Remark 3.2. We take $(S,\varphi)=(\hat{S}_{n,k},\psi_\delta^+\psi_\eta^+)$ as test function in (6.1), and we obtain

$$A_1+A_2+A_3+A_4+A_5+A_6=0,$$

where

$$A_1=-\int_Q(\psi_\delta^+\psi_\eta^+)_t\hat{S}_{n,k}(v_\epsilon)\,dx\,dt,$$

$$A_2=\int_Q(k-T_k(v_\epsilon))H_n(v_\epsilon)a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(\psi_\delta^+\psi_\eta^+)\,dx\,dt,$$

$$A_3=-\int_Q\psi_\delta^+\psi_\eta^+a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla T_k(v_\epsilon)\,dx\,dt,$$

$$A_4=\frac{2k}{n}\int_{\{-2n<v_\epsilon\le -n\}}\psi_\delta^+\psi_\eta^+a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla v_\epsilon\,dx\,dt,$$

$$A_5=-\int_Q(k-T_k(v_\epsilon))H_n(v_\epsilon)\psi_\delta^+\psi_\eta^+d\hat{\mu}_{0,\epsilon},$$

$$A_6=\int_Q(k-T_k(v_\epsilon))H_n(v_\epsilon)\psi_\delta^+\psi_\eta^+d(\lambda_\epsilon^\oplus+\lambda_\epsilon^\ominus)$$

Therefore, as in [23], using the fact that $(\hat{S}_{n,k}(v_\epsilon))$ weakly converges to $\hat{S}_{n,k}(v)$ in X, $\hat{S}_{n,k}(v)\in L^\infty(Q)$ and (4.6) we obtain

$$A_1=-\int_Q(\psi_\delta^+)_t\psi_\eta^+\hat{S}_{n,k}(v)-\int_Q\psi_\delta^+(\psi_\eta^+)_t\hat{S}_{n,k}(v)+\omega(\epsilon)=\omega(\epsilon,\delta).$$

Now since $v_\epsilon=T_{2n}(v_\epsilon)$ on $\operatorname{supp}(H_n(v_\epsilon))$ it follows from Proposition 5.2, (iv) that sequence $(a(t,x,u_\epsilon,\nabla(T_{2n}(v_\epsilon)+g_\epsilon)))\cdot\nabla(\psi_\delta^+\psi_\eta^+)$ weakly converges to $F_{2n}\cdot\nabla(\psi_\delta^+\psi_\eta^+)$ in $L^1(Q)$. From Lemma 4.1 and the convergence of $\psi_\delta^+\psi_\eta^+$ in X to 0 as δ tends to 0, we obtain

$$A_2=\int_Q(k-T_k(v_\epsilon))H_n(v_\epsilon)F_{2n}\cdot\nabla(\psi_\delta^+\psi_\eta^+)+\omega(\epsilon)=\omega(\epsilon,\delta).$$

Because $0\le\psi_\delta^+\le 1$ (resp $0\le\psi_\delta^-\le 1$). we then deduce

$$\begin{aligned}A_4&=\frac{2k}{n}\int_{-2n<v_\epsilon\le -n}a(t,x,u_\epsilon,\nabla(T_{2n}(v_\epsilon)+g_\epsilon))\cdot\nabla(T_{2n}(v_\epsilon)+g_\epsilon)\\&\quad-\nabla g_\epsilon]\psi_\delta^+\psi_\eta^+\,dx\,dt\\&\le\frac{2k}{n}\int_{\{-2n<v_\epsilon\le -n\}}a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla v_\epsilon\psi_\eta^+\,dx\,dt+\omega(\epsilon,\delta,n).\end{aligned}$$

Therefore Lemma 4.2 implies

$$A_4=\omega(\epsilon,\delta,n,\eta).$$

From the weak convergence of $((k-T_k(v_\epsilon))H_n(v_\epsilon)\psi_\delta^+\psi_\eta^+)$ to $(k-T_k(v))H_n(v)\psi_\delta^+\psi_\eta^+$ in X and of the weak_* convergence of $(k-T_k(v_\epsilon))H_n(v_\epsilon)$ to $(k-T_k(v))H_n(v)$ in $L^\infty(Q)$ and a.e. in Q, the weak convergence of (f_ϵ) to f in $L^1(Q)$ and the strong convergence of (g_ϵ) to g in $(L^{p'}(Q))^N$. From Lemma 4.1 and the convergence of $\psi_\delta^+\psi_\eta^+$ to 0 in X and a.e. in Q as $\delta\to 0$

$$A_5=\int_Q(k-T_k(v_\epsilon))H_n(v)\psi_\delta^+\psi_\eta^+d\hat{\mu}_0+\omega(\epsilon)=\omega(\epsilon,\delta),$$

We claim that the last term

$$A_6 \le 2k \int_Q \psi_\delta^+ \psi_\eta^+ d(\lambda_\epsilon^\oplus + \lambda_\epsilon^\ominus) = 2k \int_Q \psi_\delta^+ \psi_\eta^+ d(\mu_s^+ + \mu_s^-) + \omega(\epsilon).$$

Indeed, from Lemma 4.2 we have

$$A_6 \le \omega(\epsilon, \delta, \eta),$$

because A_3 does not depend on n. We then deduce from $\sum_{i=1}^6 A_i = 0$

$$A_3 = \int_Q \psi_\delta^+ \psi_\eta^+ a(t, x, u_\epsilon, \nabla u_\epsilon) \cdot \nabla T_k(v_\epsilon) \le \omega(\epsilon, \delta, \eta).$$

Similarly, we take $(S, \varphi) = (\hat{S}_{n,k}, \psi_\delta^- \psi_\eta^-)$ as test function in (6.1), where $\hat{S}_{n,k}(s) = -\hat{S}_{n,k}(-s)$, we have, as before

$$\int_Q \psi_\delta^- \psi_\eta^- a(t, x, u_\epsilon, \nabla u_\epsilon) \cdot \nabla T_k(v_\epsilon) \le \omega(\epsilon, \delta, \eta).$$

So that using the two last inequalities we obtain

$$\int_Q \Phi_{\delta,\eta} a(t, x, u_\epsilon, \nabla u_\epsilon) \cdot \nabla T_k(v_\epsilon) \le \omega(\epsilon, \nu, \delta, \eta).$$

We finally deduce

$$I_1 = \int_{\{|v_\epsilon| \le k\}} \Phi_{\delta,\eta} a(t, x, u_\epsilon, \nabla u_\epsilon) \cdot \nabla (v_\epsilon - T_k(v)_\nu) \le \omega(\epsilon, \nu, \delta, \eta).$$

□

Remark 6.3. Note that: It is precisely for this estimate that we need the double cut functions $\psi_\delta^+ \psi_\eta^+$.

This results turns out to hold true even for more general functions ψ_η^+ and ψ_η^- in $W^{1,\infty}(Q)$, which satisfy

$$0 \le \psi_\eta^+ \le 1, \quad 0 \le \psi_\eta^- \le 1,$$
$$0 \le \int_Q \psi_\eta^+ d\mu_s^- \le \eta, \quad 0 \le \int_Q \psi_\eta^- d\mu_s^+ \le \eta.$$

Lemma 6.4. *Far from E we have the estimate*

$$I_2 = \int_{\{|v_\epsilon| \le k\}} (1 - \Phi_{\delta,\eta}) a(t, x, u_\epsilon, \nabla u_\epsilon) \cdot \nabla (T_k(v_\epsilon) - T_k(v)_\nu).$$

Proof. Now we follow the ideas in [22, 24], for any $h > 2k > 0$, we define

$$w_\epsilon = T_{2k}(v_\epsilon - T_h(v_\epsilon) + T_k(v_\epsilon) - T_k(v)_\nu),$$

Note that $\nabla w_\epsilon = 0$ if $|v_\epsilon| > h + 4k$. As a consequence of the estimate on $T_k(v_\epsilon)$ in Proposition 5.2 we have w_ϵ is bounded in $L^p(0, T; W_0^{1,p}(\Omega))$; we easily obtain

$$w_\epsilon \to T_{2k}(v - T_h(v) + T_k(v) - T_k(v)_\nu))$$

since $\|T_k(v)_\nu\|_{L^\infty(Q)} \le k$, we have also

$$w_\epsilon = 2k \operatorname{sign}(v_\epsilon), \text{ in } \{|v_\epsilon| > h + 2k\}, \quad |w_\epsilon| \le 4k, \quad w_\epsilon = w(\epsilon, \nu, h) \text{ a.e. in } Q,$$
$$\lim_\epsilon w_\epsilon = T_{h+k}(v - (T_k(v))_\nu) - T_{h-k}(v - T_k(v)), \text{ a.e. in } Q \text{ and weakly in X.}$$

Let us take $w_\epsilon(1-\Phi_{\delta,\eta})$ as test functions in (5.3). We obtain

$$A_1 + A_2 + A_3 = A_4 + A_5,$$

where

$$\begin{gathered} A_1 = \int_0^T \langle v_{t,\epsilon}, w_\epsilon(1-\Phi_{\delta,\eta})\rangle \, dt, \\ A_2 = \int_Q a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot \nabla w_\epsilon(1-\Phi_{\delta,\eta}), \\ A_3 = -\int_Q a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot \nabla \Phi_{\delta,\eta} w_\epsilon dx\, dt, \\ A_4 = w_\epsilon(1-\Phi_{\delta,\eta}) d\hat{\mu_0}, \\ A_5 = \int_Q w_\epsilon(1-\Phi_{\delta,\eta}) d(\lambda_\epsilon^\oplus - \lambda_\epsilon^\ominus) \end{gathered}$$

Using the weak convergence of f_ϵ, again from the decomposition (3.2)

$$A_4 = \int_Q f_\epsilon w_\epsilon (1-\Phi_{\delta,\eta})\, dx\, dt + \int_Q G_\epsilon \cdot \nabla (w_\epsilon(1-\Phi_{\delta,\eta}))\, dx\, dt,$$

since f_ϵ converges to f weakly in $L^1(Q)$, from Lemma 4.1, we obtain

$$\int_Q f_\epsilon w_\epsilon (1-\Phi_{\delta,\eta})\, dx\, dt = \omega(\epsilon,\nu,h).$$

Lemma 6.5. *Let $h,k>0$, and u_ϵ and $\Phi_{\delta,\eta}$ as before, then*

$$\int_{\{h\le |v_\epsilon|<h+k\}} |\nabla u_\epsilon|^p (1-\Phi_{\delta,\eta}) = \omega(\epsilon,h,\delta,\eta).$$

For a proof of the above lemma see [23, Lemma 7].

Note that (g_ϵ) converges to g strongly in $(L^{p'}(Q))^N$, and $T_k(v)_\nu$ converges to $T_k(v)$ strongly in X. Then we deduce from Young's inequality and Lemma 6.5,

$$\begin{aligned} &\int_Q G_\epsilon \cdot \nabla (w_\epsilon(1-\Phi_{\delta,\eta}))\, dx\, dt \\ &= \int_Q (1-\Phi_{\delta,\eta}) G\cdot \nabla (T_{h+k}(v-T_k(v)) - T_{h-k}(v-T_k(v)))\, dx\, dt + \omega(\epsilon,\nu) \\ &= \int_{\{h\le v<h+2k\}} (1-\Phi_{\delta,\eta}) G\cdot \nabla v\, dx\, dt + \omega(\epsilon,\nu,h) \\ &= \omega(h,\delta,\eta). \end{aligned}$$

Then

$$A_4 = \omega(\epsilon,\nu,h,\delta,\eta).$$

To estimate of A_5, we have $|w_\epsilon|\le 2k$ and reasoning as in the proof of Lemma 6.5, and thanks to (4.8) - (4.11); we obtain

$$A_5 = \omega(\epsilon,\delta,\eta).$$

To estimate of A_1, we observe that, since $|T_k(v)_\nu|\le k$, w_ϵ can be written in the following way:

$$w_\epsilon = T_{h+k}(v_\epsilon - T_k(v)_\nu) - T_{h-k}(v_\epsilon - T_k(v_\epsilon)).$$

Hence, setting $G(t)=\int_0^t T_{h-k}(s-T_k(s))ds$, we have

$$\begin{aligned}
&\int_0^t \langle (v_\epsilon)_t, w_\epsilon(1-\Phi_{\delta,\eta})\rangle\,dt\\
&=\int_0^t \langle (T_k(v)_\nu)_t, T_{h+k}(v_\epsilon-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle\,dt\\
&\quad+\int_Q S_{h+k}(v_\epsilon-T_k(v)_\nu)_t(1-\Phi_{\delta,\eta})\;dx\,dt-\int_Q G(v_\epsilon)_t(1-\Phi_{\delta,\eta})\,dx\,dt
\end{aligned}$$

and since $|T_k(v)_\nu|\le k$, using the definition of $T_k(v)_\nu$ we obtain

$$\begin{aligned}
&\int_0^t \langle (T_k(v)_\nu)_t, T_{h+k}(v_\epsilon-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle\,dt\\
&=\nu\int_Q (T_k(v)-T_k(v)_\nu)T_{h+k}(v_\epsilon-T_k(v)_\nu)\,dx\,dt,
\end{aligned}$$

so that as ϵ tends to infinity, we have

$$\begin{aligned}
&\int_0^t \langle (T_k(v))_t, T_{h+k}(v_\epsilon-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle\,dt\\
&=\omega(\epsilon)+\nu\int_Q (T_k(v)-T_k(v)_\nu)T_{h+k}(v-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\,dx\,dt\\
&=\omega(\epsilon)+\nu\int_{\{|v|\le k\}} (v-T_k(v)_\nu)T_{h+k}(v-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\,dx\,dt\\
&\quad+\int_{\{v>k\}} (k-T_k(v)_\nu)T_{h+k}(v-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\,dx\,dt\\
&\quad+\int_{\{v<-k\}} (-k-T_k(v)_\nu)T_{h+k}(v-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\,dx\,dt.
\end{aligned}$$

since $|T_k(v)_\nu|\le k$, last three terms are positives, hence we deduce by letting ϵ and ν to ∞,

$$\begin{aligned}
&\int_0^t \langle (v_\epsilon)_t, w_\epsilon(1-\Phi_{\delta,\eta})\rangle\,dt\\
&=\omega(\epsilon)+\int_Q S_{h+k}(v_\epsilon-T_k(v)_\nu)_t(1-\Phi_{\delta,\eta})\,dx\,dt-\int_Q G(v_\epsilon)_t(1-\Phi_{\delta,\eta})\,dx\,dt\\
&=\omega(\epsilon)+\int_Q S_{h+k}(v_\epsilon-T_k(v)_\nu)\frac{\partial\Phi_{\delta\eta}}{dt}\,dx\,dt-\int_Q G(v_\epsilon)\frac{\partial\Phi_{\delta\eta}}{}\,dx\,dt\\
&\quad+\int_\Omega S_{h+k}(v_\epsilon-T_k(v)_\nu)(T)\,dx-\int_\Omega S_{h+k}(u_{0,\epsilon}-z_\nu)\,dx\\
&\quad-\int_\Omega G(v_\epsilon)(T)\,dx+\int_\Omega G(u_{0,\epsilon})\,dx.
\end{aligned}$$

Now we define the function $R(y)=S_{h+k}(y-z)\cdot G(y)$, with $|z|\le k$. Then

$$\begin{gathered}
R(y)=S_{h+k}(y+z)\ge 0,\quad |y|\le k,\\
R'(y)=T_{h+k}(y-z)-T_{h-k}(y-T_k(y))\ge 0,\quad y\ge k\ge z,\\
R'(y)\le 0,\quad y\le -k\le z.
\end{gathered}$$

Hence for every $z, |z| \leq k$, we have $R(y) \geq 0$ for every y in $\mathbb{R}$, we obtain

$$\int_\Omega S_{h+k}(v_\epsilon - T_k(v)_\nu)(T)\,dx - \int_\Omega G(v_\epsilon)(T)\,dx \geq 0,$$

letting ϵ and ν go to their limits,

$$\int_\Omega G(u_{u_{0,\epsilon}})\,dx - \int_\Omega S_{h+k}(u_{0,\epsilon} - z_\nu)\,dx = \int_\Omega G(u_0) - \int_\Omega S_{h+k}(u_0 - T_k(u_0)) + \omega(\epsilon,\nu),$$

Since we have $|G(u_0) - S_{h+k}(u_0 - T_k(u_0))| \leq 2k|u_0|\chi_{\{|u_0|>k\}}$, it follows that by letting h to $+\infty$,

$$\int_\Omega G(u_{0,\epsilon})\,dx - \int_\Omega S_{h+k}(u_{0,\epsilon} - z_\nu)\,dx = \omega(\epsilon,\nu,h)\,.$$

By the definition of $T_k(v)_\nu$,

$$\begin{aligned}
&\int_Q S_{h+k}(v_\epsilon - T_k(v)_\nu)\frac{d\Phi_{\delta\eta}}{dt}dx\,dt - \int_Q G(v_\epsilon)\frac{d\Phi_{\delta\eta}}{dt}dx\,dt \\
&= \int_Q (S_{h+k}(v - T_k(v) - G(v))\frac{d\Phi_{\delta\eta}}{dt}dx\,dt + \omega(\epsilon,\nu).
\end{aligned}$$

So, if $|v| \leq h-k$, $S_{h+k}(v-T_k(v)) - G(v) = 0$, then $S_{h+k}(v-T_k(v)) - G(v)$ converges a.e. to 0 on Q, and since $v \in L^1(Q)$, by dominated convergence theorem

$$\int_Q S_{h+k}(v_\epsilon - T_k(v)_\nu)\frac{d\Phi_{\delta\eta}}{dt}dx\,dt - \int_Q (v_\epsilon)\frac{d\Phi_{\delta\eta}}{dt}dx\,dt \geq \omega(\epsilon,\nu,h),$$

and so

$$\int_0^T \langle (v_\epsilon)_t, w_\epsilon(1-\Phi_{\delta\eta})\rangle \geq \omega(\epsilon,\nu,h).$$

Now we estimate of A_2. Note that $\nabla w_\epsilon = 0$ if $|v_\epsilon| > h+4k$; then if we set $M = h+4k$, splitting the integral (A_2) on the sets $\{|v_\epsilon| > k\}$ and $\{|v_\epsilon| \leq k\}$, using the fact that $T_h(v_\epsilon) = T_k(v_\epsilon) = v_\epsilon$ in $\{|v_\epsilon| \leq k\}$ and $\nabla T_k(v_\epsilon)\chi_{|v_\epsilon|>k} = 0$. Then for $\{|v_\epsilon| \leq M\}$ and $h \geq 2k$, we have

$$\begin{aligned}
A_2 &= \int_Q a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla w_\epsilon(1-\Phi_{\delta\eta})\,dx\,dt \\
&= \int_{\{|v_\epsilon|\leq k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(v_\epsilon - T_k(v)_\nu)(1-\Phi_{\delta\eta})\,dx\,dt \\
&\quad + \int_{\{|v_\epsilon|>k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla[(v_\epsilon - T_h(v_\epsilon)) - (T_k(v)_\nu)](1-\Phi_{\delta\eta})\,dx\,dt \\
&= \int_{\{|v_\epsilon|\leq k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(v_\epsilon - T_k(v)_\nu)(1-\Phi_{\delta\eta})\,dx\,dt \\
&\quad + \int_{\{|v_\epsilon|>h\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla[(v_\epsilon - T_h(v_\epsilon))(1-\Phi_{\delta\eta})\,dx\,dt \\
&\quad + \int_{\{|v_\epsilon|>k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(T_k(v)_\nu - T_k(v)) + \nabla T_k(v)(1-\Phi_{\delta\eta})\,dx\,dt \\
&= \int_{\{|v_\epsilon|\leq k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(v_\epsilon - T_k(v)_\nu)(1-\Phi_{\delta\eta})\,dx\,dt \\
&\quad + \int_{\{h<|v_\epsilon|>h+4k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla v_\epsilon(1-\Phi_{\delta\eta})\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\{|v_\epsilon|>k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(T_k(v)_\nu - T_k(v))(1-\Phi_{\delta\eta})\,dx\,dt\\
&+\int_{\{|v_\epsilon|>k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla T_k(v)(1-\Phi_{\delta\eta})\,dx\,dt\,.
\end{aligned}
$$

Using assumption (2.2), young's inequality, equi-integrability and Lemma 6.5, we see that for some $C=C(p,c_2)$,

$$
\begin{aligned}
&\int_{\{h\le|v_\epsilon|<h+4k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla v_\epsilon(1-\Phi_{\delta\eta})\,dx\,dt\\
&\le C\int_{\{h\le|v_\epsilon|<h+4k\}} (|\nabla u_\epsilon|^p+|\nabla g|^p+|b_0(t,x)|^{p'})(1-\Phi_{\delta\eta})\,dx\,dt\\
&\le \omega(\epsilon,h,\delta,\eta)\,.
\end{aligned}
$$

Thanks to Proposition 5.2 and the fact that $T_k(v)_\nu$ converges strongly to $T_k(v)$ in $L^p(0,T;W_0^{1,p}(\Omega))$, we have

$$
\int_{\{|v_\epsilon|>k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla T_k(v)(1-\Phi_{\delta\eta})\,dx\,dt=\omega(\epsilon),
$$

$$
\int_{\{|v_\epsilon|>k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(T_k(v)_\nu - T_k(v))(1-\Phi_{\delta\eta})\,dx\,dt=\omega(\epsilon,\nu),
$$

Therefore,

$$
A_2=\int_{\{|v_\epsilon|\le k\}} a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(v_\epsilon - T_k(v)_\nu)(1-\Phi_{\delta\eta})\,dx\,dt+\omega(\epsilon,\nu,h,\delta,\eta).
$$

□

Next we conclude the proof of Theorem 3.5.

Lemma 6.6. *The function u is a renormalized solution of* (1.1).

Proof. (i) Let $\varphi\in X\cap L^\infty(Q)$ such that $\varphi_t\in X'+L^1(Q)$, $\varphi(\cdot,T)=0$, and $S\in W^{2,\infty}(\mathbb{R})$, such that S' has compact support on $\mathbb{R}$, $S(0)=0$. Let $M>0$ such that $\operatorname{supp} S'\subset[-M,M]$. Taking successively (φ,S), (φ,ψ_δ^+) and (φ,ψ_δ^-) as test functions in (6.1) applied to u_ϵ, we can write

$$
\begin{gathered}
A_1+A_2)+A_3+A_4=A_5+A_6+A_7,\\
(A_2)_\delta^+ +(A_3)_\delta^+ +(A_4)_\delta^+ =(A_5)_\delta^+ +(A_6)_\delta^+ +(A_7)_\delta^+,\\
(A_2)_\delta^- +(A_3)_\delta^- +(A_4)_\delta^- =(A_5)_\delta^- +(A_6)_\delta^- +(A_7)_\delta^-
\end{gathered}
$$

where

$$
\begin{gathered}
A_1=-\int_\Omega \varphi(0)S(u_{0,\epsilon})dt,\quad A_2=-\int_Q \varphi_t S(v_\epsilon)\,dx\,dt,\\
A_3=\int_Q S'(v_\epsilon)a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla\varphi\,dx\,dt,\\
A_4=\int_Q S''(v_\epsilon)\varphi a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla v_\epsilon dx\,dt,\\
A_5=\int_Q S'(v_\epsilon)\varphi\hat\mu_\epsilon,\quad A_6=\int_Q S'(v_\epsilon)\varphi d\lambda_\epsilon^\oplus
\end{gathered}
$$

$$A_7 = -\int_Q S'(v_\epsilon)\varphi d\lambda_\epsilon^\ominus,$$

and

$$\begin{gathered}
(A_2)_\delta^+ = -\int_Q (\varphi\psi_\delta^+)_t S(v_\epsilon)\,dx\,dt,\\
(A_3)_\delta^+ = \int_Q S'(v_\epsilon)a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(\varphi\psi_\delta^+)\,dx\,dt,\\
(A_4)_\delta^+ = \int_Q S''(v_\epsilon)\varphi\psi_\delta^+ a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla v_\epsilon dx\,dt,\\
(A_5)_\delta^+ = \int_Q S'(v_\epsilon)\varphi\psi_\delta^+ d\lambda_\epsilon^\oplus,\\
(A_6)_\delta^+ = -\int_Q S'(v_\epsilon)\varphi\psi_\delta^+ d\lambda_\epsilon^\ominus\,.
\end{gathered}$$

Since $(u_{0,\epsilon})$ converges to u_0 in $L^1(\Omega)$, and $(S(v_\epsilon))$ converges to $S(v)$, strongly in X, and weak_* in $L^\infty(Q)$, it follows that

$$\begin{gathered}
A_1 = \int_\Omega \varphi(0)S(u_0)\,dx + \omega(\epsilon), \quad A_2 = -\int_Q \varphi_t S(v) + \omega(\epsilon),\\
(A_2)_\delta^+ = \omega(\epsilon,\delta), \quad (A_2)_\delta^- = \omega(\epsilon,\delta)\,.
\end{gathered}$$

Moreover, $T_M(v_\epsilon)$ converges to $T_M(v)$, then $T_M(v_\epsilon)+h_\epsilon$ converges to $T_k(v)+h$ strongly in X. Therefore,

$$\begin{aligned}
A_3 &= \int_Q S'(v_\epsilon)a(t,x,u_\epsilon,\nabla(T_M(v_\epsilon)+h_\epsilon)\cdot\nabla\varphi,\\
&= \omega(\epsilon) + \int_Q S'(v)a(t,x,u_\epsilon,\nabla(T_M(v)+h))\cdot\nabla\varphi,\\
&= \omega(\epsilon) + \int_Q S'(v)a(t,x,u,\nabla u)\cdot\nabla\varphi,
\end{aligned}$$

and

$$\begin{aligned}
A_4 &= \int_Q S''(v_\epsilon)\varphi a(t,x,u_\epsilon,\nabla(T_M(v_\epsilon)+h_\epsilon))\cdot\nabla T_M(v_\epsilon)\\
&= \omega(\epsilon) + \int_Q S''(v)\varphi a(t,x,u,\nabla(T_M(v)+h))\cdot\nabla T_M(v)\\
&= \omega(\epsilon) + \int_Q S''(v)\varphi a(t,x,u,\nabla u)\cdot\nabla v\,.
\end{aligned}$$

In the same way, since $\psi_\delta^+,\psi_\delta^-$ converges to 0 in X,

$$\begin{gathered}
(A_3)_\delta^+ = \omega(\epsilon) + \int_Q S'(v)a(t,x,u,\nabla u)\cdot\nabla(\varphi\psi)_\delta^+ = \omega(\epsilon,\delta),\\
(A_3)_\delta^- = \omega(\epsilon) + \int_Q S'(v)a(t,x,u,\nabla u)\cdot\nabla(\varphi\psi_\delta^-) = \omega(\epsilon,\delta),\\
(A_4)_\delta^+ = \omega(\epsilon) + \int_Q S''(v)\varphi\psi_\delta^+ a(t,x,u,\nabla u)\cdot\nabla v = \omega(\epsilon,\delta),
\end{gathered}$$

$$(A_4)^-_\delta = \omega(\epsilon) + \int_Q S''(v)\varphi\psi^-_\delta a(t,x,u,\nabla u)\cdot\nabla v = \omega(\epsilon,\delta),$$

and (g_ϵ) strongly converges to g in $(L^{p'}(\Omega))^N$. Therefore,

$$\begin{aligned}
(A_5) &= \int_Q S'(v_\epsilon)\varphi f_\epsilon + \int_Q S'(v_\epsilon)g_\epsilon\cdot\nabla\varphi + \int_Q S''(v_\epsilon)\varphi g_\epsilon\cdot\nabla T_M(v_\epsilon)\\
&= \omega(\epsilon) + \int_Q S'(v)\varphi f + \int_Q S'(v)g\cdot\nabla\varphi + \int_Q S''(v)\varphi g\cdot\nabla T_M(v)\\
&= \omega(\epsilon) + \int_Q S'(v)\varphi d\hat{\mu}_0
\end{aligned}$$

Now, thanks to Proposition 5.2 and the proprieties of ψ^+_δ and ψ^-_δ, we readily have

$$\begin{gathered}
(A_5)^+_\delta = \omega(\epsilon) + \int_Q S'(v)\varphi\psi^+_\delta d\hat{\mu}_\epsilon = \omega(\epsilon,\delta),\\
(A_5)^-_\delta = \omega(\epsilon) + \int_Q S'(v)\varphi\psi^-_\delta d\hat{\mu}_\epsilon = \omega(\epsilon,\delta).
\end{gathered}$$

Then

$$(A_6)^+_\delta + (A_7)^+_\delta = \omega(\epsilon,\delta),$$

and thanks to (4.9),

$$\begin{gathered}
(A_7)^+_\delta \le |\int_Q S'(v_\epsilon)\varphi\psi^+_\delta d\lambda^\ominus_\epsilon| \le c\int_Q \psi^+_\delta d\lambda^\ominus_\epsilon = \omega(\epsilon,\delta),\\
(A_7)^-_\delta = \omega(\epsilon,\delta)\,.
\end{gathered}$$

Then

$$(A_6)^+_\delta = \int_Q S'(v_\epsilon)\varphi\psi^+_\delta d\lambda^\oplus_\epsilon = \omega(\epsilon,\delta).$$

Moreover,

$$\begin{aligned}
A_6 &= \int_Q S'(v_\epsilon)\varphi d\lambda^\ominus_\epsilon\\
&= \int_Q S'(v_\epsilon)\varphi\psi^+_\delta d\lambda^\oplus_\epsilon + \int_Q S'(v_\epsilon)\varphi(1-\psi^+_\delta)d\lambda^\oplus_\epsilon\\
&\le \omega(\epsilon,\delta) + \int_Q |S'(v_\epsilon)\varphi|(1-\psi^+_\delta)d\lambda^\oplus_\epsilon\\
&\le \omega(\epsilon,\delta) + \|S\|_{W^{2,\infty}(\mathbb{R})}\|\varphi\|_{L^\infty(Q)}\int_Q (1-\psi^+_\delta)d\lambda^\oplus_\epsilon\\
&\le \omega(\epsilon,\delta)\,.
\end{aligned}$$

Hence

$$A_6 = \omega(\epsilon) \text{ and } (A_7) = \omega(\epsilon).$$

Therefore, we finally obtain

$$\begin{aligned}
&-\int_\Omega \varphi(0)S(u_0)\,dx - \int_Q \varphi_t S(v) + \int_Q S'(v)a(t,x,u,\nabla u)\cdot\nabla\varphi\\
&+\int_Q S''(v)\varphi a(t,x,u,\nabla u)\cdot\nabla v
\end{aligned}$$

$$= \int_Q S'(v)\varphi d\hat{\mu}_0$$

with $\varphi \in C_0^1([0,T]\times\Omega)$. By density argument we have (2.8) for any $\varphi \in X \cap L^\infty(Q)$ such that $\varphi_t \in X' + L^1(Q)$ and $\varphi(\cdot,T) = 0$.

(ii) Next, we prove (2.9). We take $\varphi \in C_c^\infty(Q)$ and $(\varphi, S) = ((1-\psi_\delta-)\varphi, \overline{H}_n)$ as test functions in (2.8) and the same test functions in (6.1) applied to u_ϵ, we can write

$$\begin{gathered} B_1^n + B_2^n = B_3^n + B_4^n + B_5^n, \\ B_{1,\epsilon}^n + B_{2,\epsilon}^n = B_{3,\epsilon}^n + B_{4,\epsilon}^n + B_{5,\epsilon}^n, \end{gathered}$$

where

$$\begin{gathered} B_1^n = -\int_Q ((1-\psi_\delta^-)\varphi)_t \overline{H}_n(v)\,dx\,dt, \\ B_2^n = \int_Q H_n(v) a(t,x,u,\nabla u)\cdot\nabla((1-\psi_\delta^-)\varphi)\,dx\,dt, \\ B_3^n = \int_Q H_n(v)(1-\psi_\delta^-)\varphi d\hat{\mu}_0, \\ B_4^n = \frac{1}{n}\int_{\{n<v\le 2n\}} (1-\psi_\delta^-)\varphi a(t,x,u,\nabla u)\cdot\nabla v\,dx\,dt, \\ B_5^n = -\frac{1}{n}\int_{\{-2n\le v<-n\}} (1-\psi_\delta^-)\varphi a(t,x,u,\nabla u)\cdot\nabla v\,dx\,dt, \end{gathered}$$

and

$$\begin{gathered} B_{1,\epsilon}^n = -\int_Q ((1-\psi_\delta^-)\varphi)_t \overline{H}_n(v_\epsilon)\,dx\,dt, \\ B_{2,\epsilon}^n = \int_Q H_n(v_\epsilon) a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla((1-\psi_\delta^-)\varphi)\,dx\,dt, \\ B_{3,\epsilon}^n = \int_Q H_n(v_\epsilon)(1-\psi_\delta^-)\varphi d(\hat{\mu}_{\epsilon,0} + \lambda_\epsilon^\oplus - \lambda_\epsilon^\ominus), \\ B_{4,\epsilon}^n = \frac{1}{n}\int_{\{n<v_\epsilon\le 2n\}} (1-\psi_\delta^-)\varphi a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla v_\epsilon\,dx\,dt, \\ B_{5,\epsilon}^n = -\frac{1}{n}\int_{\{-2n\le v_\epsilon<-n\}} (1-\psi_\delta^-)\varphi a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla v_\epsilon\,dx\,dt\,. \end{gathered}$$

In the last terms, we go to the limit as $n \to +\infty$, since $(\overline{H}_n(v_\epsilon))$ converges to 0, weakly in $(L^p(Q))^N$, we obtain the relation

$$B_{1,\epsilon} + B_{2,\epsilon} = B_{3,\epsilon} + B_\epsilon$$

where

$$\begin{gathered} B_{1,\epsilon} = -\int_Q ((1-\psi_\delta^-)\varphi)_t v_\epsilon, \\ B_{2,\epsilon} = \int_Q a(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla((1-\psi_\delta^-\varphi), \\ B_{3,\epsilon} = \int_Q (1-\psi_\delta^-)\varphi d\hat{\mu}_{\epsilon,0}, \end{gathered}$$

$$B_\epsilon = \int_Q (1-\psi_\delta^-)\varphi d(\lambda_{\epsilon,0}^{\oplus} - \lambda_{\epsilon,0}^{\ominus}) + \int_Q (1-\psi_\delta^-)\varphi d(\lambda_{\epsilon,s}^{\oplus} - \lambda_{\epsilon,s}^{\ominus})\,.$$

Clearly, $(B_{i,\epsilon}) - (B_i^n) = \omega(\epsilon, n)$ for $i = 1, 3$, from (4.9) - (4.11), we obtain

$$B_5^n = \omega(\epsilon, n, \delta),$$
$$\frac{1}{n}\int_{\{n<v\le 2n\}} \psi_\delta^- \varphi a(t,x,u,\nabla u)\cdot\nabla v = \omega(\epsilon,n,\delta)\,.$$

Thus

$$B_4^n = \frac{1}{n}\int_{\{n<v\le 2n\}} \varphi a(t,x,u,\nabla u)\cdot\nabla v\,dx\,dt + \omega(\epsilon,n,\delta)$$

since

$$|\int_Q (1-\psi_\delta^-)\varphi d\lambda_\epsilon^{\ominus}| \le \|\varphi\|_{L^\infty}\int_Q (1-\psi_\delta^-) d\lambda_\epsilon^{\ominus},$$

it follows that $\int_Q (1-\psi_\delta^-)\varphi d\lambda_\epsilon^{\ominus} = \omega(\epsilon,n,\delta)$ from (4.11). And $|\int_Q \psi_\delta^- \varphi d\lambda_\epsilon^{\oplus}| \le \|\varphi\|_{L^\infty}\int_Q \psi_\delta^- d\lambda_\epsilon^{\oplus}$. Thus from (4.8) and (4.9), $\int_Q (1-\psi_\delta^-)\varphi d\lambda_\epsilon^{\oplus} = \int_Q \varphi d\mu_s^+ + \omega(\epsilon,n,\delta)$. Then

$$B_\epsilon = \int_Q \varphi d\mu_s^+ + \omega(\epsilon,n,\delta).$$

Therefore, by subtraction, we obtain successively

$$\frac{1}{n}\int_{\{n<v\le 2n\}} \varphi a(t,x,u,\nabla u)\cdot\nabla v\,dx\,dt = \int_Q \varphi d\mu_s^+ + \omega(\epsilon,n,\delta),$$
$$\lim_{n\to+\infty}\frac{1}{n}\int_{\{n<v\le 2n\}} \varphi a(t,x,u,\nabla u)\cdot\nabla v = \int_\varphi d\mu_s^+,$$

which proves (2.9) when $\varphi \in C_c^\infty(Q)$. Next assume only $\varphi \in C^\infty(\overline{Q})$. Then

$$\begin{aligned}
&\lim_{n\to+\infty}\frac{1}{n}\int_{\{n\le v<2n\}} \varphi a(t,x,u,\nabla u)\cdot\nabla v\,dx\,dt\\
&= \lim_{n\to+\infty}\frac{1}{n}\int_{\{n\le v<2n\}} \varphi\psi_\delta^+ a(t,x,u,\nabla u)\cdot\nabla v\,dx\,dt\\
&\quad + \lim_{n\to+\infty}\frac{1}{n}\int_{\{n\le v<2n\}} \varphi(1-\psi_\delta^+) a(t,x,u,\nabla u)\cdot\nabla v\,dx\,dt\\
&= \int_Q \varphi\psi_\delta^+ d\mu_s^+ + \lim_{n\to+\infty}\frac{1}{n}\int_{\{n\le v<2n\}} \varphi(1-\psi_\delta^+) a(t,x,u,\nabla u)\cdot\nabla v\,dx\,dt\\
&= \int_Q \varphi d\mu_s^+ + D
\end{aligned}$$

where

$$D = \int_Q \varphi(1-\psi_\delta^+) d\mu_s^+ + \lim_{n\to+\infty}\frac{1}{n}\int_{\{n\le v<2n\}} \varphi(1-\psi_\delta^+) a(t,x,u,\nabla u)\nabla v\,dx\,dt = \omega(\epsilon).$$

Therefore, (2.9) still holds for $\varphi \in C^\infty(\overline{Q})$, and we deduce (2.9) by density, and similarly the second convergence. This complete the proof of Theorem 3.5. $\square$

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Mohammed Abdellaoui
University of Fez, Faculty of Sciences Dhar El Mahraz, Laboratory LAMA, Department of Mathematics, B.P. 1796, Atlas Fez, Morocco
E-mail address: mohammed.abdellaoui3@usmba.ac.ma

Elhoussine Azroul
University of Fez, Faculty of Sciences Dhar El Mahraz, Laboratory LAMA, Department of Mathematics, B.P. 1796, Atlas Fez, Morocco
E-mail address: elhoussine.azroul@usmba.ac.ma