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FINAL-VALUE PROBLEM FOR A WEAKLY-COUPLED SYSTEM OF STRUCTURALLY DAMPED WAVES

NGUYEN HUY TUAN, VO VAN AU, NGUYEN HUU CAN, MOKHTAR KIRANE

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Abstract. We consider the final-value problem of a system of strongly-damped wave equations. First of all, we find a solution of the system, then by an example we show the problem is ill-posed. Next, by using a filter method, we propose stable approximate (regularized) solutions. The existence, uniqueness of the corresponding regularized solutions are obtained. Furthermore, we show that the corresponding regularized solutions converge to the exact solutions in L^2 uniformly with respect to the space coordinate under some a priori assumptions on the solutions.

1. Introduction

Let T be a positive number and $\Omega \subset \mathbb{R}^n, n \geq 1$, be an open bounded domain with a smooth boundary Γ. Set $D_T = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$. In this article, we consider the question of finding a couple of functions $(u,v)(x,t)$, $(x,t) \in \overline{\Omega} \times [0,T]$, satisfying the Cauchy problem for the weakly-coupled system of nonlinear structurally damped wave equations

$$
\begin{gathered}
u_{tt} - \Delta u + 2a(-\Delta)^\gamma u_t = F(u,v), \quad \text{in } D_T, \\
v_{tt} - \Delta v + 2a(-\Delta)^\gamma v_t = G(u,v), \quad \text{in } D_T, \\
u = v = 0, \quad \text{on } \Sigma,
\end{gathered}
\tag{1.1}
$$

subject to the final observation

$$
\begin{gathered}
u(x,T) = u_T(x),\ u_t(x,T) = \widetilde{u}_T(x), \quad \text{in } \Omega, \\
v(x,T) = v_T(x),\ v_t(x,T) = \widetilde{v}_T(x), \quad \text{in } \Omega,
\end{gathered}
\tag{1.2}
$$

where $\gamma > 1/2$ and $a > 0$ is a damping constant, the functions $u_T, \widetilde{u}_T, v_T, \widetilde{v}_T$ are given in $L^2(\Omega)$. The source functions F and G will be defined later. The damped wave equations and systems occur in a wide range of applications modelling the motion of viscoelastic materials. Some more physical applications of strongly damped waves can be found in [13]. The initial-value problem for damped wave equations (or pseudo-hyperbolic equations) have been widely studied, see for example Pata et al [12, 13], Thomee et al [14], Liu et al [10], Guo [6], Zelik et al [7], Yang et al [18]. However, studies of the initial-value problem for strongly damped wave systems are

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limited. Recently, Hayashi et al [4] studied the existence of small global solutions to the initial-value problem for system (1.1) in $\mathbb{R}^n$, $n \geq 4$, assuming that

$$a = \frac{1}{2}, \quad \gamma = 1, \quad F(u,v) = F(v), \quad G(u,v) = F(u).$$

D'Abbicco [16] studied the system of structurally damped waves (1.1) in $\mathbb{R}^n$ assuming that

$$\gamma = \frac{1}{2}, \quad F(u,v) = |v|^p, \quad G(u,v) = |u|^q,$$

where p, q are chosen suitably.

To the best of our knowledge, the final-value (backward) problem for the system (1.1) has not been studied yet. The final-value problem for systems of partial differential equations play an important role in engineering areas, which aims to obtain the previous data of a physical field from a given state. The first work on the regularization result for the strongly damped wave equation seems the one by Lesnic et al [17].

In practice, the exact data $u_T, \widetilde{u}_T, v_T, \widetilde{v}_T$ can only be measured with errors, and we thus would have as data some function $u_T^\delta, \widetilde{u}_T^\delta, v_T^\delta, \widetilde{v}_T^\delta$ that belong to $L^2(\Omega)$, for which

$$\|u_T^\delta - u_T\|_{L^2(\Omega)} + \|\widetilde{u}_T^\delta - \widetilde{u}_T\|_{L^2(\Omega)} + \|v_T^\delta - v_T\|_{L^2(\Omega)} + \|\widetilde{v}_T^\delta - \widetilde{v}_T\|_{L^2(\Omega)} \leq \delta,$$

where the constant $\delta > 0$ represents a bound on the measurement errors.

It is well-known that problem (1.1)-(1.2) is ill-posed in the sense of Hadamard for data $u_T^\delta, \widetilde{u}_T^\delta, v_T^\delta, \widetilde{v}_T^\delta$ in any reasonable topology (see [4]). More details of ill-posedness of the solution are given in Section 2.2. In general, no solution which satisfies the system with final data and the boundary conditions exists. Even if a solution exists, it does not depend continuously on the final data and any small perturbation in the given data may cause large change to the solution. So we need some regularization methods to deal with this problem.

This article is organized as follows. In Section 2, we present the mild solution and the ill-posedness of system (1.1)-(1.2). In Section 3, we establish a regularized solution in the case of a global Lipschitz source function F, G. In Section 4, we extend Section 3 to the situation of the locally Lipschitz sources. Furthermore, we also obtain the convergence rate between the regularized solution and the exact solution in L^2 norm.

2. Solution of the initial inverse problem (1.1)-(1.2)

We begin by introducing some notation needed for our analysis throughout this paper.

2.1. **Notation.** We denote by $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ the inner product in $L^2(\Omega)$.

- For $w \in C([0,T]; L^2(\Omega))$, we define
$$\|w\|_{C([0,T];L^2(\Omega))} = \sup_{0 \leq t \leq T} \|w(t)\|_{L^2(\Omega)}.$$
- Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces; $\mathcal{X} \times \mathcal{Y}$ is also a Banach space and its norm is defined as
$$\|(w_1, w_2)\|_{\mathcal{X} \times \mathcal{Y}} = \|w_1\|_{\mathcal{X}} + \|w_2\|_{\mathcal{Y}},$$
for any $(w_1, w_2) \in \mathcal{X} \times \mathcal{Y}$.

$\diamond$ When Ω is bounded, the system (1.1) can also be solved by a decomposition in a Hilbert basis of $L^2(\Omega)$.

- For this purpose, it is very convenient to choose a basis $\{\xi_p\}_{p\in\mathbb{N}^*}$ of $L^2(\Omega)$ composed of eigenfunctions of $-\Delta$ (with zero Dirichlet condition), i.e.,
$$\begin{gathered} -\Delta\xi_p(x) = \lambda_p\xi_p(x), \quad \text{in } \Omega,\\ \xi_p(x) = 0, \quad \text{on } \Gamma, \end{gathered} \tag{2.1}$$
which admits a family of eigenvalues $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots \le \lambda_p \cdots$ and $\lambda_p \to \infty$ as $p \to \infty$, see [3, p. 335].
- Via the spectral decomposition of $w \in L^2(\Omega)$, for each $\gamma > \frac{1}{2}$, we define the fractional Laplacian using the spectral theorem as follows
$$(-\Delta)^\gamma w = \sum_{p=1}^{\infty} \lambda_p^\gamma \langle w, \xi_p\rangle_{L^2(\Omega)} \xi_p(x). \tag{2.2}$$
More details on this fractional Laplacian can be found in [8].

$\diamond$ In addition, we introduce the abstract Gevrey class of functions of index $m, n > 0$, see e.g., [1], defined by
$$\mathbb{G}^\gamma_{m,n} = \Big\{ w \in L^2(\Omega) : \sum_{p=1}^{\infty} (\lambda_p^\gamma)^{2m} \exp(2n\lambda_p^\gamma) \langle w, \xi_p\rangle^2_{L^2(\Omega)} < \infty \Big\},$$
for $2\gamma > 1$, which is a Hilbert space equipped with the inner product
$$\begin{aligned} &\langle w_1, w_2\rangle_{\mathbb{G}^\gamma_{m,n}} \\ &:= \Big\langle ((-\Delta)^{\gamma/2})^m \exp\big(n(-\Delta)^{\gamma/2}\big) w_1, ((-\Delta)^{\gamma/2})^m \exp\big(n(-\Delta)^{\gamma/2}\big) w_2 \Big\rangle_{L^2(\Omega)}, \end{aligned}$$
for all $w_1, w_2 \in \mathbb{G}^\gamma_{m,n}$, and its corresponding norm
$$\|w\|^2_{\mathbb{G}^\gamma_{m,n}} = \sum_{p=1}^{\infty} (\lambda_p^\gamma)^{2m} \exp(2n\lambda_p^\gamma) \langle w, \xi_p\rangle^2_{L^2(\Omega)} < \infty.$$

2.2. **Mild solution of** (1.1)**-**(1.2)**.** We look for a solution of problem (1.1)-(1.2) of the form
$$u(x,t) = \sum_{p=1}^{\infty} u_p(t)\xi_p(x), \quad v(x,t) = \sum_{p=1}^{\infty} v_p(t)\xi_p(x), \tag{2.3}$$
where $u_p(t) = \langle u(x,t), \xi_p(x)\rangle_{L^2(\Omega)}$, $v_p(t) = \langle v(x,t), \xi_p(x)\rangle_{L^2(\Omega)}$.

Put $F_p(u,v)(t) = \langle F(u,v), \xi_p(x)\rangle_{L^2(\Omega)}$. We consider the problem of finding a function $u_p(t)$ satisfying
$$\begin{gathered} \frac{d^2}{dt^2} u_p(t) + 2a\lambda_p^\gamma \frac{d}{dt} u_p(t) + \lambda_p u_p(t) = F_p(u,v)(t), \quad t \in (0,T),\\ u_p(T) = \langle u(x,T), \xi_p(x)\rangle_{L^2(\Omega)}, \quad \frac{d}{dt} u_p(T) = \langle \widetilde{u}(x,T), \xi_p(x)\rangle_{L^2(\Omega)}. \end{gathered} \tag{2.4}$$
The quadratic characteristic polynomial of (2.4) is
$$Z^2 + 2a\lambda_p^\gamma Z + \lambda_p = 0.$$
With the notation $\alpha_p = a^2\lambda_p^{2\gamma} - \lambda_p$, for any $a > 0$ and $\gamma > 1/2$, we consider three cases

Case 1. $p \in \mathbb{N}_1 = \{p \in \mathbb{N}^* : \lambda_p > a^{\frac{2}{1-2\gamma}}\}$. We put $\mu_j = \mu_{j,p} := a\lambda_p^\gamma + (-1)^j \sqrt{\alpha_p}$, $j = 1, 2$. Multiplying the first equation in (2.4) by $\frac{\exp\big(\mu_1(s-t)\big) - \exp\big(\mu_2(s-t)\big)}{2\sqrt{\alpha_p}}$ and integrating both sides from t to T, we obtain

$$\begin{aligned} u_p(t) =& \frac{\mu_2 \exp\big(\mu_1(T-t)\big) - \mu_1 \exp\big(\mu_2(T-t)\big)}{2\sqrt{\alpha_p}} \langle u(x,T), \xi_p(x)\rangle_{L^2(\Omega)} \\ &+ \frac{\exp\big(\mu_1(T-t)\big) - \exp\big(\mu_2(T-t)\big)}{2\sqrt{\alpha_p}} \langle \widetilde{u}(x,T), \xi_p(x)\rangle_{L^2(\Omega)} \\ &+ \int_t^T \frac{\exp\big(\mu_1(s-t)\big) - \exp\big(\mu_2(s-t)\big)}{2\sqrt{\alpha_p}} F_p(u,v)(s)\mathrm{d}s. \end{aligned} \tag{2.5}$$

Case 2. $p \in \mathbb{N}_2 = \{p \in \mathbb{N}^* : \lambda_p = a^{\frac{2}{1-2\gamma}}\}$. Multiplying the first equation in (2.4) by $(s-t)\exp\big(a^{\frac{1}{1-2\gamma}}(s-t)\big)$, and integrating both sides from t to T, we have

$$\begin{aligned} u_p(t) =& \exp\big(a\lambda_p^\gamma(T-t)\big)(1 - a\lambda_p^\gamma(T-t))\langle u(x,T), \xi_p(x)\rangle_{L^2(\Omega)} \\ &- \exp\big(a\lambda_p^\gamma(T-t)\big)(T-t)\langle \widetilde{u}(x,T), \xi_p(x)\rangle_{L^2(\Omega)} \\ &- \int_t^T (s-t)\exp\big(a\lambda_p^\gamma(s-t)\big) F_p(u,v)(s)\mathrm{d}s. \end{aligned} \tag{2.6}$$

Case 3. $p \in \mathbb{N}_3 = \{p \in \mathbb{N}^* : \lambda_p < a^{\frac{2}{1-2\gamma}}\}$. Multiplying the first equation in (2.4) by

$$\frac{\exp\big(a\lambda_p^\gamma\big)}{\sqrt{-\alpha_p}} \sin(\sqrt{\alpha_p}(s-t)),$$

and integrating both sides from t to T, we have

$$\begin{aligned} u_p(t) =& \frac{\exp\big(a\lambda_p^\gamma(T-t)\big)}{\sqrt{-\alpha_p}} \big[a\lambda_p^\gamma \sin(\sqrt{-\alpha_p}(T-t)) - \cos(\sqrt{-\alpha_p}(T-t))\big] \\ &\times \langle u(x,T), \xi_p(x)\rangle_{L^2(\Omega)} \\ &+ \frac{\exp\big(a\lambda_p^\gamma(T-t)\big)}{\sqrt{-\alpha_p}} \sin(\sqrt{-\alpha_p}(T-t))\langle \widetilde{u}(x,T), \xi_p(x)\rangle_{L^2(\Omega)} \\ &- \int_t^T \frac{\exp\big(a\lambda_p^\gamma\big)}{\sqrt{-\alpha_p}} \sin(\sqrt{\alpha_p}(s-t)) F_p(u,v)(s)\mathrm{d}s. \end{aligned} \tag{2.7}$$

Similar considerations apply to $v_p(t)$ that satisfies

$$\begin{gathered} \frac{\mathrm{d}^2}{\mathrm{d}t^2} v_p(t) + 2a\lambda_p^\gamma \frac{\mathrm{d}}{\mathrm{d}t} v_p(t) + \lambda_p v_p(t) = G_p(u,v)(t), \quad t \in (0,T), \\ v_p(T) = \langle v(x,T), \xi_p(x)\rangle_{L^2(\Omega)}, \quad \frac{\mathrm{d}}{\mathrm{d}t} v_p(T) = \langle \widetilde{v}(x,T), \xi_p(x)\rangle_{L^2(\Omega)}, \end{gathered} \tag{2.8}$$

where $v_p = \langle v, \xi_p\rangle_{L^2(\Omega)}$, $G_p(u,v) = \langle G(u,v), \xi_p\rangle_{L^2(\Omega)}$. We also have three cases.

Case 1. $p \in \mathbb{N}_1$. We obtain

$$\begin{aligned} v_p(t) &= \frac{\mu_2 \exp\big(\mu_1(T-t)\big) - \mu_1 \exp\big(\mu_2(T-t)\big)}{2\sqrt{\alpha_p}} \langle v(x,T), \xi_p(x)\rangle_{L^2(\Omega)} \\ &\quad + \frac{\exp\big(\mu_1(T-t)\big) - \exp\big(\mu_2(T-t)\big)}{2\sqrt{\alpha_p}} \langle \widetilde{v}(x,T), \xi_p(x)\rangle_{L^2(\Omega)} \\ &\quad + \int_t^T \frac{\exp\big(\mu_1(s-t)\big) - \exp\big(\mu_2(s-t)\big)}{2\sqrt{\alpha_p}} G_p(u,v)(s)\mathrm{d}s. \end{aligned} \tag{2.9}$$

Case 2. $p \in \mathbb{N}_2$. We obtain

$$\begin{aligned} v_p(t) &= \exp\big(a\lambda_p^\gamma(T-t)\big)(1 - a\lambda_p^\gamma(T-t))\langle v(x,T), \xi_p(x)\rangle_{L^2(\Omega)} \\ &\quad - \exp\big(a\lambda_p^\gamma(T-t)\big)(T-t)\langle \widetilde{v}(x,T), \xi_p(x)\rangle_{L^2(\Omega)} \\ &\quad - \int_t^T (s-t)\exp\big(a\lambda_p^\gamma(s-t)\big) G_p(u,v)(s)\mathrm{d}s. \end{aligned} \tag{2.10}$$

Case 3. $p \in \mathbb{N}_3$. We have

$$\begin{aligned} v_p(t) &= \frac{\exp\big(a\lambda_p^\gamma(T-t)\big)}{\sqrt{-\alpha_p}} (a\lambda_p^\gamma \sin(\sqrt{-\alpha_p}(T-t)) \\ &\quad - \cos(\sqrt{-\alpha_p}(T-t)))\langle v(x,T), \xi_p(x)\rangle_{L^2(\Omega)} \\ &\quad + \frac{\exp\big(a\lambda_p^\gamma(T-t)\big)}{\sqrt{-\alpha_p}} \sin(\sqrt{-\alpha_p}(T-t))\langle \widetilde{v}(x,T), \xi_p(x)\rangle_{L^2(\Omega)} \\ &\quad - \int_t^T \frac{\exp(a\lambda_p^\gamma)}{\sqrt{-\alpha_p}} \sin(\sqrt{\alpha_p}(s-t)) G_p(u,v)(s)\mathrm{d}s. \end{aligned} \tag{2.11}$$

Hence, the solution of (1.1) is

$$\begin{aligned} u(x,t) &= \sum_{p\in\mathbb{N}_1} u_p(t)\xi_p + \sum_{p\in\mathbb{N}_2} u_p(t)\xi_p + \sum_{p\in\mathbb{N}_3} u_p(t)\xi_p, \\ v(x,t) &= \sum_{p\in\mathbb{N}_1} v_p(t)\xi_p + \sum_{p\in\mathbb{N}_2} v_p(t)\xi_p + \sum_{p\in\mathbb{N}_3} v_p(t)\xi_p. \end{aligned} \tag{2.12}$$

Let $z \in (0,T), w \in L^2(\Omega)$, $w_p = \langle w, \xi_p\rangle_{L^2(\Omega)}$, we define

$$\begin{aligned} \mathcal{A}(z)w &= \sum_{p\in\mathbb{N}_1}^\infty \frac{\mu_2\exp(z\mu_1) - \mu_1\exp(z\mu_2)}{2\sqrt{\alpha_p}} w_p\xi_p \\ &\quad + \sum_{p\in\mathbb{N}_2}^\infty \exp(a\lambda_p^\gamma z)(1 - a\lambda_p^\gamma z) w_p\xi_p \\ &\quad + \sum_{p\in\mathbb{N}_3}^\infty \frac{\exp(a\lambda_p^\gamma z)}{\sqrt{-\alpha_p}} [a\lambda_p^\gamma \sin(\sqrt{-\alpha_p}z) - \cos(\sqrt{-\alpha_p}z)] w_p\xi_p, \end{aligned} \tag{2.13}$$

$$\begin{aligned} \mathcal{B}(z)w &= \sum_{p\in\mathbb{N}_1}^\infty \frac{\exp(z\mu_1) - \exp(z\mu_2)}{2\sqrt{\alpha_p}} w_p\xi_p + \sum_{p\in\mathbb{N}_2}^\infty z\exp(a\lambda_p^\gamma z) w_p\xi_p \\ &\quad + \sum_{p\in\mathbb{N}_3}^\infty \frac{\exp(a\lambda_p^\gamma z)}{\sqrt{-\alpha_p}} \sin(\sqrt{-\alpha_p}z) w_p\xi_p. \end{aligned} \tag{2.14}$$

Then, we can rewrite (2.12) as

$$\begin{aligned} u(x,t) &= \mathcal{A}(T-t)u_T + \mathcal{B}(T-t)\widetilde{u}_T + \int_t^T \mathcal{B}(s-t)F(u,v)(s)\mathrm{d}s, \\ v(x,t) &= \mathcal{A}(T-t)v_T + \mathcal{B}(T-t)\widetilde{v}_T + \int_t^T \mathcal{B}(s-t)G(u,v)(s)\mathrm{d}s. \end{aligned} \tag{2.15}$$

We expressed the solution of problem (1.1) with the final observation in an integral formulation (2.15). In the next section, we indicate the reasons which make the solution (2.15) ill-posed in the Hadamard sense. For clarity, we give an example to show that the regularization method is necessary.

2.3. **Ill-posedness of the inverse problem for**(1.1)-(1.2)**.** We first observe that if $p \in \mathbb{N}_2 \cup \mathbb{N}_3$ then $\lambda_p \le a^{\frac{2}{1-2\gamma}}$. It is obvious that the terms $\sum_{p\in\mathbb{N}_2} u_p(t)\xi_p + \sum_{p\in\mathbb{N}_3} u_p(t)\xi_p$ and $\sum_{p\in\mathbb{N}_2} v_p(t)\xi_p + \sum_{p\in\mathbb{N}_3} v_p(t)\xi_p$ are bounded and stable in L^2 norm. However, since $p \in \mathbb{N}_1$ implies that $\lambda_p > a^{\frac{2}{1-2\gamma}}$ then exponential functions in the right-hand sides of (2.5) and (2.9) tend to infinity as p tends to infinity. Therefore, the terms $\sum_{p\in\mathbb{N}_1} u_p(t)\xi_p$ and $\sum_{p\in\mathbb{N}_1} v_p(t)\xi_p$ are unbounded. From the above arguments, we take $\mathbb{N}_2 = \mathbb{N}_3 = \emptyset$ by assuming that $a^2\lambda_1^{2\gamma-1} > 1$. Note that this also implies $a^2\lambda_p^{2\gamma} - \lambda_p > 0$ for all $p \in \mathbb{N}^*$, and hence the root of α_p are real and distinct.

Next, we give an example which shows the the solution of problem (1.1) is not stable.

Let $a = \lambda_1^{1/2-\gamma} + 1$, $\partial_t u^{(k)}(x,T) = \partial_t v^{(k)}(x,T) = \frac{1}{\sqrt{\lambda_k}}\xi_k(x) := \Psi^{(k)}$, $u(x,T) = v(x,T) = 0$, for any $k \in \mathbb{N}^*$. Let us define functions

$$\begin{aligned} F(w_1,w_2)(t) &= \sum_{p=1}^{\infty} \frac{\exp(-2(\lambda_1^{1/2-\gamma}+1)T\lambda_p^{\gamma})}{2^3\sqrt{2}T^2} \\ &\quad \times \Big(\langle w_1(t),\xi_p\rangle_{L^2(\Omega)}\xi_p + \langle w_2(t),\xi_p\rangle_{L^2(\Omega)}\xi_p \Big), \\ G(w_1,w_2)(t) &= \sum_{p=1}^{\infty} \frac{\exp(-2(\lambda_1^{1/2-\gamma}+1)T\lambda_p^{\gamma})}{2^3\sqrt{2}T^2} \\ &\quad \times \Big(\langle w_1(t),\xi_p\rangle_{L^2(\Omega)}\xi_p + \langle w_2(t),\xi_p\rangle_{L^2(\Omega)}\xi_p \Big). \end{aligned}$$

Let $u^{(k)}, v^{(k)}$ satisfy the system

$$\begin{aligned} u^{(k)}(x,t) &= \mathcal{B}(T-t)\Psi^{(k)} + \int_t^T \mathcal{B}(s-t)F(u^{(k)}(x,s),v^{(k)}(x,s))\mathrm{d}s, \\ v^{(k)}(x,t) &= \mathcal{B}(T-t)\Psi^{(k)} + \int_t^T \mathcal{B}(s-t)G(u^{(k)}(x,s),v^{(k)}(x,s))\mathrm{d}s, \end{aligned} \tag{2.16}$$

with $a = \lambda_1^{1/2-\gamma} + 1$; recalling that

$$\mathcal{B}(z)w = \sum_{p=1}^{\infty} \frac{\exp(z\mu_1) - \exp(z\mu_2)}{2\sqrt{\alpha_p}} \langle w,\xi_p\rangle_{L^2(\Omega)}\xi_p, \tag{2.17}$$

and, for $j = 1,2$,

$$\alpha_p = (\lambda_1^{1/2-\gamma}+1)^2\lambda_p^{2\gamma} - \lambda_p, \quad \mu_j = \mu_{j,p} := (\lambda_1^{1/2-\gamma}+1)\lambda_p^{\gamma} + (-1)^j\sqrt{\alpha_p}. \tag{2.18}$$

Step 1. We show that (2.16) has a unique solution $(u^{(k)}, v^{(k)}) \in [C([0,T]; L^2(\Omega))]^2$. Indeed, we consider for $(r_1, r_2) \in [C([0,T]; L^2(\Omega))]^2$ the function

$$\mathbb{E}(r_1, r_2)(t) = \Big(\mathcal{E}(r_1, r_2)(t), \widetilde{\mathcal{E}}(r_1, r_2)(t)\Big),$$

where

$$\begin{aligned}
\mathcal{E}(r_1, r_2)(t) &= \mathcal{B}(T-t)\Psi^{(k)} + \int_t^T \mathcal{B}(\tau - t) F(r_1(x,\tau), r_2(x,\tau)) \mathrm{d}\tau, \\
\widetilde{\mathcal{E}}(r_1, r_2)(t) &= \mathcal{B}(T-t)\Psi^{(k)} + \int_t^T \mathcal{B}(\tau - t) G(r_1(x,\tau), r_2(x,\tau)) \mathrm{d}\tau.
\end{aligned}$$

Then for any $(r_1, r_2), (s_1, s_2) \in [C([0,T]; L^2(\Omega))]^2$, we obtain

$$\begin{aligned}
&\|\mathcal{E}(r_1, r_2)(t) - \mathcal{E}(s_1, s_2)(t)\|_{L^2(\Omega)} \\
&\le \int_t^T \|\mathcal{B}(\tau - t)[F(r_1, r_2) - F(s_1, s_2)]\|_{L^2(\Omega)} \mathrm{d}\tau \\
&= \int_t^T \Big\{ \sum_{p=1}^{\infty} \Big[\frac{\exp((\tau - t)\mu_2) - \exp((\tau - t)\mu_1)}{\sqrt{\alpha_p}}\Big]^2 \\
&\quad \times \langle F(r_1, r_2) - F(s_1, s_2), \xi_p\rangle_{L^2(\Omega)}^2 \Big\}^{1/2} \mathrm{d}\tau \\
&\le \int_t^T \Big\{ \sum_{p=1}^{\infty} \Big[\frac{\exp((\tau - t)\mu_2) - \exp((\tau - t)\mu_1)}{\sqrt{\alpha_p}}\Big]^2 \frac{\exp(-4(\lambda_1^{1/2-\gamma} + 1)T\lambda_p^\gamma)}{2^7 T^4} \\
&\quad \times 2\langle r_1(x,\tau) - s_1(x,\tau), \xi_p\rangle_{L^2(\Omega)}^2 + 2\langle r_2(x,\tau) - s_2(x,\tau), \xi_p\rangle_{L^2(\Omega)}^2 \Big\}^{1/2} \mathrm{d}\tau.
\end{aligned} \tag{2.19}$$

Moreover, using the the inequality $|\exp(-b) - \exp(-c)| \le |b - c|$ for $b, c > 0$, we obtain the estimate

$$\begin{aligned}
&\Big[\frac{\exp((\tau - t)\mu_2) - \exp((\tau - t)\mu_1)}{\sqrt{\alpha_p}}\Big]^2 \frac{\exp(-4(\lambda_1^{1/2-\gamma} + 1)T\lambda_p^\gamma)}{2^7 T^4} \\
&= \Big[\exp\big((\tau - t)(\mu_1 + \mu_2)\big) \frac{\exp\big(-(\tau - t)\mu_1\big) - \exp\big(-(\tau - t)\mu_2\big)}{\sqrt{\alpha_p}}\Big]^2 \\
&\quad \times \frac{\exp(-4(\lambda_1^{1/2-\gamma} + 1)T\lambda_p^\gamma)}{2^7 T^4} \\
&\le \exp\Big(4(\lambda_1^{1/2-\gamma} + 1)(\tau - t)\lambda_p^\gamma\Big) \Big[\frac{(-2\sqrt{\alpha_p})(\tau - t)}{\sqrt{\alpha_p}}\Big]^2 \frac{\exp(-4(\lambda_1^{1/2-\gamma} + 1)T\lambda_p^\gamma)}{2^7 T^4} \\
&\le \Big[\frac{(-2\sqrt{\alpha_p})(\tau - t)}{\sqrt{\alpha_p}}\Big]^2 \frac{1}{2^7 T^4} \\
&= 4(\tau - t)^2 \frac{1}{2^7 T^4} \le \frac{1}{2^5 T^2},
\end{aligned}$$

where we have used $\mu_1 + \mu_2 = 2a\lambda_p^\gamma = 2(\lambda_1^{1/2-\gamma} + 1)\lambda_p^\gamma$ and $\mu_2 - \mu_1 = 2\sqrt{\alpha_p}$. According to the above observations, we deduce that for all $t \in [0,T]$

$$\|\mathcal{E}(r_1, r_2)(t) - \mathcal{E}(s_1, s_2)(t)\|_{L^2(\Omega)} \le \frac{1}{4} \int_t^T \frac{1}{T} \|R(\cdot, \tau) - S(\cdot, \tau)\|_{[L^2(\Omega)]^2} \mathrm{d}\tau$$

$$\leq \frac{1}{4}\|R-S\|_{[C([0,T];L^2(\Omega))]^2},$$

where $R := (r_1, r_2)$, $S := (s_1, s_2) \in [C([0,T];L^2(\Omega))]^2$. Whereupon

$$\|\mathcal{E}(r_1,r_2) - \mathcal{E}(s_1,s_2)\|_{L^2(\Omega)} \leq \frac{1}{4}\|R-S\|_{[C([0,T];L^2(\Omega))]^2}. \tag{2.20}$$

Similarly,

$$\|\widetilde{\mathcal{E}}(r_1,r_2) - \widetilde{\mathcal{E}}(s_1,s_2)\|_{L^2(\Omega)} \leq \frac{1}{4}\|R-S\|_{[C([0,T];L^2(\Omega))]^2}. \tag{2.21}$$

Combining (2.20) and (2.21), we obtain

$$\|\mathbb{E}(R) - \mathbb{E}(S)\|_{[C([0,T];L^2(\Omega))]^2} \leq \frac{1}{2}\|R-S\|_{[C([0,T];L^2(\Omega))]^2}.$$

Hence $\mathbb{E}$ is a contraction. Using the Banach fixed-point theorem, we conclude that $\mathbb{E}(R) = R$ has a unique solution $(u^{(k)}, v^{(k)}) \in [C([0,T];L^2(\Omega))]^2$.

Step 2. Problem (2.16) is ill-posed in the sense of Hadamard. We have

$$\|u^{(k)}(t)\|_{L^2(\Omega)} \geq \|\mathcal{B}(T-t)\Psi^{(k)}\|_{L^2(\Omega)} - \|\int_t^T \mathcal{B}(s-t)F(\mathbf{w}^{(k)})(s)\mathrm{d}s\|_{L^2(\Omega)}, \tag{2.22}$$

where $\mathbf{w}^{(k)} = (u^{(k)}, v^{(k)}) \in [C([0,T];L^2(\Omega))]^2$.

Firstly, it is easy to see that (noting that $F(0,0) = 0$ and using (2.20))

$$\begin{aligned} \|\int_t^T \mathcal{B}(s-t)F(\mathbf{w}^{(k)}(s))ds\|_{L^2(\Omega)} &= \|\mathcal{E}(u^{(k)}, v^{(k)})(t) - \mathcal{E}(0,0)(t)\|_{L^2(\Omega)} \\ &\leq \frac{1}{4}\|\mathbf{w}^{(k)}\|_{[C([0,T];L^2(\Omega))]^2}. \end{aligned} \tag{2.23}$$

Hence

$$\|u^{(k)}(t)\|_{L^2(\Omega)} \geq \Big\|\mathcal{B}(T-t)\Psi^{(k)}\Big\|_{L^2(\Omega)} - \frac{1}{4}\|\mathbf{w}^{(k)}\|_{[C([0,T];L^2(\Omega))]^2}.$$

This leads to

$$\|u^{(k)}\|_{C([0,T];L^2(\Omega))} \geq \sup_{0\leq t\leq T}\Big\|\mathcal{B}(T-t)\Psi^{(k)}\Big\|_{L^2(\Omega)} - \frac{}{4}\|\mathbf{w}^{(k)}\|_{[C([0,T];L^2(\Omega))]^2}. \tag{2.24}$$

By an argument analogous to the previous one. We get

$$\|v^{(k)}\|_{C([0,T];L^2(\Omega))} \geq \sup_{0\leq t\leq T}\Big\|\mathcal{B}(T-t)\Psi^{(k)}\Big\|_{L^2(\Omega)} - \frac{}{4}\|\mathbf{w}^{(k)}\|_{[C([0,T];L^2(\Omega))]^2}. \tag{2.25}$$

Combining (2.24) and (2.25) yields

$$\|\mathbf{w}^{(k)}\|_{[C([0,T];L^2(\Omega))]^2} \geq \frac{4}{3}\sup_{0\leq t\leq T}\Big\|\mathcal{B}(T-t)\Psi^{(k)}\Big\|_{L^2(\Omega)}. \tag{2.26}$$

Secondly, we have

$$\begin{aligned} \|\mathcal{B}(T-t)\Psi^{(k)}\|^2_{L^2(\Omega)} &= \Big[\frac{\exp\big((T-t)\mu_{2k}\big) - \exp\big((T-t)\mu_{1k}\big)}{2\sqrt{\alpha_k}}\Big]^2\frac{1}{\lambda_k} \\ &= \frac{\exp\big(2(T-t)\mu_{2k}\big)\big(1-\exp\big(-2(T-t)\sqrt{\alpha_k}\big)\big)^2}{4\lambda_k\alpha_k} \\ &\geq \frac{\exp\big(2(T-t)\mu_{2k}\big)\big(1-\exp\big(-2(T-t)\sqrt{\alpha_k}\big)\big)^2}{4\lambda_k\alpha_k}, \end{aligned}$$

where we set $\mu_{j,k} := (\lambda_1^{1/2-\gamma}+1)\lambda_k^\gamma+(-1)^j\sqrt{\alpha_k}$, $j=1,2$, $k\in\mathbb{N}^*$. Since the function $\Theta(t)=\exp\big(2(T-t)\mu_{2k}\big)\big(1-\exp\big(-2(T-t)\sqrt{\alpha_k}\big)\big)^2$ is a decreasing function with respect to the variable t, noting that $\mu_{2k}\approx 2(\lambda_1^{1/2-\gamma}+1)\lambda_k^\gamma$, we deduce that

$$
\begin{aligned}
&\sup_{0\le t\le T}\|\mathcal{B}(T-t)\Psi^{(k)}\|^2_{L^2(\Omega)}\\
&=\sup_{0\le t\le T}\|\mathcal{B}(T-t)\Psi^{(k)}\|^2_{L^2(\Omega)}\\
&\ge \sup_{0\le t\le T}\frac{\exp\big(2(T-t)\mu_{2k}\big)\big(1-\exp\big(-2(T-t)\sqrt{\alpha_k}\big)\big)^2}{4\lambda_k\alpha_k}\\
&\ge \frac{\exp(2T\mu_{2k})\big(1-\exp\big(-2T\sqrt{\alpha_k}\big)\big)^2}{4\lambda_k\alpha_k}\\
&\ge \frac{\exp(4T(\lambda_1^{1/2-\gamma}+1)\lambda_k^\gamma)\big(1-\exp\big(-2T\sqrt{\alpha_k}\big)\big)^2}{4\lambda_k\alpha_k}.
\end{aligned}
\tag{2.27}
$$

Next we estimate the right-hand side of the latter inequality. Indeed, combining (2.26) and (2.27) yields

$$
\|\mathbf{w}^{(k)}\|_{[C([0,T];L^2(\Omega))]^2}\ge\frac{2}{3}\frac{\exp\big(2T(\lambda_1^{1/2-\gamma}+1)\lambda_k^\gamma\big)\big(1-\exp\big(-2T\sqrt{\alpha_k}\big)\big)}{\sqrt{\lambda_k\alpha_k}}. \tag{2.28}
$$

As $k\to+\infty$, we see that

$$
\begin{aligned}
&\lim_{k\to\infty}\Big(\|u^{(k)}(x,T)\|_{L^2(\Omega)}+\|v^{(k)}(x,T)\|_{L^2(\Omega)}\Big)=\lim_{k\to\infty}\frac{2}{\sqrt{\lambda_k}}=0,\\
&\lim_{k\to\infty}\|\mathbf{w}^{(k)}\|_{[C([0,T];L^2(\Omega))]^2}\\
&\ge\lim_{k\to\infty}\frac{2}{3}\frac{\exp\big(2T(\lambda_1^{1/2-\gamma}+1)\lambda_k\big)\big(1-\exp\big(-2T\sqrt{\alpha_k}\big)\big)}{\sqrt{\lambda_k\alpha_k}}=\infty.
\end{aligned}
\tag{2.29}
$$

Thus, Problem (2.16) is ill-posed in the sense of Hadamard in L^2-norm.

3. Regularization and error estimate in for globally Lipschitz nonlinearities

Observe that when $p\to\infty$ the operators $\mathcal{A},\mathcal{B}$ are unbounded; to establish a regularized solution, we need to find new operators which are bounded operators, more specifically,

$$
\mathcal{A}^\Lambda(t)w := \mathcal{H}^\Lambda\mathcal{A}(t)w, \tag{3.1}
$$

$$
\mathcal{B}^\Lambda(t)w := \mathcal{H}^\Lambda\mathcal{B}(t)w, \tag{3.2}
$$

$$
\mathcal{H}^\Lambda w := \sum_{p=1}^\infty[1+\Lambda C_p e^{C_pT}]^{-1}\langle w,\xi_p\rangle_{L^2(\Omega)}\xi_p, \tag{3.3}
$$

where $C_p=2a\lambda_p^\gamma$. Here $\Lambda:=\Lambda(\delta)>0$ is a parameter regularization which satisfies $\lim_{\delta\to0^+}\Lambda=0$. The function $\mathcal{H}^\Lambda$ is called the *filter function*. The regularized problem is

$$
\begin{aligned}
U_\delta^\Lambda(x,t) &= \mathcal{A}^\Lambda(T-t)u_T^\delta(x) + \mathcal{B}^\Lambda(T-t)\widetilde{u}_T^\delta(x) \\
&\quad + \int_t^T \mathcal{B}^\Lambda(s-t)F(U_\delta^\Lambda(x,s),V_\delta^\Lambda(x,s))\mathrm{d}s, \\
V_\delta^\Lambda(x,t) &= \mathcal{A}^\Lambda(T-t)v_T^\delta(x) + \mathcal{B}^\Lambda(T-t)\widetilde{v}_T^\delta(x) \\
&\quad + \int_t^T \mathcal{B}^\Lambda(s-t)G(U_\delta^\Lambda(x,s),V_\delta^\Lambda(x,s))\mathrm{d}s.
\end{aligned} \tag{3.4}
$$

The following technical lemma plays a key role in our analysis.

Lemma 3.1. *Let $t \in [0,T]$, $\frac{1}{2} \le \gamma \le 1$ and $a > 0$. Then*

$$
\|\mathcal{A}^\Lambda(t)\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))} \le T_a \Big[\frac{T}{\Lambda \log(\frac{T}{\Lambda})}\Big]^{t/T}, \tag{3.5}
$$

$$
\|\mathcal{B}^\Lambda(t)\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))} \le T_a \Big[\frac{T}{\Lambda \log(\frac{T}{\Lambda})}\Big]^{t/T}, \tag{3.6}
$$

where $T_a := \max\{2, T, 1 + Ta^{\frac{1}{1-2\gamma}}\}$.

Proof. To show (3.5), letting $w \in L^2(\Omega)$, we have

$$
\begin{aligned}
&\|\mathcal{A}^\Lambda(t)w\|^2_{L^2(\Omega)} \\
&= \sum_{p\in\mathbb{N}_1} \Big[\frac{\mu_2 e^{\mu_1 t} - \mu_1 e^{\mu_2 t}}{2\sqrt{\alpha_p}[1 + C_p\Lambda e^{C_pT}]}\Big]^2 \langle w, \xi_p\rangle^2_{L^2(\Omega)} \\
&\quad + \sum_{p\in\mathbb{N}_2} \frac{e^{C_pt}(1 - \frac{C_p}{2}t)^2}{[1 + C_p\Lambda e^{C_pT}]^2} \langle w, \xi_p\rangle^2_{L^2(\Omega)} \\
&\quad + \sum_{p\in\mathbb{N}_3} \frac{e^{C_pt}}{(-\alpha_p)[1 + C_p\Lambda e^{C_pT}]^2} \\
&\quad \times \Big[\frac{C_p}{2}\sin(\sqrt{-\alpha_p}t) - \cos(\sqrt{-\alpha_p}t)\Big]^2 \langle w, \xi_p\rangle^2_{L^2(\Omega)} \\
&\le \sum_{p\in\mathbb{N}_1} \Big[\frac{\mu_2 e^{-\mu_2 t} - \mu_1 e^{-\mu_1 t}}{2\sqrt{\alpha_p}}\Big]^2 \Big[\frac{1}{e^{-C_pt} + C_p\Lambda e^{C_p(T-t)}}\Big]^2 \langle w, \xi_p\rangle^2_{L^2(\Omega)} \\
&\quad + \sum_{p\in\mathbb{N}_2} e^{-C_pt}(1 - \frac{C_p}{2}t)^2 \Big[\frac{1}{e^{-C_pt} + C_p\Lambda e^{C_p(T-t)}}\Big]^2 \langle w, \xi_p\rangle^2_{L^2(\Omega)} \\
&+ \sum_{p\in\mathbb{N}_3} \Big[\frac{1}{e^{-C_pt} + C_p\Lambda e^{C_p(T-t)}}\Big]^2 e^{-C_pt} \frac{[\frac{C_p}{2}\sin(\sqrt{-\alpha_p}t) - \cos(\sqrt{-\alpha_p}t)]^2}{(-\alpha_p)} \\
&\quad \times \langle w, \xi_p\rangle^2_{L^2(\Omega)}.
\end{aligned} \tag{3.7}
$$

Now, we continue estimating the terms in (3.7): First, we have

$$
\begin{aligned}
\frac{1}{e^{-C_pt} + C_p\Lambda e^{C_p(T-t)}} &= \frac{e^{-C_p(T-t)}}{[C_p\Lambda + e^{-C_pT}]^{\frac{T-t}{T}}[C_p\Lambda + e^{-C_pT}]^{t/T}} \\
&\le \frac{1}{[C_p\Lambda + e^{-C_pT}]^{t/T}}.
\end{aligned} \tag{3.8}
$$

On other hand, it is easy to see that $h(y) = \frac{1}{by+e^{-yT}} \le \frac{T}{b\log(\frac{T}{b})}$ for $0 < b < Te$. Hence if $\Lambda < Te$, then we obtain

$$\frac{1}{C_p\Lambda + e^{-C_pT}} \le \frac{T}{\Lambda\log(\frac{T}{\Lambda})}.$$

It follows from (3.8) that

$$\frac{1}{e^{-C_pt} + C_p\Lambda e^{C_p(T-t)}} \le \big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\big]^{t/T}. \tag{3.9}$$

If $p \in \mathbb{N}_2$, then $\lambda_p = a^{\frac{2}{1-2\gamma}}$ implies $C_p = 2a^{\frac{1}{1-2\gamma}}$,

$$[1 - \frac{C_p}{2}t] \le 1 + a^{\frac{1}{1-2\gamma}}T. \tag{3.10}$$

If $p \in \mathbb{N}_3$, then $\lambda_p < a^{\frac{2}{1-2\gamma}}$; using $\sin(z) \le z$ for any $z \ge 0$, we have

$$\frac{\frac{C_p}{2}\sin(\sqrt{-\alpha_p}t) - \cos(\sqrt{-\alpha_p}t)}{\sqrt{-\alpha_p}} \le 1 + a^{\frac{1}{1-2\gamma}}T. \tag{3.11}$$

If $p \in \mathbb{N}_1$, then $\lambda_p > a^{\frac{2}{1-2\gamma}}$; using the inequalities $1 - e^{-z} \le z$ and $ze^{-z} \le 1$ for $z > 0$, we obtain (noting that $\mu_2 - \mu_1 = 2\sqrt{\alpha_p}$)

$$\begin{aligned}
\big|\frac{\mu_2 e^{-t\mu_2} - \mu_1 e^{-t\mu_1}}{2\sqrt{\alpha_p}}\big| &= e^{-t\mu_2} + \mu_1\frac{|e^{-t\mu_2} - e^{-t\mu_1}|}{2\sqrt{\alpha_p}} \\
&\le e^{-t\mu_2} + \mu_1 e^{-t\mu_1}\frac{|1 - e^{-t(\mu_2-\mu_1)}|}{2\sqrt{\alpha_p}} \\
&\le e^{-t\mu_2} + \mu_1 e^{-t\mu_1}\frac{t(\mu_2-\mu_1)}{2\sqrt{\alpha_p}} \\
&\le e^{-t\mu_2} + t\mu_1 e^{-t\mu_1} \le 2.
\end{aligned} \tag{3.12}$$

Combining (3.7), (3.9), (3.10), (3.11), (3.12), we conclude that

$$\|\mathcal{A}^\Lambda(t)w\|^2_{L^2(\Omega)} \le \Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{2t/T} T_a^2\|w\|^2_{L^2(\Omega)}. \tag{3.13}$$

To show (3.6), letting $w \in L^2(\Omega)$, we have

$$\begin{aligned}
&\|\mathcal{B}^\Lambda(t)w\|^2_{L^2(\Omega)} \\
&= \sum_{p\in\mathbb{N}_1}\Big[e^{t(\mu_1+\mu_2)}\frac{e^{-\mu_2 t} - e^{-\mu_1 t}}{2\sqrt{\alpha_p}}\frac{1}{1 + C_p\Lambda e^{C_pT}}\Big]^2\langle w, \xi_p\rangle^2_{L^2(\Omega)} \\
&\quad + \sum_{p\in\mathbb{N}_2}\Big[e^{\frac{C_p}{2}t}t\frac{1}{1 + C_p\Lambda e^{C_pT}}\Big]^2\langle w, \xi_p\rangle^2_{L^2(\Omega)} \\
&\quad + \sum_{p\in\mathbb{N}_3}\Big[\frac{1}{1 + C_p\Lambda e^{C_pT}}\frac{e^{\frac{C_p}{2}t}}{\sqrt{-\alpha_p}}\sin(\sqrt{-\alpha_p}t)\Big]^2\langle w, \xi_p\rangle^2_{L^2(\Omega)} \\
&\le \sum_{p\in\mathbb{N}_1}\Big[\frac{1}{e^{-C_pt} + C_p\Lambda e^{C_p(T-t)}}\Big]^2\Big[\frac{e^{-\mu_2 t} - e^{-\mu_1 t}}{2\sqrt{\alpha_p}}\Big]^2\langle w, \xi_p\rangle^2_{L^2(\Omega)} \\
&\quad + \sum_{p\in\mathbb{N}_2}\Big[\frac{1}{e^{-C_pt} + C_p\Lambda e^{C_p(T-t)}}\Big]^2 e^{-C_pt}t^2\langle w, \xi_p\rangle^2_{L^2(\Omega)}
\end{aligned}$$

$$+\sum_{p\in\mathbb{N}_3}\Big[\frac{1}{e^{-C_pt}+C_p\Lambda e^{C_p(T-t)}}\Big]^2\frac{e^{-C_pt}}{(-\alpha_p)}\sin^2(\sqrt{-\alpha_p}t)\langle w,\xi_p\rangle^2_{L^2(\Omega)}.$$

As $|e^{-c}-e^{-d}|\le|c-d|$ for $c,d>0$, and noting that $e^{-C_pt}\le 1$, for all $t\in[0,T]$ and $\sin z\le z$, for $z>0$, we obtain

$$\begin{aligned}
\|\mathcal{B}^\Lambda(t)w\|^2_{L^2(\Omega)} &\le T^2\sum_{p\in\mathbb{N}_1}\Big[\frac{1}{e^{-C_pt}+C_p\Lambda e^{C_p(T-t)}}\Big]^2\langle w,\xi_p\rangle^2_{L^2(\Omega)}\\
&\quad + T^2\sum_{p\in\mathbb{N}_2}\Big[\frac{1}{e^{-C_pt}+C_p\Lambda e^{C_p(T-t)}}\Big]^2\langle w,\xi_p\rangle^2_{L^2(\Omega)}\\
&\quad + T^2\sum_{p\in\mathbb{N}_3}\Big[\frac{1}{e^{-C_pt}+C_p\Lambda e^{C_p(T-t)}}\Big]^2\langle w,\xi_p\rangle^2_{L^2(\Omega)}\\
&\le T_a^2\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{2t/T}\|w\|^2_{L^2(\Omega)}.
\end{aligned}$$

This completes the proof. □

Now we are ready to state and prove the main results of this paper.

3.1. **Existence and uniqueness for problem** (2.16).

Theorem 3.2. *The nonlinear integral system* (2.16) *has a solution* $(U^\Lambda_\delta,V^\Lambda_\delta)\in[C([0,T];L^2(\Omega))]^2$.

Proof. For any $w_1,w_2\in[C([0,T];L^2(\Omega))]^2$, we define the function $\mathbb{L}\colon[C([0,T];L^2(\Omega))]^2\to[C([0,T];L^2(\Omega))]^2$ as

$$\mathbb{L}(w_1,w_2)(t):=(\mathbb{X}(w_1,w_2)(t),\mathbb{Y}(w_1,w_2)(t)),$$

where

$$\begin{aligned}
\mathbb{X}(w_1,w_2)(t) &:= \mathcal{A}^\Lambda(T-t)u^\delta_T(x)+\mathcal{B}^\Lambda(T-t)\widetilde{u}^\delta_T(x)\\
&\quad+\int_t^T\mathcal{B}^\Lambda(s-t)F(w_1(x,s),w_2(x,s))\mathrm{d}s,
\end{aligned}\tag{3.14}$$

$$\begin{aligned}
\mathbb{Y}(w_1,w_2)(t) &:= \mathcal{A}^\Lambda(T-t)v^\delta_T(x)+\mathcal{B}^\Lambda(T-t)\widetilde{v}^\delta_T(x)\\
&\quad+\int_t^T\mathcal{B}^\Lambda(s-t)G(w_1(x,s),w_2(x,s))\mathrm{d}s.
\end{aligned}\tag{3.15}$$

Then for any $W=(w_1,w_2),\overline{W}=(\overline{w}_1,\overline{w}_2)\in[C([0,T];L^2(\Omega))]^2$, we obtain

$$\begin{aligned}
&\|\mathbb{L}(W)(t)-\mathbb{L}(\overline{W})(t)\|_{[L^2(\Omega)]^2}\\
&\le\|\mathbb{X}(W)(t)-\mathbb{X}(\overline{W})(t)\|_{L^2(\Omega)}+\|\mathbb{Y}(W)(t)-\mathbb{Y}(\overline{W})(t)\|_{L^2(\Omega)}.
\end{aligned}\tag{3.16}$$

Let the functions $F,G\colon[L^2(\Omega)]^2\to L^2(\Omega)$ satisfy the global Lipschitz condition

$$\|F(W)-F(\overline{W})\|_{L^2(\Omega)}\le K_F\|W-\overline{W}\|_{[L^2(\Omega)]^2},\tag{3.17}$$

$$\|G(W)-G(\overline{W})\|_{L^2(\Omega)}\le K_G\|W-\overline{W}\|_{[L^2(\Omega)]^2},\tag{3.18}$$

where K_F,K_G are constants which are independent of $W,\overline{W}$. We shall prove the estimate

$$\|\mathbb{X}^n(W)(t)-\mathbb{X}^n(\overline{W})(t)\|_{L^2(\Omega)}\le\frac{\mathbf{E}^n_{\Lambda,\delta}(t)}{n!}\|W-\overline{W}\|_{[C([0,T];L^2(\Omega))]^2},\tag{3.19}$$

where $\mathbf{E}^n_{\Lambda,\delta}(t) := \big[\frac{\mathbb{K}T_a(T-t)}{\Lambda}\big]^n$, $n \geq 1$, and $\mathbb{K} = \max\{K_F, K_G\}$, by induction.

• For $n = 1$, using Lemma (3.1) and the global Lipschitz condition of the function F, we obtain

$$
\begin{aligned}
&\|\mathbb{X}(W)(t) - \mathbb{X}(\overline{W})(t)\|_{L^2(\Omega)}\\
&\leq \int_t^T \|\mathcal{B}^\Lambda(s-t)(F(W)(s) - F(\overline{W})(s))\|_{L^2(\Omega)}\mathrm{d}s\\
&\leq \int_t^T \|\mathcal{B}^\Lambda(s-t)\|_{\mathcal{L}(L^2(\Omega)\times L^2(\Omega))}\|F(W)(s) - F(\overline{W})(s)\|_{L^2(\Omega)}\mathrm{d}s\\
&\leq \int_t^T T_a\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{\frac{s-t}{T}} K_F\|W(s) - \overline{W}(s)\|_{[L^2(\Omega)]^2}\mathrm{d}s\\
&\leq \mathbb{K}T_a\Lambda^{-1}\int_t^T \|W(s) - \overline{W}(s)\|_{[L^2(\Omega)]^2}\mathrm{d}s\\
&\leq \mathbb{K}T_a\Lambda^{-1}(T-t)\|W - \overline{W}\|_{[C([0,T];L^2(\Omega))]^2}\\
&= \mathbf{E}_{\Lambda,\delta}(t)\|W - \overline{W}\|_{[C([0,T];L^2(\Omega))]^2},
\end{aligned}
$$

where $\mathbf{E}_{\Lambda,\delta}(t) := \frac{\mathbb{K}T_a(T-t)}{\Lambda}$.

• Assume that (3.19) holds for $n = k$. Then we obtain

$$
\|\mathbb{X}^k(W)(t) - \mathbb{X}^k(\overline{W})(t)\|_{L^2(\Omega)} \leq \frac{\mathbf{E}^k_{\Lambda,\delta}(t)}{k!}\|W - \overline{W}\|_{[C([0,T];L^2(\Omega))]^2}. \tag{3.20}
$$

• We show that (3.19) holds for $n = k+1$. In fact, we have

$$
\begin{aligned}
&\|\mathbb{X}^{k+1}(W)(t) - \mathbb{X}^{k+1}(\overline{W})(t)\|_{L^2(\Omega)}\\
&= \Big\|\int_t^T \mathcal{B}^\Lambda(s-t)[F(\mathbb{X}^k(W)(s)) - F(\mathbb{X}^k(\overline{W})(s))]\mathrm{d}s\Big\|_{L^2(\Omega)}\\
&\leq \int_t^T T_a\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{\frac{s-t}{T}} K_F\|\mathbb{X}^k(W)(s) - \mathbb{X}^k(\overline{W})(s)\|_{L^2(\Omega)}\mathrm{d}s\\
&\leq \mathbb{K}T_a\Lambda^{-1}\int_t^T \|\mathbb{X}^k(W)(s) - \mathbb{X}^k(\overline{W})(s)\|_{L^2(\Omega)}\mathrm{d}s\\
&\leq \mathbb{K}T_a\Lambda^{-1}\int_t^T \frac{\big[\mathbb{K}T_a\Lambda^{-1}(T-s)\big]^k}{k!}\|W - \overline{W}\|_{[C([0,T];L^2(\Omega))]^2}\mathrm{d}s\\
&= \frac{(\mathbb{K}T_a\Lambda^{-1})^{k+1}}{k!}\|W - \overline{W}\|_{[C([0,T];L^2(\Omega))]^2}\int_t^T (T-s)^k\mathrm{d}s\\
&= \frac{\mathbf{E}^{k+1}_{\Lambda,\delta}(t)}{(k+1)!}\|W - \overline{W}\|_{[C([0,T];L^2(\Omega))]^2}.
\end{aligned}
$$

Therefore, (3.19) holds for $n \geq 1$.

Secondly, we estimate $\|\mathbb{Y}(W)(t) - \mathbb{Y}(\overline{W})(t)\|_{L^2(\Omega)}$. Using similar arguments, we infer that if $W := (w_1, w_1), \overline{W} := (\overline{w}_1, \overline{w}_2) \in [L^2(\Omega)]^2$ then

$$
\|\mathbb{Y}^n(W)(t) - \mathbb{Y}^n(\overline{W})(t)\|_{L^2(\Omega)} \leq \frac{\mathbf{E}^n_{\Lambda,\delta}(t)}{n!}\|W - \overline{W}\|_{[C([0,T];L^2(\Omega))]^2}, \tag{3.21}
$$

where $\mathbf{E}^n_{\Lambda,\delta}(t) := \big[\frac{\mathbb{K}T_a(T-t)}{\Lambda}\big]^n$.

Combining (3.16), (3.19) and (3.21), we obtain

$$\|\mathbb{L}^n(W)(t) - \mathbb{L}^n(\overline{W})(t)\|_{[L^2(\Omega)]^2} \le \frac{2\mathbf{E}^n_{\Lambda,\delta}(t)}{n!}\|W - \overline{W}\|_{[C([0,T];L^2(\Omega)]^2} \tag{3.22}$$

On the other hand,

$$\mathbf{E}^n_{\Lambda,\delta}(t) \le \mathbf{E}^n_{\Lambda,\delta}(0) = \Big(\frac{\mathbb{K}T_aT}{\Lambda}\Big)^n, \quad n \ge 1.$$

This implies

$$\|\mathbb{L}^n(W) - \mathbb{L}^n(\overline{W})\|_{[C([0,T];L^2(\Omega)]^2} \le \frac{2\mathbf{E}^n_{\Lambda,\delta}(0)}{n!}\|W - \overline{W}\|_{[C([0,T];L^2(\Omega)]^2},$$
$$\lim_{n\to\infty}\frac{2\mathbf{E}^n_{\Lambda,\delta}(0)}{n!} = 0.$$

There exits a positive integer n_0 such that $\mathbb{L}^{n_0}$ is a contraction. Thus, the existence and uniqueness arguments are obtained by the Banach fixed-point theorem, i.e, $\mathbb{L}(w_1, w_2) = (w_1, w_2)$ has a unique solution $(w_1, w_2) \in [C([0,T];L^2(\Omega)]^2$. Hence, $\mathbb{X}(U^\Lambda_\delta, V^\Lambda_\delta) = U^\Lambda_\delta$ and $\mathbb{Y}(U^\Lambda_\delta, V^\Lambda_\delta) = V^\Lambda_\delta$. $\square$

3.2. **Error estimate.** Now, we shall state (and prove) some regularization results under some conditions on the exact solution (u, v) of system (2.15).

Theorem 3.3. *Let $\Lambda(\delta)$ be a regularization parameter such that*

$$\lim_{\delta\to 0^+}\Lambda = \lim_{\delta\to 0^+}\delta/\Lambda = 0. \tag{3.23}$$

For $m \ge 1, n \ge 2aT$, we assume that system (2.16) has a unique solution

$$\mathcal{S} := (u, v) \in [C([0,T];L^2(\Omega)) \cap L^\infty(0,T;\mathbb{G}^\gamma_{m,n})]^2$$

which satisfies

$$\|\mathcal{S}(t)\|_{[C([0,T];L^2(\Omega))]^2} + \|\mathcal{S}(t)\|_{[L^\infty(0,T;\mathbb{G}^\gamma_{m,n})]^2} \le Q. \tag{3.24}$$

Then we have the estimate

$$\|\mathcal{S}^\Lambda_\delta(t) - \mathcal{S}(t)\|_{[L^2(\Omega)]^2} \le \Big(2aQ + T_a\delta\Lambda^{-1}\Big)e^{2T_a\mathbb{K}(T-t)}\Big[\frac{T}{\log(\frac{T}{\Lambda})}\Big]^{1-\frac{t}{T}}\Lambda^{t/T}, \tag{3.25}$$

where $\mathcal{S}^\Lambda_\delta := (U^\Lambda_\delta, V^\Lambda_\delta) \in [C([0,T];L^2(\Omega))]^2$ and $\mathbb{K} := \max\{K_F, K_G\}$.

Remark 3.4. In (3.25), the error estimate is of order $\Lambda^{t/T}[\frac{T}{\log(\frac{T}{\Lambda})}]^{1-\frac{t}{T}}$. If $t \approx T$, the first term $\Lambda^{t/T}$ tends to zero quickly, and if $t \approx 0$, the second term $[\frac{T}{\log(\frac{T}{\Lambda})}]^{1-\frac{t}{T}}$ tends to zero as $\delta \to 0^+$. And if $t = 0$, the error (3.25) becomes

$$\|\mathcal{S}^\Lambda_\delta(t) - \mathcal{S}(t)\|_{[L^2(\Omega)]^2} \le C[\log(\frac{T}{\Lambda})]^{-1}. \tag{3.26}$$

We also note that the right-hand side of (3.26) tends to zero when $\delta \to 0^+$.

Proof. Let $\mathcal{S}^\Lambda := (U^\Lambda, V^\Lambda) \in [C([0,T]; L^2(\Omega))]^2$ satisfy the nonlinear integral equations

$$
\begin{aligned}
U^\Lambda(x,t) &= \mathcal{A}^\Lambda(T-t)u_T(x) + \mathcal{B}^\Lambda(T-t)\widetilde{u}_T(x)\\
&\quad + \int_t^T \mathcal{B}^\Lambda(s-t)F(U^\Lambda(x,s), V^\Lambda(x,s))\mathrm{d}s,\\
V^\Lambda(x,t) &= \mathcal{A}^\Lambda(T-t)v_T(x) + \mathcal{B}^\Lambda(T-t)\widetilde{v}_T(x)\\
&\quad + \int_t^T \mathcal{B}^\Lambda(s-t)G(U^\Lambda(x,s), V^\Lambda(x,s))\mathrm{d}s.
\end{aligned}
\tag{3.27}
$$

Using the triangle inequality

$$
\|\mathcal{S}_\delta^\Lambda(t) - \mathcal{S}(t)\|_{[L^2(\Omega)]^2} \le \|\mathcal{S}_\delta^\Lambda(t) - \mathcal{S}^\Lambda(t)\|_{[L^2(\Omega)]^2} + \|\mathcal{S}^\Lambda(t) - \mathcal{S}(t)\|_{[L^2(\Omega)]^2}, \tag{3.28}
$$

the proof of (3.25) can be completed in two steps.

Step 1. Estimate of $\|\mathcal{S}_\delta^\Lambda(t) - \mathcal{S}^\Lambda(t)\|_{[L^2(\Omega)]^2}$. From (3.4) and (3.27), we have

$$
\begin{aligned}
U_\delta^\Lambda(t) - U^\Lambda(t) &= \mathcal{A}^\Lambda(T-t)(u_T^\delta - u_T) + \mathcal{B}^\Lambda(T-t)(\widetilde{u}_T^\delta - \widetilde{u}_T)\\
&\quad + \int_t^T \mathcal{B}^\Lambda(s-t)[F(\mathcal{S}_\delta^\Lambda(s)) - F(\mathcal{S}^\Lambda(s))]\mathrm{d}s.
\end{aligned}
$$

Using Lemma (3.1) and that F satisfies the global Lipschitz condition, we obtain

$$
\begin{aligned}
&\|U_\delta^\Lambda(t) - U^\Lambda(t)\|_{L^2(\Omega)}\\
&\le \|\mathcal{A}^\Lambda(T-t)(u_T^\delta - u_T)\|_{L^2(\Omega)} + \|\mathcal{B}^\Lambda(T-t)(\widetilde{u}_T^\delta - \widetilde{u}_T)\|_{L^2(\Omega)}\\
&\quad + \Big\|\int_t^T \mathcal{B}^\Lambda(s-t)[F(\mathcal{S}_\delta^\Lambda(s)) - F(\mathcal{S}^\Lambda(s))]\mathrm{d}s\Big\|_{L^2(\Omega)}\\
&\le \|\mathcal{A}^\Lambda(T-t)\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))}\|u_T^\delta - u_T\|_{L^2(\Omega)}\\
&\quad + \|\mathcal{B}^\Lambda(T-t)\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))}\|\widetilde{u}_T^\delta - \widetilde{u}_T\|_{L^2(\Omega)}\\
&\quad + \int_t^T \|\mathcal{B}^\Lambda(s-t)\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))}\|F(\mathcal{S}_\delta^\Lambda(s)) - F(\mathcal{S}^\Lambda(s))\|_{L^2(\Omega)}\mathrm{d}s\\
&\le T_a\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{\frac{T-t}{T}}\|u_T^\delta - u_T\|_{L^2(\Omega)} + T_a\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{\frac{T-t}{T}}\|\widetilde{u}_T^\delta - \widetilde{u}_T\|_{L^2(\Omega)}\\
&\quad + \int_t^T TK_F\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{\frac{s-t}{T}}\|\mathcal{S}_\delta^\Lambda(s) - \mathcal{S}^\Lambda(s)\|_{[L^2(\Omega)]^2}\mathrm{d}s\\
&\le \Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{\frac{T-t}{T}} T_a\Big(\|u_T^\delta - u_T\|_{L^2(\Omega)} + \|\widetilde{u}_T^\delta - \widetilde{u}_T\|_{L^2(\Omega)}\Big)\\
&\quad + T_aK_F\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{-t/T}\int_t^T \Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{s/T}\|\mathcal{S}_\delta^\Lambda(s) - \mathcal{S}^\Lambda(s)\|_{[L^2(\Omega)]^2}\mathrm{d}s.
\end{aligned}
\tag{3.29}
$$

Similarly, one has

$$
\begin{aligned}
&\|V_\delta^\Lambda(t) - V^\Lambda(t)\|_{L^2(\Omega)}\\
&\le \Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{\frac{T-t}{T}} T_a\Big(\|v_T^\delta - v_T\|_{L^2(\Omega)} + \|\widetilde{v}_T^\delta - \widetilde{v}_T\|_{L^2(\Omega)}\Big)\\
&\quad + T_aK_G\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{-t/T}\int_t^T \Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{s/T}\|\mathcal{S}_\delta^\Lambda(s) - \mathcal{S}^\Lambda(s)\|_{[L^2(\Omega)]^2}\mathrm{d}s.
\end{aligned}
\tag{3.30}
$$

Combining (3.29) and (3.30) yields

$$
\begin{aligned}
&\|\mathcal{S}_\delta^\Lambda(t)-\mathcal{S}^\Lambda(t)\|_{[L^2(\Omega)]^2}\\
&\le \Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{\frac{T-t}{T}} T_a\Big(\|u_T^\delta-u_T\|_{L^2(\Omega)}+\|\widetilde{u}_T^\delta-\widetilde{u}_T\|_{L^2(\Omega)}\\
&\quad +\|v_T^\delta-v_T\|_{L^2(\Omega)}+\|\widetilde{v}_T^\delta-\widetilde{v}_T\|_{L^2(\Omega)}\Big)\\
&\quad + T_a(K_F+K_G)\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{-t/T}\int_t^T \Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{s/T}\|\mathcal{S}_\delta^\Lambda(s)-\mathcal{S}^\Lambda(s)\|_{[L^2(\Omega)]^2}\mathrm{d}s.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\|\mathcal{S}_\delta^\Lambda(t)-\mathcal{S}^\Lambda(t)\|_{[L^2(\Omega)]^2} &\le \Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{\frac{T-t}{T}} T_a\delta + 2T_a\mathbb{K}\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{-t/T}\\
&\quad\times\int_t^T \Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{s/T}\|\mathcal{S}_\delta^\Lambda(s)-\mathcal{S}^\Lambda(s)\|_{[L^2(\Omega)]^2}\mathrm{d}s,
\end{aligned}
\tag{3.31}
$$

where $\mathbb{K}:=\max\{K_F,K_G\}$. Multiplying both sides of (3.31) by $\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{t/T}$, we obtain

$$
\begin{aligned}
&\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{t/T}\|\mathcal{S}_\delta^\Lambda(t)-\mathcal{S}^\Lambda(t)\|_{[L^2(\Omega)]^2}\\
&\le \frac{T}{\Lambda\log(\frac{T}{\Lambda})}T_a\delta + 2T_a\mathbb{K}\int_t^T \Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{s/T}\|\mathcal{S}_\delta^\Lambda(s)-\mathcal{S}^\Lambda(s)\|_{[L^2(\Omega)]^2}\mathrm{d}s.
\end{aligned}
$$

Then Gronwall's inequality yields

$$
\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{t/T}\|\mathcal{S}_\delta^\Lambda(t)-\mathcal{S}^\Lambda(t)\|_{[L^2(\Omega)]^2}\le e^{2T_a\mathbb{K}(T-t)}\frac{T}{\Lambda\log(\frac{T}{\Lambda})}T_a\delta.
$$

From this we have

$$
\|\mathcal{S}_\delta^\Lambda(t)-\mathcal{S}^\Lambda(t)\|_{[L^2(\Omega)]^2}\le e^{2T_a\mathbb{K}(T-t)}\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{1-\frac{t}{T}}T_a\delta. \tag{3.32}
$$

Step 2. Estimate of $\|\mathcal{S}^\Lambda(t)-\mathcal{S}(t)\|_{[L^2(\Omega)]^2}$. First, we note that

$$
\begin{aligned}
\mathcal{H}^\Lambda u(x,t) &= \mathcal{H}^\Lambda\mathcal{A}(T-t)u_T-\mathcal{H}^\Lambda\mathcal{B}(T-t)\widetilde{u}_T-\int_t^T\mathcal{H}^\Lambda\mathcal{B}(s-t)F(\mathcal{S})(s)\mathrm{d}s\\
&= \mathcal{A}^\Lambda(T-t)u_T-\mathcal{B}^\Lambda(T-t)\widetilde{u}_T-\int_t^T\mathcal{B}^\Lambda(s-t)F(\mathcal{S})(s)\mathrm{d}s.
\end{aligned}
\tag{3.33}
$$

Thanks to the triangle inequality, we conclude that

$$
\|U^\Lambda(t)-u(t)\|_{L^2(\Omega)}\le \underbrace{\|U^\Lambda(t)-\mathcal{H}^\Lambda u(t)\|_{L^2(\Omega)}}_{\mathbb{I}}+\underbrace{\|u(t)-\mathcal{H}^\Lambda u(t)\|_{L^2(\Omega)}}_{\mathbb{II}}. \tag{3.34}
$$

One has

$$
\begin{aligned}
|I| &\le \int_t^T\|\mathcal{B}^\Lambda(s-t)\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))}\|[F(\mathcal{S}^\Lambda)(s)-F(\mathcal{S})(s)]\|_{L^2(\Omega)}\mathrm{d}s\\
&\le K_FT_a\int_t^T\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{\frac{s-t}{T}}\|\mathcal{S}^\Lambda(s)-\mathcal{S}(s)\|_{L^2(\Omega)}\mathrm{d}s.
\end{aligned}
\tag{3.35}
$$

Now we have

$$\begin{aligned}|II|^2 &= \|(1-\mathcal{H}^\Lambda)u(t)\|^2_{L^2(\Omega)}\\ &= \sum_{p=1}^\infty \big[1-(1+\Lambda C_p e^{C_pT})^{-1}\big]^2 \langle u(x,t),\xi_p\rangle^2_{L^2(\Omega)}\\ &= \sum_{p=1}^\infty \Lambda^2 C_p^2\Big[\frac{1}{e^{-C_pT}+\Lambda C_p}\Big]^2 \langle u(x,t),\xi_p\rangle^2_{L^2(\Omega)}\\ &= \sum_{p=1}^\infty \Big[\frac{e^{-C_pt}}{e^{-C_pT}+\Lambda C_p}\Big]^2 \Lambda^2 C_p^2 e^{2C_pt}\langle u(x,t),\xi_p\rangle^2_{L^2(\Omega)}.\end{aligned} \tag{3.36}$$

Similarly to (3.8), we infer that

$$\begin{aligned}\frac{e^{-C_pt}}{e^{-C_pT}+\Lambda C_p} &= \frac{e^{-C_pt}}{[e^{-C_pT}+\Lambda C_p]^{t/T}[e^{-C_pT}+\Lambda C_p]^{1-t/T}}\\ &\le \frac{1}{[e^{-C_pT}+\Lambda C_p]^{1-t/T}}\\ &\le \Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{1-t/T}.\end{aligned} \tag{3.37}$$

Using (3.37) in (3.36), we obtain

$$\begin{aligned}|II|^2 &\le \sum_{p=1}^\infty \Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{2-2t/T}\Lambda^2 C_p^2 e^{2C_pt}\langle u(x,t),\xi_p\rangle^2_{L^2(\Omega)}\\ &\le \Lambda^{2t/T}\Big[\frac{T}{\log(\frac{T}{\Lambda})}\Big]^{2-2t/T}\sum_{p=1}^\infty 4a^2\lambda_p^{2\gamma}e^{4a\lambda_p^\gamma T}\langle u(x,t),\xi_p\rangle^2_{L^2(\Omega)}\\ &= 4a^2\Lambda^{2t/T}\Big[\frac{T}{\log(\frac{T}{\Lambda})}\Big]^{2-2t/T}\|u(\cdot,t)\|^2_{\mathbb{G}^\gamma_{2,2aT}}.\end{aligned} \tag{3.38}$$

Consequently,

$$|II| \le 2a\Lambda^{t/T}\Big[\frac{T}{\log(\frac{T}{\Lambda})}\Big]^{1-t/T}\|u(\cdot,t)\|_{\mathbb{G}^\gamma_{2,2aT}}. \tag{3.39}$$

Combining (3.34), (3.35) and (3.39), we deduce that

$$\begin{aligned}\|U^\Lambda(t)-u(t)\|_{L^2(\Omega)} &\le K_F T_a\int_t^T \Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{\frac{s-t}{T}}\|\mathcal{S}^\Lambda(s)-\mathcal{S}(s)\|_{[L^2(\Omega)]^2}\mathrm{d}s\\ &\quad + 2a\Lambda[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}]^{1-t/T}\|u(\cdot,t)\|_{\mathbb{G}^\gamma_{2,2aT}}.\end{aligned} \tag{3.40}$$

Multiplying both sides of (3.40) by $\big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\big]^{t/T}$, we obtain

$$\begin{aligned}&\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{t/T}\|U^\Lambda(t)-u(t)\|_{L^2(\Omega)}\\ &\le \frac{2aT}{\log(\frac{T}{\Lambda})}\|u(\cdot,t)\|_{\mathbb{G}^\gamma_{2,2aT}}\\ &\quad + K_F T_a\int_t^T \Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{s/T}\|\mathcal{S}^\Lambda(s)-\mathcal{S}(s)\|_{[L^2(\Omega)]^2}\mathrm{d}s.\end{aligned} \tag{3.41}$$

Similarly,

$$
\begin{aligned}
&\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{t/T}\|V^\Lambda(t)-v(t)\|_{L^2(\Omega)}\\
&\le \frac{2aT}{\log(\frac{T}{\Lambda})}\|v(\cdot,t)\|_{\mathbb{G}^\gamma_{2,2aT}}\\
&\quad + K_G T_a\int_t^T\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{s/T}\|\mathcal{S}^\Lambda(s)-\mathcal{S}(s)\|_{[L^2(\Omega)]^2}\mathrm{d}s.
\end{aligned}
\tag{3.42}
$$

From (3.41) and (3.42), we have

$$
\begin{aligned}
&\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{t/T}\|\mathcal{S}^\Lambda(t)-\mathcal{S}(t)\|_{[L^2(\Omega)]^2}\\
&\le \frac{2aT}{\log(\frac{T}{\Lambda})}\|\mathcal{S}(\cdot,t)\|_{[\mathbb{G}^\gamma_{2,2aT}]^2}+2\mathbb{K}T_a\int_t^T\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{s/T}\|\mathcal{S}^\Lambda(s)-\mathcal{S}(s)\|_{[L^2(\Omega)]^2}\mathrm{d}s.
\end{aligned}
$$

Applying Gronwall's inequality, we obtain

$$
\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{t/T}\|\mathcal{S}^\Lambda(t)-\mathcal{S}(t)\|_{[L^2(\Omega)]^2}\le e^{2T_a\mathbb{K}(T-t)}\ \frac{2aT}{\log(\frac{T}{\Lambda})}\|\mathcal{S}(\cdot,t)\|_{[\mathbb{G}^\gamma_{2,2aT}]^2}.
$$

It follows that

$$
\|\mathcal{S}^\Lambda(t)-\mathcal{S}(t)\|_{[L^2(\Omega)]^2}\le 2a\Lambda e^{2T_a\mathbb{K}(T-t)}\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{1-\frac{t}{T}}\|\mathcal{S}(\cdot,t)\|_{[\mathbb{G}^\gamma_{2,2aT}]^2}. \tag{3.43}
$$

Combining (3.32) and (3.43), we conclude that

$$
\begin{aligned}
&\|\mathcal{S}_\delta^\Lambda(t)-\mathcal{S}(t)\|_{[L^2(\Omega)]^2}\\
&\le 2a\Lambda e^{2T_a\mathbb{K}(T-t)}\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{1-\frac{t}{T}}\|\mathcal{S}(\cdot,t)\|_{[\mathbb{G}^\gamma_{2,2aT}]^2}+e^{2T_a\mathbb{K}(T-t)}\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{1-\frac{t}{T}}T_a\delta.
\end{aligned}
$$

This completes the proof. □

4. Locally Lipschitz source functions

In the rest of this paper, for solving System (1.1), we concentrate on the case of locally Lipschitz functions. In many ways, the locally Lipschitz functions are more natural. For example, $h(u)=u^2,u^3,u\sin u$, etc, are locally Lipschitz functions but not globally ones. Results for the locally Lipschitz case are still very scarce. The local Lipschitz condition (coercive-type)

$$
\|h(u_1,v_1)-h(u_2,v_2)\|_{L^2(\Omega)}\le K(\mathcal{R})(\|u_1-u_2\|_{L^2(\Omega)}+\|v_1-v_2\|_{L^2(\Omega)}),
$$

for $\|u_1\|_{L^2(\Omega)},\|u_2\|_{L^2(\Omega)},\|v_1\|_{L^2(\Omega)},\|u_2\|_{L^2(\Omega)}\le\mathcal{R}$. The conditions hold for the following source.

Example. Let

$$
h_1(u,v)=u\|u\|^2_{L^2(\Omega)}+v\|v\|^2_{L^2(\Omega)}.
$$

By direct computations, we obtain

$$
\begin{aligned}
&\|h_1(u_1,v_1)-h_1(u_2,v_2)\|_{L^2(\Omega)}\\
&=\|u_1\|u_1\|^2_{L^2(\Omega)}-u_2\|u_2\|^2_{L^2(\Omega)}+v_1\|v_1\|^2_{L^2(\Omega)}-v_2\|v_2\|^2_{L^2(\Omega)}\|_{L^2(\Omega)}\\
&\le\Big\|\|u_1\|^2_{L^2(\Omega)}(u_1-u_2)+u_2\Big(\|u_1\|^2_{L^2(\Omega)}-\|u_2\|^2_{L^2(\Omega)}\Big)\Big\|_{L^2(\Omega)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \big\|\|v_1\|^2_{L^2(\Omega)}(v_1-v_2) + v_2\Big(\|v_1\|^2_{L^2(\Omega)} - \|v_2\|^2_{L^2(\Omega)}\Big)\big\|_{L^2(\Omega)} \\
&\le \|u_1\|^2_{L^2(\Omega)}\|u_1-u_2\|_{L^2(\Omega)} + \|u_2\|_{L^2(\Omega)}\big(\|u_1\|_{L^2(\Omega)} + \|u_2\|_{L^2(\Omega)}\big) \\
&\quad \times \big|\|u_1\|_{L^2(\Omega)} - \|u_2\|_{L^2(\Omega)}\big| + \|v_1\|^2_{L^2(\Omega)}\|v_1-v_2\|_{L^2(\Omega)} \\
&\quad + \|v_2\|_{L^2(\Omega)}\big(\|v_1\|_{L^2(\Omega)} + \|v_2\|_{L^2(\Omega)}\big)\big|\|v_1\|_{L^2(\Omega)} - \|v_2\|_{L^2(\Omega)}\big| \\
&\le \Big(\|u_1\|^2_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)}\|u_2\|_{L^2(\Omega)} + \|u_2\|^2_{L^2(\Omega)}\Big)\|u_1-u_2\|_{L^2(\Omega)} \\
&\quad + \Big(\|v_1\|^2_{L^2(\Omega)} + \|v_1\|_{L^2(\Omega)}\|v_2\|_{L^2(\Omega)} + \|v_2\|^2_{L^2(\Omega)}\Big)\|v_1-v_2\|_{L^2(\Omega)}.
\end{aligned}
$$

It is easy to check that h_1 is not global Lipschitz. Let $\mathcal{R} > 0$. For each u_1, u_2, v_1, v_2 such that $\mathcal{R} \ge \max\{\|u_1\|_{L^2(\Omega)}, \|u_2\|_{L^2(\Omega)}, \|v_1\|_{L^2(\Omega)}, \|v_2\|_{L^2(\Omega)}\}$, we can choose $K(\mathcal{R}) = 3\mathcal{R}^2$.

However, this is not satisfied in many cases, e.g. $h_2(u,v) = a(u+v) - b(u^3+v^3)$, $(a, b > 0)$. Hence, we have to find another regularization method to study the problem with the locally Lipschitz source which is similar to the latter source. We assume that the functions $F, G : \overline{D_T} \times \mathbb{R}^2 \to \mathbb{R}$, are locally Lipschitz i.e., for each $\mathcal{R} > 0$, there exists $K_F(\mathcal{R}), K_G(\mathcal{R}) > 0$ such that for all $(x,t) \in D_T$, we have

$$
\begin{aligned}
|F(x,t;u_1;v_1) - F(x,t;u_2;v_2)| &\le K_F(\mathcal{R})(|u_1-u_2| + |v_1-v_2|), \\
|G(x,t;u_1;v_1) - G(x,t;u_2;v_2)| &\le K_G(\mathcal{R})(|u_1-u_2| + |w_1-w_2|),
\end{aligned} \tag{4.1}
$$

if $u_i, v_i \in \overline{\mathbb{S}}(\mathcal{R})$, $i = 1,2$, where $\overline{\mathbb{S}}(\mathcal{R})$ is the closed ball in $L^2(\Omega)$ of center zero and radius $\mathcal{R}$, and

$$
\begin{aligned}
K_F(\mathcal{R}) := \sup_{(x,t)\in D_T} \Big\{ \Big|\frac{F(x,t;u_1;v_1) - F(x,t;u_2;v_2)}{|u_1-u_2| + |v_1-v_2|}\Big| \\
: (u_1,v_1) \ne (u_2,v_2),\ u_i, w_i \in \overline{\mathbb{S}}(\mathcal{R}),\ i = 1,2 \Big\} < \infty, \\
K_G(\mathcal{R}) := \sup_{(x,t)\in D_T} \Big\{ \Big|\frac{G(x,t;u_1;v_1) - G(x,t;u_2;v_2)}{|u_1-u_2| + |v_1-v_2|}\Big| \\
: (u_1,v_1) \ne (u_2,v_2),\ u_i, v_i \in \overline{\mathbb{S}}(\mathcal{R}),\ i = 1,2 \Big\} < \infty.
\end{aligned}
$$

Notice that $K_F(\mathcal{R}), K_G(\mathcal{R})$ are increasing. The main idea is to approximate the locally Lipschitz functions F, G by the sequences $\mathcal{F}_{\mathcal{R}^\delta}, \mathcal{G}_{\mathcal{R}^\delta}$ of globally Lipschitz functions:

$$
\mathcal{F}_{\mathcal{R}^\delta}(x,t;u;v) := F(x,t;\widetilde{u};\widetilde{v}), \quad \mathcal{G}_{\mathcal{R}^\delta}(x,t;u;v) := G(x,t;\widetilde{u};\widetilde{v}), \tag{4.2}
$$

where

$$
\widetilde{w} := \begin{cases} -\mathcal{R}^\delta, & \text{if } w \in (-\infty, -\mathcal{R}^\delta), \\ w, & \text{if } w \in [-\mathcal{R}^\delta, \mathcal{R}^\delta], \\ \mathcal{R}^\delta, & \text{if } w \in (\mathcal{R}^\delta, +\infty). \end{cases} \tag{4.3}
$$

Here, the term $\mathcal{R}^\delta$ is positive and depends on δ and satisfies $\lim_{\delta\to 0} \mathcal{R}^\delta = +\infty$. Moreover, for δ sufficiently small, we have

$$
\mathcal{R}^\delta \ge \sup_{(x,t)\in\overline{D_T}} \Big(|u(x,t)| + |v(x,t)|\Big).
$$

This implies immediately

$$
\mathcal{F}_{\mathcal{R}^\delta}(x,t;u;v) := F(x,t;u;v), \quad \mathcal{G}_{\mathcal{R}^\delta}(x,t;u;v) := G(x,t;u;v). \tag{4.4}
$$

Remark 4.1. The locally Lipschitz constants K_F, K_G depend on δ. It is also interesting that $\mathcal{R}^\delta$ is chosen suitable in order to obtain a convergence rate (our purpose is to improve the rate of convergence).

Before presenting the main results, we need to some auxiliary results. We do not claim that these auxiliary results are new, but for completeness of the presentation we give their proofs here.

Lemma 4.2. *Let $\mathcal{F}_{\mathcal{R}^\delta}, \mathcal{G}_{\mathcal{R}^\delta} \in L^\infty(\overline{D_T} \times \mathbb{R}^2)$ given as in (4.2). Then we have*

$$|\mathcal{F}_{\mathcal{R}^\delta}(x,t;u_1;v_1) - \mathcal{F}_{\mathcal{R}^\delta}(x,t;u_2;v_2)| \le K_F(\mathcal{R}^\delta)(|u_1-u_2| + |v_1-v_2|), \tag{4.5}$$

$$|\mathcal{G}_{\mathcal{R}^\delta}(x,t;u_1;v_1) - \mathcal{G}_{\mathcal{R}^\delta}(x,t;u_2;v_2)| \le K_G(\mathcal{R}^\delta)(|u_1-u_2| + |v_1-v_2|), \tag{4.6}$$

for any $(x,t) \in D_T$, $u_i, v_i \in \mathbb{R}$, $i = 1,2$.

Proof. First, we show that for any $w_1, w_2 \in \mathbb{R}$ and $\widetilde{w}_1, \widetilde{w}_2$ satisfying (4.3) then

$$|\widetilde{w}_1 - \widetilde{w}_2| \le |w_1 - w_2|. \tag{4.7}$$

The proof (4.7) is divided into three cases.

Case 1. $w_1 < -\mathcal{R}^\delta$.

- ▷ If $w_2 < -\mathcal{R}^\delta$ then $|\widetilde{w}_1 - \widetilde{w}_2| = 0$.
- ▷ If $-\mathcal{R}^\delta \le w_2 \le \mathcal{R}^\delta$ then $|\widetilde{w}_1 - \widetilde{w}_2| = w_2 + \mathcal{R}^\delta < w_2 - w_1 = |w_1 - w_2|$.
- ▷ If $w_2 > \mathcal{R}^\delta$ then $|\widetilde{w}_1 - \widetilde{w}_2| = 2\mathcal{R}^\delta \le w_2 - w_1 = |w_1 - w_2|$.

Case 2. $-\mathcal{R}^\delta \le w_1 \le \mathcal{R}^\delta$.

- ▷ If $w_2 < -\mathcal{R}^\delta$ then $|\widetilde{w}_1 - \widetilde{w}_2| = |w_1 + \mathcal{R}^\delta| = w_1 + \mathcal{R}^\delta < w_1 - w_2 = |w_1 - w_2|$.
- ▷ If $-\mathcal{R}^\delta \le w_2 \le \mathcal{R}^\delta$ then $|\widetilde{w}_1 - \widetilde{w}_2| = |w_1 - w_2|$.
- ▷ If $w_2 > \mathcal{R}^\delta$ then $|\widetilde{w}_1 - \widetilde{w}_2| = \mathcal{R}^\delta - w_1 \le w_2 - w_1 = |w_1 - w_2|$.

Case 3. $w_1 > \mathcal{R}^\delta$.

- ▷ If $w_2 < -\mathcal{R}^\delta$ then $|\widetilde{w}_1 - \widetilde{w}_2| = 2\mathcal{R}^\delta < w_1 - w_2 = |w_1 - w_2|$.
- ▷ If $-\mathcal{R}^\delta \le w_2 \le \mathcal{R}^\delta$ then $|\widetilde{w}_1 - \widetilde{w}_2| = \mathcal{R}^\delta - w_2 < w_1 - w_2 = |w_1 - w_2|$.
- ▷ If $w_2 > \mathcal{R}^\delta$ then $|\widetilde{w}_1 - \widetilde{w}_2| = 0 \le |w_1 - w_2|$.

Summarizing the above discussions, we arrive at (4.7). Now we return to the proof of Lemma 4.2. Since $\widetilde{u}$, $\widetilde{v} \le \mathcal{R}^\delta$ and using (4.1), we have

$$\begin{aligned}
|\mathcal{F}_{\mathcal{R}^\delta}(x,t;u_1;v_1) - \mathcal{F}_{\mathcal{R}^\delta}(x,t;u_2;v_2)| &= |F(x,t;\widetilde{u}_1;\widetilde{v}_1) - F(x,t;\widetilde{u}_2;\widetilde{v}_2)| \\
&\le K_F(\mathcal{R}^\delta)(|\widetilde{u}_1 - \widetilde{u}_2| + |\widetilde{v}_1 - \widetilde{v}_2|) \\
&\le K_F(\mathcal{R}^\delta)(|u_1 - u_2| + |v_1 - v_2|),
\end{aligned}$$

where we have used (4.7) in the last estimate. We use a similar argument to ensure the local Lipschitzian condition f the function $\mathcal{G}_{\mathcal{R}^\delta}$. □

We first consider a perturbed model yielding a well-posed system whose solution will approximate u, v. In particular, we define the approximate system

$$\begin{aligned}
I_\delta^\Lambda(x,t) &= \mathcal{A}^\Lambda(T-t)u_T^\delta(x) + \mathcal{B}^\Lambda(T-t)\widetilde{u}_T^\delta(x) \\
&\quad + \int_t^T \mathcal{B}^\Lambda(\tau - t)\mathcal{F}_{\mathcal{R}^\delta}(I_\delta^\Lambda(x,\tau), J_\delta^\Lambda(x,\tau))\mathrm{d}\tau, \\
J_\delta^\Lambda(x,t) &= \mathcal{A}^\Lambda(T-t)v_T^\delta(x) + \mathcal{B}^\Lambda(T-t)\widetilde{v}_T^\delta(x) \\
&\quad + \int_t^T \mathcal{B}^\Lambda(\tau - t)\mathcal{G}_{\mathcal{R}^\delta}(I_\delta^\Lambda(x,\tau), J_\delta^\Lambda(x,\tau))\mathrm{d}\tau.
\end{aligned} \tag{4.8}$$

Our principal result, based on the analysis above, is then the following theorem.

Theorem 4.3. *Let $m \geq 1$, $n \geq 2aT$ and $\Lambda := \Lambda(\delta)$ be as in Theorem 3.3. Assume that the system (2.16) has a unique solution $\mathcal{S} := (u, v) \in [C([0,T]; L^2(\Omega)) \cap L^\infty(0,T; \mathbb{G}^\gamma_{m,n})]^2$ which satisfies*

$$\|\mathcal{S}(t)\|_{[C([0,T];L^2(\Omega))]^2} + \|\mathcal{S}(t)\|_{[L^\infty(0,T;\mathbb{G}^\gamma_{m,n})]^2} \leq Q. \tag{4.9}$$

Assuming that we can choose a sequence $\mathcal{R}^\delta > 0$ such that $\lim_{\delta\to 0^+} \mathcal{R}^\delta = \infty$ and

$$\mathbb{K}(\mathcal{R}^\delta) \leq \frac{\eta}{TT_a} \log[\log(\Lambda^{-1})], \quad \textit{for some } \eta \in (0, \frac{1}{2}), \tag{4.10}$$

where, $\mathbb{K}(\mathcal{R}^\delta) = \max\{K_F(\mathcal{R}^\delta), K_G(\mathcal{R}^\delta)\}$, and T_a is defined in Lemma 3.1.

Suppose that System (4.8) has a unique solution $(I^\Lambda_\delta, J^\Lambda_\delta) \in [C([0,T]; L^2(\Omega))]^2$. Then, the error estimate between the solution $(I^\Lambda_\delta, J^\Lambda_\delta) \in [C([0,T]; L^2(\Omega))]^2$ of problem (4.8) and the sought solution (u, v) to (1.1) is given by

$$\|Z^\Lambda_\delta(t) - \mathcal{S}(t)\|_{[L^2(\Omega)]^2} \leq \left(2aQ + T_a\delta\Lambda^{-1}\right) \log^{2\eta}(\Lambda^{-1}) \Big[\frac{T}{\log(\frac{T}{\Lambda})}\Big]^{1-\frac{t}{T}} \Lambda^{t/T}, \tag{4.11}$$

where $Z^\Lambda_\delta := (I^\Lambda_\delta, J^\Lambda_\delta) \in [C([0,T]; L^2(\Omega))]^2$.

Remark 4.4. In (4.11), if $t > 0$, the error estimate is of order $\Lambda^{t/T}[\frac{T}{\log(\frac{T}{\Lambda})}]^{2\eta+\frac{t}{T}-1}$ which tends to zero as $\delta \to 0^+$. And if $t = 0$, the error (4.11) becomes

$$\|Z^\Lambda_\delta(0) - \mathcal{S}(0)\|_{[L^2(\Omega)]^2} \leq C\Big[\frac{T}{\log(\frac{T}{\Lambda})}\Big]^{2\eta-1}, \quad 0 < \eta < \frac{1}{2}. \tag{4.12}$$

We also note that the term $[\frac{T}{\log(\frac{T}{\Lambda})}]^{2\eta-1}$, $\eta \in (0, \frac{1}{2})$ tends to zero when $\delta \to 0^+$. From above observations, we conclude that the right-hand side in estimation (4.11) tends to zero for all $t \in [0,T]$.

Proof. First, we note that the proof of the existence and uniqueness of the solution to problem (4.8) is the same as in Theorem 3.2. Next, we denote $Z^\Lambda := (I^\Lambda, J^\Lambda)$ the solution of system (4.8) with exact data $(u_T, \widetilde{u}_T)$ and $(v_T, \widetilde{v}_T)$. We know that

$$\|Z^\Lambda_\delta(t) - \mathcal{S}(t)\|_{[L^2(\Omega)]^2} \leq \|Z^\Lambda_\delta(t) - Z^\Lambda(t)\|_{[L^2(\Omega)]^2} + \|Z^\Lambda(t) - \mathcal{S}(t)\|_{[L^2(\Omega)]^2}. \tag{4.13}$$

By an argument analogous to that used for the proof of Theorem 3.3, we emphasize that the proof in Step 1 and Step 2 of Theorem 3.3 remains valid. Also, replace the globally Lipschitz conditions (3.17) and (3.18) by the locally Lipschitz conditions (4.5) and (4.6), respectively. Since $\lim_{\delta\to 0^+} \mathcal{R}^\delta = \infty$, for a sufficiently small $\delta > 0$, there is an $\mathcal{R}^\delta > 0$ such that $\mathcal{R}^\delta \geq \|\mathcal{S}\|_{[L^\infty(0,T;L^2(\Omega))]^2}$. For this value of $\mathcal{R}^\delta$ (from (4.3)) we have

$$\mathcal{F}_{\mathcal{R}^\delta}(x, t; u; v) = F(x, t; u; v), \quad \mathcal{G}_{\mathcal{R}^\delta}(x, t; u; v) = G(x, t; u; v). \tag{4.14}$$

Using the global Lipschitz property of $\mathcal{F}_{\mathcal{R}^\delta}, \mathcal{G}_{\mathcal{R}^\delta}$ (see Lemma 4.2), yields

$$\|Z^\Lambda_\delta(t) - Z^\Lambda(t)\|_{[L^2(\Omega)]^2} \leq \exp\Big(2T_a\mathbb{K}(\mathcal{R}^\delta)(T - t)\Big) \Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{1-\frac{t}{T}} T_a\delta. \tag{4.15}$$

Also, one has

$$\|Z^\Lambda(t) - \mathcal{S}(t)\|_{[L^2(\Omega)]^2} \leq 2a\Lambda \exp\Big(2T_a\mathbb{K}(\mathcal{R}^\delta)(T - t)\Big) \Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{1-\frac{t}{T}} Q. \tag{4.16}$$

Combining (4.15) and (4.16), we obtain

$$\|Z_\delta^\Lambda(t) - \mathcal{S}(t)\|_{[L^2(\Omega)]^2} \le \big(2aQ + T_a\delta\Lambda^{-1}\big) \exp\Big(2T_a\mathbb{K}(\mathcal{R}^\delta)(T-t)\Big)\Big[\frac{T}{\Lambda\log(\frac{T}{\Lambda})}\Big]^{1-\frac{t}{T}}.$$

Using (4.10) in this inequality, estimate (4.11) follows. □

Conclusion. In this article, we showed that the inverse backward system (1.1) is ill-posed in the sense of Hadamard. To stabilize the solution, we developed a regularization method based on the filtering method for which a stability estimate of logarithmic type is established in the cases the source terms F, G are global and local Lipschitz reactions.

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Nguyen Huy Tuan

Applied Analysis Research Group Faculty of Mathematics and Statistics Ton Duc Thang University Ho Chi Minh City, Vietnam

E-mail address: `nguyenhuytuan@tdt.edu.vn`

Vo Van Au
Faculty of General Sciences, Can Tho University of Technology, Can Tho City, Vietnam
E-mail address: vvau@ctuet.edu.vn

Nguyen Huu Can
Faculty of Mathematics and Computer Science, University of Science, Vietnam National University, (VNU-HCMC), Ho Chi Minh City, Vietnam
E-mail address: nguyenhuucan@gmail.com

Mokhtar Kirane
LaSIE, Faculté des Sciences, Pôle Sciences et Technologies, Université de La Rochelle, Avenue M. Crepeau, 17042 La Rochelle Cedex, France.
NAAM Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: mkirane@univ-lr.fr