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EXISTENCE OF MINIMIZERS OF MULTI-CONSTRAINED VARIATIONAL PROBLEMS FOR PRODUCT FUNCTIONS

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Abstract. We prove the existence of minimizers of a class of multi-constrained variational problems in which the non linearity involved is a product function not satisfying compactness, monotonicity, neither symmetry properties. Our result cannot be covered by previous studies that considered only a particular class of integrands. A key step is establishing the strict sub-additivity condition in the vectorial setting. This inequality is also interesting in itself.

1. Introduction

For $c_1, \dots, c_m > 0$, we consider the minimization problem

$$\inf\{J(\vec{u}) : \vec{u} \in S_c\} =: I_{c_1,\dots,c_m}, \tag{1.1}$$

$$J(\vec{u}) = \frac{1}{2}\int |\nabla \vec{u}|^2 - \int F(x, \vec{u}) \tag{1.2}$$

where $\vec{u} = (u_1, \dots, u_m), u_i \in H^1$ and F is a Carathéodory function and

$$\begin{gathered} S_c = \{\vec{u} = (u_1, \dots, u_m) \in H^1 \times \dots \times H^1 : \int u_i^2 = c_i^2, 1 \le i \le m\}, \\ c^2 = \sum_{i=1}^m c_i^2 \,. \end{gathered} \tag{1.3}$$

Under some additional regularity assumptions on F, solutions of (1.1) satisfy the elliptic system

$$\begin{gathered} \Delta u_1 + \partial_1 F(x, u_1, \dots, u_m) + \lambda_1 u_1 = 0 \\ \dots \\ \Delta u_m + \partial_m F(x, u_1, \dots, u_m) + \lambda_m u_m = 0 \,, \end{gathered} \tag{1.4}$$

where λ_i are Lagrange multipliers.

When $\partial_i F(x, u_1, \dots, u_m) = \partial_i F(x, |u_1|, \dots, |u_m|)$, the solutions of (1.1) can also be viewed as standing waves of the non-linear Schrödinger system

$$\begin{gathered} i\partial_t \Phi_1(t, x) + \partial_1 F(x, |\Phi_1|, .. + |\Phi_m|) + \Delta_{xx}\Phi_1 = 0 \\ \dots \end{gathered}$$

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$$\begin{gathered} i\partial_t\Phi_m(t,x)+\partial_m F(x,|\Phi_1|,\dots,+|\Phi_m|)+\Delta_{xx}\Phi_m=0\,,\\ \Phi_i(0,x)=\Phi_i^0(x)\quad 1\le i\le m\,. \end{gathered}$$

To the best of our knowledge, there are no pubications about (1.1) when $m\geq 2$ and the non-linearity F does not satisfy the standard convexity, compactness, symmetry or monotonicity properties. This happens despite the importance of such problem in many domains such as mechanics, engineering and especially non-linear optics, see [1] and references therein. Only very particular cases have been addressed.

The purpose of this paper is to establish the existence of minimizers of (1.1) under following assumptions.

Suppose that the function $F:\mathbb{R}^N\times\mathbb{R}^m\to\mathbb{R}$ is such that $F\in D(\mathbb{R}^N\times\mathbb{R}^m)$:

(F0) for all $x\in\mathbb{R}^N$, $\vec{s}\in\mathbb{R}^m$, there exist $A,B>0$ such that $0\le F(x,\vec{s})\le A(|\vec{s}|^2+|\vec{s}|^{\ell+2})$, and for all $1\le i\le m$, $\partial_i F(x,\vec{s})\le B(|\vec{s}|+|\vec{s}|^{\ell+1})$ where $0<\ell<\frac{4}{N}$.

(F1) There exist $\Delta>0$, $S>0$, $R>0$, $\alpha_1,\dots,\alpha_m>0$, and $t\in[0,2)$ such that $F(x,\vec{s})>\Delta|x|^{-t}|s_1|^{\alpha_1}\dots|s_m|^{\alpha_m}$ for all $|x|\ge R$ and $|\vec{s}|<S$ where $N+2>\frac{N}{2}\alpha+t$; and $\alpha=\sum_{i=1}^m\alpha_i$.

(F2) $F(x,\theta_1 s_1,\theta_m s_m)\ge\theta_{\max}^2 F(x,s_1,\dots,s_m)$ for all $x\in\mathbb{R}^N$, $s_i\in\mathbb{R},\theta_i\ge 1$, where $\theta_{\max}=\max_{1\le i\le m}\theta_i$.

Also we assume that there exists function $F^\infty(x,\vec{s})$ periodic in x; i.e. there exists $z\in\mathbb{Z}^N$ for which $F^\infty(x+z,\vec{s})=F^\infty(x,\vec{s}),\forall x\in\mathbb{R}^N,\vec{s}\in\mathbb{R}^m$ satisfying (F1) and the following properties:

(F3) There exists $0<\alpha<4/N$ such that

$$\lim_{|x|\to\infty}\frac{F(x,\vec{s})-F^\infty(x,\vec{s})}{|\vec{s}|^2+|\vec{s}|^{\alpha+2}}=0$$

uniformly for any $\vec{s}$.

(F4) There exist $A',B'>0$ and $0<\beta<\ell<4/N$ such that $0\le F^\infty(x,\vec{s})\le A'(|\vec{s}|^{\beta+2}+|\vec{s}|^{\ell+2})$ and for $1\le i\le m$,

$$\partial_i F^\infty(x,\vec{s})\le B'(|\vec{s}|^{\beta+1}+|\vec{s}|^{\ell+1})\quad\forall x\in\mathbb{R}^N,\vec{s}\in\mathbb{R}^m.$$

(F5) There exists $\sigma\in[0,4/N)$ such that

$$F^\infty(x,\theta_1 s_1,\dots,\theta_m s_m)\ge\theta_{\max}^{\sigma+2}F^\infty(x,s_1,\dots,s_m)$$

for any $\theta_i\ge 1,x\in\mathbb{R}^N,\vec{s}\in\mathbb{R}^m$, where $\theta_{\max}=\max_{1\le i\le m}\theta_i$.

(F6) $F^\infty(x,\vec{s})\le F(x,\vec{s})$ for any $x\in\mathbb{R}^N$ and $s\in\mathbb{R}^m$, with strict inequality in a measurable set having a positive Lebesgue measure.

Theorem 1.1. *Under Assumptions* (F0)–(F6) *there exists* $\vec{u}_c\in S_c$ *such that* $J(\vec{u}_c)=I_{c_1,\dots,c_m}$.

When $n=2$, as an example of functions satisfying (F0)–(F6), we have

$$F(r,s)=q(r)\sum_{i\neq j}^m a_{ij}|s_i|^{n_i}|s_j|^{n_j}$$

where $a_{ij}>0$, $0<n_i+n_j<\frac{4}{N}$ and $q\in L_+^\infty(0,\infty)$. This class of functions arises in nonlinear optics; see or example [2, 3, 6].

The following is our intermediate result, which is interesting in itself.

Theorem 1.2. *If* (F1) *holds for a function* F^∞*, and* (F4) *and* (F5) *are satisfied, then there exists* $\vec{u} \in S_c$ *such that* $J^\infty(\vec{u}_c) = I^\infty_{c_1,\dots,c_m}$*, where*

$$\begin{gathered} J^\infty(\vec{u}) = \frac{1}{2}\int |\nabla \vec{u}|^2 - \int F^\infty(x,\vec{u}), \\ I^\infty_{c_1,\dots,c_m} = \inf\{J^\infty(\vec{u}) : \vec{u} \in S_c\} \end{gathered} \tag{1.5}$$

Our proofs of Theorems 1.1 and 1.2 are based on the concentration-compactness principle [4, 5]. In the one-constrained setting

$$m_c = \inf\{j(u) : \int u^2 = c^2\}, \tag{1.6}$$

where $j(u) = \frac{1}{2}\int |\nabla u|^2 - \int f(x,u(x))$, the principle states that if (u_n) is a minimizing sequence of problem (1.6), then only one of the three following phenomena can occur.

(1) Vanishing: $\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} u_n^2(x)dx = 0$.
(2) Dichotomy: There exists $a \in (0,c)$ such that for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ and two bounded sequences in H^1, $\{u_{n,1}\}$ and $\{u_{n,2}\}$ (all depending on ε) such that for every $n \geq n_0$.
$$|\int u_{n,1}^2 - a^2| < \varepsilon, \quad |\int u_{n,2}^2 - (c^2 - a^2)| < \varepsilon$$
with $\lim_{n\to\infty} \operatorname{dist}\operatorname{supp}(u_{n,1}, u_{n,2}) = \infty$.
(3) Compactness: There exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that, for all $\varepsilon > 0$, there exists $R(\varepsilon)$ such that
$$\int_{B(y_n,R(\varepsilon))} u_n^2(x)dx \geq c^2 - \varepsilon \quad \forall n \in \mathbb{N}.$$

The seminal work by Lions states a general line of attack to exclude the two first alternatives. When one knows that compactness in the only possible case, (1.6) becomes much easier to handle.

Now, to rule out vanishing the main ingredient is to get a strict sign of the value of m_c (let us say $m_c < 0$ without loss of generality). This can be obtained by dilatations arguments or test function techniques .

The most difficult point is to prove that dichotomy cannot occur. To achieve this objective, Lions suggested a heuristical approach based on the strict subadditivity inequality

$$m_c < m_a + m^\infty_{c-a} \quad \forall a \in (0,c), \tag{1.7}$$

where $m^\infty_c = \inf\{j^\infty(u) : u \in s_c\}$ and $j^\infty(u) = \frac{1}{2}\int |\nabla u|^2 - \int f^\infty(x,u(x))$ and f^∞ is defined as in (F3).

On the other hand, we should establish suitable assumptions on f for which $j(u_n) \geq j(u_{n,1}) + j^\infty(u_{n,2}) - g(\delta)$ where $g(\delta) \to 0$ as $\delta \to 0$. The latter requires a deep study of the functionals j and j^∞. The continuity of m_c and m^∞_c also play a crucial role to show that dichotomy cannot occur. When one knows that compactness is the only plausible alternative, the strict inequality

$$m_c < m^\infty_c \tag{1.8}$$

is very helpful for proving that (1.6) admits a solution.

Equations (1.7) and (1.8) seem to be inescapable to rule out dichotomy in Lions method. In the most interesting cases ($m^\infty_c \neq 0$), in order to get (1.8), we need first to apply the concentration-compactness method to the problem at infinity. This problem is less complicated than the original one since it has translation invariance properties.

The key tool to prove that m_c^∞ is achieved

$$\exists u_\infty \in s_c \quad \text{such that} \quad j^\infty(u_\infty) = m_c^\infty \tag{1.9}$$

is to strict the subadditivity inequality:

$$m_c^\infty < m_a^\infty + m_{c-a}^\infty\,. \tag{1.10}$$

On the other hand, it is quite easy to establish assumptions on f such that

$$j(u) < j^\infty(u) \quad \forall u \in H^1 \tag{1.11}$$

and therefore $m_c \leq m_c^\infty$. Combining (1.9) and (1.11) leads to (1.8). Hence to obtain (1.7), it suffices to prove that

$$m_c \leq m_a + m_{c-a} \tag{1.12}$$

This inequality can be derived immediately from the following property,

$$f(x, \theta s) \geq \theta^2 f(x, s) \quad \forall s \in \mathbb{R}_+, x \in \mathbb{R}^N \text{ and } \theta > 1.$$

To study the multi-constrained variational problem (1.1), we will follow the same line of attack described in details above. Let us mention that to our knowledge, there are no previous results dealing with (1.1) when $m \geq 2$ and the non-linearity F is a product function not satisfying the classical convexity, compactness, monotonicity properties. Quite recently, in [1, 2, 3], the second author was able to generalize and extend previous results addressed to (1.1) when F is radial and supermodular (i.e $\partial_i \partial_j F \geq 0$ $\forall 1 \leq i \neq j \leq m$ when F is smooth).

In the vectorial context, the equivalent of (1.7) is

$$I_{c_1,\dots,c_m} < I_{a_1,\dots,a_m} + I^\infty_{c_1-a_1,\dots,c_m-a_m} \quad \forall 0 < a_i < c_i \ \forall 1 \leq i \leq m\,. \tag{1.13}$$

We will first prove that $I_{c_1,\dots,c_m} < 0$ in lemma 3.2. This property together with (F2) permit us to have

$$I_{c_1,\dots,c_m} \leq I_{a_1,\dots,a_m} + I_{c_1-a_1,\dots,c_m-a_m} \quad \forall 0 < a_i < c_i,\ \forall 1 \leq i \leq m\,. \tag{1.14}$$

It turns out that (1.14) is a subtle combinatorial inequality (part (a) of Lemma 3.3).

Following the same approach detailed for the scalar case, we will then study (1.5) and prove that this variational problem has a minimum: There exists $\vec{u}_c^\infty \in S_c$ such that

$$J^\infty(\vec{u}_c^\infty) = I^\infty_{c_1,\dots,c_m} \tag{1.15}$$

This equality is obtained thanks to the subadditivity condition

$$I^\infty_{c_1,\dots,c_m} < I^\infty_{a_1,\dots,a_m} + I^\infty_{c_1-a_1\dots,c_m-a_m} \quad \forall 0 < a_i < c_i,\ \forall 1 \leq i \leq m\,, \tag{1.16}$$

which is proved in part (b) of Lemma 3.3.

On the other hand, (F6) tells us that

$$J(\vec{u}) < J^\infty(\vec{u}) \quad \forall \vec{u} \in \vec{H}^1\,. \tag{1.17}$$

From (1.15) and (1.17) it follows that

$$I_{c_1,\dots,c_m} < I^\infty_{c_1,\dots,c_m}\,. \tag{1.18}$$

Inequalities (1.14) and (1.18) lead to (1.13). Then using the properties of the splitting sequences $\vec{v}_n$ and $\vec{w}_n$ (see appendix) and those of the functionals J and J^∞ (Lemma 3.1) we prove that any minimizing sequence of (1.1) is such that

$$J(\vec{u}_n) \geq J(\vec{v}_n) + J^\infty(\vec{w}_n) - \delta \quad \delta \to 0$$

or

$$J(\vec{u}_n) \geq J^\infty(\vec{v}_n) + J(\vec{w}_n) - \delta$$

and we find a contradiction with (1.13). Therefore compactness occurs and we can conclude that Theorem 1.1 holds using (1.18).

For the convenience of the reader, we summarize our approach (inspired by Lions principle) into the following steps:

(1) Obtain useful properties about the functionals J and J^∞ (Lemma 3.1);
(2) Prove that $I_{c_1,\dots,c_m} < 0$ and $I^\infty_{c_1,\dots,c_m} < 0$ (Lemma 3.2);
(3) $I_{c_1,\dots,c_m} \leq I_{a_1,\dots,a_m} + I_{c_1-a_1,\dots,c_m-a_m}$ (Lemma 3.3);
(4) Prove that (1.5) is achieved thanks to the strict inequality
$$I^\infty_{c_1,\dots,c_m} < I^\infty_{a_1,\dots,a_m} + I^\infty_{c_1-a_1,\dots,c_m-a_m}$$
(5) $I_{c_1,\dots,c_m} < I^\infty_{c_1,\dots,c_m}$ (Lemma 3.4)
(6) $I_{c_1,\dots,c_m} < I_{a_1,\dots,a_m} + I^\infty_{c_1-a_1,\dots,c_m-a_m}$ follows from step (3) and step (5);
(7) Only compactness can occur. In fact step (2) permits us to rule out vanishing. step 1 and step (6) will be crucial to eliminate dichotomy.

2. Notation

Let N, m be two integers ≥ 1.

- $\vec{s} = (s_1, \dots, s_m)$; where $s_i \in \mathbb{R}$; $|\vec{s}|$ denotes its modulus.
- For $\vec{u} = (u_1, \dots, u_m) \in L^p \times \dots \times L^p = \vec{L}^p$, let $2 \leq p \leq 2^*$, where 2^* is the critical Sobolev component, and $|\vec{u}|_{\vec{L}_p} = \sum_{i=1}^m |u_i|_p$.
- For $\vec{u} \in H^1 \times H^1 = \vec{H}^1$, let $|\vec{u}|_{\vec{H}^1} = \sum_{i=1}^m |u_i|_{H^1}$.
- In integrals where no domain is specified, it is understood that it extends over $\mathbb{R}^N$.
- We will make frequent use of the inequality
$$\int \big||\vec{u}|\big|^p \leq c_p \sum_{i=1}^m |u_i|_p$$
- Sometimes we keep the same constants in different inequalities, even if they change value form line to line.
- In the following, we fix $c_1, \dots, c_m > 0$, $c^2 = \sum_{i=1}^m c_i^2$.

3. Proof of our main result

Lemma 3.1. *If F satisfies* (F0), *then*

(i) (a) $J \in C^1(\vec{H}^1, \mathbb{R})$ *and there exists a constant $E > 0$ such that*
$$|J'(\vec{u})|_{\vec{H}^{-1}} \leq E\Big(|\vec{u}|_{\vec{H}^1} + |\vec{u}|_{\vec{H}^1}^{1+\frac{4}{N}}\Big)$$
for any $\vec{u} \in \vec{H}^1$.

(b) $J^\infty \in C^1(\vec{H}^1, \mathbb{R})$ *and there exists a constant $E_\infty > 0$ such that*
$$|{J^\infty}'(\vec{u})|_{\vec{H}^{-1}} \leq E_\infty\Big(|\vec{u}|_{\vec{H}^1} + |\vec{u}|_{\vec{H}^1}^{1+\frac{4}{N}}\Big) \quad \textit{for any } \vec{u} \in \vec{H}^1.$$

(ii) *There exist constants $A_i, B_i > 0$ such that for any $\vec{u} \in S_c$, we have*
$$J(\vec{u}) \geq A_1|\nabla \vec{u}|_2^2 - A_2c^2 - A_3c^{(1-\sigma)(\ell+2)q},$$
$$J^\infty(\vec{u}) \geq B_1|\nabla \vec{u}|_2^2 - B_2c^{(1-\sigma_1)(\beta+2)q_1} - B_3c^{(1-\sigma)(\ell+2)q},$$
where σ, σ_1 and q, q_1 are defined in the proof below.

(iii) (a) *$I_{c_1,\dots,c_m} > -\infty$ and any minimzing sequence of (1.1) is bounded in $\vec{H}^1$;*
(b) *$I^\infty_{c_1,\dots,c_m} > -\infty$ and any minimizing sequence of (1.5) is bounded in $\vec{H}^1$.*

(iv) (a) *$(c_1,\dots,c_m) \to I_{c_1,\dots,c_m}$ and*
(b) *$(c_1,\dots,c_m) \to I^\infty_{c_1,\dots,c_m}$ are continuous on $(0,\infty)^m$.*

Proof. (i) (a) Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be the function defined by
$$\varphi(\vec{s}) = 1 \text{ if } |\vec{s}| \leq 1$$
$$\varphi(\vec{s}) = -|\vec{s}| + 2 \text{ if } 1 \leq |\vec{s}| \leq 2$$
$$\varphi(\vec{s}) = 0 \text{ if } |\vec{s}| \geq 2$$
Let $1 \leq i \leq m$.
$$\partial_i^1 F(x,\vec{s}) = \varphi(\vec{s})\partial_i F(x,\vec{s}), \quad |\partial_i^1 F(x,\vec{s})| \leq B(1+2^{\ell+1})|\vec{s}|, \tag{3.1}$$
$$\partial_i^2 F(x,\vec{s}) = (1-\varphi(\vec{s}))\partial_i F(x,\vec{s}), \quad |\partial_i^2 F(x,\vec{s})| \leq 2B|\vec{s}|^{1+\frac{4}{N}}. \tag{3.2}$$
Let
$$p = \begin{cases} \frac{2N}{N+2} & \text{for } N \geq 3 \\ \frac{4}{3} & \text{for } N \leq 2 \end{cases} \quad \text{and } q = (1+\frac{4}{N})p\,.$$
Note that (3.1) and (3.2) imply $\partial_i^1 F(x,\cdot) \in C(\vec{L}^2, L^2)$ and $\partial_i^2 F(x,\cdot) \in C(\vec{L}^q, L^p)$ and there exists a constant $K > 0$ such that
$$|\partial_i^1 F(x,\vec{u})|_2 \leq K(|\vec{u}|_2), \quad \forall \vec{u} \in \vec{L}^2,$$
$$|\partial_i^2 F(x,\vec{u})|_p \leq K(|\vec{u}|_q^{1+\frac{4}{N}}); \quad \forall \vec{u} \in \vec{L}^q.$$
Noticing that $\vec{H}^1$ is continuously embedded in $\vec{L}^q$ since $q \in [2, \frac{2N}{N-2}]$ for $N \geq 3$ and $q \in [2,\infty)$ for $N \leq 2$, and $\vec{L}^p$ is continuously embedded in $\vec{H}^{-1}$ since $p' \in [2, \frac{2N}{N-2}]$ for $N \geq 3$ and $p' \in [2,\infty)$ for $N \leq 2$. We can assert that $\partial_i F(x,\cdot) + \partial_i^2 F(x,\cdot) \in C(\vec{H}^1, \vec{H}^{-1})$ and there exists a constant $C > 0$ such that
$$|\partial_i F(x,\vec{u})|_{\vec{H}^{-1}} \leq C\{|\vec{u}|_{\vec{H}^1} + |\vec{u}|_{\vec{H}^1}^{1+\frac{4}{N}}\} \tag{3.3}$$
for all $\vec{u} \in \vec{H}^1$. On the other hand
$$\int F(x,\vec{u}) \leq A(|\vec{u}|_2^2 + |\vec{u}|_{\ell+2}^{\ell+2}) \leq C(|\vec{u}|_{\vec{H}^1}^2 + |\vec{u}|_{\vec{H}^1}^{\ell+2})$$
which implies that $J \in C^1(\vec{H}^1, \mathbb{R})$ by standard arguments of differential calculus. Thus
$$J'(\vec{u})\vec{v} = \int \sum_{i=1}^m \nabla u_i \, \nabla v_i - \partial_i F(x,\vec{u})v_i \quad \forall \vec{u}, \vec{v} \in \vec{H}^1,$$
$$|J'(\vec{u})|_{\vec{H}^{-1}} \leq C\{|\vec{u}|_{\vec{H}^1} + |\vec{u}|_{\vec{H}^1}^{1+\frac{4}{N}}\} \forall \vec{u} \in \vec{H}^1.$$

(i) (b) It is easy to deduce the estimates (3.1) and (3.2) for $\partial_i F^\infty$ and (i) (b) follows using the same approach.

(ii) Let $\vec{u} \in S_c; \vec{u} = (u_1, \dots, u_m)$. Using (F0), we have $\int F(x, \vec{u}) \le Ac^2 + A\sum_{i=1}^m \int |u_i(x)|^{\ell+2}$. For $1 \le i \le m$, the Gagliardo-Nirenberg inequality tells us that

$$|u_i|_{\ell+2}^{\ell+2} \le A" |u_i|_2^{(1-\sigma)(\ell+2)} |\nabla u_i|_2^{\sigma(\ell+2)} \tag{3.4}$$

where $\sigma = \frac{N}{2}\frac{\ell}{\ell+2}$.

Now let $\varepsilon > 0$, $p = \frac{4}{N\ell}$, q is such that $\frac{1}{p} + \frac{1}{q} = 1$. Applying Young's inequality, we obtain

$$|u_i|_{\ell+2}^{\ell+2} \le \big\{\frac{A"}{\varepsilon} |u_i|_2^{(1-\sigma)(\ell+2)}\big\}^q \frac{1}{q} + \frac{N\ell}{4}\{\varepsilon^{\frac{4}{N\ell}} |\nabla u_i|_2^2\}.$$

Consequently,

$$J(\vec{u}) \ge \big\{\frac{1}{2} - \frac{AN\ell}{4}\varepsilon^{\frac{4}{N\ell}}\big\} |\nabla \vec{u}|_2^2 - A^2c^2 - \frac{AA"^q}{q\varepsilon^q} mc^{(1-\sigma)(\ell+2)q} \tag{3.5}$$

Taking ε such that $\frac{1}{2} - \frac{AN\ell}{4}\varepsilon^{\frac{4}{N\ell}} \ge 0$, we prove that J is bounded from below in $\vec{H}^1$. To show that all minimizing sequence of (1.1) is bounded in $\vec{H}^1$, it suffices to take the latter inequality with a strict sign.

remark (1) If we allow $\ell = 4/N$ in (F0), the minimization problem (1.1) makes sense for sufficiently small values of c since in (3.4), we then have $\sigma = \frac{2}{\ell+2}$ and $(1-\sigma)(\ell+2) = \frac{4}{N}$. Therefore,

$$|u_i|_{\ell+2}^{\ell+2} \le A" c^{(1-\sigma)(\ell+2)} |\nabla u_i|_2^2 \le A" c^{4/N} |\nabla u_i|_2^2,$$
$$J(\vec{u}) \ge \left\{\frac{1}{2} - AA" c^{4/N}\right\} |\nabla \vec{u}|_2^2 - Ac^2$$

Thus if $c < (\frac{1}{2AA''})^{N/4}$, the minimization problem (1.1) is still well-posed.

(2) If $\ell > 4/N$, we can prove that $I_{c_1,\dots,c_m} = -\infty$.

(b) (ii) Under slight modifications in the proof of (ii) (a) we can easily obtain

$$J^\infty(\vec{u}) \ge \left\{\frac{1}{2} - A^{(3)}\varepsilon^{\frac{4}{N\ell}}\right\} |\nabla \vec{u}|_2^2 - \frac{A^{(4)}m}{q_1\varepsilon^{q_1}} c^{(1-\sigma_1)(\beta+2)q_1} - \frac{A^{(5)}mc^{(1-\sigma)(\ell+2)q}}{q\varepsilon^q},$$

where $\sigma_1 = \frac{N}{2}\frac{\beta}{\beta+2}$, $\sigma_1 = \frac{N}{2}\frac{\ell}{\ell+2}$, and q_1 is also defined as in the previous proof.

Statement (iii) is a direct consequence of (ii).

(iv) Consider $c = (c_1, \dots, c_m)$, $c_i > 0$ and a sequence $(c_1^n, \dots, c_m^n)$ such that $c_i^n \to c_i$ for $1 \le i \le m$. For any n, there exist $u_{n,i} \in S_{c_i^n}$, $\int u_{n,i}^2 = (c_i^n)^2$ and

$$I_{c_1^n,\dots,c_m^n} \le J(u_{n,1}, \dots, u_{n,m}) \le I_{c_1^n,\dots,c_n^m} + \frac{1}{n}.$$

Now by (ii) (a), we can easily see that there exists a constant $K > 0$ such that $|\vec{u}_n|_{\vec{H}^1} \le K$ for all $n \in \mathbb{N}$.

Let $w_{n,i} = \frac{c_i}{c_i^n} u_{n,i}$, $\vec{w}_n = (w_{n,1}, \dots, w_{n,m})$, then $\vec{w}_n \in S_c$ and

$$|\vec{u}_n - \vec{w}_n|_{\vec{H}^1} = \sum_{i=1}^m |u_{n,i} - w_{n,i}|_{H^1} \le \sum_{i=1}^m |\frac{c_i}{c_i^n} u_{n,i} - u_{n,i}|_{H^1} \le \sum_{i=1}^m |\frac{c_i}{c_i^n} - 1||u_{n,i}|_{H^1}$$

In particular, there exists n_1 such that

$$|\vec{u}_n - \vec{w}_n|_{\vec{H}^1} \le K + 1 \quad \text{for } n \ge n_1.$$

Now it follows from i) a) that

$$|J'(\vec{u})|_{\vec{H}^{-1}} \leq L(K) \quad \text{for } |\vec{u}|_{\vec{H}^1} \leq 2K+1. \tag{3.6}$$

Therefore, for any $n \geq n_1$,

$$\begin{aligned} |J(\vec{w}_n) - J(\vec{u}_n)| &= |\int_0^1 \frac{d}{dt} J(t\vec{w}_n + (1-t)\vec{u}_n) dt| \\ &\leq \sup_{|\vec{u}|_{\vec{H}^1} \leq 2K+1} |J'(\vec{u})|_{\vec{H}^{-1}} |\vec{u}_n - \vec{w}_n|_{\vec{H}^1} \\ &\leq L(K) K \sum_{i=1}^m |1 - \frac{c_i}{c_i^n}| \end{aligned}$$

Finally, we have

$$I_{c_1^n,\dots,c_m^n} \geq J(\vec{u}_n) - \frac{1}{n} \geq J(\vec{w}_n) + KL(K) \sum_{i=1}^m |1 - \frac{c_i}{c_i^n}| - \frac{1}{n}$$

Thus $\liminf I_{c_1^n,\dots,c_m^n} \geq I_{c_1,\dots,_m}$.

On the other hand, there exists a sequence $\vec{u}_n \in S_c$ such that $J(\vec{u}_n) \to I_{c_1,\dots,c_m}$ and by i) a) there exists $K > 0$ such that $|\vec{u}_n|_{\vec{H}^1} \leq K$. Now set $w_{n,i} = \frac{c_n^i}{c_i} u_{n,i}$ (c_n^i is such that $c_n^i \to c_i$ as $n \to \infty$). As done above, we certainly have

$$\begin{gathered} \vec{w}_n = (w_{n,1,\dots,}w_{n,m}) \in S_{c_n}, \quad c_n = (c_n^1, \dots, c_n^m), \\ |\vec{u}_n - \vec{w}_n|_{\vec{H}^1} \leq \sum_{i=1}^m K|1 - \frac{c_n^i}{c_i}||u_{n,i}|_{H^1}. \end{gathered}$$

As done previously, we obtain

$$|J(\vec{w}_n) - J(\vec{u}_n)| \leq KL(K) \sum_{i=1}^m |1 - \frac{c_n^i}{c_i}|,$$

which implies

$$I_{c_n^1,\dots,c_n^m} \leq J(\vec{w}_n) \leq J(\vec{u}_n) + L(K)K \sum_{i=1}^m |1 - \frac{c_n^i}{c_i}|.$$

Thus $\limsup I_{c_n^1,\dots,c_n^m} \leq I_{c_1,\dots,c_m}$ and we have the conclusion.

Statement (iv) b) follows in a similar manner. □

Lemma 3.2. *If F satisfies* (F0) *and* (F1), *then*

$$I_{c_1,\dots,c_m} < 0 \quad \textit{for any} c_i > 0;\ 1 \leq i \leq m.$$

Proof. Let φ be a radial and radially decreasing function such that $|\varphi|_2 = 1$. Set $\varphi_i = c_i\varphi$. Let $0 < \lambda \ll 1$ and

$$\vec{\Phi}_\lambda(x) = \lambda^{N/2}\vec{\Phi}(\lambda x) = \lambda^{N/2}(\varphi_1(\lambda x)), \dots, \varphi_m(\lambda x))$$

Then

$$\begin{aligned} J(\vec{\Phi}_\lambda) &= \lambda^2 |\nabla\vec{\Phi}|_2^2 - \int F(x, \lambda^{N/2}\varphi_1(\lambda x), \dots, \lambda^{N/2}\varphi_m(\lambda x)) dx \\ &\leq \lambda^2 |\nabla\vec{\Phi}|_2^2 - \int_{|x|\geq R} F(x, \lambda^{N/2}\varphi_1(\lambda x), \dots, \lambda^{N/2}\varphi_m(\lambda x)) dx \end{aligned}$$

$$\leq \lambda^2|\nabla\vec{\Phi}|_2^2 - \lambda^{\frac{N}{2}\alpha}\Delta\int_{|x|\geq R}|x|^{-t}\varphi_1^{\alpha_1}(\lambda x)\dots\varphi_m^{\alpha_m}(\lambda x)dx$$

By the change of variable $y=\lambda x$, it follows that

$$J(\vec{\Phi}_\lambda)\leq \lambda^2|\nabla\vec{\Phi}|_2^2 - \lambda^{\frac{N}{2}\alpha}\lambda^{-N}\Delta\lambda^t\int_{|y|\geq\lambda R}|y|^{-t}\varphi_1^{\alpha_1}(y),\dots,\varphi_m^{\alpha_m}(y)dy$$

Since $0<\lambda\ll 1$, we obtain

$$\begin{aligned}J(\vec{\Phi}_\lambda)&\leq \lambda^2|\nabla\vec{\Phi}|_2^2 - \lambda^{\frac{N}{2}\alpha-N+t}\int_{|y|\geq R}|y|^{-t}\varphi_1^{\alpha_1}(y)\dots\varphi_m^{\alpha_m}(y)dy\\ &\leq \lambda^2\{C_1-\lambda^{\frac{N}{2}\alpha-N+t-2}C_2\},\end{aligned}$$

because $\lambda\ll 1$ and $\frac{N}{2}\alpha-N+t-2>0$. $\square$

The strict negativity of the infinimum is also discussed in [1] where the author provides other type of assumptions ensuring this.

Lemma 3.3. *(1) If F satisfies* (F0)-(F2)*, then for any $c_1,\dots,c_m>0$*

$$I_{c_1,\dots,c_m}\leq I_{a_1,\dots,a_m}+I_{c_1-a_1,\dots,c_m-a_m} \tag{3.7}$$

for all $a_i\in(0,c_i)$ where $1\leq i\leq m$.

(2) If F satisfies (F1), (F24), (F4) *hold for F^∞, then for any $c_1,\dots,c_m>0$,*

$$I^\infty_{c_1,\dots,c_m}< I^\infty_{a_1,\dots,a_m}+I^\infty_{c_1-a_1,\dots,c_m-a_m} \tag{3.8}$$

for all $a_i\in(0,c_i)$ where $1\leq i\leq m$.

Proof. (1) By (F2), we certainly have

$$I_{\theta_1c_1,\dots,\theta_mc_m}\leq\theta^2_{\max}I_{c_1,\dots,c_m}\quad\forall\theta_i\geq 1$$

which implies by Lemma3.1 that

$$I_{\theta_1c_1,\dots,\theta_mc_m}\leq\theta^2_{i}I_{c_1,\dots,c_m}\quad\forall\theta_i\geq 1 \tag{3.9}$$

For the convenience of the reader, we will start by proving the result in a specific case then we will explain the proof in the general setting. Suppose first that

$$\frac{a_m}{c_m-a_m}\leq 1\quad\text{and}\quad\frac{a_j}{c_j-a_j}\geq 1\quad\forall 1\leq j\leq m-1\,.$$

Then

$$I_{(c_1,\dots,c_m)}=I_{(\frac{c_1}{a_1}a_1,\dots,\frac{c_m}{c_m-a_m}c_m-a_m)} \tag{3.10}$$

Thus using (3.9), it follows that

$$I_{(c_1,\dots,c_m)}\leq\frac{c_1}{a_1}I_{(a_1,\dots,c_m-a_m)}=\frac{c_1-a_1}{a_1}I_{(a_1,\dots,c_m-a_m)}+I_{(a_1,\dots,c_m-a_m)} \tag{3.11}$$

However,

$$I_{(a_1,\dots,c_m-a_m)}=I_{(\frac{a_1}{c_1-a_1}c_1-a_1,\dots,\frac{c_m-a_m}{a_m}a_m)}$$

Again applying (3.9), we obtain

$$I_{(a_1,..,c_m-a_m)}\leq\frac{a_1}{c_1-a_1}I_{(c_1-a_1,\dots,a_m)} \tag{3.12}$$

Combining (3.11) and (3.12), we have

$$I_{(c_1,\dots,c_m)}\leq I_{(a_1,\dots,a_m)}+I_{(c_1-a_1,\dots,c_m-a_m)} \tag{3.13}$$

which concludes the proof in this case.

Now in the general setting, we follow the same approach. If $1 \le i \ne j \le m$ such that $\frac{a_i}{c_i - a_i}$ and $\frac{a_j}{c_j - a_j} > 1$ and $1 \le k \ne \ell \le m$ such that $\frac{a_k}{c_k - a_k}$ and $\frac{a_\ell}{c_\ell - a_\ell} < 1$, then these indices will appear in (3.7) in the following manner

$$I_{c_1,\dots,c_m} \le I_{\dots,a_i,\dots,c_k-a_k,\dots,c_\ell-a_\ell,\dots,a_j,\dots} + I_{\dots,c_i-a_i,\dots,a_k,\dots,,a_\ell,\dots,c_j-a_j\dots}$$

Now if F satisfies (F1), (F4) and (F5) hold for F^∞ then by part (1), we have

$$I^\infty_{c_1,\dots,c_m} \le I^\infty_{a_1,\dots,a_m} + I^\infty_{c_1-a_1,\dots,c_m-a_m}$$

for any $c_i > 0$, $0 < a_i < c_i$, $1 \le i \le m$.

Following the same steps as in the previous part, we conclude that (2) is true if $I^\infty_{\theta_1 c_1,\dots,\theta_m c_m,\dots,\theta_m c_m} < \theta^2_{\max} I^\infty_{c_1,\dots,c_m}$.

For any $c_1,\dots,c_m > 0$ and $\theta_1,\dots,\theta_m > 1$, we can choose $\varepsilon > 0$ such that $\varepsilon < -I^\infty_{c_1,\dots,c_m}(1-\theta^{-\sigma}_{\max})$ and there exists $\vec{v}$ such that $\int v_i^2 = c_i^2$ verifying

$$I^\infty_{c_1,\dots,c_m} \le J^\infty(\vec{v}) < I^\infty_{c_1,\dots,c_m} + \varepsilon.$$

Hence

$$I^\infty_{\theta_1 c_1,\dots,\theta_m c_m} \le J^\infty(\theta_1 v_1,\dots,\theta_m v_m) \le \theta^{\sigma+2}_{\max} J^\infty(\vec{v}),$$
$$I^\infty_{\theta_1 c_1,\dots,\theta_m c_m} \le \theta^{\sigma+2}_{\max}\{I^\infty_{c_1,\dots,c_m} + \varepsilon\} < \theta^2_{\max} I^\infty_{c_1,\dots,c_m} \quad \text{by the choice of } \varepsilon$$

□

Lemma 3.4. *If F satisfies* (F0)–(F2)*, and* (F1), (F5) *hold for for F^∞, then*

$$I_{c_1,\dots,c_m} < I_{a_1,\dots,a_m} + I^\infty_{c_1-a_1,\dots,c_m-a_m} \quad \forall 0 < a_i < c_i \quad \forall 1 \le i \le m.$$

Proof of Theorem 1.2. Let $(\vec{u}_n)$ be a minimzing sequence of the problem (1.5).

Vanishing does not occur. If it occurs, from [5, Lemma I.1] it follows that $\big||\vec{u}_n|\big|_p \to 0$ as $n \to +\infty$ for $p \in (2, 2^*)$. By (F4),

$$\int F^\infty(x, \vec{u}_n(x)) \le \left\{\big||\vec{u}_n|\big|^{\beta+2}_{\beta+2} + \big||\vec{u}_n|\big|^{\ell+2}_{\ell+2}\right\}.$$

Thus $\lim_{n\to+\infty} \int F^\infty(x, \vec{u}_n(x) = 0$, which implies that liminf $J^\infty(\vec{u}_n) \ge 0$, contradicting the fac that $I^\infty_{c_1,\dots,c_m} < 0$.

Dichotomy does not occur. The notation used here, is stated in the appendix. For $n \ge n_0$ and since $\operatorname{supp}\vec{v}_n \cap \operatorname{supp}\vec{w}_n = \emptyset$,

$$\begin{aligned}
&J^\infty(\vec{u}_n) - J^\infty(\vec{v}_n) - J^\infty(\vec{w}_n)\\
&= \frac{1}{2}\int |\nabla\vec{u}_n|^2 - |\nabla\vec{v}_n|^2 - |\nabla\vec{w}_n|^2 - \int F^\infty(x,\vec{u}_n) - F^\infty(x,\vec{v}_n) - F^\infty(x,\vec{w}_n)\\
&= \frac{1}{2}\int |\nabla\vec{u}_n|^2 - |\nabla\vec{v}_n|^2 - |\nabla\vec{w}_n|^2 - \int F^\infty(x,\vec{u}_n) - F^\infty(x,\vec{v}_n+\vec{w}_n)\\
&\ge -\varepsilon - \int F^\infty(x,\vec{u}_n) - F^\infty(x,\vec{v}_n+\vec{w}_n)
\end{aligned}$$

Now since $\{\vec{w}_n\}$, $\{\vec{v}_n\}$ and $\{\vec{w}_n\}$ are bounded in $\vec{H}^1$, it follows from the proof of Lemma 3.1 that there exist $C, K > 0$ such that

$$\begin{aligned}
&\Big|\int F^\infty(x,\vec{u}_n) - F^\infty(x,\vec{v}_n+\vec{w}_n)\Big|\\
&\le \sup_{|\vec{u}|_{\vec{H}^1}\le K} \sum_{i=1}^m |\partial_i F^\infty(x,\vec{u})|_{\vec{H}^{-1}} |\vec{u}_n - (\vec{v}_n+\vec{w}_n)|_{\vec{H}^1}
\end{aligned}$$

$$
\begin{aligned}
&\le \sup_{|\vec{u}|_{\vec{H}^1}\le K} \sum_{i=1}^{m} |\partial_i^1 F^\infty(x,\vec{u})|_{\vec{L}^2} |\vec{u}_n - (\vec{v}_n + \vec{w}_n)|_{\vec{L}^2} \\
&\quad + \sup_{|\vec{u}|_{\vec{H}^1}\le K} \sum_{i=1}^{m} |\partial_i^2 F^\infty(x,\vec{u})|_{\vec{L}^p} |\vec{u}_n - (\vec{v}_n + \vec{w}_n)_{\vec{p'}} \\
&\le C \sup_{|\vec{u}|_{\vec{H}^1}\le K} |\vec{u}|_{\vec{L}^2} |\vec{u}_n - (\vec{v}_n + \vec{w}_n)|_{\vec{L}^2} + C \sup_{|\vec{u}|_{\vec{H}^1}\le K} |\vec{u}|_{L^q}^{1+\frac{4}{N}} |\vec{u}_n - (\vec{v}_n + \vec{w}_n)|_{\vec{L}^{p'}} \\
&\le C_1 K |\vec{u}_n - (\vec{v}_n + \vec{w}_n)|_{\vec{L}^2} + C_2 K^{1+\frac{4}{N}} |\vec{u}_n - (\vec{v}_n + \vec{w}_n)|_{\vec{L}^{p'}};
\end{aligned}
$$

therefore,

$$
\begin{aligned}
&J^\infty(\vec{v}_n) - J^\infty(\vec{v}_n) - J^\infty(\vec{w}_n) \\
&\ge -\varepsilon - C_1 K |\vec{u}_n - (\vec{v}_n + \vec{w}_n)|_{\vec{L}^2} - C_2 K^{1+\frac{4}{N}} |\vec{u}_n - (\vec{v}_n + \vec{w}_n)|_{\vec{L}^{p'}}
\end{aligned}
$$

Given any $\delta > 0$, we can find $\varepsilon_\delta \in (0,\delta)$ such that $J^\infty(\vec{u}_n) - J^\infty(\vec{v}_n) - J^\infty(\vec{w}_n) \ge -\delta$. Now let

$$
\begin{aligned}
a_{n,i}^2(\delta) &= \int v_{n,i}^2 \quad i = 1,\dots,m; \\
b_{n,i}^2(\delta) &= \int w_{n,i}^2 \quad i = 1,\dots,m.
\end{aligned}
$$

Passing to a subsequence, we may suppose that

$$
a_{n,i}^2(\delta) \to a_i^2(\delta) \quad \text{and} \quad b_{n,i}^2(\delta) \to b_i^2(\delta)
$$

where $|a_i^2(\delta) - a_i^2| \le \varepsilon_\delta < \delta$ and $|b_i^2(\delta) - (c_i^2 - a_i^2)| \le \varepsilon_\delta < \delta$. Recalling that $I_{c_1,\dots,c_m}^\infty$ is continuous, we find that

$$
\begin{aligned}
I_{c_1,\dots,c_m}^\infty &\ge \lim_{n\to+\infty} J^\infty(\vec{u}_n) \ge \liminf\{J^\infty(\vec{v}_n) + J^\infty(\vec{w}_n)\} - \delta \\
&\ge \liminf\{I_{a_{n,1}(\delta),\dots,a_{n,m}(\delta)}^\infty + I_{b_{n,1}(\delta),\dots,b_{n,m}(\delta)}^\infty\} - \delta \\
&\ge I_{a_1(\delta),\dots,a_m(\delta)}^\infty + I_{b_1(\delta),\dots,b_m(\delta)}^\infty - \delta
\end{aligned}
$$

Letting δ approach zero and using again the continuity of $I_{c_1,\dots,c_m}^\infty$, we obtain

$$
I_{c_1,\dots,c_m}^\infty \ge I_{a_1,\dots,a_m}^\infty + I_{\sqrt{c_1^2-a_1^2},\dots,\sqrt{c_m^2-a_m^2}}^\infty
$$

contracting Lemma 3.3 part (2). Hence compactness occurs; so there exists $\{y_n\} \subset \mathbb{R}^N$ such that for all $\varepsilon > 0$ such that

$$
\int_{B(y_n,R(\varepsilon))} u_{n,1}^2 + \dots + u_{n,m}^2 \ge c_1^2 + \dots + c_m^2 - \varepsilon.
$$

For each $n \in \mathbb{N}$, we can choose $z_n \in \mathbb{Z}^N$ such that $z_n \in \mathbb{Z}^N$ such that $y_n - z_n \in [0,1]^N$.

Now set $\vec{v}_n(x) = \vec{u}_n(x + z_n)$, we certainly have that $|\vec{v}_n|_{\vec{H}^1} = |\vec{u}_n|_{\vec{H}^1}$ is bounded and so passing to a subsequence, we may assume that $\vec{v}_n \rightharpoonup \vec{v}$ in $\vec{H}^1$, in particular $\vec{v}_n \rightharpoonup \vec{v}$ weakly in $\vec{L}^2$ and

$$
|v_{n,i}|_2^2 = c_i^2 \quad \forall 1 \le i \le m.
$$

However,

$$
\int |\vec{v}|^2 \ge \int_{B(0,R(\varepsilon)+\sqrt{N})} |\vec{v}|^2
$$

$$= \lim_{n\to+\infty} \int_{B(0,R(\varepsilon)+\sqrt{N})} |\vec{v}_n|^2 = \lim \int_{B(z_n,R(\varepsilon)+\sqrt{N})} |\vec{v}_n|^2$$

and

$$\int_{B(z_n,R(\varepsilon)+\sqrt{N})} |\vec{u}_n|^2 \geq \int_{B(y_n,R(\varepsilon))} |\vec{u}_n|^2 \geq c_1^2 + \dots + c_m^2 - \varepsilon$$

since $|y_n - z_n| \leq \sqrt{N}$. Hence $|\vec{v}|_{\vec{L}^2}^2 \geq c_1^2 + \dots + c_m^2 - \varepsilon$ for all $\varepsilon > 0$ which implies

$$|\vec{v}|_2^2 \geq c_1^2 + \dots + c_m^2 \tag{3.14}$$

On the other hand $|v_i|_2 \leq \liminf |v_{n,i}|_2$ implies

$$|v_i|_2 \leq c_i \quad \forall 1 \leq i \leq m \tag{3.15}$$

Thus combining (3.14) and (3.15), we have that $|v_i|_2^2 = c_i^2$ for all $1 \leq i \leq m$ implies

$$|\vec{v} - \vec{v}_n|_{\vec{L}^2} \to 0 \quad \text{as } n \to \infty.$$

Furthermore by the periodicity of F^∞,

$$J^\infty(\vec{u}_n) = J^\infty(\vec{v}_n) \to I^\infty_{c_1,\dots,c_m}$$

and $\vec{v}_n \to \vec{v}$ in $\vec{L}^p$, $p \in [2, 2^*)$. If follows that $\vec{v}_n \to \vec{v}$ in $\vec{H}^1$ and consequently

$$\int F^\infty(x, \vec{v}_n) \to \int F^\infty(x, \vec{v})$$

which implies that $J^\infty(\vec{v}) = I^\infty_{c_1,\dots,c_m}$. □

Proof of Theorem 1.1. In the following $(\vec{u}_n)$ is a minimizing sequence of (1.1) and we will use the notation introduced in the appendix.

Vanishing does not occur. If it occurs, it follows from [5, Lemma I.1] that $\big||\vec{u}_n|\big|_p \to 0$ for $p \in (2, 2^*)$. Combining (F0) and (F3) we have: For each $\delta > 0$ there exists $R_\delta > 0$ such that

$$F(x, \vec{s}) \leq \delta(|\vec{s}|^2 + |\vec{s}|^{\alpha+2}) + A'(|\vec{s}|^{\beta+2} + |\vec{s}|^{\ell+2}) \quad \forall |x| \geq R_\delta.$$

Hence

$$\int_{|x|\geq R_\delta} F(x, \vec{u}_n) \leq \delta(|\vec{u}_n|_2^2 + |\vec{u}_n|_{\alpha+2}^{\alpha+2}) + A'(|\vec{u}_n|_{\beta+2}^{\beta+2} + |\vec{u}_n|_{\ell+2}^{\ell+2}),$$

$$\limsup_{n\to+\infty} \int_{|x|\geq R_\delta} F(x, \vec{u}_n) \leq \delta c^2.$$

On the other hand,

$$\begin{aligned}
\int_{|x|\leq R_\delta} F(x, \vec{u}_n) dx &\leq A \int_{|x|\leq R_\delta} |\vec{u}_n|^2 + |\vec{u}_n|^{\ell+2} \\
&\leq A\{|\vec{u}_n|_{\ell+2}^{\ell+2} |R_\delta|^{\frac{\ell}{\ell+2}} + |\vec{u}_n|_{\ell+2}^{\ell+2}\} \to \quad \text{as } n \to +\infty\,.
\end{aligned}$$

Hence for any $\delta > 0$ we have

$$\limsup_{n\to\infty} \int F(x, \vec{u}_n) < \delta c^2$$

and so

$$\lim \int F(x, \vec{u}_n) = 0.$$

Thus $J(\vec{u}_n) \to I_{c_1,\dots,c_m} < 0$ leads to a contradiction.

Dichotomy does not occur. Suppose first that the sequence $\{y_n\}$ is bounded and let us consider

$$\begin{aligned}
&J(\vec{u}_n) - J(\vec{v}_n) - J^\infty(\vec{w}_n)\\
&= \frac{1}{2}\int |\nabla \vec{w}_n|^2 - |\nabla \vec{v}_n|^2 - \nabla \vec{w}_n|^2 - \int F(x,\vec{u}_n) - F(x,\vec{v}_n) - F(x,\vec{w}_n)\\
&\quad + \int F^\infty(x,\vec{w}_n) - F(x,\vec{w}_n)\\
&\geq -\varepsilon - \int F(x,\vec{u}_n) - F(x,\vec{v}_n + \vec{w}_n) + \int F^\infty(x,\vec{w}_n) - F(x,\vec{w}_n)\\
&\quad (\text{since } \operatorname{supp} \vec{v}_n \cap \operatorname{supp} \vec{w}_n = \emptyset)\\
&\geq -\varepsilon - \int F(x,\vec{u}_1) - F(x,\vec{v}_n + \vec{w}_n) + \int_{|x-y_n|\geq R_n} F^\infty(x,\vec{w}_n) - F(x,\vec{w}_n)
\end{aligned}$$

Now using the same argument as before, it follows hat given $\delta > 0$, we can choose $\varepsilon = \varepsilon_\delta \in (0,\delta)$ such that

$$-\varepsilon - \int F(x,\vec{u}_n) - F(x,\vec{v}_n + \vec{w}_n) \geq -\delta$$

and hence

$$J(\vec{u}_n) - J(\vec{v}_n) - J^\infty(\vec{w}_n) \geq -\delta + \int_{|x-y_n|\geq R_n} F^\infty(x,\vec{w}_n) - F(x,\vec{w}_n)$$

Given any $\eta > 0$, we can find $R > 0$ such that for all $\vec{s}$ and $|x| \geq R$

$$|F^\infty(x,\vec{s}) - F(x,\vec{s})| \leq \eta(|\vec{s}|^2 + |\vec{s}|^{\alpha+2}).$$

Now since $R_n \to \infty$ and we are supposing that $\{y_n\}$ is bounded. We have that

$$\{x : |x - y_n| \geq R_n\} \subset \{x : |x| \geq R\}$$

for n large enough. From this and the boundedness of $\vec{w}_n$ in $\vec{H}^1$, it follows that

$$\lim_{n\to+\infty} \int_{|x-y_n|\geq R_n} F^\infty(x,\vec{w}_n) - F(x,\vec{w}_n) = 0\,.$$

Now let

$$\begin{aligned}
a_{n,i}^2(\delta) &= \int v_{n,i}^2 \quad 1 \leq i \leq m,\\
b_{n,i}^2(\delta) &= \int w_{n,i}^2 \quad 1 \leq i \leq m.
\end{aligned}$$

Passing to a subsequence, we suppose that

$$a_{n,i}^2(\delta) \to a_i^2(\delta), \quad b_{n,i}^2(\delta) \to b_i^2(\delta)$$

where $|a_i^2(\delta) - a_i^2| \leq \varepsilon_\delta < \delta$ and $|b_i^2(\delta) - (c_i^2 - a_i^2)| \leq \varepsilon_\delta < \delta$. Recalling that $I_{c_1,\dots,c_m}$ and $I^\infty_{c_1,\dots,c_m}$ are continuous we find that

$$\begin{aligned}
I_{c_1,\dots,c_m} = \lim_{n\to+\infty} J(\vec{u}_n) &\geq \liminf_{n\to+\infty} \{J(\vec{v}_n) + J^\infty(\vec{w}_n)\} - \delta\\
&\geq \liminf_{n\to+\infty} \{I_{a_{n,1}(\delta),\dots,a_{n,m}(\delta)} + I_{b_{n,1}(\delta),\dots,b_{n,m}(\delta)}\} - \delta
\end{aligned}$$

Therefore,

$$I_{c_1,\dots,c_m} \geq I_{a_1(\delta),\dots,a_m(\delta)} + I_{b_1(\delta),\dots,b_m(\delta)} - \delta$$

Letting $\delta \to 0$ we obtain

$$I_{c_1,\dots,c_m} \ge I_{a_1,\dots,a_m} + I_{\sqrt{c_1^2-a_1^2},\dots,\sqrt{c_m^2-a_m^2}}$$

Thus the sequence $\{y_n\}$ cannot be bounded and, passing to a subsequence, we may suppose that $|y_n| \to \infty$. Now we obtain a contradiction with Lemma 3.4 by using similar arguments applied to $J(\vec{u}_n) - J^\infty(\vec{v}_n) - J(\vec{w}_n)$ to show that

$$I_{c_1,\dots,c_m} \ge I^\infty_{a_1,\dots,a_m} + I_{\sqrt{c_1^2-a_1^2},\dots,\sqrt{c_m^2-a_m^2}}$$

Thus dichotomy cannot occur and we have compactness. According to the appendix, there exists $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B(y_n,R(\varepsilon))} u_{n,1}^2 + \dots + u_{n,m}^2 \ge c_1^2 + \dots + c_m^2 - \varepsilon \quad \forall \varepsilon > 0.$$

Let us first prove that the sequence $\{y_n\}$ is bounded. If it is not the case, we may assume that $|y_n| \to \infty$ by passing to a subsquence.

Now we can choose $z_n \in \mathbb{Z}^N$ such that $y_n - z_n \in [0,1]^N$. Setting $\vec{v}_n(x) = \vec{u}_n(x + z_n)$, we can suppose that

$$\vec{v}_n \rightharpoonup \vec{v} \quad \text{weakly in } \vec{H}^1$$

and

$$|\vec{v}_n - \vec{v}|_{\vec{L}^2} \to 0 \quad \text{as } n \to \infty \text{ for } 2 \le p \le 2^*,$$
$$J^\infty(\vec{v}_n) = J^\infty(\vec{u}_n);$$

on the other hand,

$$\begin{aligned} J(\vec{u}_n) - J^\infty(\vec{u}_n) &= \int F^\infty(x,\vec{u}_n) - F(x,\vec{u}_n) \\ &= \int F^\infty(x,\vec{v}_n) - F(x - z_n,\vec{v}_n) \end{aligned}$$

Now given $\varepsilon > 0$, it follows from (F3) that there exists $R > 0$ such that

$$\begin{aligned} &\Big|\int_{|x-z_n|\ge R} F^\infty(x,\vec{v}_n) - F(x-z_n,\vec{v}_n)\Big| \\ &= \Big|\int_{|x-z_n|\ge R} F^\infty(x-z_n,\vec{v}_n) - F(x-z_n,\vec{v}_n)\Big| \\ &\le \varepsilon \int_{|x-z_n|\ge R} |\vec{v}_n|^2 + |\vec{v}_n|^{\alpha+2} \\ &\le \varepsilon C\{|\vec{v}_n|^2_{\vec{H}^1} + |\vec{v}_n|^{\alpha+2}_{\vec{H}^1}\} \\ &\le \varepsilon D \end{aligned}$$

since $\vec{v}_n$ is bounded in $\vec{H}^1$. On the other hand, since $|z_n| \to \infty$, there exists $n_R > 0$ such that for all $n \ge n_R$,

$$\begin{aligned} &\Big|\int_{|x-z_n|\le R} F^\infty(x,\vec{v}_n) - F(x-z_n,\vec{v}_n)\Big| \\ &\le \Big|\int_{|x|\ge \frac12 |z_n|} F^\infty(x,\vec{v}_n) - F(x-z_n,\vec{v}_n)\Big| \\ &\le K \int_{|x|\ge \frac12 |z_n|} |\vec{v}_n|^2 + |\vec{v}_n|^{\ell+2} \end{aligned}$$

$$
\begin{aligned}
&\le K\int_{|x|\ge\frac12|z_n|}|\vec v|^2+|\vec v|^{\ell+2}+K\int_{|x|\ge\frac12|z_n|}|\vec v-\vec v_n|^2+|\vec v-\vec v_n|^{\ell+2}\\
&\le K\int_{|x|\ge\frac12|z_n|}|\vec v|^2+|\vec v|^{\ell+2}+K\int_{\mathbb{R}^N}|\vec v-\vec v_n|^2+|\vec v-\vec v_n|^{\ell+2}
\end{aligned}
$$

and hence

$$
\lim|\int_{|x-z_n|\ge R_n}F^\infty(x,\vec v_n)-F(x-z_n,\vec v_n)=0.
$$

Thus $\liminf\{J(\vec u_n)-J^\infty(\vec u_n)\}\ge-\varepsilon D$ for all $\varepsilon>0$. So that

$$
I_{c_1,\dots,c_m}=\lim J(\vec u_n)\ge\liminf J^\infty(\vec u_n)\ge I^\infty_{c_1,\dots,c_m}
$$

which contradicts that $I_{c_1,\dots,c_m}<I^\infty_{c_1,\dots,c_m}$. Hence $\{y_n\}$ is bounded. Setting $\rho=\sup_{n\in\mathbb{N}}|y_n|$, it follows that

$$
\begin{aligned}
\int_{B(0,R(\varepsilon)+\rho)}u_{n,1}^2+\dots+u_{n,m}^2&\ge\int_{B(y_n,R(\varepsilon))}u_{n,1}^2+\dots+u_{n,m}^2\\
&\ge c^2-\varepsilon.\quad\forall\varepsilon>0.
\end{aligned}
$$

Thus

$$
\int|\vec u|^2\ge\int_{B(0,R(\varepsilon)+\rho)}|\vec u|^2=\lim_{n\to+\infty}\int_{B(0,R(\varepsilon)+\rho)}|\vec u_n|^2\ge c^2-\varepsilon\quad\forall\varepsilon>0.
$$

Hence $\int u_1^2+..+u_m^2\ge c^2$.

On the other hand, $\int u_i^2\le c_i^2$ for $1\le i\le m$. Thus $\vec u\in S_c$ and $|\vec u_n-\vec u|_{\vec L^2}\to0$. By the boundedness of $\vec u_n$ in $\vec H^1$, it follows that $\vec u_n\to\vec u$ in $\vec L^p$ for $p\in[2,2^*]$, therefore

$$
\lim_{n\to\infty}\int F(x,\vec u_n)=\int F(x,\vec u)
$$

which implies $J(\vec u)=I_{c_1,\dots,c_m}$. □

4. Appendix

The concentration-compactness lemma in the multi-constrained setting states that: If $\{\vec u_n\}$ is a minimizing sequence of the problem (1.1), we associate to it the concentration function

$$
Q_n(R)=\sup_{y\in\mathbb{R}^N}\int_{B_R+y}\rho_n^2(\xi)d\xi
$$

where $\rho_n^2(\xi)=|\vec u_n|^2=\sum_{i=1}^m u_{n,i}^2(\xi)$. And applying the concentration compactness method, see [4, page 136-137] and [5, page 272-273]), one of the following alternatives occurs:

(1) Vanishing. $\limsup_{y\in\mathbb{R}^N}\int_{y+B_R}|\vec u_n|^2=0$.

(2) Dichotomy. For all $1\le i\le m$, there exists $a_i\in(0,c_i)$ such that for all $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ and two bounded sequences in $\vec H^1$ denoted by $\vec v_n$ and $\vec w_n$ (all depending on ε) such that for every $n\ge n_0$, we have

$$
\begin{gathered}
\big|\int v_{n,i}^2-a_i^2\big|<\varepsilon\quad\text{and}\quad\big|\int w_{n,i}^2-(c_i^2-a_i^2)\big|<\varepsilon\quad\forall1\le i\le m,\\
\int|\nabla\vec u_n|^2-|\nabla\vec v_n|^2-|\nabla\vec w_n|^2\ge-2\varepsilon,\\
|u_{n,i}-(v_{n,i}+w_{n,i})|_p\le4\varepsilon\quad\forall p\in(2,2^*].
\end{gathered}
$$

Furthermore, there exists $\{y_n\} \subset \mathbb{R}^N$ and $\{R_n\} \subset (0,\infty)$ such that $\lim_{n\to+\infty} R_n = +\infty$,

$$\begin{gathered} v_{n,i} = u_{n,i} \quad \text{if } |x-y_n| \le R_0,\\ |v_{n,i}| \le |u_{n,i}| \quad \text{if } R_0 \le |x-y_n| \le 2R_0,\\ v_{n,i} = 0 \quad \text{if } |x-y_n| \ge 2R_0; \end{gathered}$$

$$\begin{gathered} w_{n,i} = 0 \quad \text{if } |x-y_n| \le R_n,\\ |w_{n,i}| \le |u_{n,i}| \quad \text{if } R_n \le |x-y_n| \le< 2R_n,\\ w_{n,i} = u_{n,i} \quad \text{if } |x-y_n| \ge 2R_n; \end{gathered}$$

and $\operatorname{dist}\operatorname{supp}|v_{n,i}|, \operatorname{supp}|w_{n,i}|) \to \infty$ as $n \to \infty$.

(3) Compactness. There exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that for all $\varepsilon > 0$, there exists $R(\varepsilon) > 0$ such that

$$\int_{B(y_n,R(\varepsilon))} |\vec{u}_n|^2 \ge \sum_{i=1}^m c_i^2 - \varepsilon.$$

As suggested and stated by Lions in [5, pages 137-138], to get the above properties, it suffices to apply his method to ρ_n. Decomposing ρ_n in the classical setting and thus similtaneously $u_{n,i}$, leads to the properties of the splitting sequences $\vec{v}_n$ and $\vec{w}_n$, mentioned above.

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