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EXISTENCE OF GROUND STATES FOR FRACTIONAL KIRCHHOFF EQUATIONS WITH GENERAL POTENTIALS VIA NEHARI-POHOZAEV MANIFOLD

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Abstract. We consider the nonlinear fractional Kirchhoff equation

$$\Big(a+b\int_{\mathbb{R}^3}|(-\Delta)^{\alpha/2}u|^2\,\mathrm{d}x\Big)(-\Delta)^\alpha u+V(x)u=f(u)\quad \text{in } \mathbb{R}^3, u\in H^\alpha(\mathbb{R}^3),$$

where $a>0$, $b\geq 0$, $\alpha\in(3/4,1)$ are three constants, $V(x)$ is differentiable and $f\in C^1(\mathbb{R},\mathbb{R})$. Our main results show the existence of ground state solutions of Nehari-Pohozaev type, and the existence of the least energy solutions to the above problem with general superlinear and subcritical nonlinearity. These results are proved by applying variational methods and some techniques from [27].

1. Introduction

In this article, we study the fractional Kirchhoff equation

$$\begin{gathered}\Big(a+b\int_{\mathbb{R}^3}|(-\Delta)^{\alpha/2}u|^2\,\mathrm{d}x\Big)(-\Delta)^\alpha u+V(x)u=f(u)\quad \text{in } \mathbb{R}^3,\\ u\in H^\alpha(\mathbb{R}^3),\end{gathered}\tag{1.1}$$

where $a>0$, $b\geq 0$, $\alpha\in(3/4,1)$ are three constants, the operator $(-\Delta)^\alpha$ is the fractional Laplacian defined as $\mathcal{F}((-\Delta)^\alpha\phi)(\xi)=|\xi|^{2\alpha}\mathcal{F}(\phi)(\xi)$, where $\mathcal{F}$ is the Fourier transform. $V:\mathbb{R}^3\to\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$ satisfy the following assumptions:

- (A1) $V\in C(\mathbb{R}^3,[0,\infty))$;
- (A2) $V_\infty:=\liminf_{|y|\to\infty}V(y)\geq(\not\equiv)V(x)$ for all $x\in\mathbb{R}^3$;
- (A3) $f\in C^1(\mathbb{R},\mathbb{R})$ and there exist constants $C_0>0$ and $p\in(2,2_\alpha^*)$, $2_\alpha^*:=\frac{6}{3-2\alpha}$, such that
$$|f(t)|\leq C_0(1+|t|^{p-1}),\quad \forall t\in\mathbb{R};$$
- (A4) $f(t)=o(t)$ as $t\to 0$.

When $a=1$ and $b=0$, problem (1.1) reduces to the fractional Schrödinger equation

$$(-\Delta)^\alpha u+V(x)u=f(u)\quad \text{in } \mathbb{R}^3.\tag{1.2}$$

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As we all known, the fractional Laplacian $(-\Delta)^\alpha$ is now experiencing impressive applications in different subjects, such as phase transition, anomalous diffusion, fractional quantum mechanics and so on, see [22]. Equation (1.2) concerns fractional quantum mechanics, exactly, in the study of particles on stochastic fields modeled by Lévy processes. For this kind of fractional and nonlocal problems, Caffarelli and Silvestre [6] constructed general fractional power of Laplacian from an extension problem to the upper half space by mapping the Dirichlet condition to Neumann condition for a specific elliptic equation. Their work devotes efforts to dealing with nonlinear variational problems with fractional Laplacian by standard local perturbation method from variational method. After that, many results on the existence of ground state and multiplicity for solutions of (1.2) have been obtained. Just to mention a few, we recall, for instance, the following papers and the references therein [33, 12, 34, 25, 7]. Moreover, if $\alpha = 1$, (1.2) reduces to the classic nonlinear Schrödinger equation, we refer to [4, 29, 30] and the references therein for the recent research progress on this field.

If $\alpha = 1$, then (1.1) formally reduces to the well-known Kirchhoff equation

$$-\Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x\Big)\Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^3. \tag{1.3}$$

This equation is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \Big(a + b\int_{\Omega} |\nabla u|^2\,\mathrm{d}x\Big)\Delta u = f(x,u), \tag{1.4}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain. In 1883, Kirchhoff [18] proposed (1.4) as a generalization of the well-known D'Alembert wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{r_0}{\lambda} + \frac{K}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2\,\mathrm{d}x\Big)\frac{\partial^2 u}{\partial x^2} = f(x,u)$$

for describing the changes in length of the elastic string arising from transversal oscillations. Here, L is the length of the string, K is the Young modulus of the material, ρ is the mass density, r_0 is the initial tension and f denotes the area of the cross section, see [23]. In addition, Kirchhoff-type equation also models several physical and biological systems for the applications of the nonlocal effect. For more detail on the background of Kirchhoff type problems, we refer the readers to [1, 3] and the reference therein. With the widespread applications of Kirchhoff type equations, abundant results on the solvability of this kind of equations have brought out after the pioneer work [20]. Without attempts to provide a complete list of references, we refer the reader to [9, 19, 14, 31, 10, 15, 32, 26, 27] for the existence of ground state solutions, nodal solutions, sign-changing solutions, positive solutions and the concentration phenomena of solutions.

In the context of fractional quantum mechanics, a great interest has been devoted to fractional Kirchhoff type equations in recent years. Similar with classical Kirchhoff's model, Fiscella and Valdinoci [13] first proposed the stationary Kirchhoff equation involving nonlocal integro-differential operators

$$\begin{gathered} M\Big(\int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2}u|^2\,\mathrm{d}x\Big)(-\Delta)^\alpha u = f(x,u), \quad x \in \Omega, \\ u = 0, \quad x \in \mathbb{R}^N \setminus \Omega, \end{gathered} \tag{1.5}$$

which takes into account the nonlocal aspect of the tension produced by nonlocal measurement of the fractional length of the string. Ambrosio and Isernia [2] investigated the fractional Kirchhoff equation

$$\Big(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\alpha/2}u|^2\,\mathrm{d}x\Big)(-\Delta)^\alpha u=f(u)\quad\text{in }\mathbb{R}^N,\tag{1.6}$$

when f is an odd subcritical nonlinearity satisfying the well known Berestycki-Lions assumptions introduced in [4]. By minimax arguments, the authors established a multiplicity result in the fractional radial function space $H^\alpha_{\rm rad}(\mathbb{R}^N)$ while the parameter b is sufficiently small. Jin and Liu [17] considered the Kirchhoff equation with the critical growth

$$\Big(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\alpha/2}u|^2\,\mathrm{d}x\Big)(-\Delta)^\alpha u+u=f(u)\quad\text{in }\mathbb{R}^N,\tag{1.7}$$

when $N>2\alpha$. By using a perturbation approach, they proved the existence of positive radial solutions to (1.7) without the (AR) condition when the parameter b is small. A natural question now arises on whether (1.1) has ground state solution if the potential function $V(x)$ is not a constant. Recently, Liu, Squassina and Zhang [21] studied the equation with general potential

$$\Big(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\alpha/2}u|^2\,\mathrm{d}x\Big)(-\Delta)^\alpha u+V(x)u=f(u)\quad\text{in }\mathbb{R}^N\tag{1.8}$$

in low dimension $2\alpha<N<4\alpha$, that is $N=2,3$. When $N=3$ with $\alpha\in(3/4,1)$, the potential V in (1.8) satisfied not only the assumptions (A1)-(A2) but also the following assumptions:

- (A5) $V\in C^1(\mathbb{R}^3,\mathbb{R})$ and $\|\max\{\nabla V(x)\cdot x,0\}\|_{L^{\frac{3}{2\alpha}}(\mathbb{R}^3)}<2a\alpha S_\alpha$, where $\cdot$ denotes the inner product in $\mathbb{R}^3$ and S_α will be defined in Section 2.

Moreover, the nonlinearity f in (1.8) satisfies the following assumptions in addition to (A4):

- (A6) $f\in C^1(\mathbb{R}^+,\mathbb{R})$, $f(t)=0$ for all $t\le 0$;
- (A7) $\lim_{t\to\infty}\frac{f(t)}{t^{2^*_\alpha-1}}=1$;
- (A8) there are $D>0$ and $2<q<2^*_\alpha$ such that $f(t)\ge t^{2^*_\alpha-1}+Dt^{q-1}$ for any $t\ge 0$.

Clearly, when $N=3$, weak solutions to (1.8) correspond to critical points of the energy functional defined in $H^\alpha(\mathbb{R}^3)$ by

$$\begin{aligned}I(u)=&\frac12\int_{\mathbb{R}^3}[a|(-\Delta)^{\alpha/2}u|^2+V(x)u^2]\,\mathrm{d}x+\frac b4\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\alpha/2}u|^2\,\mathrm{d}x\Big)^2\\&-\int_{\mathbb{R}^3}F(u)\,\mathrm{d}x.\end{aligned}\tag{1.9}$$

Here, $H^\alpha(\mathbb{R}^3)$ denotes the usual fractional Sobolev space and $F(u):=\int_0^u f(s)\,\mathrm{d}s$. Under above assumptions on V and f, Liu et al. proved that (1.8) has a positive least energy solution when D is large enough. In fact, under the assumptions on N and α, it follows that $2^*_\alpha>4$. Thus $f(x,t)$ is critical growth at $t=\infty$ and

$$\lim_{t\to\infty}\frac{F(t)}{t^4}=\infty\quad\text{uniformly in }x\in\mathbb{R}^3.$$

Then it easily follows that the energy functional I possesses a mountain-pass geometry. But it is difficult to get a bounded (PS) sequence because of without (AR) condition. To overcome this difficulty, Liu et al. used Jeanjean's monotonicity [16] to show that there exists a bounded (PS) sequence $\{u_n(\lambda)\}$ at the level c_λ for almost $\lambda \in [1/2, 1]$. However, by the presence of the Kirchhoff term, it is not easy to show that I has a critical point since in general, for any $v \in C_0^\infty(\mathbb{R}^3)$, we do not know that

$$\begin{aligned}&\int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2} u_n|^2 \,\mathrm{d}x \int_{\mathbb{R}^3} (-\Delta)^{\alpha/2} u_n (-\Delta)^{\alpha/2} v \,\mathrm{d}x\\ &\to \int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2} u|^2 \,\mathrm{d}x \int_{\mathbb{R}^3} (-\Delta)^{\alpha/2} u (-\Delta)^{\alpha/2} v \,\mathrm{d}x\end{aligned}$$

from $u_n \rightharpoonup u$ in $H^\alpha(\mathbb{R}^3)$. Although $I'_\lambda(u) = 0$ from $u_n \rightharpoonup u$ in $H^\alpha(\mathbb{R}^3)$ can not be directly concluded, partly inspired by [19], they consider a family of related functionals J_λ, whose corresponding problem is a non-Kirchhoff equation. Through establishing a profile decomposition of the (PS) sequence $\{u_n(\lambda)\}$ related to J_λ, they obtained a nontrivial critical point u_λ of I_λ at the level c_λ. Subsequently, by choosing a sequence $\{\lambda_n\} \subset [1/2, 1]$ with $\lambda_n \to 1$, thanks to the Pohozaev identity they obtain a bounded $(PS)_{c_1}$ sequence of the original functional I. We point out that $f \in C^1$ is very crucial to get the profile decomposition of the (PS) sequence in [21]. In addition, the result is heavily dependent on the existence of a constant $D > 0$ large enough in (A8).

This article is motivated by [8, 28, 27, 29, 21]. Provided $f \in C^1(\mathbb{R}, \mathbb{R})$ with superlinear growth at infinity, under some mild assumptions on V and f, we obtain the existence of a ground solution of Nehari-Pohozaev type for (1.1) which is a minimizer of I on the Nehari-Pohozaev manifold $\mathcal{M}$. Moreover, we prove the existence of the least energy solutions for (1.1).

To state our results, in addition to (A1)–(A4), we make the following assumptions on V and f.

(A9) $V(x)$ is weakly differentiable, and there exists $\theta \in [0, 1)$ such that

$$(\nabla V(x), x) \le \frac{2\alpha\theta a K(\alpha)}{|x|^{2\alpha}}, \quad \text{a.e. } x \in \mathbb{R}^3 \backslash \{0\},$$

where $K(\alpha) := \frac{1}{\pi}\Gamma^2(\frac{1+2\alpha}{2})$ and Γ is the Gamma function;

(A10) $V \in C^1(\mathbb{R}^3, \mathbb{R})$ and there exists $\theta \in [0, 1)$ such that

$$4\alpha t^{4\alpha}[V(x) - V(tx)] - (1 - t^{4\alpha})(\nabla V(x), x) \ge -\frac{2\alpha\theta a K(\alpha)(1 - t^{2\alpha})^2}{|x|^{2\alpha}},$$

for all $t > 0$, $x \in \mathbb{R}^3 \backslash \{0\}$;

(A11) $\lim_{|t|\to\infty} \frac{F(t)}{|t|^2} = \infty$;

(A12) $\frac{(4\alpha-3)f(t)t+6F(t)}{(4\alpha-3)t|t|}$ is nondecreasing on $(-\infty, 0) \cup (0, \infty)$.

Remark 1.1. Note that the monotonicity condition

(A13) $\frac{f(t)}{|t|}$ is nondecreasing on $(-\infty, 0) \cup (0, \infty)$

is the weakened Nehari type condition. When $\alpha \in (3/4, 1)$, it is easy to prove that (A13) implies (A12). In fact, there are many functions satisfying (A12) but not (A13). For example,

$$f(t) = \Big(\frac{t^2}{5} + 3|t| + 2\sin t + t\cos t\Big)t.$$

Furthermore by (A3) and (A4), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(t)| \le \varepsilon|t| + C_\varepsilon|t|^{p-1}, \quad \forall t \in \mathbb{R}. \tag{1.10}$$

Before starting our main results, we define a functional on $H^\alpha(\mathbb{R}^3)$ as follows

$$\begin{aligned} J(u) :=&\alpha a \int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2} u|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} [4\alpha V(x) + (\nabla V(x), x)]u^2 \,\mathrm{d}x \\ &+ \alpha b\Big(\int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2} u|^2 \,\mathrm{d}x\Big)^2 - \frac{1}{2}\int_{\mathbb{R}^3} [(4\alpha-3)f(u)u + 6F(u)] \,\mathrm{d}x. \end{aligned} \tag{1.11}$$

In addition, we set

$$\mathcal{M} := \{u \in H^\alpha(\mathbb{R}^3)\setminus\{0\} : J(u) = 0\} \tag{1.12}$$

as Nehari-Pohozaev manifold. We are now in a position to state the main results of this article.

Theorem 1.2. *Assume that V and f satisfy* (A1)-(A4), (A10)–(A12). *Then Problem* (1.1) *has a solution $\bar{u} \in H^\alpha(\mathbb{R}^3)$ such that $I(\bar{u}) = \inf_{\mathcal{M}} I > 0$.*

Theorem 1.3. *Assume that V and f satisfy* (A1)–(A4), (A9), (A11), (A12). *Then Problem* (1.1) *has a least energy solution $\bar{u} \in H^\alpha(\mathbb{R}^3)\setminus\{0\}$.*

To prove Theorem 1.2, we first look for a minimizer of the functional restricted to Nehari-Pohozaev manifold $\mathcal{M}$ which is defined by a condition through combining the Nehari equation with the Pohozaev equality, for the usual method of Nehari manifold becomes invalid in this case. Note that such type of manifold was first introduced by Ruiz [24] for the study of the Schrödinger-Poisson problem. Then we prove that the minimizer on $\mathcal{M}$ is a critical point. To show Theorem 1.3, we use Jeanjeans monotonicity tricks, which is partly followed by [21]. These results will greatly improve the existing ones on fractional Kirchhoff problems.

To illustrate conveniently, we introduce some useful notation. In the sequel, $\|u\|_s = (\int_{\mathbb{R}^3} |u|^s \,\mathrm{d}x)^{1/s}$ denotes the norm of the Lebesgue space $L^s(\mathbb{R}^3)(1 \le s \le \infty)$. we set $(\cdot)_t := t^{4\alpha-3}[(\cdot)(\frac{x}{t})]$ for $t > 0$, where $(\cdot)$ denotes any function belongs to $H^\alpha(\mathbb{R}^3)\setminus\{0\}$. For $x \in \mathbb{R}^3$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^3 : |y - x| < r\}$. Throughout the paper, $C_1, C_2, \dots$ denote various positive constants.

This article is organized as follows. In Section 2, we give some preliminaries. In Section 3, the limited problem is discussed and the proof of Theorem 1.2 is given. Section 4 is devoted to finding a least energy solution for (1.1). Theorem 1.3 will be proved in this section.

2. Preliminaries and variational setting

The fractional Sobolev space $H^\alpha(\mathbb{R}^3)$ is defined by

$$H^\alpha(\mathbb{R}^3) := \Big\{u \in L^2(\mathbb{R}^3) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{3+2\alpha}{2}}} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)\Big\}.$$

It is known that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2\alpha}} \,\mathrm{d}x\,\mathrm{d}y = 2C_\alpha^{-1}\int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2}u|^2 \,\mathrm{d}x,$$

where

$$C_\alpha = \Big(\int_{\mathbb{R}^3} \frac{1-\cos\zeta_1}{|\zeta|^{3+2\alpha}} \,\mathrm{d}\zeta\Big)^{-1}.$$

We endow the space $H^\alpha(\mathbb{R}^3)$ with the norm

$$\|u\|_{H^\alpha(\mathbb{R}^3)} := \Big(\int_{\mathbb{R}^3} |u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2}u|^2 \,\mathrm{d}x\Big)^{1/2}.$$

The space $H^\alpha(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with $\|\cdot\|_{H^\alpha(\mathbb{R}^3)}$ and it is continuously embedded into $L^q(\mathbb{R}^3)$ for $q\in[2,2_\alpha^*]$ and compactly embedded into $L^q_{\rm loc}(\mathbb{R}^3)$ for any $q\in[2,2_\alpha^*)$. Moreover, the best embedding constant is

$$S_\alpha := \inf_{u\in D^{\alpha,2}(\mathbb{R}^3),\, u\neq 0} \frac{\int_{\mathbb{R}^3}|(-\Delta)^{\alpha/2}u|^2\,\mathrm{d}x}{\big(\int_{\mathbb{R}^3}|u|^{2_\alpha^*}\,\mathrm{d}x\big)^{2/2_\alpha^*}}.$$

Let

$$\mathcal{H} := \Big\{u\in H^\alpha(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2\,\mathrm{d}x<\infty\Big\}$$

be a Hilbert space equipped with the inner product

$$\langle u,v\rangle_{\mathcal{H}} := a\int_{\mathbb{R}^3}(-\Delta)^{\alpha/2}u(-\Delta)^{\alpha/2}v\,\mathrm{d}x + \int_{\mathbb{R}^3}V(x)uv\,\mathrm{d}x$$

and the corresponding induced norm

$$\|u\| := \Big(\int_{\mathbb{R}^3} a|(-\Delta)^{\alpha/2}u|^2\,\mathrm{d}x + \int_{\mathbb{R}^3}V(x)u^2\Big)^{1/2}.$$

The homogeneous Sobolev space $D^{\alpha,2}(\mathbb{R}^3)$ is defined by

$$D^{\alpha,2}(\mathbb{R}^3) := \Big\{u\in L^{2_\alpha^*}(\mathbb{R}^3) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{3+2\alpha}{2}}}\in L^2(\mathbb{R}^3\times\mathbb{R}^3)\Big\},$$

which is also the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|_{D^{\alpha,2}(\mathbb{R}^3)} := \Big(\int_{\mathbb{R}^3}|(-\Delta)^{\alpha/2}u|^2\,\mathrm{d}x\Big)^{1/2} = \|(-\Delta)^{\alpha/2}u\|_2.$$

A function $u\in\mathcal{H}$ is a weak solution to problem (1.1) if, for every $\phi\in\mathcal{H}$, we have

$$\begin{aligned}&\Big(a+b\int_{\mathbb{R}^3}|(-\Delta)^{\alpha/2}u|^2\,\mathrm{d}x\Big)\int_{\mathbb{R}^3}(-\Delta)^{\alpha/2}u(-\Delta)^{\alpha/2}\phi\,\mathrm{d}x + \int_{\mathbb{R}^3}V(x)u\phi\,\mathrm{d}x\\ &= \int_{\mathbb{R}^3}f(u)\phi\,\mathrm{d}x.\end{aligned}$$

Lemma 2.1 ([21]). *Assume that V satisfies* (A2). *Then for every $\epsilon>0$ there exists $\tau_\epsilon>0$ such that*

$$\int_{\mathbb{R}^3}((a-\epsilon)|(-\Delta)^{\alpha/2}u|^2+V(x)u^2)\,\mathrm{d}x \ge \tau_\epsilon\int_{\mathbb{R}^3}u^2\,\mathrm{d}x,\quad u\in H^\alpha(\mathbb{R}^3).$$

From the above lemma, it easily follows that the norm $\|\cdot\|$ is equivalent to $\|\cdot\|_{H^\alpha}$.

Lemma 2.2 ([25]). *Assume that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$ and*

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_r(y)}|u_n|^2\,\mathrm{d}x = 0,$$

for some $r>0$. Then $u_n\to 0$ in $L^s(\mathbb{R}^3)$ for all $s\in(2,2_\alpha^*)$.

3. Limit problem and proof of Theorem 1.2

In this section, through discussing the corresponding limit problem for (1.1), we will obtain the proof of Theorem 1.2. To this aim, we define two new functionals on $H^\alpha(\mathbb{R}^3)$ as follows:

$$\begin{aligned} I^\infty(u) &= \frac{1}{2}\int_{\mathbb{R}^3}\big[a|(-\Delta)^{\alpha/2}u|^2 + V_\infty u^2\big]\,\mathrm{d}x \\ &\quad + \frac{b}{4}\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\alpha/2}u|^2\,\mathrm{d}x\Big)^2 - \int_{\mathbb{R}^3}F(u)\,\mathrm{d}x, \end{aligned} \tag{3.1}$$

$$\begin{aligned} J^\infty(u) &= \alpha a\|(-\Delta)^{\alpha/2}u\|_2^2 + 2\alpha V_\infty\|u\|_2^2 + \alpha b\|(-\Delta)^{\alpha/2}u\|_2^4 \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^3}[(4\alpha-3)f(u)u + 6F(u)]\,\mathrm{d}x. \end{aligned} \tag{3.2}$$

Set

$$\begin{gathered} \mathcal{M}^\infty := \{u \in H^\alpha(\mathbb{R}^3)\backslash\{0\} : J^\infty(u) = 0\}, \\ m_0 := \inf_{u\in\mathcal{M}} I(u), \quad m^\infty := \inf_{u\in\mathcal{M}^\infty} I^\infty(u). \end{gathered}$$

Lemma 3.1. *Assume that* (A3) *and* (A12) *hold and* $\alpha \in (3/4, 1)$. *Then*

$$\frac{(4\alpha-3)(1-t^{4\alpha})}{8\alpha}f(w)w - \frac{4\alpha-3+3t^{4\alpha}}{4\alpha}F(w) + t^3F(t^{\frac{4\alpha-3}{2}}w) \ge 0, \tag{3.3}$$

for all $t \ge 0$ *and* $w \in \mathbb{R}$.

Proof. It is clear that (3.3) holds for $w = 0$. For $w \neq 0$, let

$$h(t) = \frac{4\alpha-3}{8\alpha}(1-t^{4\alpha})f(w)w - \frac{4\alpha-3+3t^{4\alpha}}{4\alpha}F(w) + t^3F(t^{\frac{4\alpha-3}{2}}w), \quad t \ge 0. \tag{3.4}$$

Then from (A12), one has

$$\begin{aligned} h'(t) &= \frac{t^{4\alpha-1}w^2}{2}\Big[\frac{f(t^{\frac{4\alpha-3}{2}}w)(t^{\frac{4\alpha-3}{2}}w) + \frac{6}{4\alpha-3}F(t^{\frac{4\alpha-3}{2}}w)}{(t^{\frac{4\alpha-3}{2}}w)^2} \\ &\quad - \frac{f(w)w + \frac{6}{4\alpha-3}F(w)}{w^2}\Big] \\ &= \begin{cases} \ge 0, & t \ge 1, \\ \le 0, & 0 < t < 1. \end{cases} \end{aligned} \tag{3.5}$$

From this and the continuity of h it follows that $h(t) \ge h(1) = 0$ for $t \ge 0$. This implies (3.3) holds. $\square$

Lemma 3.2. *Assume that* (A1), (A3), (A4), (A10), (A12) *hold. Then*

$$I(u) \ge I(u_t) + \frac{1-t^{4\alpha}}{4\alpha}J(u) + \frac{a(1-\theta)(1-t^{2\alpha})^2}{4}\|(-\Delta)^{\alpha/2}u\|_2^2, \tag{3.6}$$

for all $u \in H^\alpha(\mathbb{R}^3)$, $t > 0$.

Proof. According to the fractional Hardy inequality in [5], we have

$$\|(-\Delta)^{\alpha/2}u\|_2^2 \ge \frac{1}{\pi}\Gamma^2\big(\frac{1+2\alpha}{2}\big)\int_{\mathbb{R}^3}\frac{u^2}{|x|^{2\alpha}}\,\mathrm{d}x. \tag{3.7}$$

Note that

$$\begin{aligned} I(u_t) = \frac{at^{2\alpha}}{2}\|(-\Delta)^{\alpha/2}u\|_2^2 + \frac{t^{4\alpha}}{2}\int_{\mathbb{R}^3} V(tx)u^2\,\mathrm{d}x + \frac{bt^{4\alpha}}{4}\|(-\Delta)^{\alpha/2}u\|_2^4 \\ - t^3\int_{\mathbb{R}^3} F\big(t^{\frac{4\alpha-3}{2}}u\big)\,\mathrm{d}x. \end{aligned} \tag{3.8}$$

Thus, in view of (A10), (1.9), (3.3), (3.7) and (3.8), one has

$$\begin{aligned} &I(u) - I(u_t) \\ &= \frac{a(1-t^{2\alpha})}{2}\|(-\Delta)^{\alpha/2}u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3}[V(x) - t^{4\alpha}V(tx)]u^2\,\mathrm{d}x \\ &\quad + \frac{b(1-t^{4\alpha})}{4}\|(-\Delta)^{\alpha/2}u\|_2^4 + \int_{\mathbb{R}^3}\Big[t^3F(t^{\frac{4\alpha-3}{2}}u) - F(u)\Big]\,\mathrm{d}x \\ &= \frac{1-t^{4\alpha}}{4\alpha}\Big\{\alpha(a + b\|(-\Delta)^{\alpha/2}u\|_2^2)\|(-\Delta)^{\alpha/2}u\|_2^2 \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^3}[4\alpha V(x) + (\nabla V(x), x)]\,u^2\,\mathrm{d}x \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^3}[(4\alpha-3)f(u)u + 6F(u)]\,\mathrm{d}x\Big\} + \frac{a(1-t^{2\alpha})^2}{4}\|(-\Delta)^{\alpha/2}u\|_2^2 \\ &\quad + \frac{1}{8\alpha}\int_{\mathbb{R}^3}\{4\alpha t^{4\alpha}[V(x) - V(tx)] - (1-t^{4\alpha})(\nabla V(x), x)\}u^2\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3}\Big[\frac{(4\alpha-3)(1-t^{4\alpha})}{8\alpha}f(u)u - \frac{4\alpha-3+3t^{4\alpha}}{4\alpha}F(u) + t^3F(t^{\frac{4\alpha-3}{2}}u)\Big]\,\mathrm{d}x \\ &\geq \frac{1-t^{4\alpha}}{4\alpha}J(u) + \frac{a(1-\theta)(1-t^{2\alpha})^2}{4}\|(-\Delta)^{\alpha/2}u\|_2^2. \end{aligned}$$

This shows that (3.6) holds. □

From Lemma 3.2, we have the following two corollaries.

Corollary 3.3. *Assume that* (A3), (A4), (A12) *hold. Then*

$$I^\infty(u) \geq I^\infty(u_t) + \frac{1-t^{4\alpha}}{4\alpha}J^\infty(u) + \frac{a(1-t^{2\alpha})^2}{4}\|(-\Delta)^{\alpha/2}u\|_2^2, \tag{3.9}$$

for all $u \in H^\alpha(\mathbb{R}^3)$ *and* $t > 0$.

Corollary 3.4. *Assume that* (A1), (A3), (A4), (A10), (A12) *hold. Then for* $u \in \mathcal{M}$,

$$I(u) = \max_{t>0} I(u_t). \tag{3.10}$$

Lemma 3.5. *Assume that* (A1), (A2), (A10) *hold. Then there exist two constants* $\omega_1, \omega_2 > 0$ *such that*

$$\omega_1\|u\|^2 \leq \alpha a\|(-\Delta)^{\alpha/2}u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3}[4\alpha V(x) + (\nabla V(x), x)]u^2\,\mathrm{d}x \leq \omega_2\|u\|^2, \tag{3.11}$$

for all $u \in H^\alpha(\mathbb{R}^3)$.

Proof. Using (A2) and Lemma 2.1, in a same way as in [27, Lemma 2.5], we can prove (3.11) holds. □

Lemma 3.6. *Assume that Assumptions* (A1)-(A4), (A10)–(A12) *hold. Then for any* $u \in H^\alpha(\mathbb{R}^3) \setminus \{0\}$, *there exists a unique* $t(u) > 0$ *such that* $u_{t(u)} \in \mathcal{M}$.

Proof. Let $u \in H^\alpha(\mathbb{R}^3)\setminus\{0\}$ be fixed and define a function $g(t) := I(u_t)$ on $(0,\infty)$. From (3.8), we have that

$$\begin{aligned}
&g'(t) = 0\\
&\Leftrightarrow \alpha a t^{2\alpha}\|(-\Delta)^{\alpha/2}u\|_2^2 + \frac{t^{4\alpha}}{2}\int_{\mathbb{R}^3}[4\alpha V(tx) + (\nabla V(tx), tx)]u^2\,\mathrm{d}x\\
&\quad + \alpha b t^{4\alpha}\|(-\Delta)^{\alpha/2}u\|_2^4 - \frac{t^3}{2}\int_{\mathbb{R}^3}[(4\alpha-3)f(t^{\frac{4\alpha-3}{2}}u)t^{\frac{4\alpha-3}{2}}u + 6F(t^{\frac{4\alpha-3}{2}}u)]\,\mathrm{d}x = 0\\
&\Leftrightarrow J(u_t) = 0\\
&\Leftrightarrow u_t \in \mathcal{M}.
\end{aligned} \tag{3.12}$$

It is easy to verify, using (A1), (A2), (A3), (A4) and (A11), that $g(0)=0$, $g(t)>0$ for $t>0$ small and $g(t)<0$ for t large. Therefore $\max_{t\in[0,\infty)} g(t)$ is achieved at $t_0 = t(u) > 0$ so that $g'(t_0)=0$ and $u_{t_0}\in\mathcal{M}$.

Next we claim that $t(u)$ is unique for any $u \in H^\alpha(\mathbb{R}^3)\setminus\{0\}$. In fact, for any given $u\in H^\alpha(\mathbb{R}^3)\setminus\{0\}$, let $t_1, t_2 > 0$ such that $u_{t_1}, u_{t_2}\in\mathcal{M}$. Then $J(u_{t_1}) = J(u_{t_2}) = 0$. Jointly with (3.6), we have

$$\begin{aligned}
I(u_{t_1}) &\geq I(u_{t_2}) + \frac{t_1^{4\alpha}-t_2^{4\alpha}}{4\alpha t_1^{4\alpha}}J(u_{t_1}) + \frac{(1-\theta)a(t_1^{2\alpha}-t_2^{2\alpha})^2}{4t_1^{2\alpha}}\|(-\Delta)^{\alpha/2}u_{t_1}\|_2^2\\
&= I(u_{t_2}) + \frac{(1-\theta)a(t_1^{2\alpha}-t_2^{2\alpha})^2}{4t_1^{2\alpha}}\|(-\Delta)^{\alpha/2}u_{t_1}\|_2^2
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
I(u_{t_2}) &\geq I(u_{t_1}) + \frac{t_2^{4\alpha}-t_1^{4\alpha}}{4\alpha t_2^{4\alpha}}J(u_{t_2}) + \frac{a(t_2^{2\alpha}-t_1^{2\alpha})^2}{4t_2^{2\alpha}}\|(-\Delta)^{\alpha/2}u_{t_2}\|_2^2\\
&= I(u_{t_1}) + \frac{(1-\theta)a(t_2^{2\alpha}-t_1^{2\alpha})^2}{4t_2^{2\alpha}}\|(-\Delta)^{\alpha/2}u_{t_2}\|_2^2.
\end{aligned} \tag{3.14}$$

Inequalitites (3.13) and (3.14) imply $t_1 = t_2$. Therefore, $t(u)>0$ is unique for any $u\in H^\alpha(\mathbb{R}^3)\setminus\{0\}$. ☐

Lemma 3.7. *Assume that* (A1)-(A4), (A10)–(A12) *hold. Then*

$$\inf_{u\in\mathcal{M}} I(u) := m_0 = \inf_{u\in H^\alpha(\mathbb{R}^3)\setminus\{0\}}\max_{t>0} I(u_t).$$

Note that Corollary 3.4 and Lemma 3.6 imply the above lemma.

Lemma 3.8. *Assume that* (A1)-(A4), (A10)–(A12) *hold. Then*

(i) *there exists $\rho_0>0$ such that $\|u\|\geq\rho_0$ for all $u\in\mathcal{M}$;*
(ii) $m_0 = \inf_{u\in\mathcal{M}} I(u) > 0$.

Proof. (i) Since $J(u)=0$ for all $u\in\mathcal{M}$, by (A3), (1.10), (1.11), (3.2), (3.11) and Lemma 2.1, one has

$$\begin{aligned}
\omega_1\|u\|^2 &\leq \alpha a\|(-\Delta)^{\alpha/2}u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3}[4\alpha V(x) + (\nabla V(x), x)]u^2\,\mathrm{d}x + \alpha b\|(-\Delta)^{\alpha/2}u\|_2^4\\
&= \frac{1}{2}\int_{\mathbb{R}^3}[(4\alpha-3)f(u)u + 6F(u)]\,\mathrm{d}x\\
&\leq \frac{\omega_1}{2}\|u\|^2 + C_1\|u\|^p.
\end{aligned}$$

This implies

$$\|u\| \geq \rho_0 := \big(\frac{\omega_1}{2C_1}\big)^{1/(p-2)}, \quad \forall\, u \in \mathcal{M}. \tag{3.15}$$

(ii). By (A2), there exists $R>0$ such that $V(x) \geq \frac{V_\infty}{2}$ for $|x| \geq R$. From (1.10), there exists $C_2>0$ such that

$$|F(t)| \leq \frac{1}{4}\min\Big\{aS_\alpha\Big(\frac{3}{4\pi R^3}\Big)^{2\alpha/3}, V_\infty\Big\}|t|^2 + C_2|t|^{2^*_\alpha}, \quad \forall t \in \mathbb{R}. \tag{3.16}$$

For $u \in \mathcal{M}$, let $t_u = \big(\frac{aS_\alpha^{\frac{2^*_\alpha}{2}}}{12C_2}\big)^{\frac{3-2\alpha}{4\alpha^2}}\|(-\Delta)^{\alpha/2}u\|_2^{-\frac{1}{\alpha}}$. Using Hölder's inequality and the fractional Sobolev inequality, we obtain

$$\begin{aligned}\int_{|t_u x|<R} u^2\,\mathrm{d}x &\leq \Big(\frac{4\pi R^3}{3t_u^3}\Big)^{2\alpha/3}\Big(\int_{|t_u x|<R} u^{2^*_\alpha}\,\mathrm{d}x\Big)^{2/2^*_\alpha}\\ &\leq \Big(\frac{4\pi R^3}{3t_u^3}\Big)^{2\alpha/3} S_\alpha^{-1}\|(-\Delta)^{\alpha/2}u\|_2^2.\end{aligned} \tag{3.17}$$

Then from (3.6), (3.8), (3.16), (3.17) and the fractional Sobolev inequality, we have

$$\begin{aligned}
I(u) &\geq I(u_{t_u})\\
&= \frac{at_u^{2\alpha}}{2}\|(-\Delta)^{\alpha/2}u\|_2^2 + \frac{t_u^{4\alpha}}{2}\int_{\mathbb{R}^3} V(t_u x)u^2\,\mathrm{d}x + \frac{bt_u^{4\alpha}}{4}\|(-\Delta)^{\alpha/2}u\|_2^4\\
&\quad - t_u^3\int_{\mathbb{R}^3} F(t_u^{\frac{4\alpha-3}{2}}u)\,\mathrm{d}x\\
&\geq \frac{at_u^{2\alpha}}{4}\|(-\Delta)^{\alpha/2}u\|_2^2 + \frac{aS_\alpha}{4}\Big(\frac{3}{4\pi R^3}\Big)^{2\alpha/3} t_u^{4\alpha}\int_{|t_u x|<R} u^2\,\mathrm{d}x\\
&\quad + \frac{V_\infty t_u^{4\alpha}}{4}\int_{|t_u x|\geq R} u^2\,\mathrm{d}x - t_u^3\int_{\mathbb{R}^3} F(t_u^{\frac{4\alpha-3}{2}}u)\,\mathrm{d}x\\
&\geq \frac{at_u^{2\alpha}}{4}\|(-\Delta)^{\alpha/2}u\|_2^2 + \frac{1}{4}\min\Big\{aS_\alpha\Big(\frac{3}{4\pi R^3}\Big)^{2\alpha/3}, V_\infty\Big\}t_u^{4\alpha}\|u\|_2^2\\
&\quad - t_u^3\int_{\mathbb{R}^3} F(t_u^{\frac{4\alpha-3}{2}}u)\,\mathrm{d}x\\
&\geq \frac{at_u^{2\alpha}}{4}\|(-\Delta)^{\alpha/2}u\|_2^2 - C_2 t_u^{\frac{6\alpha}{3-2\alpha}}\|u\|_{2^*_\alpha}^{2^*_\alpha}\\
&\geq \frac{at_u^{2\alpha}}{4}\|(-\Delta)^{\alpha/2}u\|_2^2 - C_2 S_\alpha^{-\frac{2^*_\alpha}{2}} t_u^{\frac{6\alpha}{3-2\alpha}}\|(-\Delta)^{\alpha/2}u\|_2^{2^*_\alpha}\\
&= \frac{a}{6}\Big(\frac{aS_\alpha^{\frac{2^*_\alpha}{2}}}{12C_2}\Big)^{\frac{3-2\alpha}{2\alpha}}, \quad \forall u \in \mathcal{M}.
\end{aligned}$$

This shows that $m_0 = \inf_{u\in\mathcal{M}} I(u) > 0$. □

Lemma 3.9. *Assume that* (A3) *and* (A4) *hold. If $u_n \rightharpoonup \bar{u}$ in $H^\alpha(\mathbb{R}^3)$, then along a subsequence of $\{u_n\}$,*

$$\lim_{n\to\infty}\sup_{\varphi\in H^\alpha(\mathbb{R}^3),\|\varphi\|\leq 1}\Big|\int_{\mathbb{R}^3}[f(u_n) - f(u_n-\bar{u}) - f(\bar{u})]\varphi\,\mathrm{d}x\Big| = 0. \tag{3.18}$$

The above lemma was proved in [11, Page 77-80] and a simpler and more direct proof has been given in [28, Lemma 2.7].

Lemma 3.10. *Assume that* (A1), (A3), (A4), (A10) *hold. If $u_n \rightharpoonup \bar{u}$ in $H^\alpha(\mathbb{R}^3)$, then*

$$I(u_n) = I(\bar{u}) + I(u_n - \bar{u}) + \frac{\alpha b}{2}\|(-\Delta)^{\alpha/2}\bar{u}\|_2^2\|(-\Delta)^{\alpha/2}(u_n-\bar{u})\|_2^2 + o(1), \tag{3.19}$$

$$\begin{aligned}\langle I'(u_n), u_n\rangle &= \langle I'(\bar{u}), \bar{u}\rangle + \langle I'(u_n-\bar{u}), u_n-\bar{u}\rangle \\ &\quad + 2\alpha b\|(-\Delta)^{\alpha/2}\bar{u}\|_2^2\|(-\Delta)^{\alpha/2}(u_n-\bar{u})\|_2^2 + o(1),\end{aligned} \tag{3.20}$$

$$J(u_n) = J(\bar{u}) + J(u_n-\bar{u}) + 2\alpha b\|(-\Delta)^{\alpha/2}\bar{u}\|_2^2\|(-\Delta)^{\alpha/2}(u_n-\bar{u})\|_2^2 + o(1). \tag{3.21}$$

The proof of the above lemma is similar to the one in [27, Lemma 2.10], we omit it here.

Lemma 3.11. *Assume that* (A3), (A4), (A11), (A12) *hold. Then m^∞ is achieved.*

Proof. Let $t \to 0$ in (3.3), then we have

$$f(w)w - 2F(w) \ge 0, \quad \forall w \in \mathbb{R}. \tag{3.22}$$

We introduce a new functional $\Phi_\infty : H^\alpha(\mathbb{R}^3) \to \mathbb{R}$ as follows

$$\Phi_\infty(u) = \frac{a}{4}\|(-\Delta)^{\alpha/2}u\|_2^2 + \frac{4\alpha-3}{8\alpha}\int_{\mathbb{R}^3}[f(u)u - 2F(u)]\,\mathrm{d}x. \tag{3.23}$$

For any $u \in \mathcal{M}^\infty$, we have $\Phi_\infty(u) = I^\infty(u) \ge m^\infty$. Let $\{u_n\} \subset \mathcal{M}^\infty$ be such that $I^\infty(u_n) \to m^\infty$. Since $J^\infty(u_n) = 0$, then it follows from (3.9) with $t \to 0$ that

$$m^\infty + o(1) = I^\infty(u_n) \ge \frac{a}{4}\|(-\Delta)^{\alpha/2}u_n\|_2^2. \tag{3.24}$$

This shows that $\{\|(-\Delta)^{\alpha/2}u_n\|_2\}$ is bounded. Next, we prove that $\{\|u_n\|\}$ is also bounded. By (1.10), (3.2) and Sobolev embedding theorem, it holds

$$\begin{aligned}&\min\{\alpha a, 2V_\infty\}\|u_n\|^2 \\ &\le \int_{\mathbb{R}^3}(\alpha a|(-\Delta)^{\alpha/2}u_n|^2 + 2V_\infty u_n^2)\,\mathrm{d}x + \alpha b\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\alpha/2}u_n|^2\,\mathrm{d}x\Big)^2 \\ &= \frac{1}{2}\int_{\mathbb{R}^3}[(4\alpha-3)f(u_n)u_n + 6F(u_n)]\,\mathrm{d}x \\ &\le \frac{1}{2}\min\{\alpha a, 2V_\infty\}\|u_n\|^2 + C_3\|u_n\|_{2_\alpha^*}^{2_\alpha^*} \\ &\le \frac{1}{2}\min\{\alpha a, 2V_\infty\}\|u_n\|^2 + C_3 S_\alpha^{-\frac{2_\alpha^*}{2}}\|(-\Delta)^{\alpha/2}u_n\|_2^{2_\alpha^*}.\end{aligned} \tag{3.25}$$

This shows that $\{u_n\}$ is bounded in $H^\alpha(\mathbb{R}^3)$. By Lemma 2.2, one easily prove that

$$\delta := \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_1(y)}|u_n|^2\,\mathrm{d}x > 0.$$

Going to a subsequence, if necessary, we may assume the existence of $\{y_n\} \subset \mathbb{R}^3$ such that

$$\int_{B_1(y_n)}|u_n|^2\,\mathrm{d}x > \frac{\delta}{2}.$$

Let $\widetilde{u}_n(x) = u_n(x+y_n)$. Then $\|\widetilde{u}_n\| = \|u_n\|$,

$$\int_{B_1(0)}|\widetilde{u}_n|^2\,\mathrm{d}x > \frac{\delta}{2}, \tag{3.26}$$

$$I^\infty(\widetilde{u}_n) \to m^\infty, \qquad J^\infty(\widetilde{u}_n) = 0. \tag{3.27}$$

Then, there exists $\bar{u} \in H^\alpha(\mathbb{R}^3) \setminus \{0\}$ such that

$$\begin{gathered} \widetilde{u}_n \rightharpoonup \bar{u}, \quad \text{in } H^\alpha(\mathbb{R}^3); \\ \widetilde{u}_n \to \bar{u}, \quad \text{in } L^s_{\rm loc}(\mathbb{R}^3),\ \forall\, s \in [1, 2^*_\alpha); \\ \widetilde{u}_n \to \bar{u}, \quad \text{a.e. on } \mathbb{R}^3. \end{gathered} \tag{3.28}$$

Let $\hat{u}_n = \widetilde{u}_n - \bar{u}$. Then (3.28) and Lemma 3.10 yield

$$\Phi_\infty(\widetilde{u}_n) = \Phi_\infty(\bar{u}) + \Phi_\infty(\hat{u}_n) + o(1), \tag{3.29}$$

$$J^\infty(\widetilde{u}_n) = J^\infty(\bar{u}) + J^\infty(\hat{u}_n) + 2\alpha b\|(-\Delta)^{\alpha/2}\bar{u}\|_2^2\|(-\Delta)^{\alpha/2}\hat{u}_n\|_2^2 + o(1). \tag{3.30}$$

From (3.1), (3.2), (3.27), (3.29) and (3.30), one has

$$\Phi_\infty(\hat{u}_n) = m^\infty - \Phi_\infty(\bar{u}) + o(1), \quad J^\infty(\hat{u}_n) \le -J^\infty(\bar{u}) + o(1). \tag{3.31}$$

If there exists a subsequence $\{\hat{u}_{n_i}\}$ of $\{\hat{u}_n\}$ such that $\hat{u}_{n_i} = 0$, then going to this subsequence, we have

$$I^\infty(\bar{u}) = m^\infty, \quad J^\infty(\bar{u}) = 0, \tag{3.32}$$

which implies that Lemma 3.11 holds. Next, we assume that $\hat{u}_n \ne 0$. In view of Lemma 3.6, there exists $t_n > 0$ such that $(\hat{u}_n)_{t_n} \in \mathcal{M}^\infty$. We claim that $J^\infty(\bar{u}) \le 0$. Otherwise, if $J^\infty(\bar{u}) > 0$, then (3.31) implies $J^\infty(\hat{u}_n) < 0$ for large n. From (3.1), (3.2), (3.6), (3.23) and (3.31), we obtain

$$\begin{aligned} m^\infty - \Phi_\infty(\bar{u}) + o(1) &= \Phi_\infty(\hat{u}_n) \\ &= \frac{a}{4}\|(-\Delta)^{\alpha/2}u\|_2^2 + \frac{4\alpha-3}{8\alpha}\int_{\mathbb{R}^3}[f(u)u - 2F(u)]\,\mathrm{d}x \\ &= I^\infty(\hat{u}_n) - \frac{1}{4\alpha}J^\infty(\hat{u}_n) \\ &\ge I^\infty((\hat{u}_n)_{t_n}) - \frac{t_n^{4\alpha}}{4\alpha}J^\infty(\hat{u}_n) \\ &\ge m^\infty - \frac{t_n^{4\alpha}}{4\alpha}J^\infty(\hat{u}_n) \ge m^\infty, \end{aligned}$$

which implies $J^\infty(\bar{u}) \le 0$ due to $\Phi_\infty(\bar{u}) > 0$. Since $\bar{u} \ne 0$, in view of Lemma 3.6, there exists $\bar{t} > 0$ such that $\bar{u}_{\bar{t}} \in \mathcal{M}^\infty$. From (3.1), (3.2), (3.6), (3.22), the weak semicontinuity of norm and Fatou's lemma, one has

$$\begin{aligned} m^\infty &= \lim_{n\to\infty}\big[I^\infty(\widetilde{u}_n) - \frac{1}{4\alpha}J^\infty(\widetilde{u}_n)\big] \\ &= \lim_{n\to\infty}\Big\{\frac{a}{4}\|(-\Delta)^{\alpha/2}\widetilde{u}_n\|_2^2 + \frac{4\alpha-3}{8\alpha}\int_{\mathbb{R}^3}[f(\widetilde{u}_n)\widetilde{u}_n - 2F(\widetilde{u}_n)]\,\mathrm{d}x\Big\} \\ &\ge \frac{a}{4}\|(-\Delta)^{\alpha/2}\bar{u}\|_2^2 + \frac{1}{8\alpha}\int_{\mathbb{R}^3}[(4\alpha-3)f(\bar{u})\bar{u} - 2F(\bar{u})]\,\mathrm{d}x \\ &= I^\infty(\bar{u}) - \frac{1}{4\alpha}J^\infty(\bar{u}) \\ &\ge I(\bar{u}_{\bar{t}}) - \frac{\bar{t}^{4\alpha}}{4\alpha}J^\infty(\bar{u}) \\ &\ge m^\infty - \frac{\bar{t}^{4\alpha}}{4\alpha}J^\infty(\bar{u}) \ge m^\infty, \end{aligned}$$

which implies

$$J^\infty(\bar{u}) = 0, \quad I^\infty(\bar{u}) = m^\infty.$$

□

Lemma 3.12. *Assume that* (A1)–(A4), (A10)–(A12) *hold. If* $\bar{u} \in \mathcal{M}$ *and* $I(\bar{u}) = m_0$*, then* $\bar{u}$ *is a critical point of* I*.*

Proof. Assume that $I'(\bar{u}) \neq 0$. Then there exist $\delta > 0$ and $\varrho > 0$ such that

$$\|u - \bar{u}\| \leq 3\delta \Rightarrow \|I'(u)\| \geq \varrho. \tag{3.33}$$

Firstly, in the same way as [27, Lemma 2.13], we can prove that

$$\lim_{t\to 1} \|\bar{u}_t - \bar{u}\| = 0. \tag{3.34}$$

Thus, there exists $\delta_1 > 0$ such that

$$|t^\alpha - 1| < \delta_1 \Rightarrow \|\bar{u}_t - \bar{u}\| < \delta. \tag{3.35}$$

In view of Lemma 3.2, one has

$$\begin{aligned} I(\bar{u}_t) &\leq I(\bar{u}) - \frac{a(1-\theta)(1-t^{2\alpha})^2}{4}\|(-\Delta)^{\alpha/2}\bar{u}\|_2^2 \\ &= m_0 - \frac{a(1-\theta)(1-t^{2\alpha})^2}{4}\|(-\Delta)^{\alpha/2}\bar{u}\|_2^2, \quad \forall t > 0. \end{aligned} \tag{3.36}$$

It follows from (3.12) that there exist $T_1 \in (0,1)$ and $T_2 \in (1,\infty)$ such that

$$J(\bar{u}_{T_1}) > 0, \quad J(\bar{u}_{T_2}) < 0.$$

Let

$$\begin{aligned} \varepsilon := \min\Big\{&\frac{a(1-\theta)(1-T_1^{2\alpha})^2\|(-\Delta)^{\alpha/2}\bar{u}\|_2^2}{24}, \\ &\frac{a(1-\theta)(1-T_2^{2\alpha})^2\|(-\Delta)^{\alpha/2}\bar{u}\|_2^2}{24}, 1, \varrho\delta/8\Big\}. \end{aligned}$$

The rest of the proof is similar to the proof of [29, Lemma 2.13]. □

Now we can draw a conclusion on the existence of a ground state solution of Nehari-Pohozaev type to the "limit problem" of Problem (1.1)

$$\begin{gathered} \Big(a + b\int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2}u|^2 \,\mathrm{d}x\Big)(-\Delta)^\alpha u + V_\infty u = f(u), \quad x \in \mathbb{R}^3; \\ u \in H^\alpha(\mathbb{R}^3). \end{gathered} \tag{3.37}$$

Theorem 3.13. *Assume that* f *satisfies* (A3), (A4), (A11), (A12) *hold. Then Problem* (3.37) *has a solution* $\bar{u} \in H^\alpha(\mathbb{R}^3)\backslash\{0\}$ *such that* $I^\infty(\bar{u}) = \inf_{\mathcal{M}^\infty} I^\infty > 0$.

Corollary 3.14. *Let* $\widetilde{f}(t) = 0$ *for* $t < 0$ *and* $\widetilde{f}(t) = f(t)$ *for* $t \geq 0$*, use* $\widetilde{f}$ *to take place of* f *in* (3.37) *and assume* (A3), (A4), (A11), (A12) *hold, then Problem* (3.37) *has a positive solution* $\bar{u} \in H^\alpha(\mathbb{R}^3)\backslash\{0\}$ *such that* $I^\infty(\bar{u}) = \inf_{\mathcal{M}^\infty} I^\infty > 0$.

Lemma 3.15. *Assume that* (A1)–(A4), (A10)–(A12) *hold. Then* $m_0 < m^\infty$.

Proof. In view of Theorem 3.13 and Corollary 3.14, I^∞ has a minimizer $u^\infty > 0$ on $\mathcal{M}^\infty$, i.e.

$$u^\infty \in \mathcal{M}^\infty \quad \text{and} \quad m^\infty = I^\infty(u^\infty).$$

In view of Lemma 3.6, there exists $t_0 > 0$ such that $(u^\infty)_{t_0} \in \mathcal{M}$. Thus, it follows from (A2), (1.9), (3.1), (3.9) that

$$m^\infty = I^\infty(u^\infty) \geq I^\infty((u^\infty)_{t_0}) > I((u^\infty)_{t_0}) \geq m_0.$$

□

Lemma 3.16. *Assume that* (A1)–(A4), (A10)–(A12) *hold. Then m_0 is achieved.*

The proof of this lemma is analogous to the one of [27, Lemma 3.2], so we omit it here.

Proof of Theorem 1.2. It is a direct corollary of Lemmas 3.8, 3.12 and 3.16. □

4. Proof of Theorem 1.3

To obtain the boundedness of the Palais-Smale sequence, we adopt a monotonicity technique due to Jeanjean.

Proposition 4.1 ([16]). *Let X be a Banach space and let $\Lambda \subset \mathbb{R}^+$ be an interval. We consider a family $\{\Phi_\lambda\}$ of C^1-functionals on X of the form*

$$\Phi_\lambda(u) = A(u) - \lambda B(u), \quad \forall \lambda \in J,$$

where $B(u) \geq 0$ for all $u \in X$, and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$, as $\|u\| \to \infty$. We assume that there are two points v_1, v_2 in X such that

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_\lambda(\gamma(t)) > \max\{\Phi_\lambda(v_1), \Phi_\lambda(v_2)\}, \tag{4.1}$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost every $\lambda \in J$, there is a bounded $(PS)_{c_\lambda}$ sequence for Φ_λ; that is, there exists a sequence such that

- (i) *$\{u_n(\lambda)\}$ is bounded in X;*
- (ii) *$\Phi_\lambda(u_n(\lambda)) \to c_\lambda$;*
- (iii) *$\Phi'_\lambda(u_n(\lambda)) \to 0$ in X^*, where X^* is the dual of X.*

To apply Proposition 4.1, we introduce two families of perturbed functionals

$$\begin{aligned} I_\lambda(u) = &\frac{1}{2} \int_{\mathbb{R}^3} (a|(-\Delta)^{\alpha/2} u|^2 + V(x) u^2) \,\mathrm{d}x \\ &+ \frac{b}{4} \Big(\int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2} u|^2 \,\mathrm{d}x \Big)^2 - \lambda \int_{\mathbb{R}^3} F(u) \,\mathrm{d}x \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} I_\lambda^\infty(u) = &\frac{1}{2} \int_{\mathbb{R}^3} (a|(-\Delta)^{\alpha/2} u|^2 + V_\infty u^2) \,\mathrm{d}x + \frac{b}{4} \Big(\int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2} u|^2 \,\mathrm{d}x \Big)^2 \\ &- \lambda \int_{\mathbb{R}^3} F(u) \,\mathrm{d}x, \end{aligned} \tag{4.3}$$

for $\lambda \in [1/2, 1]$.

Lemma 4.2 ([14]). *Assume that* (A1)–(A4) *amd* (A9) *hold. Let u be a critical point of I_λ in $H^\alpha(\mathbb{R}^3)$, then we have the following Pohozaev type identity*

$$\begin{aligned} P_\lambda(u) := &\frac{(3-2\alpha)a}{2} \|(-\Delta)^{\alpha/2} u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} [3V(x) + (\nabla V(x), x)]\, u^2 \,\mathrm{d}x \\ &+ \frac{(3-2\alpha)b}{2} \Big(\int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2} u|^2 \,\mathrm{d}x \Big)^2 - 3\lambda \int_{\mathbb{R}^3} F(u) \,\mathrm{d}x = 0. \end{aligned} \tag{4.4}$$

Note that $P_\lambda(u) = \frac{\mathrm{d}}{\mathrm{d}t} I_\lambda(u(x/t))|_{t=1}$.

We set $J_\lambda(u) := \frac{4\alpha-3}{2}\langle I'_\lambda(u), u\rangle + P_\lambda(u)$, then

$$\begin{aligned} J_\lambda(u) &= \alpha a\|(-\Delta)^{\alpha/2}u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3}[4\alpha V(x) + (\nabla V(x), x)]u^2\,\mathrm{d}x \\ &\quad + \alpha b\|(-\Delta)^{\alpha/2}u\|_2^4 - \frac{\lambda}{2}\int_{\mathbb{R}^3}[(4\alpha-3)f(u)u + 6F(u)]\,\mathrm{d}x, \end{aligned} \tag{4.5}$$

for $\lambda \in [1/2, 1]$. Correspondingly, we let

$$\begin{aligned} J_\lambda^\infty(u) &= \alpha a\|(-\Delta)^{\alpha/2}u\|_2^2 + 2\alpha V_\infty\|u\|_2^2 + \alpha b\|(-\Delta)^{\alpha/2}u\|_2^4 \\ &\quad - \frac{\lambda}{2}\int_{\mathbb{R}^3}[(4\alpha-3)f(u)u + 6F(u)]\,\mathrm{d}x \end{aligned} \tag{4.6}$$

for $\lambda \in [1/2, 1]$. Set

$$\mathcal{M}_\lambda^\infty := \{u \in H^\alpha(\mathbb{R}^3) \setminus \{0\} : J_\lambda^\infty(u) = 0\}, \quad m_\lambda^\infty := \inf_{u\in\mathcal{M}_\lambda^\infty} I_\lambda^\infty(u).$$

By Corollary 3.3, we have the following lemma.

Lemma 4.3. *Assume that* (A3), (A4), (A12) *hold. Then*

$$I_\lambda^\infty(u) \ge I_\lambda^\infty(u_t) + \frac{1-t^{4\alpha}}{4\alpha}J_\lambda^\infty(u) + \frac{a(1-t^{2\alpha})^2}{4}\|(-\Delta)^{\alpha/2}u\|_2^2, \tag{4.7}$$

for all $u \in H^\alpha(\mathbb{R}^3)$, $t > 0$, $\lambda \ge 0$.

In view of Theorem 3.13 and Corollary 3.14, I_1^∞ has a minimizer $u_1^\infty > 0$ on $\mathcal{M}_1^\infty$, i.e.,

$$u_1^\infty \in \mathcal{M}_1^\infty, \quad (I_1^\infty)'(u_1^\infty) = 0, \quad m_1^\infty = I_1^\infty(u_1^\infty). \tag{4.8}$$

Under the assumptions of Theorem 1.2, we first show that I_λ with $\lambda \in [\frac{1}{2}, 1]$ has the mountain pass geometry.

Lemma 4.4. *Assume that* (A1)–(A4), (A9), (A11), (A12) *hold. Then*

(i) *there exists $T_0 > 0$ independent of λ such that $I_\lambda((u_1^\infty)_{T_0}) < 0$ for all $\lambda \in [1/2, 1]$;*

(ii) *there exists $\kappa_0 > 0$ independent of λ such that for all $\lambda \in [\frac{1}{2}, 1]$,*

$$c_\lambda := \inf_{\gamma\in\Gamma}\max_{\varsigma\in[0,1]} I_\lambda(\gamma(\varsigma)) > \max\{I_\lambda(0), I_\lambda((u_1^\infty)_{T_0})\}$$

where

$$\Gamma = \{\gamma \in C([0,1], H^\alpha(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(1) = (u_1^\infty)_{T_0}\};$$

(iii) *c_λ and m_λ^∞ are non-increasing on $\lambda \in [\frac{1}{2}, 1]$.*

Proof. (i) For fixed $u \in H^\alpha(\mathbb{R}^3) \setminus \{0\}$ and any $\lambda \in [\frac{1}{2}, 1]$, one has

$$\begin{aligned} I_\lambda(u) \le I_{\frac{1}{2}}(u) &= \frac{1}{2}\int_{\mathbb{R}^3}(a|(-\Delta)^{\alpha/2}u|^2 + V(x)u^2)\,\mathrm{d}x + \frac{b}{4}\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\alpha/2}u|^2\,\mathrm{d}x\Big)^2 \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^3}F(u)\,\mathrm{d}x. \end{aligned}$$

It is easy to verify that

$$\begin{aligned} I_{\frac{1}{2}}(u_t) &= \frac{at^{2\alpha}}{2}\|(-\Delta)^{\alpha/2}u\|_2^2 + \frac{t^{4\alpha}}{2}\int_{\mathbb{R}^3}V(tx)u^2\,\mathrm{d}x + \frac{bt^{4\alpha}}{4}\|(-\Delta)^{\alpha/2}u\|_2^4 \\ &\quad - \frac{1}{2}t^3\int_{\mathbb{R}^3}F(t^{\frac{4\alpha-3}{2}}u)\,\mathrm{d}x. \end{aligned}$$

Note that $F(t^{\frac{4\alpha-3}{2}}u)/|t^{\frac{4\alpha-3}{2}}u|^2 \to +\infty$. Then by Fatou's lemma, $I_{\frac{1}{2}}(u_t) \to -\infty$ as $t \to +\infty$. Then, combining Lemma 3.6, there exists $T_0 > 0$ sufficiently large, independent of $\lambda \in [\frac{1}{2}, 1]$, such that $I_\lambda((u_1^\infty)_{T_0}) < 0$.

(ii) In view of (4.2), (1.10) and the Sobolev embedding, there exist $\sigma_1, \sigma_2 > 0$ such that

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2}\int_{\mathbb{R}^3}(a|(-\Delta)^{\alpha/2}u|^2 + V(x)u^2)\,\mathrm{d}x + \frac{b}{4}\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\alpha/2}u|^2\,\mathrm{d}x\Big)^2 \\ &\quad - \lambda\int_{\mathbb{R}^3}(\epsilon|u|^2 + C_\epsilon|u|^p)\,\mathrm{d}x \\ &\geq \sigma_1\|u\|^2 - \sigma_2\|u\|^p, \end{aligned}$$

Recalling that $p > 2$, then there exist $\rho_1, \kappa_0 > 0$ independent of λ, such that for $\|u\| = \rho_1, I_\lambda(u) \geq \kappa_0$. For any $\lambda \in [\frac{1}{2}, 1]$ and $\gamma \in \Gamma$, it is easily seen that $\|\gamma(1)\| > \rho_1$. By continuity, there exists $\bar{\varsigma} \in (0,1)$ such that $\|\gamma(\bar{\varsigma})\| = \rho_1$, which implies that

$$c_\lambda \geq \inf_{\gamma\in\Gamma} I_\lambda(\gamma(\bar{\varsigma})) \geq \kappa_0 > \max\{I_\lambda(0), I_\lambda((u_1^\infty)_{T_0})\}, \quad \forall\lambda \in [\frac{1}{2}, 1].$$

(iii) For any $u \in H^\alpha(\mathbb{R}^3)\backslash\{0\}$, since $c_\lambda \leq \max_{t>0} I_\lambda(u_t) \leq \max_{t>0} I_{\frac{1}{2}}(u_t)$ for all $\lambda \in [\frac{1}{2}, 1]$, we obtain the conclusion. □

Lemma 4.5. *Assume that* (A1)–(A4), (A9), (A11), (A12) *hold. Then there exists* $\lambda_1 \in [\frac{1}{2}, 1)$ *such that* $c_\lambda < m_\lambda^\infty$ *for* $\lambda \in [\lambda_1, 1]$.

This lemma can be proved similarly as in [27, Lemma 4.5], we omit the proof here. In what follows we use profile decomposition to obtain the compactness for any bounded (PS) sequence of the perturbed functional, which is crucial in our proof.

Lemma 4.6. *Assume that* (A1)–(A4), (A9), (A11) *hold. Let* $\{u_n\}$ *be a bounded (PS)*$_c$ *sequence for* I_λ *with* $\lambda \in [\frac{1}{2}, 1]$. *Then there exist a subsequence of* $\{u_n\}$, *still denoted by* $\{u_n\}$, *and* $u_0 \in H^\alpha(\mathbb{R}^3)$ *such that* $B^2 := \lim_{n\to\infty}\|(-\Delta)^{\alpha/2}u\|_2^2$ *exist,* $u_n \rightharpoonup u_0$ *in* $H^\alpha(\mathbb{R}^3)$ *and* $\mathcal{I}'_\lambda(u_0) = 0$, *where*

$$\mathcal{I}_\lambda(u) = \frac{a + bB^2}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\alpha/2}u|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}V(x)u^2\,\mathrm{d}x - \lambda\int_{\mathbb{R}^3}F(u)\,\mathrm{d}x, \tag{4.9}$$

and either

(i) $u_n \to u_0$ *in* $H^\alpha(\mathbb{R}^3)$; *or*

(ii) *there exist an integer* $l \in \mathbb{N}$ *and* $w^1, \dots, w^l \in H^\alpha(\mathbb{R}^3) \setminus \{0\}$ *such that* $(\mathcal{I}_\lambda^\infty)'(w^k) = 0$ *for* $1 \leq k \leq l$, *and*

$$c + \frac{bB^4}{4} = \mathcal{I}_\lambda(u_0) + \sum_{k=1}^{l}\mathcal{I}_\lambda^\infty(w^k); B^2 = \|(-\Delta)^{\alpha/2}u_0\|_2^2 + \sum_{k=1}^{l}\|(-\Delta)^{\alpha/2}w^k\|_2^2,$$

where

$$\mathcal{I}_\lambda^\infty(u) = \frac{a + bB^2}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\alpha/2}u|^2\,\mathrm{d}x + \frac{V_\infty}{2}\int_{\mathbb{R}^3}u^2\,\mathrm{d}x - \lambda\int_{\mathbb{R}^3}F(u)\,\mathrm{d}x. \tag{4.10}$$

Since (3.18) and (3.19) hold, we can prove the above lemma as in [19, Lemma 3.4]. We omit it here.

Lemma 4.7. *Assume that* (A1)–(A4), (A9), (A11) *hold. Then for almost every* $\lambda \in [\lambda_1, 1]$*, there exists* $u_\lambda \in H^\alpha(\mathbb{R}^3) \setminus \{0\}$ *such that*

$$I'_\lambda(u_\lambda) = 0, \quad I_\lambda(u_\lambda) = c_\lambda. \tag{4.11}$$

Proof. Under Assumptions (A3), (A4), (A11), Lemma 4.4 implies that $I_\lambda(u)$ satisfies the assumptions of Proposition 4.1 with $X = H^\alpha(\mathbb{R}^3)$ and $\Phi_\lambda = I_\lambda$. So for almost every $\lambda \in [1/2, 1]$, there exists a bounded sequence $\{u_n(\lambda)\} \subset H^\alpha(\mathbb{R}^3)$ (for simplicity, we denote the sequence by $\{u_n\}$ instead of $\{u_n(\lambda)\}$ such that

$$I_\lambda(u_n) \to c_\lambda > 0, \quad \|I'_\lambda(u_n)\| \to 0. \tag{4.12}$$

By Lemma 4.6, there exist a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and $u_\lambda \in H^\alpha(\mathbb{R}^3)$ such that $B_\lambda^2 := \lim_{n\to\infty} \|(-\Delta)^{\alpha/2} u_n\|_2^2$ exists, $u_n \rightharpoonup u_\lambda$ in $H^\alpha(\mathbb{R}^3)$ and $(\widehat{\mathcal{I}}_\lambda)'(u_\lambda) = 0$, and either (i) or (ii) occurs, where

$$\widehat{\mathcal{I}}_\lambda(u) = \frac{a + bB_\lambda^2}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2} u|^2 \,\mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^3} V(x) u^2 \,\mathrm{d}x - \lambda \int_{\mathbb{R}^3} F(u) \,\mathrm{d}x. \tag{4.13}$$

If (ii) occurs, i.e. there exists $l \in \mathbb{N}$ and $w^1, \dots, w^l \in H^\alpha(\mathbb{R}^3) \setminus \{0\}$ such that $(\widehat{\mathcal{I}}_\lambda^\infty)'(w^k) = 0$ for $1 \le k \le l$,

$$c_\lambda + \frac{bB_\lambda^4}{4} = \widehat{\mathcal{I}}_\lambda(u_\lambda) + \sum_{k=1}^{l} \widehat{\mathcal{I}}_\lambda^\infty(w^k), \tag{4.14}$$

$$B_\lambda^2 = \|(-\Delta)^{\alpha/2} u_\lambda\|_2^2 + \sum_{k=1}^{l} \|(-\Delta)^{\alpha/2} w^k\|_2^2, \tag{4.15}$$

where

$$\widehat{\mathcal{I}}_\lambda^\infty(u) = \frac{a + bB_\lambda^2}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2} u|^2 \,\mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^3} V_\infty u^2 \,\mathrm{d}x - \lambda \int_{\mathbb{R}^3} F(u) \,\mathrm{d}x. \tag{4.16}$$

Since $(\widehat{\mathcal{I}}_\lambda)'(u_\lambda) = 0$, then we have the Pohozaev identity of the functional $\widehat{\mathcal{I}}_\lambda$

$$\begin{aligned} \widetilde{P}_\lambda(u_\lambda) &:= \frac{(3-2\alpha)(a + bB_\lambda^2)}{2} \|(-\Delta)^{\alpha/2} u_\lambda\|_2^2 \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^3} \left[3V(x) + (\nabla V(x), x)\right] u_\lambda^2 \,\mathrm{d}x - 3\lambda \int_{\mathbb{R}^3} F(u_\lambda) \,\mathrm{d}x = 0. \end{aligned} \tag{4.17}$$

From (A9) and fractional Hardy inequality,

$$a\|(-\Delta)^{\alpha/2} u_\lambda\|_2^2 \ge \frac{a\Gamma^2(\frac{1+2\alpha}{2})}{\pi} \int_{\mathbb{R}^3} \frac{u_\lambda^2}{|x|^{2\alpha}} \,\mathrm{d}x \ge \frac{1}{2\alpha} \int_{\mathbb{R}^3} (\nabla V(x), x) u_\lambda^2 \,\mathrm{d}x. \tag{4.18}$$

From (3.22), (4.13), (4.17) and (4.18) it follows that

$$\begin{aligned} \widehat{\mathcal{I}}_\lambda(u_\lambda) &= \widehat{\mathcal{I}}_\lambda(u_\lambda) - \frac{1}{4\alpha} \Big[\frac{4\alpha - 3}{2} \langle \widehat{\mathcal{I}}'_\lambda(u_\lambda), u_\lambda \rangle + \widetilde{P}_\lambda(u_\lambda) \Big] \\ &= \frac{a + bB_\lambda^2}{4} \|(-\Delta)^{\alpha/2} u_\lambda\|_2^2 - \frac{1}{8\alpha} \int_{\mathbb{R}^3} (\nabla V(x), x) u_\lambda^2 \,\mathrm{d}x \\ &\quad + \frac{\lambda(4\alpha - 3)}{8\alpha} \int_{\mathbb{R}^3} [f(u_\lambda) u_\lambda - 2F(u_\lambda)] \,\mathrm{d}x \\ &\ge \frac{bB_\lambda^2}{4} \|(-\Delta)^{\alpha/2} u_\lambda\|_2^2. \end{aligned} \tag{4.19}$$

Since $(\widehat{\mathcal{I}}_\lambda^\infty)'(w^k)=0$, we have the Pohozaev identity of the functional $\widehat{\mathcal{I}}_\lambda^\infty$,

$$\begin{aligned}\widetilde{P}_\lambda^\infty(u_\lambda):=\frac{(3-2\alpha)}{2}(a+bB_\lambda^2)\|(-\Delta)^{\alpha/2}u_\lambda\|_2^2+\frac{3}{2}\int_{\mathbb{R}^3}V_\infty u_\lambda^2\,\mathrm{d}x\\ -3\lambda\int_{\mathbb{R}^3}F(u_\lambda)\,\mathrm{d}x=0.\end{aligned}\tag{4.20}$$

Thus, from (4.6), (4.15), (4.16) and (4.20), we have

$$\begin{aligned}0&=\frac{4\alpha-3}{2}\langle(\widehat{\mathcal{I}}_\lambda^\infty)'(w^k),w^k\rangle+\widetilde{P}_\lambda^\infty(w^k)\\ &=\alpha(a+bB_\lambda^2)\|(-\Delta)^{\alpha/2}w^k\|_2^2+2\alpha V_\infty\|w^k\|_2^2\\ &\quad-\frac{\lambda}{2}\int_{\mathbb{R}^3}[(4\alpha-3)f(w^k)w^k+6F(w^k)]\,\mathrm{d}x\\ &\geq J_\lambda^\infty(w^k).\end{aligned}\tag{4.21}$$

Since $w^k\in H^\alpha(\mathbb{R}^3)\setminus\{0\}$, in view of Lemma 3.6, there exists $t_k>0$ such that $(w^k)_{t_k}\in\mathcal{M}_\lambda^\infty$. From (4.3), (4.6), (4.7), (4.16) and (4.21), one has

$$\begin{aligned}&\widehat{\mathcal{I}}_\lambda^\infty(w^k)\\ &=\widehat{\mathcal{I}}_\lambda^\infty(w^k)-\frac{1}{4\alpha}\big[\frac{4\alpha-3}{2}\langle(\widehat{\mathcal{I}}_\lambda^\infty)'(w^k),w^k\rangle+\widetilde{P}_\lambda^\infty(w^k)\big]\\ &=\frac{a+bB_\lambda^2}{4}\|(-\Delta)^{\alpha/2}w^k\|_2^2+\frac{\lambda(4\alpha-3)}{8\alpha}\int_{\mathbb{R}^3}[f(w^k)w^k-2F(w^k)]\,\mathrm{d}x\\ &=\frac{bB_\lambda^2}{4}\|(-\Delta)^{\alpha/2}w^k\|_2^2+I_\lambda^\infty(w^k)-\frac{1}{4\alpha}J_\lambda^\infty(w^k)\\ &\geq\frac{bB_\lambda^2}{4}\|(-\Delta)^{\alpha/2}w^k\|_2^2+I_\lambda^\infty((w^k)_{t_k})-\frac{t_k^{4\alpha}}{4\alpha}J_\lambda^\infty(w^k)\\ &\geq\frac{bB_\lambda^2}{4}\|(-\Delta)^{\alpha/2}w^k\|_2^2+m_\lambda^\infty.\end{aligned}\tag{4.22}$$

It follows from (4.14), (4.15), (4.19) and (4.22) that

$$\begin{aligned}c_\lambda+\frac{bB_\lambda^4}{4}&=\widehat{\mathcal{I}}_\lambda(u_\lambda)+\sum_{k=1}^{l}\widehat{\mathcal{I}}_\lambda^\infty(w^k)\\ &\geq lm_\lambda^\infty+\frac{bB_\lambda^2}{4}\big[\|(-\Delta)^{\alpha/2}u_\lambda\|_2^2+\sum_{k=1}^{l}\|(-\Delta)^{\alpha/2}w^k\|_2^2\big]\\ &\geq m_\lambda^\infty+\frac{bB_\lambda^4}{4},\quad\forall\lambda\in[\lambda_1,1],\end{aligned}$$

which contradicts Lemma 4.5. Thus $u_n\to u_\lambda$ in $H^\alpha(\mathbb{R}^3)$ and $I_\lambda(u_\lambda)=c_\lambda$. □

Proof of Theorem 1.3. In view of Lemma 4.7, there exist two sequences of $\{\lambda_n\}\subset[\lambda_1,1]$ and $\{u_{\lambda_n}\}\subset H^\alpha(\mathbb{R}^3)$, denoted by $\{u_n\}$, such that

$$\lambda_n\to1,\quad I'_{\lambda_n}(u_n)=0,\quad I_{\lambda_n}(u_n)=c_{\lambda_n}.\tag{4.23}$$

From (A9), (3.7), (3.22), (4.2), (4.5) and (4.23), one has

$$\begin{aligned} c_{1/2} \ge c_{\lambda_n} &= I_{\lambda_n}(u_n) - \frac{1}{4\alpha} J_{\lambda_n}(u_n) \\ &= \frac{a}{4}\|(-\Delta)^{\alpha/2}u_n\|_2^2 - \frac{1}{8\alpha}\int_{\mathbb{R}^3}(\nabla V(x), x)u_n^2 \,\mathrm{d}x \\ &\quad + \frac{\lambda_n(4\alpha-3)}{8\alpha}\int_{\mathbb{R}^3}[f(u_n)u_n - 2F(u_n)]\,\mathrm{d}x \\ &\ge \frac{(1-\theta)a}{4}\|(-\Delta)^{\alpha/2}u_n\|_2^2. \end{aligned} \tag{4.24}$$

This shows that $\{\|(-\Delta)^{\alpha/2}u_n\|_2\}$ is bounded. Next, we show that $\{u_n\}$ is bounded in $H^\alpha(\mathbb{R}^3)$. Combining Lemma 2.1, (1.10), (4.2), (4.23), (4.24) and the fractional Sobolev embedding theorem, we have

$$\begin{aligned} \omega_1'\|u_n\|^2 &\le \int_{\mathbb{R}^3}\Big[a|(-\Delta)^{\alpha/2}u_n|^2 + V(x)u_n^2\Big]\,\mathrm{d}x \\ &\le 2c_{\lambda_n} + 2\lambda_n\int_{\mathbb{R}^3}F(u_n)\,\mathrm{d}x \\ &\le 2c_{1/2} + \frac{\omega_1'}{4}\|u_n\|^2 + C_5\|u_n\|_{2_\alpha^*}^{2_\alpha^*} \\ &\le 2c_{1/2} + \frac{\omega_1'}{4}\|u_n\|^2 + C_5 S_\alpha^{-\frac{2_\alpha^*}{2}}\|(-\Delta)^{\alpha/2}u_n\|_2^{2_\alpha^*}. \end{aligned}$$

where $\omega_1' > 0$ is a constant. This shows that $\{u_n\}$ is bounded in $H^\alpha(\mathbb{R}^3)$. The rest of the proof is the same as the one in [14], so we omit it. □

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