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ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS OF A SEMILINEAR DIRICHLET PROBLEM IN EXTERIOR DOMAINS

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Abstract. In this article, we study the existence, uniqueness and the asymptotic behavior of a positive classical solution to the semilinear boundary-value problem

$$-\Delta u = a(x)u^{\sigma} \quad \text{in } D,$$
$$u|_{\partial D} = 0, \quad \lim_{|x|\to\infty} u(x) = 0.$$

Here D is an unbounded regular domain in $\mathbb{R}^n$ $(n \geq 3)$ with compact boundary, $\sigma < 1$ and the function a is a nonnegative function in $C^{\gamma}_{\rm loc}(D)$, $0 < \gamma < 1$, satisfying an appropriate assumption related to Karamata regular variation theory.

1. Introduction

The semilinear elliptic equation

$$-\Delta u = a(x)u^{\sigma}, \quad \sigma < 1, \quad x \in \Omega \subset \mathbb{R}^n, \tag{1.1}$$

has been extensively studied for both bounded and unbounded domains Ω in $\mathbb{R}^n$ $(n \geq 2)$. We refer to [1, 3, 4, 5, 6, 13, 16, 17, 18, 20, 22, 23, 24, 25, 26, 27, 28, 32, 38, 39] and the references therein, for various existence and uniqueness results related to solutions for the above equation with homogeneous Dirichlet boundary conditions.

Most recently, applying regular variation theory, many authors have studied the exact asymptotic behavior of solutions of equation (1.1). In fact, the combined use of regular variation theory and the Karamata theory has been introduced by Cîrstea and Rădulescu [10, 11, 12, 13, 14] in the study of various qualitative and asymptotic properties of solutions of nonlinear differential equations. Then, this setting becomes a powerful tool in describing the asymptotic behavior of solutions of large classes of nonlinear equations (see [2, 4, 7, 8, 9, 13, 15, 19, 21, 23, 29, 30, 31, 35, 36, 40]).

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For example, Mâagli [29] considered the problem

$$
\begin{gathered}
-\Delta u=a(x)u^{\sigma} \quad \text{in } \Omega,\\
u>0 \quad \text{in } \Omega,\\
u\big|_{\partial\Omega}=0,
\end{gathered} \tag{1.2}
$$

where Ω is a bounded $C^{1,1}$-domain, $\sigma<1$ and a satisfies some appropriate conditions with reference to $\mathcal{K}_0$, the set of all Karamata functions L regularly varying at zero, defined on $(0,\eta]$ by

$$
L(t):=c\exp\Big(\int_t^{\eta}\frac{z(s)}{s}ds\Big),
$$

for some $\eta>0$, where $c>0$ and z is a continuous function on $[0,\eta]$, with $z(0)=0$. As a typical example of function $L\in\mathcal{K}_0$, we have

$$
L(t)=\prod_{k=1}^{m}(\log_k(\frac{\omega}{t}))^{\xi_k},
$$

where $m\in\mathbb{N}^*$, $\log_k x=\log\circ\log\circ\dots\circ\log x$ (k times), $\xi_k\in\mathbb{R}$ and ω is a sufficiently large positive real number such that the function L is defined and positive on $(0,\eta]$.

Thanks to the sub-supersolution method and using some potential theory tools, Mâagli showed in [29] that (1.2) has a unique positive classical solution and gave sharp estimates on the solution. These estimates improve and extend those stated in [16, 23, 28, 32, 40]. In order to describe the result of [29] in more details, we need some notations.

For two nonnegative functions f and g defined on a set S, the notation $f(x)\approx g(x)$, $x\in S$, means that there exists a constant $c>0$ such that for each $x\in S$, $\frac{1}{c}\,g(x)\le f(x)\le c\,g(x)$. Further, for a domain Ω of $\mathbb{R}^n$ $(n\ge 2)$, $d_\Omega(x)$ denotes the Euclidean distance from $x\in\Omega$ to the boundary of Ω. Also for $\lambda\le 2$, $\sigma<1$ and $L\in\mathcal{K}_0$ defined on $(0,\eta]$, $\eta>0$ such that $\int_0^{\eta}t^{1-\lambda}L(t)dt<\infty$, we put $\Phi_{L,\lambda,\sigma}$ the function defined on $(0,\nu]$, $0<\nu<\eta$, by

$$
\Phi_{L,\lambda,\sigma}(t):=\begin{cases}
1, & \text{if } \lambda<1+\sigma,\\
\big(\int_t^{\eta}\frac{L(s)}{s}ds\big)^{\frac{1}{1-\sigma}}, & \text{if } \lambda=1+\sigma,\\
(L(t))^{\frac{1}{1-\sigma}}, & \text{if } 1+\sigma<\lambda<2,\\
\big(\int_0^{t}\frac{L(s)}{s}ds\big)^{\frac{1}{1-\sigma}}, & \text{if } \lambda=2.
\end{cases}
$$

Now, let us present the result by Mâagli [29].

Theorem 1.1. *Let $a\in C^{\gamma}_{\rm loc}(\Omega)$, $0<\gamma<1$, satisfying for $x\in\Omega$,*

$$
a(x)\approx(d_\Omega(x))^{-\lambda}L(d_\Omega(x)),
$$

where $\lambda\le 2$, $L\in\mathcal{K}_0$ defined on $(0,\eta]$, $(\eta>\operatorname{diam}(\Omega))$ such that $\int_0^{\eta}t^{1-\lambda}L(t)dt<\infty$. Then problem (1.2) has a unique positive classical solution u satisfying for each $x\in\Omega$,

$$
u(x)\approx(d_\Omega(x))^{\min(\frac{2-\lambda}{1-\sigma},1)}\Phi_{L,\lambda,\sigma}(d_\Omega(x)).
$$

On the other hand, Chemmam et al. [8] were concerned with $\mathcal{K}_\infty$ the set of Karamta functions regularly varying at infinity consisting of functions L defined on $[1,\infty)$ by

$$
L(t):=c\exp\Big(\int_1^{t}\frac{z(s)}{s}ds\Big),
$$

where $c > 0$ and z is a continuous function on $[1, \infty)$ such that $\lim_{t\to\infty} z(t) = 0$. As a standard example of functions belonging to the class $\mathcal{K}_\infty$, we have

$$L(t) = \exp\Big(\prod_{k=1}^{m} (\log_k(\omega t))^{\tau_k}\Big),$$

where $m \in \mathbb{N}^*$, $\tau_k \in (0, 1)$ and ω is a sufficiently large positive real number such that the function L is defined and positive on $[1, \infty)$.

By using properties of functions in $\mathcal{K}_\infty$, the authors in [8] studied the asymptotic behavior of the unique classical solution of the problem

$$\begin{gathered} -\Delta u = a(x)u^\sigma \quad \text{in } \mathbb{R}^n, \\ u > 0 \quad \text{in } \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) = 0, \end{gathered} \tag{1.3}$$

where $n \geq 3$ and $\sigma < 1$. The existence of a unique classical solution of (1.3) has been proved in [5, 27]. Namely, Chemmam et al. [8] proved the following result.

Theorem 1.2. *Let $a \in C^\gamma_{\rm loc}(\mathbb{R}^n)$, $0 < \gamma < 1$, satisfying for $x \in \mathbb{R}^n$,*

$$a(x) \approx (1 + |x|)^{-\mu} L(1 + |x|),$$

where $\mu \geq 2$, $L \in \mathcal{K}_\infty$ such that $\int_1^\infty t^{1-\mu} L(t)dt < \infty$. Then the solution u of problem (1.3) satisfies for each $x \in \mathbb{R}^n$,

$$u(x) \approx (1 + |x|)^{-\min(\frac{\mu-2}{1-\sigma}, n-2)} \Psi_{L,\mu,\sigma}(1 + |x|).$$

Here and always, for $\mu \geq 2$, $\sigma < 1$ and $L \in \mathcal{K}_\infty$ such that $\int_1^\infty t^{1-\mu} L(t)dt < \infty$, the function $\Psi_{L,\mu,\sigma}$ is defined on $[1, \infty)$ by

$$\Psi_{L,\mu,\sigma}(t) := \begin{cases} \big(\int_t^\infty \frac{L(s)}{s} ds\big)^{\frac{1}{1-\sigma}}, & \text{if } \mu = 2, \\ (L(t))^{\frac{1}{1-\sigma}}, & \text{if } 2 < \mu < n - \sigma(n-2), \\ \big(\int_1^{t+1} \frac{L(s)}{s} ds\big)^{\frac{1}{1-\sigma}}, & \text{if } \mu = n - \sigma(n-2), \\ 1, & \text{if } \mu > n - \sigma(n-2). \end{cases}$$

In [31], the authors were concerned with the existence, uniqueness and estimates of positive classical solutions to the following semilinear Dirichlet problem

$$\begin{gathered} -\Delta u = a(x)u^\sigma \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ \lim_{|x|\to 1} u(x) = \lim_{|x|\to\infty} u(x) = 0, \end{gathered} \tag{1.4}$$

where $\Omega = \{x \in \mathbb{R}^n : |x| > 1\}$ is the complementary of the closed unit ball of $\mathbb{R}^n$ $(n \geq 3)$, $\sigma < 1$. Since problem (1.4) involves homogeneous Dirichlet boundary conditions which combine those of [8] and [29], the authors in [31] imposed on the weight a an appropriate assumption related to $\mathcal{K}_0$ and $\mathcal{K}_\infty$. By means of sub-supersolution method, the authors proved that problem (1.4) has a unique positive classical solution which satisfies a specific asymptotic behavior.

Motivated by all the works above, the purpose of this paper is to establish the existence, uniqueness and the asymptotic behavior of a positive classical solution to the following semilinear boundary value problem

$$\begin{gathered} -\Delta u = a(x)u^\sigma \text{ in } D, \\ u > 0 \quad \text{in } D, \\ u\big|_{\partial D} = 0, \\ \lim_{|x|\to\infty} u(x) = 0, \end{gathered} \tag{1.5}$$

where $\sigma < 1$ and D is an unbounded regular domain in $\mathbb{R}^n$ $(n \geq 3)$, with compact boundary. The nonlinearity a is required to satisfy an appropriate condition related to Karamata classes $\mathcal{K}_0$ and $\mathcal{K}_\infty$. The characteristic of problem (1.5) that unlike [31], the domain D is not necessarily radial. This fact makes problem (1.5) more difficult and complicated and this work attempts to deal exactly with this case.

Throughout this paper, we denote by $G_\Omega(x, y)$ the Green function of the Dirichlet Laplacian in a domain Ω of $\mathbb{R}^n$. We recall that $d_\Omega(x)$ denotes the Euclidean distance from $x \in \Omega$ to the boundary of Ω.

Let $x_0 \in \mathbb{R}^n\backslash\overline{D}$ and $r > 0$ such that $\overline{B}(x_0, r) := \{x \in \mathbb{R}^n : |x-x_0| \leq r\} \subset \mathbb{R}^n\backslash\overline{D}$. Then we have

$$\begin{gathered} G_D(x, y) = r^{2-n} G_{\frac{D-x_0}{r}}\big(\frac{x - x_0}{r}, \frac{y - x_0}{r}\big), \quad \text{for } x, y \in D, \\ d_D(x) = r d_{\frac{D-x_0}{r}}\big(\frac{x - x_0}{r}\big), \text{ for } x \in D. \end{gathered}$$

So, without loss of generality, we may assume that $\overline{B}(0, 1) \subset \mathbb{R}^n\backslash\overline{D}$. Form here on, for $x \in D$, we denote by $\delta(x) = d_D(x)$ and $\rho(x) = \frac{\delta(x)}{1+\delta(x)}$.

To study problem (1.5), we suppose that the function a satisfies the following hypothesis:

(H1) a is a nonnegative function in $C^\gamma_{\rm loc}(D)$, $0 < \gamma < 1$, such that for $x \in D$,

$$a(x) \approx (\rho(x))^{-\lambda} M(\rho(x))|x|^{-\mu} N(|x|),$$

where $\lambda \leq 2 \leq \mu$, $M \in \mathcal{K}_0$ defined on $(0, \eta]$, $(\eta > 1)$, $N \in \mathcal{K}_\infty$ satisfying

$$\int_0^\eta t^{1-\lambda} M(t)dt < \infty, \quad \int_1^\infty t^{1-\mu} N(t)dt < \infty.$$

Our main result in this paper is the following.

Theorem 1.3. *Assume* (H1)*, then problem* (1.5) *has a unique classical solution* u *satisfying*

$$u(x) \approx \theta(x), \quad x \in D, \tag{1.6}$$

where

$$\theta(x) := \frac{(\rho(x))^{\min(\frac{2-\lambda}{1-\sigma},1)}}{|x|^{\min(\frac{\mu-2}{1-\sigma},n-2)}} \Phi_{M,\lambda,\sigma}(\rho(x))\Psi_{N,\mu,\sigma}(|x|). \tag{1.7}$$

The techniques used for proving Theorem 1.3 are based on the sub-supersolution method. For the convenience of the readers, we shall recall the following definitions. A positive function $v \in C^{2,\gamma}(D)$, $0 < \gamma < 1$, is called a subsolution of problem (1.5) if

$$\begin{gathered} -\Delta v \leq a(x)\, v^\sigma \quad \text{in } D, \\ v\big|_{\partial D} = 0, \\ \lim_{|x|\to\infty} v(x) = 0. \end{gathered}$$

If the inequality is reversed, v is called a supersolution of problem (1.5).

Since our approach is based on potential theory tools, we lay out some basic arguments that we are mainly concerned with in this work. For a nonnegative measurable function f defined on D, we denote by Vf the potential of f defined on D by

$$Vf(x) = \int_D G_D(x,y)f(y)\,dy.$$

Recall that for each nonnegative function f in $C^{\gamma}_{\rm loc}(D)$, $0<\gamma<1$, such that $Vf \in L^{\infty}(D)$, we have $Vf \in C^{2,\gamma}_{\rm loc}(D)$ and satisfies $-\Delta(Vf) = f$ in D; see [34, Theorem 6.6 page 119].

The outline of this article is as follows. In Section 2, we state and prove some preliminary lemmas, involving some already known results on functions in $\mathcal{K}_0$ and $\mathcal{K}_\infty$. In Section 3, we give estimates on some potential functions. Section 4 is devoted to the proof of our main result stated in Theorem 1.3.

2. Properties of the Karamata classes $\mathcal{K}_0$ and $\mathcal{K}_\infty$

We collect in this paragraph some fundamental properties of functions belonging to the Karamata classes $\mathcal{K}_0$ and $\mathcal{K}_\infty$. It is easy to verify the following results.

Proposition 2.1. (i) *A function L is in $\mathcal{K}_0$ defined on $(0,\eta]$, $\eta>0$, if and only if L is a positive function in $C^1((0,\eta])$, such that*

$$\lim_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0.$$

(ii) *A function L is in $\mathcal{K}_\infty$ if and only if L is a positive function in $C^1([1,\infty))$, such that*

$$\lim_{t\to \infty} \frac{tL'(t)}{L(t)} = 0.$$

Remark 2.2. Using Proposition 2.1, we deduce that the map $t \mapsto L(t)$ belongs to $\mathcal{K}_\infty$ if and only if the map $t \mapsto L(\frac{1}{t})$, defined on $(0,1]$, belongs to $\mathcal{K}_0$.

Lemma 2.3 ([8, 9, 37])**.** (i) *Let $p\in\mathbb{R}$ and $L_1, L_2 \in \mathcal{K}_0$ (resp. $\mathcal{K}_\infty$). Then the functions L_1+L_2, L_1L_2 and L_1^p belong to the class $\mathcal{K}_0$ (resp. $\mathcal{K}_\infty$).*

(ii) *Let $\varepsilon>0$ and $L\in\mathcal{K}_0$ (resp. $\mathcal{K}_\infty$). Then we have*

$$\lim_{t\to 0^+} t^{\varepsilon}L(t) = 0 \quad (\textit{resp. } \lim_{t\to\infty} t^{-\varepsilon}L(t) = 0).$$

Lemma 2.4 (Karamata's Theorem [8, 37])**.** (a) *Let $\gamma\in\mathbb{R}$ and $L\in\mathcal{K}_0$ defined on $(0,\eta]$, $\eta>0$. Then we have the following assertions:*

(i) *If $\gamma>-1$, then the integral $\int_0^{\eta} t^{\gamma}L(t)dt$ converges and*

$$\int_0^t s^{\gamma}L(s)ds \overset{\sim}{t\to 0^+} \frac{t^{1+\gamma}L(t)}{1+\gamma}.$$

(ii) *If $\gamma<-1$, then $\int_0^{\eta} t^{\gamma}L(t)dt$ diverges and*

$$\int_t^{\eta} s^{\gamma}L(s)ds \overset{\sim}{t\to 0^+} -\frac{t^{1+\gamma}L(t)}{1+\gamma}.$$

(b) *Let $L\in\mathcal{K}_\infty$ and $\gamma\in\mathbb{R}$. Then we have the following:*

(i) *If $\gamma < -1$, then $\int_1^\infty t^\gamma L(t)dt$ converges and*
$$\int_t^\infty s^\gamma L(s)ds \underset{t\to\infty}{\sim} -\frac{t^{1+\gamma}L(t)}{1+\gamma}.$$
(ii) *If $\gamma > -1$, then $\int_1^\infty t^\gamma L(t)dt$ diverges and*
$$\int_1^t s^\gamma L(s)ds \underset{t\to\infty}{\sim} \frac{t^{1+\gamma}L(t)}{1+\gamma}.$$

Lemma 2.5 ([9, 37])**.** *Let $L \in \mathcal{K}_0$ defined on $(0,\eta]$, $\eta > 0$, then we have*
$$\lim_{t\to 0^+} \frac{L(t)}{\int_t^\eta \frac{L(s)}{s}ds} = 0.$$
In particular, $t \mapsto \int_t^\eta \frac{L(s)}{s}ds \in \mathcal{K}_0$. If further, $\int_0^\eta \frac{L(s)}{s}ds$ converges, then
$$\lim_{t\to 0^+} \frac{L(t)}{\int_0^t \frac{L(s)}{s}ds} = 0 \quad \text{and} \quad t \mapsto \int_0^t \frac{L(s)}{s}ds \in \mathcal{K}_0.$$

Lemma 2.6. *Let $L \in \mathcal{K}_0$ defined on $(0,\eta]$, $\eta > 1$, and $a, b \in (0,1)$, $\alpha \geq 1$ such that*
$$\frac{1}{\alpha}b \leq a \leq \alpha\, b. \tag{2.1}$$
Then there exists $m \geq 0$ such that
$$\alpha^{-m}L(b) \leq L(a) \leq \alpha^m L(b).$$

Proof. Let $L \in \mathcal{K}_0$. There exists $c > 0$ and $z \in C([0,\eta])$ such that $z(0) = 0$ and satisfying for each $t \in (0,\eta]$
$$L(t) = c\exp\Big(\int_t^\eta \frac{z(s)}{s}ds\Big).$$
Let $m := \sup_{s\in[0,\eta]} |z(s)|$, then for each $s \in [0,\eta]$, $-m \leq z(s) \leq m$. This together with (2.1) imply
$$-m\ln(\alpha) \leq \int_a^b \frac{z(s)}{s}\,ds \leq m\ln(\alpha).$$
It follows that
$$\alpha^{-m}L(b) \leq L(a) \leq \alpha^m L(b).$$
□

Lemma 2.7 ([8])**.** (i) *Let $L \in \mathcal{K}_\infty$. Then we have*
$$\lim_{t\to\infty} \frac{L(t)}{\int_1^t \frac{L(s)}{s}ds} = 0 \quad \text{and} \quad t \mapsto \int_1^{t+1} \frac{L(s)}{s}ds \in \mathcal{K}_\infty.$$
If further, $\int_1^\infty \frac{L(s)}{s}ds$ converges, then
$$\lim_{t\to\infty} \frac{L(t)}{\int_t^\infty \frac{L(s)}{s}ds} = 0 \quad \text{and} \quad t \mapsto \int_t^\infty \frac{L(s)}{s}ds \in \mathcal{K}_\infty.$$
(ii) *If $L \in \mathcal{K}_\infty$ then there exists $m \geq 0$ such that for every $\alpha > 0$ and $t \geq 1$, we have*
$$(1+\alpha)^{-m}L(t) \leq L(\alpha + t) \leq (1+\alpha)^m L(t).$$

3. Asymptotic behavior of some potential functions

In what follows, we are going to give estimates on the potential functions Va and $V(a\,\theta^\sigma)$, where a is a function satisfying (H1) and θ is the function given in (1.7). These estimates will be useful in the proof of our main result. The next lemma which is due to [2], plays a capital role to establish our estimates.

Lemma 3.1. *Let Ω be a bounded regular domain in $\mathbb{R}^n$ ($n \geq 3$) containing 0. We recall that $G_\Omega(x,y)$ is the Green function of the Dirichlet Laplacian in Ω and $d_\Omega(x)$ is the Euclidean distance from $x \in \Omega$ to the boundary of Ω. If p is a positive continuous function in $\Omega\backslash\{0\}$ such that for $x \in \Omega\backslash\{0\}$,*

$$p(x) \approx (d_\Omega(x))^{-\nu_1} L_1(d_\Omega(x))|x|^{-\nu_2} L_2(|x|),$$

where $\nu_1 \leq 2$, $\nu_2 \leq n$, L_1, $L_2 \in \mathcal{K}_0$ defined on $(0,\eta]$, ($\eta > \operatorname{diam}(\Omega)$) satisfying the following conditions of integrability $\int_0^\eta t^{1-\nu_1} L_1(t)dt < \infty$ and $\int_0^\eta t^{n-1-\nu_2} L_2(t)dt < \infty$, then for $x \in \Omega\backslash\{0\}$,

$$G_\Omega p(x) := \int_\Omega G_\Omega(x,y)p(y)dy \approx (d_\Omega(x))^{\min(2-\nu_1,1)} \widetilde{L}_1(d_\Omega(x))|x|^{\min(2-\nu_2,0)} \widetilde{L}_2(|x|),$$

where

$$\widetilde{L}_1(t) = \begin{cases} 1, & \text{if } \nu_1 < 1, \\ \int_t^\eta \frac{L_1(s)}{s} ds, & \text{if } \nu_1 = 1, \\ L_1(t), & \text{if } 1 < \nu_1 < 2, \\ \int_0^t \frac{L_1(s)}{s} ds, & \text{if } \nu_1 = 2 \end{cases}$$

and

$$\widetilde{L}_2(t) = \begin{cases} \int_0^t \frac{L_2(s)}{s} ds, & \text{if } \nu_2 = n, \\ L_2(t), & \text{if } 2 < \nu_2 < n, \\ \int_t^\eta \frac{L_2(s)}{s} ds, & \text{if } \nu_2 = 2, \\ 1, & \text{if } \nu_2 < 2. \end{cases}$$

In the sequel, we denote by D^* the open set $D^* = \{x^* \in B(O,1),\ x \in D \cup \{\infty\}\}$, where $x^* = \frac{x}{|x|^2}$ is the Kelvin transformation from $D \cup \{\infty\}$ onto D^*. We note that D^* is a bounded regular domain which contains 0. Moreover, from [3], we have for each $x \in D$,

$$\rho(x) \approx \delta_{D^*}(x^*), \tag{3.1}$$

where $\delta_{D^*}(x^*) = \operatorname{dist}(x, \partial D^*)$.

Proposition 3.2. *Let a be a function satisfying* (H1). *Then for $x \in D$, we have*

$$Va(x) \approx \frac{(\rho(x))^{\min(2-\lambda,1)}}{|x|^{\min(\mu-2,n-2)}} \,\Phi_{M,\lambda,0}(\rho(x))\, \Psi_{N,\mu,0}(|x|).$$

Proof. Let a be a function satisfying (H1). For $x \in D$, we have

$$Va(x) \approx \int_D G_D(x,y)\,(\rho(y))^{-\lambda} M(\rho(y))|y|^{-\mu} N(|y|)dy.$$

From (3.1) and Lemma 2.6, we obtain that for $x \in D$,

$$M(\rho(x)) \approx M(\delta_{D^*}(x^*)). \tag{3.2}$$

Combining (3.1), (3.2) with the fact that for $x, y \in D$,

$$G_D(x,y) = |x|^{2-n}|y|^{2-n} G_{D^*}(x^*, y^*),$$

we obtain

$$Va(x) \approx |x|^{2-n} \int_{D^*} G_{D^*}(x^*, z)(\delta_{D^*}(z))^{-\lambda} M(\delta_{D^*}(z))|z|^{\mu-n-2} N(\frac{1}{|z|})dz.$$

Using (H1), Remark 2.2 and applying Lemma 3.1 with $\nu_1 = \lambda$, $\nu_2 = -\mu + n + 2$, $L_1(t) = M(t)$ and $L_2(t) = N(\frac{1}{t})$, we get

$$Va(x) \approx |x|^{2-n}(\delta_{D^*}(x^*))^{\min(2-\lambda,1)} \widetilde{L_1}(\delta_{D^*}(x^*))|x^*|^{\min(0,\mu-n)} \widetilde{L_2}(|x^*|),$$

where for $t \in (0, 1]$,

$$\widetilde{L_1}(t) = \begin{cases} 1, & \text{if } \lambda < 1, \\ \int_t^\eta \frac{M(s)}{s} ds, & \text{if } \lambda = 1, \\ M(t), & \text{if } 1 < \lambda < 2, \\ \int_0^t \frac{M(s)}{s} ds, & \text{if } \lambda = 2 \end{cases} \tag{3.3}$$

and

$$\widetilde{L_2}(t) = \begin{cases} \int_0^t \frac{N(\frac{1}{s})}{s} ds, & \text{if } \mu = 2, \\ N(\frac{1}{t}), & \text{if } 2 < \mu < n, \\ \int_t^\eta \frac{N(\frac{1}{s})}{s} ds, & \text{if } \mu = n, \\ 1, & \text{if } \mu > n. \end{cases} \tag{3.4}$$

It is obvious to see from (3.3) that on $(0, 1]$, $\widetilde{L_1} = \Phi_{M,\lambda,0}$. Furthermore, by Proposition 2.1 and Lemma 2.5, we get that $\Phi_{M,\lambda,0} \in \mathcal{K}_0$. Which gives by using (3.1) and Lemma 2.6 that for $x \in D$,

$$(\delta_{D^*}(x^*))^{\min(2-\lambda,1)} \widetilde{L_1}(\delta_{D^*}(x^*)) \approx (\rho(x))^{\min(2-\lambda,1)}\, \Phi_{M,\lambda,0}(\rho(x)). \tag{3.5}$$

On the other hand, from (3.4), we obtain that for $t \in (0, 1]$,

$$\widetilde{L_2}(t) = \begin{cases} \int_{1/t}^\infty \frac{N(s)}{s} ds, & \text{if } \mu = 2, \\ N(\frac{1}{t}), & \text{if } 2 < \mu < n, \\ \int_{1/\eta}^{\frac{1}{t}} \frac{N(s)}{s} ds, & \text{if } \mu = n, \\ 1, & \text{if } \mu > n. \end{cases}$$

By Proposition 2.1 and Lemma 2.7, we deduce that the function $t \mapsto \widetilde{L_2}(\frac{1}{t})$ is in $\mathcal{K}_\infty$ and for $t \in [1, \infty)$,

$$\widetilde{L_2}(\frac{1}{t}) \approx \Psi_{N,\mu,0}(t).$$

This with the fact that for $x \in D$, $|x^*| = \frac{1}{|x|}$ implies that

$$\begin{aligned} |x|^{2-n}|x^*|^{\min(0,\mu-n)} \widetilde{L_2}(|x^*|) &= |x|^{2-n-\min(0,\mu-n)} \widetilde{L_2}(\frac{1}{|x|}) \\ &\approx |x|^{2-n-\min(0,\mu-n)} \Psi_{N,\mu,0}(|x|). \end{aligned} \tag{3.6}$$

Since $2 - n - \min(0, \mu - n) = -\min(\mu - 2, n - 2)$, we finally obtain by combining (3.5) and (3.6) that for $x \in D$,

$$Va(x) \approx \frac{(\rho(x))^{\min(2-\lambda,1)}}{|x|^{\min(\mu-2,n-2)}}\, \Phi_{M,\lambda,0}(\rho(x))\, \Psi_{N,\mu,0}(|x|).$$

This completes the proof. □

The following proposition plays a crucial role in the proof of Theorem 1.3.

Proposition 3.3. *Let a be a function satisfying* (H1) *and let θ be the function given by* (1.7). *Then for $x \in D$, we have*

$$V(a\theta^\sigma)(x) \approx \theta(x).$$

Proof. Let a be a function satisfying (H1). Then for $x \in D$,

$$\begin{aligned}
&a(x)\theta^\sigma(x)\\
&\approx (\rho(x))^{-\lambda+\sigma\min(\frac{2-\lambda}{1-\sigma},1)}|x|^{-\mu-\sigma\min(\frac{\mu-2}{1-\sigma},n-2)}(M\Phi^\sigma_{M,\lambda,\sigma})(\rho(x))(N\Psi^\sigma_{N,\mu,\sigma})(|x|)\\
&:= (\rho(x))^{-\lambda_1}|x|^{-\mu_1}\widetilde{M}(\rho(x))\widetilde{N}(|x|).
\end{aligned}$$

Here $\lambda_1 = \lambda - \sigma\min(\frac{2-\lambda}{1-\sigma},1)$ and $\mu_1 = \mu + \sigma\min(\frac{\mu-2}{1-\sigma}, n-2)$. We can easily see that $\lambda_1 \le 2 \le \mu_1$. By Proposition 2.1 and Lemmas 2.3 and 2.5, the function $\widetilde{M} := M\Phi^\sigma_{M,\lambda,\sigma}$ is in $\mathcal{K}_0$. Besides, from Lemma 2.4 and hypothesis (H1), we reach the condition of integrability $\int_0^\eta t^{1-\lambda_1}\widetilde{M}(t)dt < \infty$. On the other hand, applying Proposition 2.1 and Lemmas 2.3 and 2.7, we deduce that the function $\widetilde{N} := N\Psi^\sigma_{N,\mu,\sigma}$ belongs to $\mathcal{K}_\infty$. By Lemma 2.4 and hypothesis (H1), we obtain that $\int_1^\infty t^{1-\mu_1}\widetilde{N}(t)dt$ converges. Hence, it follows from Proposition 3.2, that for $x \in D$

$$V(a\theta^\sigma)(x) \approx \frac{(\rho(x))^{\min(2-\lambda_1,1)}}{|x|^{\min(\mu_1-2,n-2)}}\Phi_{\widetilde{M},\lambda_1,0}(\rho(x))\Psi_{\widetilde{N},\mu_1,0}(|x|).$$

Now, by computation we have

$$\min(2-\lambda_1,1) = \min\Big(\frac{2-\lambda}{1-\sigma},1\Big), \quad \min\Big(\mu_1-2,n-2\Big) = \min\Big(\frac{\mu-2}{1-\sigma},n-2\Big).$$

Furthermore, we obtain by elementary calculus that for $x \in D$,

$$\Phi_{\widetilde{M},\lambda_1,0}(\rho(x)) = \Phi_{M,\lambda,\sigma}(\rho(x)) \quad \text{and} \quad \Psi_{\widetilde{N},\mu_1,0}(|x|) = \Psi_{N,\mu,\sigma}(|x|).$$

This completes the proof. $\square$

4. Proof of Theorem 1.3

4.1. **Existence and asymptotic behavior.** Let a be a function satisfying (H1). We look now at the existence of positive solution of problem (1.5) satisfying (1.6). The main idea is to find a subsolution and a supersolution to problem (1.5) of the form $cV(a\omega^\sigma)$, where $c > 0$ and $\omega(x) = (\rho(x))^\alpha|x|^\beta L(\rho(x))K(|x|)$, which will satisfy

$$V(a\omega^\sigma) \approx \omega. \tag{4.1}$$

So the choice of the real numbers α, β and the functions L in $\mathcal{K}_0$ and K in $\mathcal{K}_\infty$ is such that (4.1) is satisfied. Setting $\omega(x) = \theta(x)$, where θ is the function given by (1.7), we have by Proposition 3.3, that the function θ satisfies (4.1). Hence, let $v = V(a\theta^\sigma)$ and let $m \ge 1$ be such that

$$\frac{1}{m}\theta \le v \le m\theta. \tag{4.2}$$

This implies that for $\sigma < 1$, we have

$$\frac{1}{m^{|\sigma|}}\theta^\sigma \le v^\sigma \le m^{|\sigma|}\theta^\sigma.$$

Put $c = m^{\frac{|\sigma|}{1-\sigma}}$, then it is easy to show that $\underline{u} = \frac{1}{c}v$ and $\overline{u} = cv$ are respectively a subsolution and a supersolution of problem (1.5).

Now, since $c \geq 1$, we get $\underline{u} \leq \overline{u}$ on D. Thanks to the method of sub-supersolution (see [33]), we deduce that problem (1.5) has a classical solution u such that $\underline{u} \leq u \leq \overline{u}$ in D. By using (4.2), we conclude that u satisfies (1.6). This completes the proof.

4.2. **Uniqueness.** Let a be a function satisfying (H1) and θ be the function defined in (1.7). We aim to show that problem (1.5) has a unique positive solution in the cone

$$\Gamma = \{u \in C^{2,\gamma}(D) : u(x) \approx \theta(x)\}.$$

To this end, we need the following lemma.

Lemma 4.1. *Let a be a function satisfying* (H1)*. If $u \in \Gamma$ is a solution of problem* (1.5)*, then u satisfies the integral equation*

$$u = V(au^\sigma). \tag{4.3}$$

Proof. Let $u \in \Gamma$ be a solution of problem (1.5). It is obvious that the function au^σ is in $C^\gamma_{\rm loc}(D)$, $0 < \gamma < 1$. Since $u \approx \theta$, then by Proposition 3.3, we have $V(au^\sigma) \approx \theta$. So using (1.7) and by the virtue of Proposition 2.1 and Lemmas 2.3, 2.5 and 2.7, we obtain that $V(au^\sigma)$ is in $L^\infty(D)$ and satisfies

$$V(au^\sigma)\big|_{\partial D} = 0 \quad \text{and} \quad \lim_{|x|\to\infty} V(au^\sigma)(x) = 0.$$

So, we deduce that $V(au^\sigma)$ is a classical solution of problem (1.5). Therefore, we conclude that the function $v = u - V(au^\sigma)$ is a classical solution of the following Dirichlet problem

$$\begin{gathered} -\Delta h = 0 \quad \text{in } D, \\ h\big|_{\partial D} = 0, \\ \lim_{|x|\to\infty} h(x) = 0. \end{gathered}$$

Which implies that $v = 0$ and so u satisfies (4.3). $\square$

Now to prove the uniqueness, we consider the following cases.

4.2.1. *Case $\sigma < 0$.* Let u and v be two solutions of (1.5) in Γ and put $w = u - v$. Then by applying Lemma 4.1, we get that the function w satisfies

$$w + V(hw) = 0 \quad \text{in } D,$$

where h is the nonnegative measurable function defined in D by

$$h(x) = \begin{cases} a(x)\frac{(v(x))^\sigma - (u(x))^\sigma}{u(x)-v(x)} & \text{if } u(x) \neq v(x), \\ 0 & \text{if } u(x) = v(x). \end{cases}$$

Furthermore, it is clear to see that $V(h|w|) < \infty$. Then, by [3, lemma 4.1] it follows that $w = 0$. This proves the uniqueness.

4.2.2. *Case* $0 \le \sigma < 1$. Let us now assume that u and v are arbitrary solutions of problem (1.5) in Γ. Since $u, v \in \Gamma$, then there exists a constant $m \ge 1$ such that

$$\frac{1}{m} \le \frac{u}{v} \le m, \text{ in } D.$$

This implies that the set

$$J := \{t \in (0,1] : tu \le v\}$$

is not empty. Now put $c := \sup J$. It is easy to see that $0 < c \le 1$. On the other hand, we have

$$\begin{gathered} -\Delta(v - c^\sigma u) = a(x)(v^\sigma - c^\sigma u^\sigma) \ge 0 \text{ in } D, \\ (v - c^\sigma u)\big|_{\partial D} = 0, \\ \lim_{|x|\to\infty} (v - c^\sigma u)(x) = 0. \end{gathered}$$

Then, by the maximum principle, we deduce that $c^\sigma u \le v$. Which implies that $c^\sigma \le c$. Using the fact that $\sigma < 1$, we get that $c \ge 1$. Hence, we arrive at $u \le v$ and by symmetry, we obtain that $u = v$. This completes the proof.

As applications of Theorem 1.3, we give the following examples.

Example 4.2. Let $\sigma < 1$ and a be a nonnegative function in $C^\gamma_{\rm loc}(D)$, $0 < \gamma < 1$, such that for $x \in D$,

$$a(x) \approx (\rho(x))^{-\lambda}(\log(\frac{4}{\rho(x)}))^{-\alpha}(1 + |x|)^{-\mu}(\log(2(1 + |x|)))^{-\beta},$$

where $\lambda \le 2 \le \mu$, $\alpha > 1$ and $\beta > 1$. Then by Theorem 1.3, problem (1.5) has a unique positive classical solution u satisfying, for $x \in D$,

$$u(x) \approx \Phi(\rho(x))\Psi(|x|),$$

where

$$\Phi(\rho(x)) = \begin{cases} \rho(x), & \text{if } \lambda \le 1 + \sigma, \\ (\rho(x))^{\frac{2-\lambda}{1-\sigma}}(\log(\frac{4}{\rho(x)}))^{\frac{-\alpha}{1-\sigma}}, & \text{if } 1 + \sigma < \lambda < 2, \\ (\log(\frac{4}{\rho(x)}))^{\frac{1-\alpha}{1-\sigma}}, & \text{if } \lambda = 2 \end{cases}$$

and

$$\Psi(|x|) = \begin{cases} (\log(2|x|))^{\frac{1-\beta}{1-\sigma}}, & \text{if } \mu = 2, \\ |x|^{\frac{2-\mu}{1-\sigma}}(\log(2|x|))^{\frac{-\beta}{1-\sigma}}, & \text{if } 2 < \mu < n - \sigma(n-2), \\ |x|^{2-n}, & \text{if } \mu \ge n - \sigma(n-2). \end{cases}$$

Example 4.3. Let $\sigma < 1$ and a be a nonnegative function in $C^\gamma_{\rm loc}(D)$, $0 < \gamma < 1$, such that for $x \in D$,

$$a(x) \approx (\rho(x))^{-\lambda}(\log(\frac{4}{\rho(x)}))^{-\alpha}(1 + |x|)^{-2}(\log(2(1 + |x|)))^{-2},$$

where $\lambda < 2$ and $\alpha \in \mathbb{R}$. Then by Theorem 1.3, problem (1.5) has a unique positive classical solution u satisfying the following estimates.

(i) If $\lambda < 1 + \sigma$ and $\alpha \in \mathbb{R}$ or $\lambda = 1 + \sigma$ and $\alpha > 1$, then for $x \in D$,

$$u(x) \approx \rho(x)(\log(2|x|))^{\frac{-1}{1-\sigma}}.$$

(ii) If $\lambda = 1+\sigma$ and $\alpha = 1$, then for $x \in D$,
$$u(x) \approx \rho(x)(\log_2(\frac{4}{\rho(x)}))^{\frac{1}{1-\sigma}}(\log(2|x|))^{\frac{-1}{1-\sigma}}.$$

(iii) If $\lambda = 1+\sigma$ and $\alpha < 1$, then for $x \in D$,
$$u(x) \approx \rho(x)(\log(\frac{4}{\rho(x)}))^{\frac{1-\alpha}{1-\sigma}}(\log(2|x|))^{\frac{-1}{1-\sigma}}.$$

(iv) If $1+\sigma < \lambda < 2$ and $\alpha \in \mathbb{R}$, then for $x \in D$,
$$u(x) \approx (\rho(x))^{\frac{2-\lambda}{1-\sigma}}(\log(\frac{4}{\rho(x)}))^{\frac{-\alpha}{1-\sigma}}(\log(2|x|))^{\frac{-1}{1-\sigma}}.$$

References

[1] R. Alsaedi, H. Mâagli, N. Zeddini; *Exact behavior of the unique positive solutions to some singular elliptic problem in exterior domains*, Nonlinear Anal., 119 (2015), 186–198.
[2] I. Bachar, H. Mâagli, V. D. Rădulescu; *Singular solutions of a nonlinear elliptic equation in a punctured domain*, Elect. Journal of Qualitative Theory of Diff. Equat., 2017(3) (2017), 1-19.
[3] I. Bachar, H. Mâagli, N. Zeddini; *Estimates on the Green Function and Existence of Positive Solutions of Nonlinear Singular Elliptic Equations*, Commun. Contemp. Math., 3 (2003), 401–434.
[4] S. Ben Othman, H. Mâagli, S. Masmoudi, M. Zribi; *Exact asymptotic behavior near the boundary to the solution for singular nonlinear Dirichlet problems*, Nonlinear Anal., 71 (2009), 4137–4150.
[5] H. Brezis, S. Kamin; *Sublinear elliptic equation in $\mathbb{R}^n$*, Manuscripta Math., 74 (1992), 87–106.
[6] A. J. Callegari, M. B. Friedman; *An analytical solution of a nonlinear singular boundary value problem in the theory of viscous fluids*, J. Math. Anal. Appl., 21 (1968), 510–529.
[7] M. Chaieb, A. Dhifli, S. Zermani; *Existence and asymptotic behavior of positive solutions of a semilinear elliptic system in a bounded domain*, Opuscula Math., 36 (2016), no. 3, 315–336.
[8] R. Chemmam, A. Dhifli, H. Mâagli; *Asymptotic behavior of ground state solutions for sublinear and singular nonlinear Dirichlet problem*, Electronic J. Diff. Eqs., 2011 no. 88 (2011), 1–12.
[9] R. Chemmam, H. Mâagli, S. Masmoudi, M. Zribi; *Combined effects in nonlinear singular elliptic problems in a bounded domain*, Advances in Nonlinear Analysis, 1 (2012), 301–318.
[10] F. Cîrstea, V. D. Rădulescu; *Existence and uniqueness of blow-up solutions for a class of logistic equations*, Commun. Contem. Math., 4 (2002), 559–585.
[11] F. Cîrstea, V. D. Rădulescu; *Uniqueness of the blow-up boundary solution of logistic equations with absorption*, C. R. Acad. Sci. Paris, Ser. I, 335 (2002), 447–452.
[12] F. Cîrstea, V. D. Rădulescu; *Asymptotics for the blow-up boundary solution of logistic equations with absorption*, C. R. Acad. Sci. Paris, Ser. I, 336 (2003), 231–236.
[13] F. Cîrstea, V. D. Rădulescu; *Extremal singular solutions for degenerate logistic-type equations in anisotropic media*, C. R. Acad. Sci. Paris Sér. I, 339 (2004), 119–124.
[14] F. Cîrstea, V. D. Rădulescu; *Nonlinear problems with boundary blow-up: a Karamata regular variation theory approch*, Asympt. Anal., 46 (2006), 275–298.
[15] F. Cîrstea, V. D. Rădulescu; *Boundary blow-up in nonlinear elliptic equations of Bieberbach-Rademacher*, Transactions Amer. Math. Soc., 356 (2006), 3275–3286.
[16] M. G. Crandall, P. H. Rabinowitz, L. Tartar; *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations 2 (1977), 193–222.
[17] R. Dalmasso; *Solutions d'équations elliptiques semi-linéaires singulières*, Ann. Mat. Pura. Appl., 153 (1988), 191–201.
[18] J. I. Díaz, J. Hernández, J. M. Rakotoson; *On very weak positive solutions to some semilinear elliptic problems with simultaneous singular nonlinear and partial dependence terms*, Milan J. Maths., 79 (2011), 233-245.
[19] S. Dridi, B. Khamessi; *Asymptotic behavior of positive solutions of a semilinear Dirichlet problems in the annulus*, Opuscula Math., 35 (2015), 21–36.

[20] A. Edelson; *Entire solutions of singular elliptic equations*, J. Math. Anal. Appl., 139 (1989), 523–532.
[21] M. Ghergu, V. D. Rădulescu, Singular Elliptic Problems. Bifurcation and Asymptotic Analysis, Oxford Lecture Series in Mathematics and Applications, Vol. 37, Oxford University Press, 2008.
[22] M. Ghergu, V. D. Rădulescu; *Nonlinear PDEs: Mathematical Models in Biology, Chemistry and Population Genetics*, Springer Monographs in Mathematics, Springer Verlag, Heidelberg 2012.
[23] S. Gontara, H. Mâagli, S. Masmoudi, S. Turki; *Asymptotic behavior of positive solutions of a singular nonlinear Dirichlet problems*, J. Math. Anal. Appl., 369 (2010), 719–729.
[24] J. Hernández, F. J. Mancebo, J. M. Vega; *Positive solutions for singular nonlinear elliptic equations*, Proc. Roy. Soc. Edinburgh, 137 A (2007), 41–62.
[25] T. Kusano, C. A. Swanson; *Entire positive solutions of singular semilinear elliptic equations*, Japon J. Math., 11 (1985), 145–155.
[26] A. V. Lair, A. W. Shaker; *Entire solution of a singular semilinear elliptic problem*, J. Math. Anal. Appl., 200 (1996), 498–505.
[27] A. V. Lair, A. W. Shaker; *Classical and weak solutions of a singular semilinear elliptic problem*, J. Math. Anal. Appl., 211 (1997), 371–385.
[28] A. C. Lazer, P. J. McKenna; *On a singular nonlinear elliptic boundary-value problem*, Proc. Amer. Math. Soc., 111 (1991), 721–730.
[29] H. Mâagli; *Asymptotic behavior of positive solutions of a semilinear Dirichlet problems*, Nonlinear Anal., 74 (2011), 2941–2947.
[30] H. Mâagli, S. Ben Othman, S. Dridi; *Existence and asymptotic behavior of ground state solutions of semilinear elliptic system*, Adv. Nonlinear Anal. 6 (2017), no. 3, 301–315.
[31] H. Mâagli, S. Turki, Z. Zine El Abidine; *Asymptotic behavior of positive solutions of semilinear Dirichlet problem outside the unit ball*, Electronic J. Diff. Eqs., 2013 No. 95 (2013), 1–14.
[32] H. Mâagli, M. Zribi; *Existence and estimates of solutions for singular nonlinear elliptic problems*, J. Math. Anal. Appl., 263 (2001), 522–542.
[33] E. S. Noussair, C. A. Swanson; *Positive solutions of quasilinear elliptic equations in exterior domains*, J. Math. Anal. Appl., 75 (1980), 121–133.
[34] S. C. Port, C. J. Stone; *Brownian Motion and Classical Potential Theory*, Academic Press, 1978.
[35] V. D. Rădulescu; *Qualitative analysis of nonlinear elliptic partial differential equations: monotonicity, analytic and variational methods*, Contemporary Mathematics and Its Applications, Vol. 6, Hindawi Publishing Corporation, New York 2012.
[36] D. Repovš; *Asymptotics for singular solutions of quasilinear elliptic equations with an absorption term*, J. Math. Anal. Appl., 395 (2012), no. 1, 78–85.
[37] R. Seneta; *Regular varying functions*, Lectures Notes in Math. 508 Springer-Verlag, Berlin 1976.
[38] A. W. Shaker; *On singular semilinear elliptic equations*, J. Math. Anal. Appl., 173 (1993), 222–228.
[39] J. S. Wong; *On the generalized Emden-Fowler equation*, SIAM Rev., 17 (1975), 339–360.
[40] Z. Zhang; *The asymptotic behavior of the unique solution for the singular Lane-Emden-Fowler equation*, J. Math. Anal. Appl., 312 (2005), 33–43.

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