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# STABILITY OF GROUND STATES FOR A NONLINEAR PARABOLIC EQUATION

LUCA BISCONTI, MATTEO FRANCA

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ABSTRACT. We consider the Cauchy-problem for the parabolic equation

$$u_t = \Delta u + f(u, |x|),$$

where $x \in \mathbb{R}^n$, $n > 2$, and $f(u, |x|)$ is either critical or supercritical with respect to the Joseph-Lundgren exponent. In particular, we improve and generalize some known results concerning stability and weak asymptotic stability of positive ground states.

## 1. Introduction

In this article we discuss the stability properties of positive radial solutions of the equation

$$\Delta u + f(u, |x|) = 0, \tag{1.1}$$

where $x \in \mathbb{R}^n$, $n > 2$, and $f = f(u, |x|)$ is a potential (which is null for $u = 0$) superlinear in $u$, and supercritical in a sense that will be specified just below. Such solutions correspond to the positive steady states of the following Cauchy problem

$$u_t = \Delta u + f(u, |x|), \tag{1.2}$$

$$u(x, 0) = \phi(x), \tag{1.3}$$

where $\phi$ is the initial value.

Let $u(x, t; \phi)$ be the solution of (1.2)–(1.3). The analysis of the long time behavior of $u(x, t; \phi)$ is strongly based on the separation properties of the radial solutions of (1.1). If $u(x) = U(|x|)$ is a radial solutions of (1.1), we find that $U = U(r)$ solves

$$U'' + \frac{n-1}{r} U' + f(U, r) = 0\,, \tag{1.4}$$

where "$'$" denotes the derivative with respect to $r$. In the whole paper we denote by $U(r, \alpha)$ the unique solution of (1.4) with the initial condition $U(0, \alpha) = \alpha > 0$.

In the previous decades the Cauchy problem (1.2)–(1.3) has raised a great interest in the mathematical community, starting from the model case $f(u, |x|) = u^{q-1}$, and it has been analyzed by several authors (see, e.g., [4, 12, 21, 22, 23, 31, 32, 33, 35, 36]).

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Since in the whole paper we are interested in positive solutions, there is no ambiguity in using the notation $u^{q-1}$. It is well known that the behavior of solutions of (1.4), and consequently of (1.2), changes drastically as $q$ passes through some critical values. Here we focus on the case where $q > 2^* := 2n/(n-2)$, so that for any $\alpha > 0$ the solution $U(r, \alpha)$ of (1.4) is positive and bounded for any $r > 0$, i.e. it is a Ground State (GS), and especially in the case $q \geq \sigma^*$, where

$$\sigma^* := \begin{cases} \frac{(n-2)^2-4n+8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n > 10, \\ +\infty & \text{if } n \leq 10, \end{cases} \tag{1.5}$$

so that GSs gain some stability properties (see [36]). Let us recall that $2^*$ is the Sobolev critical exponent, while $\sigma^*$ is the Joseph-Lundgren exponent (see [26]).

When $2^* < q < \sigma^*$ all the GSs intersect each other infinitely many times, and this fact is used to construct suitable sub- and super-solutions for (1.1). Then, it is possible to show that, in this range of parameters, GSs determine the threshold between solutions of (1.2) that blow up in finite time, and solutions that exist for any time $t$ and fade away.

**Theorem 1.1** ([36, 21])**.** *Assume $f(u, r) = u^{q-1}$, $2^* < q < \sigma^*$.*

1. *If there is $\alpha > 0$ such that $\phi(x) \gneqq U(|x|, \alpha)$, then there is $T(\phi)$ such that $\lim_{t\to T(\phi)^-} \|u(t, x; \phi)\|_\infty = +\infty$.*
2. *If there is $\alpha > 0$ such that $\phi(x) \lneqq U(|x|, \alpha)$, then $\lim_{t\to+\infty} \|u(t, x; \phi)\|_\infty = 0$.*

On the other hand, when $q \geq \sigma^*$, GSs are well ordered, and gain some stability properties as we will see below. In fact, already in [36], the whole argument was generalized to embrace the so called Henon-equation, i.e. when $f(u, r) = r^\delta u^{q-1}$, and $\delta > -2$. In this case there is a shift in the critical exponents, so we find convenient to introduce the following parameters (see Section 2 below, see also [4] for more details) which will be widely used through this article:

$$l_s := 2\frac{q+\delta}{2+\delta} \quad \text{and} \quad m(l_s) := \frac{2}{l_s-2} = \frac{2+\delta}{q-2}. \tag{1.6}$$

In this context, the previous discussion is still valid, but we have stability whenever $l_s \geq \sigma^*$, and we lose it for $2^* < l_s < \sigma^*$ (see [36]). Notice that $l_s$ reduces to $q$ for $\delta = 0$. In both cases the GSs, $U$, decay as $U(r) \sim U(r, +\infty) = P_1 r^{-m(l_s)}$ for $r \to +\infty$, and $U(r, +\infty)$ is the unique singular solution of (1.4).

To clarify the notion of stability we will use in the sequel, we recall the definition of the following weighted norms (see, e.g., [21]), i.e.

$$\|\psi\|_\lambda := \sup_{x\in\mathbb{R}^n} |(1 + |x|^\lambda)\psi(x)|,$$

$$|||\psi|||_\lambda := \sup_{x\in\mathbb{R}^n} \Big|\frac{(1 + |x|^\lambda)}{[\ln(2 + |x|)]}\psi(x)\Big|,$$

where $\psi$ is continuous, $\lambda \in \mathbb{R}$, and $k \in \mathbb{N}$.

**Definition 1.2.** We say that a GS, $U(|x|) = U(|x|, \alpha)$, is *stable* with respect to the norm $\|\cdot\|_\lambda$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that, when $\|\varphi - U\|_\lambda < \delta$, we have $\|u(\cdot, t, \varphi) - U(|\cdot|)\|_\lambda < \epsilon$ for all $t > 0$.

Further, we say that $U(|x|)$ is *weakly asymptotically stable* with respect to $\|\cdot\|_\lambda$ when $U(|x|)$ is stable with respect to $\|\cdot\|_\lambda$, and there exists $\delta > 0$ such that $\|u(\cdot, t, \varphi) - U(|\cdot|)\|_{\lambda'} \to 0$ as $t \to \infty$ for all $\lambda' < \lambda$, if $\|\varphi - U\|_\lambda < \delta$.

Let us consider the quadratic equation in $\lambda$,

$$\lambda^2 + \Big(n-2-\frac{4}{q-2}\Big)\lambda + 2\Big(n-2-\frac{2}{q-2}\Big) = 0. \tag{1.7}$$

This problem admits two real and negative solutions, say $\lambda_2 \le \lambda_1 < 0$ if and only if $q \ge \sigma^*$, and they coincide if and only if $q = \sigma^*$.

Gui et al [21] proved the following theorem.

**Theorem 1.3** ([36, 21]). *Assume $f(u,r) = u^{q-1}$, $q \ge \sigma^*$. Let $\lambda_2 \le \lambda_1$ be the roots of equation* (1.7).

(1) *If $q > \sigma^*$ any GS $U(r,\alpha)$ is stable with respect to the norm $\|\cdot\|_{m(q)+|\lambda_1|}$ and weakly asymptotically stable with respect to the norm $\|\cdot\|_{m(q)+|\lambda_2|}$.*
(2) *If $q = \sigma^*$ any GS $U(r,\alpha)$ is stable with respect to $|\!|\!|\cdot|\!|\!|_{m(q)+|\lambda_1|}$ and weakly asymptotically stable with respect to the norm $\|\cdot\|_{m(q)+|\lambda_1|}$.*

There are several results aimed at extending the previous analysis to more general potentials $f = f(u,|x|)$ (see, e.g., [1, 10, 8, 38, 4]). In particular, the instability result given by Theorem 1.1, and the stability result Theorems 1.3, have been generalized also to the following equation

$$u_t = \Delta u + k(r) r^{\delta} u^{q-1}, \quad \text{where } \delta > -2 \text{ and } r = |x| \tag{1.8}$$

assuming $k(r)$ decreasing, uniformly positive and bounded, in the cases $l_s > \sigma^*$ (see [10]), and $l_s = \sigma^*$ (see [8]). In particular, these hypotheses imply that the singular radial solution $U(r,+\infty)$ of (1.1) behaves like $r^{-m(l_s)}$ both as $r \to 0$ and as $r \to +\infty$.

In such a case $q$ is replaced by $l_s$ and also the values of $\lambda_1, \lambda_2$ change accordingly, i.e. they solve

$$\lambda^2 + \Big(n-2-2\frac{2+\delta}{q-2}\Big)\lambda + \frac{2+\delta}{q-2}\Big(n-2-\frac{2+\delta}{q-2}\Big) = 0. \tag{1.9}$$

In [4] we proposed a unifying approach which allows to extend Theorem 1.1 to a more general class of nonlinearities $f$, including (1.8), but also more involved dependence on $u$.

The goal of this paper is to continue the analysis of [4], extending the stability results proved in Theorem 1.3 to a class of potentials $f = f(u,|x|)$ larger than the one considered therein. This purpose is achieved with an approach obtained through the combination of the main ideas in [36, 21, 10], techniques borrowed from the theory of non-autonomous dynamical systems (see [25, 4]), along with the use of some new arguments.

As far as (1.8) is concerned we are able to drop the assumption of boundedness on $k$, replacing it by the following:

$$k(r) \sim r^{-\eta}, \quad \text{as } r \to 0 \text{ with } 0 \le \eta < 2+\delta. \tag{1.10}$$

Then, we can allow two different behaviors for singular and slow decay solutions (see [4]), namely: $U(r) \sim r^{-m(l_s)}$ as $r \to +\infty$ and $U(r) \sim r^{-m(l_u)}$ as $r \to 0$, where

$$l_u = 2\frac{q+\delta-\eta}{2+\delta-\eta} \quad \text{and} \quad m(l_u) = \frac{2+\delta-\eta}{q-2}. \tag{1.11}$$

We prove the following result.

**Theorem 1.4.** *Let $f(u,r)$ be as in (1.8), where $k(r) \in C^1$ satisfies (1.10), is decreasing, and $\lim_{r\to+\infty} k(r) > 0$. Then*

(1) *If $l_s > \sigma^*$ any GS $U(r,\alpha)$ is stable with respect to the norm $\|\cdot\|_{m(l_s)+|\lambda_1|}$ and weakly asymptotically stable with respect to the norm $\|\cdot\|_{m(l_s)+|\lambda_2|}$.*
(2) *If $l_s = \sigma^*$ any GS $U(r,\alpha)$ is stable with respect to $|||\cdot|||_{m(l_s)+|\lambda_1|}$ and weakly asymptotically stable with respect to the norm $\|\cdot\|_{m(l_s)+|\lambda_1|}$.*

Our approach is flexible enough to consider $f = f(u,|x|)$ as a finite sum of powers in $u$, i.e.

$$f(u,|x|) = k_1(|x|)r^{\delta_1}|u|^{q_1-1} + k_2(|x|)r^{\delta_2}|u|^{q_2-1}, \tag{1.12}$$

where $q_1 < q_2$, $k_i = k_i(|x|)$, $i = 1,2$, are supposed to be $C^1$ (see Theorems 3.1 and 3.2 below).

Equation (1.12) has been already considered by Yang and Zhang [38], but just in the particular situation of $k_1(r) = k_2(r) \equiv 1$. We emphasize that, even if it is not explicitly stated, in [38] it is required that

$$\frac{2+\delta_2}{q_2-2}(q_2-q_1) + \delta_1 < 0, \tag{1.13}$$

which excludes the relevant case $\delta_1 = \delta_2 = 0$. With these assumptions, Yang and Zhang were able to prove Theorem 1.3-(1), replacing $q$ by $l_s = 2\frac{q_2+\delta_2}{2+\delta_2}$, and changing the values of $m(l_s)$ and of $\lambda_i$ accordingly. We stress that condition (1.13) is in fact needed to use the approach of [38], and also for the schema proposed in this paper; See the discussion after Lemma 2.11 for more details on this point. However we believe that (1.13) is just a technical requirement and that it might be removed with the approach used by Bae and Naito in [2].

As a consequence of our main results we are able to generalize the results in [38] and to prove Theorem 1.3, allowing $k_i$ to depend on $r$, and even to be unbounded, i.e.

$$k_1(r) \sim r^{-\eta_1} \quad \text{and} \quad k_2(r) \sim r^{-\eta_2}, \quad \text{as} \quad r \to 0, \tag{1.14}$$

with $0 \le \eta_i < 2 + \delta_i$, $i = 1,2$. However we still need to require (1.13).

**Theorem 1.5.** *Let $f(u,r)$ be as in (1.12), and assume (1.13), and (1.14). Suppose that both $k_1(r)r^{\frac{2+\delta_2}{q_2-2}(q_2-q_1)+\delta_1}$ and $k_2(r)$ are decreasing, $k_1(r)$ is positive and $k_2(r)$ is uniformly positive. Then, setting $l_s = 2\frac{q_2+\delta_2}{2+\delta_2}$, we obtain the same conclusions as in Theorem 1.4.*

Notice that we can deal with non-monotone functions $k_1(r)$. Under, these assumptions we are able to prove Theorem 1.3–(2) which is new even in the case $k_1(r) = k_2(r) \equiv 1$ considered in [38].

The main ingredients to obtain our results on (1.2) are the separation and the asymptotic properties of GSs. The separation properties are a result of independent interest, and generalize the ones obtained in [9, Theorems 1,2], [37, Theorem 2]. As a consequence we also get Proposition 2.13, which gives an insight on the behavior of the singular solution of (1.4), which seems to play a key role in determining the threshold between blowing up and fading solutions (see the Introduction in [36]).

To prove weak asymptotic stability, we need a suitable asymptotic expansion for GSs, which refines and generalizes the ones of [10, 38] (see Proposition 2.16, below). In fact in [10, 38] the highly nontrivial proof relies on an iterative scheme developed by [36] in a simpler (and still nontrivial) context. Here, we followed a different idea:

in fact we proved an asymptotic results for nonlinear systems of ODEs, which seems to be new to the best of our knowledge, and that, in our opinion, is of intrinsic mathematical interest (even for systems of ODEs). In this more general framework the statement assumes a more comprehensible aspect, and the proof is simplified, even if it is still quite cumbersome. So to keep the technical analytic machinery to the minimum, we rely on the Appendix of reference [5] for a detailed proof of this result.

Now, we briefly review some results which have been proved just in the setting of Theorems 1.1, 1.3. First, using some sub- and super-solutions constructed on the self-similar solutions, [22, 28] proved that $U(|x|, \alpha)$ is weakly asymptotically stable in the norm $\| \cdot \|_\lambda$ for any $m(q) + \lambda_1 < \lambda < m(q) + \lambda_2 + 2$. Further Naito in [28] showed that this result is optimal, i.e. in this range asymptotic stability does not hold. Moreover Gui et al. in [22] proved that GSs are not even stable if we use too coarse, but surprisingly also too fine norms, namely for $\lambda < m(q) + \lambda_1$ and for $\lambda \geq n$. Notice that we have stability for $\lambda = m(q) + \lambda_1$, but still there is a small gap for $m(q) + \lambda_2 + 2 < n$. Similarly the null solution is weakly asymptotically stable if $m(q) \leq \lambda < n$ and unstable otherwise, [22].

Moreover, in a series of papers (see [11, 27, 28]) the authors showed that the speed of convergence of solutions $u(t, x; \phi)$ depends linearly on the weight used to measure the distance with respect to the GS. Namely if $\|\phi(x) - U(|x|, \alpha)\|_\lambda$ is small enough then $t^\nu \|u(t, x; \phi) - U(|x|, \alpha)\|_{\lambda'}$ is bounded for any $t > 0$, where $\nu = \frac{1}{2} \max\{\lambda - \lambda', \lambda - m(q) - \lambda_1\}$, whenever $m(q) + \lambda_1 < \lambda < m(q) + \lambda_2 + 2$ and $0 < \lambda' < \lambda$.

If either the assumptions of Theorem 1.1 or of Theorem 1.3 are satisfied, following [4] we can construct a family of sub-solutions $\phi$ for (1.4) with arbitrarily small $L^\infty$-norm and decaying like $r^{2-n}$ for $r$ large, and such that the solution $u(t, x, \phi)$ blows up in finite time. This type of behavior contradicts the idea that the decay of the singular solution, i.e. $r^{-m(q)}$, is the critical one to determine the threshold between fading and blowing up solutions: The situation is indeed more intricate. This result in fact extends to more general non-linearities $f = f(u, |x|)$ (see [4]).

To conclude, we recall that when the non-linearity $f(u, r)$ becomes unbounded as $r \to 0$, in general it is not possible to find classical solutions of (1.2)–(1.3). However it is still possible to obtain mild solutions assuming that $f(u, r)r^\ell$ is bounded for a certain $0 < \ell < 2$, and in fact the solutions $u$ are classical for $x \neq 0$ and $t > 0$, and they are $C^{\alpha, \alpha/2}$ also for $x = 0$ and $t = 0$ for any $\alpha \in (0, 2 - \ell)$. For an exhaustive exposition about such a topic we refer to [36] (see also [4]).

This article is organized as follows: In Section 2 we collect all the preliminary results concerning the solutions of (1.4). We prove ordering properties and asymptotic estimates for positive solutions of such a problem. Section 3 is devoted to the proof of the main results of the paper (from which Theorems 1.4 and 1.5 follow directly).

## 2. Ordering results and asymptotic estimates for the stationary problem

In this section we give some preliminary results which are crucial for our analysis. These results are obtained by applying the Fowler transformation to (1.4). To this end we introduce the following quantities that will appear frequently in the whole paper, i.e.

$$m(l)=\frac{2}{l-2}\,,\quad A(l)=n-2-2m(l)\,,\quad B(l)=m(l)[n-2-m(l)], \tag{2.1}$$

where $l>2$ is a parameter (which is related to $l_s$ and $l_u$, in (1.6) and in (1.11), respectively) whose role will be explained few lines below. Set

$$\begin{gathered} r=\mathrm{e}^s\,,\quad y_1(s,l)=U(\mathrm{e}^s)\mathrm{e}^{m(l)s}\,,\quad y_2(s,l)=\dot{y}_1(s,l), \\ g(y_1,s;l)=f(y_1\mathrm{e}^{-m(l)s},\mathrm{e}^s)\mathrm{e}^{(m(l)+2)s}\,. \end{gathered} \tag{2.2}$$

In what follows with "$\cdot$" we will denote the differentiation with respect to $s$ (recall that "$'$" indicates differentiation with respect to $r$). Using these transformations we pass from (1.4) to the system

$$\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\end{pmatrix}=\begin{pmatrix}0 & 1\\ B(l) & -A(l)\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}-\begin{pmatrix}0\\ g(y_1,s;l)\end{pmatrix}. \tag{2.3}$$

Here and in the sequel, we write

$$\mathbf{y}(s,\tau;\mathbf{Q};\bar{l})=\big(y_1(s,\tau;\mathbf{Q};\bar{l}),y_2(s,\tau;\mathbf{Q};\bar{l})\big) \tag{2.4}$$

to denote a trajectory of (2.3), where $l=\bar{l}$, evaluated at $s$ and starting from $\mathbf{Q}\in\mathbb{R}^2$ for $s=\tau$.

Assume first $f(u,r)=r^\delta u^{q-1}$ and set $l=2\frac{q+\delta}{2+\delta}$, so that (2.3) reduces to the autonomous system

$$\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\end{pmatrix}=\begin{pmatrix}0 & 1\\ B(l) & -A(l)\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}-\begin{pmatrix}0\\ (y_1)^{q-1}\end{pmatrix}. \tag{2.5}$$

In this case we passed from a singular non-autonomous ODE to an autonomous system from which the singularity has been removed. Also note that when $\delta=0$ we can simply take $l=q$. The sign of the constants $A(l)$, $B(l)$ defined in (2.1) determines whether the system is sub- or supercritical, if there are slow decay solutions ($B(l)\geq 0$) or if they do not exist ($B(l)<0$).

**Remark 2.1.** Under the previous assumptions, as $r\to 0$, positive solutions $U(r)$ of (1.4) have two possible behaviors: *Regular*, i.e. $\lim_{r\to 0}U(r)=\alpha>0$, or *Singular*, i.e. $\lim_{r\to 0}U(r)=+\infty$.

Similarly, as $r\to+\infty$, we either have $\lim_{r\to+\infty}U(r)r^{n-2}=\beta>0$ and we say that $U(r)$ has *fast decay*, or $\lim_{r\to+\infty}U(r)r^{n-2}=+\infty$ and we say that $U(r)$ has *slow decay*.

In fact, the behavior of singular and slow decay solutions can be specified better, see Proposition 2.9 below), and Proposition 2.16.

In this article we restrict the whole discussion to the case $l>2^*$. Therefore $A(l)>0$ and $B(l)>0$. System (2.5) admits three critical points for $l>2^*$: The origin $O=(0,0)$, $\mathbf{P}=(P_1,0)$ and $-\mathbf{P}$, where $P_1=[B(l)]^{1/(q-2)}>0$. The origin is a saddle point and admits a one-dimensional $C^1$ stable manifold $\overline{M}^s$ and a one-dimensional $C^1$ unstable manifold $\overline{M}^u$, see Figure 1. The origin splits $\overline{M}^u$ in two relatively open components: We denote by $M^u$ the component which leaves the origin and enters into the semi-plane $y_1\geq 0$. Since we are just interested in positive solutions, with a slight abuse of notation, we will refer to $M^u$ as the unstable manifold.

**Remark 2.2.** The critical point $\mathbf{P}$ of (2.5) is a stable focus if $2^* < l < \sigma^*$ and a stable node if $l \geq \sigma^*$.

As a consequence of some asymptotic estimates we deduce the following useful fact (see, e.g. [17, 16]).

**Remark 2.3.** Let $u(r)$ be a solution of (1.4) and let $\mathbf{Y}(s;l)$ be the corresponding trajectory for the system (2.5), with $l > 2^*$. Then $u(r)$ is regular (respectively has fast decay) if and only if $\mathbf{Y}(s;l)$ converges to the origin as $s \to -\infty$ (resp. as $s \to +\infty$), $u(r)$ is singular (respectively has slow decay) if and only if $\mathbf{Y}(s;l)$ converges to $\mathbf{P}$ as $s \to -\infty$ (resp. as $s \to +\infty$).

Using the Pohozaev identity introduced in [30], and adapted to this context in [14], we can draw a picture of the phase portrait of (2.5) (see Figure 1 below) and deduce information on positive solutions of (1.4). Then it is not hard to classify positive solutions: In the supercritical case ($l > 2^*$) all the regular solutions are GSs with slow decay, and there is a unique Singular Ground State (SGS) with slow decay.

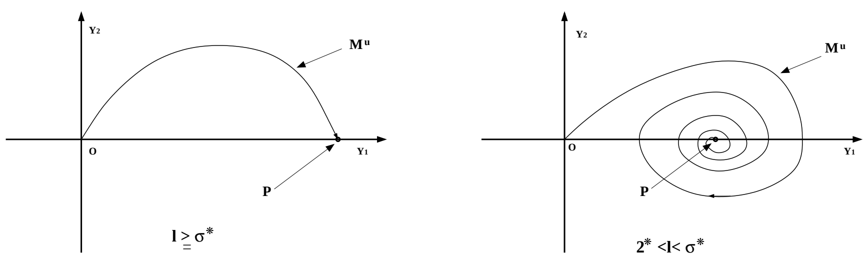


FIGURE 1. Sketches of the phase portrait of (2.3), for $q > 2$ fixed.

We stress that all the previous arguments concerning the autonomous Equation (2.3) still hold for any autonomous super-linear system (2.3). More precisely, whenever $g(y_1, s; l) \equiv g(y_1; l)$ and $g(y_1; l)$ has the following property, denoted by (A0) (see [15] for a proof in the general $p$-Laplace context, see also [4]).

- (A0) There is $l > 2^*$ such that $g(0; l) = 0 = \partial_{y_1} g(0, l)$ and $\partial_{y_1} g(y_1, l)$ is a positive strictly increasing function for $y_1 > 0$ and $\lim_{y_1 \to +\infty} \partial_{y_1} g(y_1, l) = +\infty$.

When (A0) holds, we denote by $P_1$ the unique positive solution in $y_1$ of $g(y_1; l) = B(l) y_1$. Hence $(P_1, 0)$ is again a critical point for (2.5). Further, we let $\sigma_* < \sigma^*$ be the real solutions of the equation in $l$ given by

$$A(l)^2 - 4\big[\partial_y g(P_1, l) - B(l)\big] = 0, \tag{2.6}$$

which reduces to $A(l)^2 - 4(q-2)B(l) = 0$ for $g(y_1) = (y_1)^{q-1}$. We emphasize that when $f(u, r) = u^{q-1}$ the value of $\sigma^*$ coincides with the one given in (1.5). Notice that Remarks 2.2, 2.3 continue to hold in this slightly more general context (see [16, 17]).

2.1. **Main assumptions and preliminaries.** We collect below the assumptions used in our main results.

(A1) There is $l_u \geq \sigma^*$ such that for any $y_1 > 0$ the function $g(y_1, s; l_u)$ converges to a $s$-independent $C^1$ function $g(y_1, -\infty; l_u) \not\equiv 0$ as $s \to -\infty$, uniformly on compact intervals. The function $g(y_1, s; l_u)$ satisfies (A0) for any $s \in \mathbb{R}$. Further, there is $\varpi > 0$ such that $\lim_{s\to-\infty} e^{-\varpi s}\partial_s g(y_1, s; l_u) = 0$.

(A2) There is $l_s \geq \sigma^*$ such that for any $y_1 > 0$ the function $g(y_1, s; l_s)$ converges to a $s$-independent $C^1$ function $g(y_1, +\infty, l_s) \not\equiv 0$ as $s \to +\infty$, uniformly on compact intervals. The function $g(y_1, s; l_s)$ satisfies (A0) for any $s \in \mathbb{R}$. Further, there is $\varpi > 0$ such that $\lim_{s\to+\infty} e^{+\varpi s}\partial_s g(y_1, s; l_s) = 0$.

(A3) Condition (A2) holds and $g(y_1, s; l_s)$ and $\partial_{y_1} g(y_1, s; l_s)$ are decreasing in $s$ for any $y_1 > 0$.

(A4) Condition (A2) is verified with $\varpi = \gamma$ satisfying

$$g(P_1^+, s; l_s) = g(P_1^+, +\infty; l_s) + ce^{-\gamma s} + o(e^{-\gamma s})$$

for a certain $c \neq 0$.

(A5) Either $f$ is as in (1.8) or $f$ is as in (1.12) and satisfies (1.13).

Assumptions (A1), (A2) are used to ensure that the phase portrait of (2.3) converges to an autonomous system of the form (2.5) (with $l \geq \sigma^*$), respectively as $s \to \pm\infty$.

Instead, (A3) is needed to prove the ordering properties of positive solutions and generalizes the condition required in [10].

Assumption (A4) is used to derive asymptotic estimates on slow decay solutions of (1.4), and it gives back the standard requirement when $f(u, r) = k(r)u^{q-1}$, i.e. $k(r) = k(\infty) + cr^{-\gamma} + o(r^{-\gamma})$ (see [10]). Condition (A4) is assumed for definiteness and may be weakened, at the price of some additional cumbersome technicalities.

Finally, condition (A5) is just a technical requirement we are not able to avoid, which is in fact implicitly assumed also in [38]. It implies that there is $c > 0$ such that

$$B(l_s) = \frac{g(P_1^+, +\infty; l_s)}{P_1^+} = c|P_1^+|^{\overline{q}-2} = \frac{\partial_{y_1} g(P_1^+, +\infty; l_s)}{\overline{q} - 1} \tag{2.7}$$

with $\overline{q} = q$ in the case of (1.8), and $\overline{q} = q_2$ for the potential (1.12).

**Remark 2.4.** Observe that (A1) and (A2) are satisfied, e.g., in the following cases:

- For equation (1.8) with $k$ satisfying (1.10): $l_s$ and $l_u$ are as in (1.6) and (1.11), respectively.
- When $f$ is as in (1.12) and (1.14) holds: In this case $l_s$ is as in Theorem 1.5, i.e. $l_s = \min\{2\frac{q_i+\delta_i}{2+\delta_i} : i = 1, 2\}$, while $l_u = \max\{2\frac{q_i+\delta_i-\eta_i}{2+\delta_i-\eta_i} : i = 1, 2\}$. We also emphasize that, if we consider (1.12), then (1.13) amounts to ask for $2\frac{q_2+\delta_2}{2+\delta_2} \leq 2\frac{q_1+\delta_1}{2+\delta_1}$; so (A5) is not satisfied if $\delta_i = \eta_i = 0$, since we find $l_s = q_1 < q_2 = l_u$.

**Lemma 2.5.** *Assume* (A2) *and* (A3). *Then we have the following condition*

(A6) *The function* $G(y_1, s; 2^*) := \int_0^{y_1} g(a, s; 2^*)da$ *is decreasing in* $s$ *for any* $y_1 > 0$ *strictly for some* $s$.

*Proof.* Set $G(z, s, l_s) = \int_0^z g(a, s, l_s)da$, $H(z, s) = G(z, s, l_s)/z$. Then

$$G(z, s, l_s) = \int_0^z \frac{g(a, s, l_s)}{a} a da \leq \frac{g(z, s, l_s)}{z} \int_0^z a da = \frac{zg(z, s, l_s)}{2}$$

Therefore $zg - G \geq zg - 2G \geq 0$. Since $\partial_z H = (zg - G)/z^2$, then $H(z, s)$ is increasing in $z$ and decreasing in $s$ for (A3). Hence

$$G(y_1, s, 2^*) = G(y_1 e^{-\delta s}, s, l_s)e^{\delta s} = H(y_1 e^{-\delta s}, s)y_1 ,$$

so we conclude that $G(y_1, s, 2^*)$ is decreasing in $s$. □

Observe that (A6) means that the system is supercritical with respect to $2^*$, and this ensures the existence of GSs for (1.4) (see e.g. [4, Proposition 2.12]). In the sequel, in some cases, it will be convenient to use (A6) along with (A2), in place of the combination of (A2) and (A3). In fact, the first couple of requirements is slightly weaker than the second.

2.2. **Stationary problem: spatial dependent case.** Now we consider (2.3) in the $s$-dependent case. The first step is to extend invariant manifold theory to the non-autonomous setting.

Assume (A1). We introduce the following 3-dimensional autonomous system, obtained from (2.3) by adding the extra variable $z = \mathrm{e}^{\varpi t}$, i.e.,

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ B(l_u) & -A(l_u) & 0 \\ 0 & 0 & \varpi \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ g(y_1, \frac{\ln(z)}{\varpi}; l_u) \\ 0 \end{pmatrix}. \tag{2.8}$$

Similarly if (A2) is satisfied we set $l = l_s$ and $\zeta(t) = \mathrm{e}^{-\varpi t}$ and we consider

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ B(l_s) & -A(l_s) & 0 \\ 0 & 0 & \varpi \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \zeta \end{pmatrix} - \begin{pmatrix} 0 \\ g(y_1, -\frac{\ln(\zeta)}{\varpi}; l_s) \\ 0 \end{pmatrix}. \tag{2.9}$$

The technical assumptions at the end of (A1), (A2) are needed in order to ensure that the systems are smooth respectively for $z = 0$ and $\zeta = 0$.

We recall that if a trajectory of (2.3) does not cross the coordinate axes indefinitely then it is continuable for any $s \in \mathbb{R}$ (see e.g. [16, Lemma 3.9], [7]). Consider (2.8) (respectively (2.9)) each trajectory corresponding to a definitively positive solution $u(r)$ of (1.4) is such that its $\alpha$-limit set is contained in the $z = 0$ plane (respectively its $\omega$-limit set is contained in the $\zeta = 0$ plane). Moreover such a plane is invariant and the dynamics reduced to $z = 0$ (respectively, $\zeta = 0$) coincides with the one of the autonomous system (2.3) where $g(y_1, s; l_u) \equiv g(y_1, -\infty; l_u)$ (respectively, $g(y_1, s; l_s) \equiv g(y_1, +\infty; l_s)$).

Note that the origin of (2.8) admits a 2-dimensional unstable manifold $\mathbf{W^u}(l_u)$ which is transversal to $z = 0$, and a 1-dimensional stable manifold $M^s$ contained in $z = 0$.

Following [18] (see also [25]), for any $\tau \in \mathbb{R}$ we have that

$$W^u(\tau; l_u) = \mathbf{W^u}(l_u) \cap \{z = \mathrm{e}^{\varpi\tau}\} \quad \text{and} \quad W^u(-\infty; l_u) = \mathbf{W^u}(l_u) \cap \{z = 0\}$$

are 1-dimensional immersed manifolds, i.e. the graph of $C^1$ regular curves. Moreover, they inherit the same smoothness as (2.8) and (2.9), that is: Let $K$ be a segment which intersects $W^u(\tau_0; l_u)$ transversally in a point $\mathbf{Q}(\tau_0)$ for $\tau_0 \in [-\infty, +\infty)$, then there is a neighborhood $I$ of $\tau_0$ such that $W^u(\tau; l_u)$ intersects $K$ in a point $\mathbf{Q}(\tau)$ for any $\tau \in I$, and $\mathbf{Q}(\tau)$ is as smooth as (2.8).

Since we need to compare $W^u(\tau; l_u)$ and $W^s(\tau; l_s)$, we introduce the following manifolds:

$$W^u(\tau; l_s) := \big\{\mathbf{R} = \mathbf{Q} \exp\big(\big(m(l_s) - m(l_u)\big)\tau\big) \in \mathbb{R}^2 : \mathbf{Q} \in W^u(\tau; l_u)\big\}. \tag{2.10}$$

Note that $W^u(\tau; l_u)$ and $W^u(\tau; l_s)$ are homothetic, since they are obtained from each other simply multiplying by an exponential scalar. However, if $l_u > l_s$, $W^u(\tau; l_s)$ becomes unbounded as $\tau \to -\infty$.

To deal with bounded sets, we also define the following manifold which will be useful in Section 3, i.e.

$$W^u(\tau;l_*) := \begin{cases} W^u(\tau;l_u) & \text{if } \tau \le 0 \\ W^u(\tau;l_s) & \text{if } \tau \ge 0 \end{cases}, \quad \xi(\tau) := \begin{cases} z(\tau) & \text{if } \tau \le 0 \\ 2-\zeta(\tau) & \text{if } \tau \ge 0 \end{cases} \tag{2.11}$$

and

$$\mathbf{W}^u(l_*) := \big\{(\mathbf{Q}, \xi(\tau)) \mid \mathbf{Q} \in W^u(\tau;l_*)\big\}.$$

The sets $W^u(\tau;l_u)$ may be constructed also using the argument of [6, §13], simply requiring that (2.3) is $C^1$ in $\mathbf{y}$ uniformly with respect to $t$ for $t \le \tau$ in a fixed neighborhood of the origin. With this second method we see that the tangent space to $W^u(\tau;l_u)$ is simply the unstable space of the linearization of (2.3) in the origin, and we obtain the following.

**Remark 2.6.** Assume (A1). Then, in the origin $W^u(\tau;l_u)$ is tangent to the line $y_2 = m(l_u)y_1$, for any $\tau \in \mathbb{R}$. Since $W^u(\tau;l_u)$, $W^u(\tau;l_s)$ and $W^u(\tau;l_*)$ are homothetic, they are all tangent to $y_2 = m(l_u)y_1$ in the origin.

As in the $s$-independent case, we see that the *regular solutions* correspond to the trajectories in $W^u$ (see [18, 16]). More precisely, from [16, Lemma 3.5], we obtain the following result.

**Lemma 2.7.** *Assume* (A1), (A2). *Consider the trajectory* $\mathbf{y}(s,\tau,\mathbf{Q};l_u)$ *of* (2.3) *with* $l=l_u$, *the corresponding trajectory* $\mathbf{y}(t,\tau,\mathbf{R};l_s)$ *of* (2.3) *with* $l=l_s$ *and let* $u(r)$ *be the corresponding solution of* (1.4). *Then* $\mathbf{R} = \mathbf{Q}\exp[(m(l_s)-m(l_u))\tau]$.

*Further* $u(r)$ *is a regular solution if and only if* $\mathbf{Q} \in W^u(\tau;l_u)$ *or equivalently* $\mathbf{R} \in W^u(\tau;l_s)$.

Now, we consider singular and slow decay solutions of (1.4). Let $P_1^-$, $P_1^+$ be the unique positive solutions in $y_1$ respectively of $B(l_u)y_1 = g(y_1,-\infty;l_u)$ and of $B(l_s)y_1 = g(y_1,+\infty;l_s)$, and set $\mathbf{P}^\pm = (P_1^\pm, 0)$. Then, it follows that $(\mathbf{P}^-,0)$ and $(\mathbf{P}^+,0)$ are respectively critical points of (2.8) and (2.9).

If $l_u \ge 2^*$, then $(\mathbf{P}^-,0)$ admits a 1-dimensional exponentially unstable manifold, transversal to $z=0$ (the graph of a trajectory which will be denoted by $\mathbf{y}^*(s,*;l_u)$) for system (2.8), while if $l_s > 2^*$ then $(\mathbf{P}^+,0)$ is stable for (2.9), so it admits a 3-dimensional stable manifold (an open set).

From [4, Proposition 2.12] we find the following proposition.

**Proposition 2.8** ([4])**.** *Assume* (A1), (A2), (A6). *Then, all the regular solutions* $U(r,\alpha)$ *of* (1.4) *are GSs with slow decay, there is a unique singular solution, say* $U(r,\infty)$, *and it is a SGS with slow decay.*

**Proposition 2.9** ([4])**.** *Assume* (A1), (A2). *Then if* $u(r)$ *and* $v(r)$ *are respectively a singular and a slow decay solution of* (1.4) *we have* $u(r)r^{m(l_u)} \to P_1^-$ *as* $r \to 0$ *and* $u(r)r^{m(l_s)} \to P_1^+$ *as* $r \to +\infty$.

2.3. **Separation properties of stationary solutions.** In this subsection we adapt the argument of [10] and of [38] to obtain separation properties of (1.4). We begin by the following Lemma which is rephrased from [37, Theorem 4.1], which is a slight adaption of [10, Lemma 2.11]. We emphasize that condition (A5) is needed to prove estimate (2.16) below, and it is in fact implicitly required in [37, Theorem 4.1], even if it is not explicitly stated.

**Lemma 2.10.** *Assume* (A1), (A2), (A3), (A5). *Let* $\bar{\mathbf{y}}(s)$ *be the trajectory of* (2.3) *corresponding to the GS* $U(r,\alpha)$ *of* (1.4). *Then, for any* $s \in \mathbb{R}$ *we have* $\bar{y}_2(s) = \dot{\bar{y}}_1(s) \geq 0$, $0 < \bar{y}_1(s) < P_1^+$ *and*

$$g(\bar{y}_1(s), s; l_s) < B(l_s)\bar{y}_1(s) \tag{2.12}$$

*Proof.* Let us recall that all the regular solutions are GSs: This is a direct consequence of Proposition 2.8 and Lemma 2.7. Let $\bar{\mathbf{y}}(s; l_u) = \bar{\mathbf{y}}(s)\mathrm{e}^{(\alpha_{l_s} - \alpha_{l_u})s}$ be the corresponding trajectory of (2.3) where $l = l_u$, then, by standard facts in dynamical system theory, see [6], we see that there are $c_i > 0$ such that $\bar{y}_i(s; l_u)\mathrm{e}^{-\alpha_{l_u}s} \to c_i$ as $s \to -\infty$ for $i = 1, 2$. Hence $\bar{y}_i(s) \sim c_i \mathrm{e}^{\alpha_{l_s}s} \to 0$ as $s \to -\infty$ for $i = 1, 2$: So (2.12) is satisfied for $s \ll 0$.

Let us set

$$s_0 := \sup\big\{S \in \mathbb{R} \mid g(\bar{y}_1(s), s; l_s) < B(l_s)\bar{y}_1(s) \text{ for any } s < S\ \big\}, \tag{2.13}$$

so that (2.12) holds for $s < s_0$.

It follows that $\dot{\bar{y}}_2(s) + A(l_s)\bar{y}_2(s) > 0$ for $s < s_0$, hence $w(s) = \bar{y}_2(s)\mathrm{e}^{A(l_s)s}$ is increasing for $s < s_0$. Since $w(s) \to 0$ as $s \to -\infty$ we find that $\bar{y}_2(s) > 0$, for $s \leq s_0$.

Further, assume by contradiction that there is $\tilde{s} < s_0$ such that $\bar{y}_1(\tilde{s}) = P_1^+$. Then, from (A3) we have

$$g(\bar{y}_1(\tilde{s}), +\infty; l_s) \leq g(\bar{y}_1(\tilde{s}), \tilde{s}; l_s) < B(l_s)\bar{y}_1(\tilde{s}) = g(P_1^+, +\infty; l_s).$$

Since $g(\cdot, +\infty; l_s)$ is increasing we obtain $\bar{y}_1(\tilde{s}) < P_1^+$, and this gives an absurd conclusion. Thus, $0 < \bar{y}_1(s) < P_1^+$ for $s < s_0$.

Now, we show that $s_0 = +\infty$, so that (2.12) holds for any $s \in \mathbb{R}$ and the Lemma is proved. Assume by contradiction that $s_0 < +\infty$. Consider the curve $\bar{\mathbf{y}}(s) = (\bar{y}_1(s), \bar{y}_2(s))$ defined for $s \leq s_0$. Since $\bar{y}_2(s) = \dot{\bar{y}}_1(s) > 0$ for $s \leq s_0$, it follows that $\bar{\mathbf{y}}(s)$ is a graph on the $y_1$-axis, and we can parametrize it by $\bar{y}_1$. Hence, we set $Q(\bar{y}_1) := \dot{\bar{y}}_1(\bar{y}_1)$ so that $\bar{\mathbf{y}}(s)$ for $s \leq s_0$ and $\mathbf{\Gamma} := \mathbf{\Gamma}(y_1) = (y_1, Q(y_1))$ for $y_1 \in (0, \bar{y}_1(s_0)]$ are reparametrization of the same curve. As a consequence we have

$$\frac{\partial Q}{\partial \bar{y}_1} = \frac{\partial Q}{\partial s}\frac{\partial s}{\partial \bar{y}_1} = \frac{\ddot{\bar{y}}_1}{\dot{\bar{y}}_1} = -A(l_s) + \frac{B(l_s)\bar{y}_1 - g(\bar{y}_1, s; l_s)}{Q(\bar{y}_1)}. \tag{2.14}$$

In the phase plane, consider the line $r(\mu)$ passing through $\mathbf{R} = (\bar{y}_1(s_0), 0)$ with angular coefficient $-\mu$, i.e.

$$r(\mu) := \big\{(y_1, y_2) \mid y_2 = \mu(\bar{y}_1(s_0) - y_1)\big\}.$$

Since $\bar{y}_2(s_0) = \dot{\bar{y}}_1(s_0) > 0$, we see that $\mathbf{\Gamma}(\bar{y}_1(s_0)) = (\bar{y}_1(s_0), \bar{y}_2(s_0))$ lies above $\mathbf{R}$. By construction $r(\mu)$ intersects $\mathbf{\Gamma}$ at least in a point, for any $\mu > 0$: We denote by $\big(Y_1(\mu), \mu(\bar{y}_1(s_0) - Y_1(\mu))\big)$ the intersection with the smallest $Y_1$. Then, it follows that $Y_1 < \bar{y}_1(s_0)$ and $\frac{\partial Q}{\partial \bar{y}_1}(Y_1) \geq -\mu$. From these inequalities, along with (2.14), and using the fact that

$$B(l_s)\bar{y}_1(s_0) = g(\bar{y}_1(s_0), s_0; l_s) \tag{2.15}$$

we obtain

$$
\begin{aligned}
-\mu &\le \frac{\partial Q}{\partial \bar{y}_1}(Y_1) \\
&= -A(l_s) + \frac{B(l_s)[Y_1 - \bar{y}_1(s_0)] + [g(\bar{y}_1(s_0), s_0; l_s) - g(Y_1, s; l_s)]}{\mu[\bar{y}_1(s_0) - Y_1]} \\
&\le -A(l_s) - \frac{B(l_s)}{\mu} + \frac{g(\bar{y}_1(s_0), s_0; l_s) - g(Y_1, s_0; l_s)}{\mu[\bar{y}_1(s_0) - Y_1]} \\
&\le -A(l_s) + \frac{1}{\mu}\Big[-B + \partial_{y_1} g(C, s_0; l_s)\Big] \\
&\le -A + \frac{1}{\mu}\Big[-B + \frac{(\bar{q}-1)g(C, s_0; l_s)}{C}\Big] \\
&< -A + \frac{1}{\mu}\Big[-B + \frac{(\bar{q}-1)g(\bar{y}_1(s_0), s_0; l_s)}{\bar{y}_1(s_0)}\Big] = -A + \frac{B(\bar{q}-2)}{\mu}
\end{aligned}
\tag{2.16}
$$

where $C \in (Y_1, \bar{y}_1(s_0))$ and we used the mean value theorem. Further $\bar{q}$ stands for $q$ if $f$ is of type (1.8) and it stands for $q_2$ if $f$ is of type (1.12). Therefore, using (2.16) along with (2.15), we obtain

$$\mu^2 - A\mu + B(\bar{q}-2) = \mu^2 - A\mu - B + \partial_{y_1} g(P_1^+, +\infty, l_s) > 0, \quad \text{for any } \mu > 0.$$

But this is verified if and only if

$$A^2 - 4B(\bar{q}-2) = A^2 - 4[\partial_{y_1} g(P_1^+, +\infty, l_s) - B] < 0,$$

which is equivalent to $l_s \in (\sigma_*, \sigma^*)$, cf (2.6), so we have found a contradiction. Hence $s_0 = +\infty$. In particular, it follows that $\bar{y}_1(s) < P_1^+$, $\dot{\bar{y}}_1(s) > 0$, for any $s \in \mathbb{R}$, and (2.12) holds. □

**Lemma 2.11.** *Assume the hypotheses of Lemma 2.10 are verified. Also, assume that condition* (A5) *holds. Then*

$$\frac{\partial g}{\partial y_1}(\bar{y}_1(s), s; l_s) < \frac{\partial g}{\partial y_1}(P_1^+, +\infty; l_s) \tag{2.17}$$

*Proof.* From a straightforward computation we see that, when $f$ is as in (1.8), then (2.12) implies (2.17). When $f$ is as in (1.12),

$$\partial_{y_1} g(y_1, s, l_s) = (q_1-1)k_1(\mathrm{e}^s)y_1^{q_1-2} + (q_2-1)k_2(\mathrm{e}^s)y_1^{q_2-2} \le (q_2-1)g(y_1, s, l_s)/y_1.$$

So, let $\bar{\mathbf{y}}(s)$ be a trajectory corresponding to a GS of (1.4) as above; If (A5) holds, from (2.12) we obtain

$$
\begin{aligned}
\frac{\partial g}{\partial y_1}(\bar{y}_1, s, l_s) &\le (q_2-1)\frac{g(\bar{y}_1(s), s, l_s)}{\bar{y}_1(s)} \\
&\le (q_2-1)\frac{g(P_1^+, +\infty; l_s)}{P_1^+} \\
&\le \frac{\partial g}{\partial y_1}(P_1^+, +\infty; l_s),
\end{aligned}
\tag{2.18}
$$

so (2.17) follows and the proof is complete. □

We emphasize that Lemma 2.11 is already contained in the last lines of the proof of Lemma 2.10. We decided to restate and prove it in details because this is the only point where the assumption (A5) is explicitly required. Such a condition is in fact needed also in [38] where the stability problem for an $f$ of the form

$$f(u,r) = r^{\delta_1} u^{q_1-1} + r^{\delta_2} u^{q_2-2} \tag{2.19}$$

is discussed. Notice that (2.19) is a special case of (1.12) considered in this paper. In [38] condition (A5), i.e. (1.13), is omitted, but in fact it is needed to prove [38, Proposition 2.3]. To be more precise: In the important case $\delta_1 = \delta_2 = 0$, for which (1.13) does not hold, we find that $l_s = q_1$, and $m(l_s) = 2/(q_1 - 2)$. In general if (1.13) does not hold we find that $m(l_s)$ (hence the asymptotic behavior of positive solutions for $r$ large, which is $m_1$ in the notation of [38]) depends on $q_1$, while if (1.13) does not hold we find that $m(l_s)$ depends on $q_2$. Consequently in the former case we find $g(y_1, +\infty; l_s) = y_1^{q_1-2}$ and in the latter $g(y_1, +\infty; l_s) = y_1^{q_2-2}$. Hence, the last estimate in (2.18) holds in the latter case but not in the former. Analogously, in the proof of [38, Proposition 2.3] (in the last two lines of page 112), using the notation of [38] it is required that $p_1$ is the largest exponent, which is indeed equivalent to (1.13).

We believe that condition (A5), i.e. (1.13) is just technical, and that it might be removed using the methods introduced by Bae and Naito in [2].

**Proposition 2.12.** *Assume* (A1)–(A3), (A5)*. Then $U(r,\alpha_1) < U(r,\alpha_2)$ for any $r > 0$, whenever $\alpha_1 < \alpha_2$.*

We emphasize that if $g(y_1, s; l)$ is $s$-independent, as in [36], Lemma 2.10 implies Proposition 2.12. This fact follows directly by noticing that $M^u$ is a graph on the $y_1$-axis, since $y_1(s) = U(\mathrm{e}^s, \alpha)\mathrm{e}^{m(l_s)s}$ is increasing in $s$, for any $\alpha > 0$. In view of Lemma 2.10, we can parametrize the manifold $M^u$ by $\alpha$, then the ordering of the regular solutions $U(r,\alpha)$ is preserved as $s$ varies (i.e. as $r$ varies), since they all move along a 1-dimensional object.

When we turn to consider an $s$-dependent function $g(y_1, s; l)$, Proposition 2.12 needs a separate proof, which can be obtained by adapting the ideas developed in [10, 38]. In fact, in such a case $W^u(\tau; l_s)$ is still one dimensional but may not be a graph on the $y_1$-axis, so a priori we may lose the ordering property.

*Proof of Proposition 2.12.* Let us set $Q(s) = \mathrm{e}^{\lambda_1 s}$ and observe that

$$\ddot{Q} + A\dot{Q} + [\partial_{y_1} g(P_1^+, +\infty; l_s) - B]Q = 0. \tag{2.20}$$

Denote by $W(s) := [U(\mathrm{e}^s, \alpha_2) - U(\mathrm{e}^s, \alpha_1)]\mathrm{e}^{m(l_s)s}$, and observe that

$$\ddot{W} + A\dot{W} - BW + D(s) = 0, \tag{2.21}$$

where

$$D(s) := g(U(\mathrm{e}^s, \alpha_2)]\mathrm{e}^{m(l_s)s}, s; l_s) - g(U(\mathrm{e}^s, \alpha_1)]\mathrm{e}^{m(l_s)s}, s; l_s). \tag{2.22}$$

Using continuous dependence on initial data we see that $U(r,\alpha_2) > U(r,\alpha_1)$ for $r$ small enough, so that $D(s) > 0$ for $s \ll 0$. Assume by contradiction that there is $\bar{r} = \mathrm{e}^{\bar{s}} > 0$ such that $U(r,\alpha_2) - U(r,\alpha_1) > 0$ for $0 \le r < \bar{r}$, and $U(\bar{r},\alpha_2) - U(\bar{r},\alpha_1) = 0$. Then, $W(s)$, and $D(s)$ are positive for $s < \bar{s}$ and they are null for $s = \bar{s}$.

Setting $Z(s) := \dot{W}(s)Q(s) - W(s)\dot{Q}(s)$, by direct calculation we can easily see that $\dot{Z}(s) = \ddot{W}(s)Q(s) - W(s)\ddot{Q}(s)$. Then from (2.20) and (2.21) we obtain

$$\dot{Z} = -AZ(s) + Q(s)[\partial_{y_1} g(P_1^+, +\infty; l_s)W(s) - D(s)]. \tag{2.23}$$

Observe now that $W(s) \sim (\alpha_2 - \alpha_1)e^{m(l_s)s}$, as $s \to -\infty$, and also that

$$\dot{W}(s) = m(l_s)W(s) + [U'(e^s, \alpha_2) - U'(e^s, \alpha_1)]e^{[1+m(l_s)]s} \sim m(l_s)(\alpha_2 - \alpha_1)e^{m(l_s)s},$$

as $s \to -\infty$. Hence, we obtain

$$Z(s) \sim (m(l_s) - \lambda_1(l_s))(\alpha_2 - \alpha_1)(e^{[m(l_s)+\lambda_1(l_s)]s}) \to 0 \quad \text{as } s \to -\infty . \tag{2.24}$$

Moreover $\lambda_1(l_s) + A(l_s) = -\lambda_2(l_s) > 0$ and $D(s) \to 0$ as $s \to -\infty$, hence $e^{As}Q(s)D(s) \in L^1(-\infty, \bar{s}]$. Since $Z(s)$ is the unique solution of (2.23) satisfying (2.24) we find

$$Z(\bar{s}) = \int_{-\infty}^{\bar{s}} e^{-A(\bar{s}-s)}Q(s)[\partial_{y_1} g(P_1^+, +\infty; l_s)W(s) - D(s)]ds. \tag{2.25}$$

From the mean value theorem we find that (see (2.22))

$$\partial_{y_1} g(P_1^+, +\infty; l_s)W(s) - D(s) = [\partial_{y_1} g(P_1^+, +\infty; l_s) - \partial_{y_1} g(U(s), s; l_s)]W(s).$$

where $U(s)$ lies between $U(r, \alpha_1)r^{m(l_s)}$ and $U(r, \alpha_2)r^{m(l_s)}$.

Since $\partial_{y_1} g(y_1, s; l_s)$ is increasing in $y_1$, and using (2.17), for $s < \bar{s}$ we find

$$\begin{aligned} &\partial_{y_1} g(P_1^+, +\infty; l_s)W(s) - D(s) \\ &\geq [\partial_{y_1} g(P_1^+, +\infty; l_s) - \partial_{y_1} g(U(e^s, \alpha_2)e^{m(l_s)s}, s; l_s)]W(s) > 0 \end{aligned} \tag{2.26}$$

Hence, from (2.25) and (2.26) we obtain

$$0 < Z(\bar{s}) = \dot{W}(\bar{s})Q(\bar{s}) - W(\bar{s})\dot{Q}(\bar{s}) = \dot{W}(\bar{s})Q(\bar{s})\,,$$

which gives $\dot{W}(\bar{s}) > 0$, but this contradicts the initial assumption that $W(s) > 0$ for $s < \bar{s}$ and $W(\bar{s}) = 0$: Hence $U(r, \alpha_2) - U(r, \alpha_1) > 0$ for any $r \geq 0$. □

Now, we consider the singular solution $U(r, \infty)$.

**Proposition 2.13.** *Under the hypotheses of Proposition 2.12, $U(r, \infty)r^{m(l_s)}$ is non-decreasing for any $r > 0$, and $U(r, \alpha) < U(r, \infty)$ for any $r > 0$, $\alpha > 0$.*

This proposition is new even for $f$ of type $f(u, r) = K(r)u^{q-1}$ or of type $f(u, r) = u^{q_1-1} + u^{q_2-1}$, which are considered, respectively, in [10, 38].

*Proof.* The result is well known when the system is autonomous: In fact in this case $U(r, \infty)r^{m(l_s)} \equiv P_1^+$ and $W^u_{l_s} = W^u_{l_u}$ is a graph on the $y_1$-axis connecting the origin and $\mathbf{P}^+$.

From the previous discussion we know that the manifold $M^u$ of the autonomous system (2.3), where $l = l_u$ and $g = g(y_1, -\infty; l_u)$, is a graph on the $y_1$-axis connecting the origin and the critical point $\mathbf{P}^-$. Now, we turn to consider the $s$-dependent setting. Let us recall first that $\mathbf{y}^*(s; l_u)$ is the trajectory corresponding to the unique singular solution $U(r, \infty)$, and that $\lim_{s\to-\infty} \mathbf{y}^*(s; l_u) = \mathbf{P}^-$. Observe that for any $\tau \in \mathbb{R}$ the manifold $W^u(\tau; l_u)$ is a graph connecting the origin and $\mathbf{y}^*(s; l_u)$.

We claim that $W^u(\tau; l_u)$ is a graph on the $y_1$-axis, for any $\tau \in \mathbb{R}$. In fact let $\mathbf{Q}, \mathbf{R} \in W^u(\tau; l_u)$, with $\mathbf{Q} = (Q_1, Q_2), \mathbf{R} = (R_1, R_2)$, and let $U(r, \alpha_Q)$ and $U(r, \alpha_R)$ be the corresponding solution of (1.4). From Proposition 2.12 we know that if $\alpha_Q < \alpha_R$, then

$$Q_1 = U(e^\tau, \alpha_Q)e^{m(l_u)\tau} < U(e^\tau, \alpha_R)e^{m(l_u)\tau} = R_1\,, \tag{2.27}$$

so the claim follows.

Moreover, we also get $Q_1 < y_1^*(\tau; l_u)$. Assume by contradiction that $Q_1 > y_1^*(\tau; l_u)$. Then we can choose $\mathbf{R}$ in the branch of $W^u(\tau; l_u)$ between $\mathbf{Q}$ and $\mathbf{y}^*(\tau; l_u)$, so that $\alpha_R > \alpha_Q$ and $Q_1 > R_1 > y_1^*(\tau; l_u)$; but this contradicts (2.27). Similarly if $Q_1 = y_1(\tau, *; l_u)$, then $\mathbf{R} \in W^u(\tau; l_u)$ is such that $\alpha_R > \alpha_Q$, and $R_1 > Q_1 = y_1(\tau, *; l_u)$. But again we can choose $\tilde{\mathbf{R}}$ in the branch of $W^u(\tau; l_u)$ between $\mathbf{R}$ and $\mathbf{y}^*(\tau; l_u)$, and reasoning as above we find again a contradiction. Therefore $U(r, \alpha) < U(r, \infty)$ for any $r > 0$, and any $\alpha > 0$.

Further, since $W^u(\tau; l_u)$ and $W^u(\tau; l_s)$ are homothetic, cf (2.10), then $W^u(\tau; l_s)$ is a graph on the $y_1$-axis, which connects the origin and the trajectory $\mathbf{y}^*(s; l_s)$ corresponding to $U(r, \infty)$. Further $W^u(\tau; l_s) \subset \{(y_1, y_2) \mid 0 < y_1 < P_1^+,\ y_2 > 0\}$ (see Lemma 2.10). Therefore $y_2^*(s; l_s) \geq 0$ for any $s \in \mathbb{R}$. Hence $U(r, \infty) r^{m(l_s)}$ is non-decreasing for any $r > 0$, and the proof is concluded. □

**Remark 2.14.** Proposition 2.13 is interpreted in terms of system (2.3): Under the hypotheses of Proposition 2.12 (hence of Proposition 2.13) we have that $W^u(\tau; l_u)$, $W^u(\tau; l_s)$, and $W^u(\tau; l_*)$ are graphs on the $y_1$-axis respectively for any $\tau \in \mathbb{R}$. Further they are contained in $y_2 \geq 0$ and connect the origin respectively with $\mathbf{y}^*(\tau; l_u)$, $\mathbf{y}^*(\tau; l_s)$, and $\mathbf{y}^*(\tau; l_*)$.

2.4. **Asymptotic estimates for slow decay solutions.** In this subsection we state the asymptotic estimates for slow decay solutions of (1.4), which are crucial to prove our main results: We always assume (A1), (A2), and (A4).

In fact, we generalize the results obtained in [10, §3] for $f(u, r) = k(r) u^{q-1}$ with $q > \sigma^*$, and in [8], for the same potential, in the case of $q = \sigma^*$. The main argument in [10] has been reused in [38], and it is an adaptation to the non-autonomous context of the scheme introduced by Li in [24] (and developed in [21]). Here, we follow a different approach: We give an interpretation of the main argument behind [10, § 3] in terms of some general facts of the ODE theory. This approach contributes to make the scheme used in [10, §3] clearer.

From assumption (A4) we can now set $\zeta = \mathrm{e}^{-\gamma s}$ in (2.9), and obtain a smooth system which has $\mathcal{P} := (P_1^+, 0, 0)$ as critical point. For the remainder of this subsection we consider this system and its linearization around $\mathcal{P}$ so we leave the explicit dependence on $l_s$ unsaid. Hence, we consider (2.9) where $\varpi = \gamma$ and the following system

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ B - \partial_{y_1} g^{+\infty}(P_1^+) & -A & 0 \\ 0 & 0 & -\gamma \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \zeta \end{pmatrix} \tag{2.28}$$

Let us denote by $\mathcal{A}$ the matrix in (2.28): It has 3 negative eigenvalues $\lambda_2 \leq \lambda_1 < 0$ and $-\gamma < 0$. Observe that (A4) is needed to guarantee smoothness of the system (2.9) for $\zeta = 0$. Therefore the critical point $\mathcal{P}$ of (2.9) is a stable node.

Assume first that the 3 eigenvalues are simple, then we have 3 eigenvectors, respectively $v_1 = (1, -m + \lambda_1, 0)$, $v_2 = (1, -m + \lambda_2, 0)$, and $v_z := v_3 = (0, 0, 1)$. Any solution $\ell(t)$ of (2.28) can be written as

$$\ell(s) = \bar{a} v_1 \mathrm{e}^{\lambda_1 s} + \bar{b} v_2 \mathrm{e}^{\lambda_2 s} + z v_z \mathrm{e}^{-\gamma s} \tag{2.29}$$

for some $\bar{a}, \bar{b}, z \in \mathbb{R}$.

By standard facts in invariant manifold theory (see, e.g., [6, §13]), any trajectory $(\mathbf{y}(s), \zeta(s))$ of (2.9) converging to $\mathcal{P}$ can be seen as a non-linear perturbation of a solution $\ell(s)$ of (2.28). More precisely set $\mathbf{n}(s) = (n_1(s), n_2(s)) = (y_1(s) - P_1^+, y_2(s))$, then $\mathbf{N}(s) := (n_1(s), n_2(s), \zeta(s)) = \ell(s) + O(|\ell(s)|^2)$. Therefore

$$n_1(s) = \bar{a}\mathrm{e}^{\lambda_1 s} + \bar{b}\mathrm{e}^{\lambda_2 s} + z\mathrm{e}^{-\gamma s} + O(\mathrm{e}^{2\lambda_1 s} + \mathrm{e}^{2\lambda_2 s} + \mathrm{e}^{-2\gamma s})$$

In [5, Appendix] we prove that the expansion can be continued to an arbitrarily large order: This is the content of Proposition 2.15 and of its general form containing resonances, i.e. Proposition 2.16. Let us rewrite (2.3) as

$$\dot{\vec{x}} = \mathcal{A}\vec{x} + \vec{N}(\vec{x}) \tag{2.30}$$

where $\vec{x} = (y_1, y_2, \zeta)$, and $\mathcal{A}$ is as the matrix in (2.28).

**Proposition 2.15.** *Assume for simplicity $\vec{N} \in C^\infty$ and that the eigenvalues of $\mathcal{A}$ are real, negative and simple and are rationally independent, i.e there is no $\chi = (\chi_1, \chi_2, \chi_3) \in \mathbb{Z}^3 \setminus \{(0,0,0)\}$ such that $\chi_1|\lambda_1| + \chi_2|\lambda_2| + \chi_3\gamma = 0$, so that no resonances are possible. Further assume for definiteness that $|\lambda_1| < \gamma$.*

*Then for any $k \in \mathbb{N}$ we can find a polynomial $P$ of degree $k$ in 3 variables such that*

$$y_1(t) = P(\mathrm{e}^{\lambda_1 t}, \mathrm{e}^{\lambda_2 t}, \mathrm{e}^{-\gamma t}) + o(\mathrm{e}^{[(k+1)\lambda_1 + \varepsilon]t})$$

*as $t \to +\infty$, for $\varepsilon > 0$ small enough.*

We refer the interested reader to [5, Appendix] for further details.

Now, we rephrase the result in a more suitable form for our purpose. Let us set

$$I_\theta = \{\chi = (\chi_1, \chi_2, \chi_3) \in \mathbb{N}^3 : \chi_1|\lambda_1| + \chi_2|\lambda_2| + \chi_3|\gamma| \le \theta\}. \tag{2.31}$$

Then, we can expand $n_1(s)$ as

$$n_1(s) = a\mathrm{e}^{\lambda_1 s} + b\mathrm{e}^{\lambda_2 s} + z\mathrm{e}^{-\gamma s} + P_\theta(s) + o(\mathrm{e}^{-\theta s}), \tag{2.32}$$

where the function $P_\theta(s)$ is completely determined by the values of the coefficients $a, b, z$.

As a first case, assume that $\gamma, |\lambda_1|, |\lambda_2|$ are rationally independent. Then, there are constants $c^\chi \in \mathbb{R}$ such that

$$P_\theta(s) = \sum_{\chi \in I_\theta,\, |\chi| \ge 2} c^\chi \mathrm{e}^{(\chi_1\lambda_1 + \chi_2\lambda_2 - \chi_3\gamma)s} \quad \text{with } \chi = (\chi_1, \chi_2, \chi_3) \tag{2.33}$$

and $|\chi| = \chi_1 + \chi_2 + \chi_3$.

Let us now consider the resonant cases, i.e. when there are $M^0, M^1, \dots, M^j$, (a $j$-ple resonance) $M^i = (\chi_1^i, \chi_2^i, \chi_3^i) \in I_\theta$, $|M^i| > 0$ for $i = 1, \dots, j$, such that

$$\chi_1^i|\lambda_1| + \chi_2^i|\lambda_2| + \chi_3^i\gamma = \bar{\theta} \le \theta\,.$$

Then, we have to replace $\sum_{i=0}^{j} c_{M^i} \mathrm{e}^{(\chi_1^i\lambda_1 + \chi_2^i\lambda_2 - \chi_3^i\gamma)s}$ by

$$\sum_{i=0}^{j} c_{M^i} s^i \mathrm{e}^{(\chi_1^i\lambda_1 + \chi_2^i\lambda_2 - \chi_3^i\gamma)s} \quad \text{in the function } P_\theta, \tag{2.34}$$

(notice that we have included the possible case of resonances with the linear terms, e.g., $\chi_2$ multiple of $\chi_1$ etc.). The same happens when we have resonances within the linear terms, e.g. $|\lambda_1| = |\lambda_2|$ (i.e. $l_s = \sigma^*$), or $|\lambda_1| = \gamma$: We replace the terms as done in (2.34).

Before collecting all these facts in Proposition 2.16 below, we need some further notation. Let us introduce the following sets:

$$J_{|\lambda_1|} = \{\chi = (0, 0, \chi_3) \in \mathbb{N}^3 : 0 < \chi_3\gamma < |\lambda_1|\}\,, \tag{2.35}$$

$$J_{|\lambda_2|} = \{\chi = (\chi_1, \chi_2, \chi_3) \in \mathbb{N}^3 : |\lambda_1| < \chi_1|\lambda_1| + \chi_3\gamma < |\lambda_2|\}. \tag{2.36}$$

Observe that $J_{|\lambda_1|}$ is empty if $|\lambda_1| \le \gamma$, and $J_{|\lambda_2|}$ is empty if $|\lambda_2| < 2|\lambda_1|$ and $|\lambda_2| \le \gamma$. We denote

$$\Psi(s) = \sum_{\chi=(0,0,\chi_3)\in J_{|\lambda_1|}} c^{\chi} \mathrm{e}^{-\chi_3\gamma s} + \chi_r(s)\mathrm{e}^{\lambda_1 s} \tag{2.37}$$

where $\chi_r(s) = 0$ if $|\lambda_1|/\gamma \notin \mathbb{N}$, and $\chi_r(s) = \chi_r s$ if $|\lambda_1|/\gamma \in \mathbb{N}$ and $l_s > \sigma^*$, while $\chi_r(s) = \chi_r s^2$ if $|\lambda_1|/\gamma \in \mathbb{N}$ and $l_s = \sigma^*$, for a certain $\chi_r \in \mathbb{R}$.

**Proposition 2.16.** *Assume* (A1), (A2), (A4). *Let* $\bar{I}_\theta = I_\theta \backslash [\{(1,0,0),(0,1,0)\} \cup J_{|\lambda_1|} \cup J_{|\lambda_2|}]$. *Any trajectory* $(y_1(s), y_2(s), \zeta(s))$ *converging to* $\mathcal{P}$ *is such that* $y_1(s)$ *has the following expansion if* $l_s > \sigma^*$*:*

$$y_1(s) = P_1^+ + \Psi(s) + a\mathrm{e}^{\lambda_1 s} + Q_\theta^1(s) + b\mathrm{e}^{\lambda_2 s} + Q_\theta^2(s) + o(\mathrm{e}^{-\theta s}), \tag{2.38}$$

*where*

$$Q_{1,\theta}(s) = \sum_{\chi\in J_{|\lambda_2|}} c^{\chi} \mathrm{e}^{(\chi_1\lambda_1+\chi_2\lambda_2-\chi_3\gamma)s}, \quad \textit{with } \chi = (\chi_1, \chi_2, \chi_3), \textit{ and}$$

$$Q_{2,\theta}(s) = \sum_{\chi\in \bar{I}_\theta} c^{\chi} \mathrm{e}^{(\chi_1\lambda_1+\chi_2\lambda_2-\chi_3\gamma)s}$$

*as* $s \to +\infty$*, if we do not have resonances; otherwise we need to replace the resonant terms in* $Q_{1,\theta}(s)$ *according to* (2.34).

*If* $l_s = \sigma^*$ *so that* $\lambda_1 = \lambda_2$ *we have*

$$y_1(s) = P_1^+ + \Psi(s) + as\mathrm{e}^{\lambda_1 s} + b\mathrm{e}^{\lambda_1 s} + Q_{2,\theta}(s) + o(\mathrm{e}^{-\theta s}) \tag{2.39}$$

*as* $s \to +\infty$*, again if we do not have resonances, otherwise we need to replace the resonant terms in* $Q_{2,\theta}(s)$ *according to* (2.34).

**Remark 2.17.** We emphasize that $Q_{1,\theta}(s)$ contains terms which are negligible with respect to $a\mathrm{e}^{\lambda_1 s}$ while $Q_{2,\theta}(s)$ contains terms which are negligible with respect to $b\mathrm{e}^{\lambda_2 s}$. Further if $|\lambda_1| < \gamma$ then $\Psi(s)$ is identically null by definition.

The proof is developed in [5, Appendix] by means of an asymptotic expansion result for ODEs, which seems to be new to the best of our knowledge. In fact, we borrow some of the ideas from [10, 38].

**Remark 2.18.** Fix $\mathbf{Q}$ and $\tau \in \mathbb{R}$; then $y_1(t, \tau, \mathbf{Q}; l_s)$ admits an expansion either of the form (2.38) or of the form (2.39). All the coefficients in the expansions are determined by the choice of $a, b$, which are in fact smooth functions of $\mathbf{Q}$, i.e. $a = a(\mathbf{Q})$, $b = b(\mathbf{Q})$.

In fact, all the coefficients in $\Psi(s)$ are determined when the non-linearity $g$ and $\tau$ are fixed; the coefficients in $Q_{1,\theta}$ are assigned (and can be determined) once $a$ is fixed, while $Q_{2,\theta}$ is assigned once $a$ and $b$ are assigned.

**Remark 2.19.** Fix $\mathbf{Q}$ and $\tau$, the coefficients $a = a(\mathbf{Q})$, $b = b(\mathbf{Q})$ may be evaluated through the method explained in [10]. However from the previous discussion we have the following. Let $a_1, b_1, z_1$ be such that $(\mathbf{Q} - \mathbf{P}^+, \mathrm{e}^{-\gamma\tau}) = a_1 v_1 + b_1 v_1 + z_1 v_z$. Then $a = a_1 + O(|\mathbf{Q} - \mathbf{P}^+|^2)$ and $b = b_1 + O(|\mathbf{Q} - \mathbf{P}^+|^2)$.

The proof of what is stated in these two remarks is provided in [5, Appendix]. For further details about these points see [5, Remarks 4.12, 4.16].

Now, we translate Proposition 2.16 for the original equation (1.4).

**Lemma 2.20.** *Assume* (A1), (A2) *with* $l_u \geq l_s \geq \sigma^*$, (A3), (A4)*. Consider either a GS* $U(r,\alpha)$ *for* $\alpha > 0$*, or the SGS* $U(r,\infty)$*; Then there are continuous functions* $\mathcal{A}\colon (0,+\infty] \to \mathbb{R}$*,* $\mathcal{B}\colon (0,+\infty] \to \mathbb{R}$*, such that* $\mathcal{A}$ *is monotone increasing, and if* $l_s > \sigma^*$

$$\begin{aligned} U(r,\alpha) = \frac{P_1^+}{r^m} + \frac{\Psi(\ln(r))}{r^m} + \mathcal{A}(\alpha) r^{\lambda_1 - m} + \frac{Q_{1,\theta}(\ln(r))}{r^m} \\ + \mathcal{B}(\alpha) r^{\lambda_2 - m} + \frac{Q_{2,\theta}(\ln(r))}{r^m} + o(r^{-\theta - m}) \end{aligned} \tag{2.40}$$

*as* $r \to +\infty$*. If* $l_s = \sigma^*$ *we have*

$$\begin{aligned} U(r,\alpha) = \frac{P_1^+}{r^m} + \frac{\Psi(\ln(r))}{r^m} + \mathcal{A}(\alpha)\ln(r) r^{\lambda_1 - m} + \mathcal{B}(\alpha) r^{\lambda_2 - m} \\ + \frac{Q_{2,\theta}(\ln(r))}{r^m} + o(r^{-\theta - m}). \end{aligned} \tag{2.41}$$

**Remark 2.21.** If we replace (A3) with the weaker assumption (A6) in Lemma 2.20, then we still have the expansions in (2.40), (2.41), but we cannot ensure that $\mathcal{A}$ is monotone decreasing.

*Proof of Lemma 2.20.* Fix $\tau \in \mathbb{R}$; let $\mathbf{y}(s,\tau,\mathbf{Q}(\alpha); l_s)$ be the trajectory of (2.3) corresponding to $U(r,\alpha)$, so that $\mathbf{Q}(\alpha) \in W^u_{l_s}(\tau)$. Then we can apply Proposition 2.16 to $y_1(s,\tau,\mathbf{Q}(\alpha); l_s)$ and we find the expansions (2.40), (2.41), where, according to Remark 2.18, the coefficients $a, b$ are $a = a(\mathbf{Q}(\alpha))$ and $b = b(\mathbf{Q}(\alpha))$. We set

$$\mathcal{A}(\alpha) = a(\mathbf{Q}(\alpha)), \text{ and } \mathcal{B}(\alpha) = b(\mathbf{Q}(\alpha)). \tag{2.42}$$

It follows that $\mathcal{A}\colon (0,+\infty) \to \mathbb{R}$ and $\mathcal{B} : (0,+\infty) \to \mathbb{R}$ are continuous functions. Finally if (A3) holds, then $U(r,\alpha_1) < U(r,\alpha_2)$ if $\alpha_1 < \alpha_2$ for any $r > 0$, and in particular for $r$ large, so $\mathcal{A}(\alpha)$ is monotone increasing. □

## 3. Main results: Stability and asymptotic stability

Let us state Theorems 3.1 and 3.2 from which Theorems 1.3, 1.4, 1.5 follow directly. Let $r > 0$, we denote by $[[r]] := \{k \in \mathbb{N} k - 1 < r \leq k\}$. We have the following results

**Theorem 3.1.** *Suppose* $f$ *is* $C^k$ *where* $k = [[|\lambda_1|/\gamma]]$*. Assume* (A1)–(A5)*. Then any radial GS* $U(r,\alpha)$ *of* (1.2) *is stable with respect to the norm* $\|\cdot\|_{m(l_s)+\lambda_1}$ *if* $l_s > \sigma^*$*, and with respect to the norm* $|||\cdot|||_{m(l_s)+|\lambda_1|}$ *if* $l_s = \sigma^*$*.*

**Theorem 3.2.** *Assume the hypotheses of Theorem 3.1. Then any radial GS* $U(r,\alpha)$ *of* (1.2) *is weakly asymptotically stable with respect to the norm* $\|\cdot\|_{m(l_s)+|\lambda_2|}$ *if* $l_s > \sigma^*$*, and with respect to the norm* $\|\cdot\|_{m(l_s)+|\lambda_1|}$ *if* $l_s = \sigma^*$*.*

Let us recall that the stability of positive GS $U(|x|,\alpha)$ of (1.2) has been analyzed in a number of papers, (see [10, 21, 22, 36]). In [21], when $f(u,|x|) = u^{q-1}$ and $q > \sigma^*$, the authors proved that the positive GS of (1.2) are stable in the norm $\|\cdot\|_{m+|\lambda_1|}$, and weakly asymptotically stable with respect to $\|\cdot\|_{m+|\lambda_2|}$. These results have been subsequently extended in [10] to functions $f(u,|x|)$ of the form $k(|x|)r^{\delta}|u|^{q-1}$ where $K$ is a monotone decreasing uniformly positive and bounded function. Here we drop the assumption that $k$ is bounded: This will allow us to consider potentials giving rise to singular solutions $U(r,\infty)$ having two different behaviors as $r \to 0$ (i.e. $U(r,\infty) \sim P^- r^{-m(l_u)}$) and as $r \to \infty$ (i.e. $U(r,\infty) \sim P^+ r^{-m(l_s)}$).

3.1. **Proof of Theorem 3.1.** We first introduce some standard definitions.

**Definition 3.3.** We say that $\overline{\phi}$ is a *super-solution* of (1.1) if $\Delta\overline{\phi} + f(\overline{\phi}, |x|) \leq 0$; analogously $\underline{\phi}$ is a *sub-solution* of (1.1) if $\Delta\underline{\phi} + f(\underline{\phi}, |x|) \geq 0$.

We refer to [36] or to [4, §3] for an extension of these definitions to weak and mild solutions. Also, depending on a number of very relevant factors (for instance, the type of domain and of the boundary conditions, the regularity of the forcing term, etc.) the notion of weak solution for parabolic equations can change considerably as described, e.g., in [23, 19, 20, 35]. In particular, we mention that, a dynamical approach to study a generalized parabolic equation on an unbounded strip-like domain is given in [34]: In this case a suitable definition of weak solutions, on weighted Sobolev (and Bochner) spaces, is considered and the author proved the existence of a global attractor. Then, this situation is further generalized in [3].

Both Theorems 3.1, 3.2 depend strongly on the following well known fact, proved in [36, Theorem 2.4], see also [4, Theorem 3.10].

**Lemma 3.4.** *Assume* (A1), (A2) *and let $U_1(r)$ and $U_2(r)$ be positive solutions of* (1.4) *respectively for $r \leq R_1$ and for $r \geq R_2$, where $R_1 > R_2$, and let $R \in (R_2, R_1)$ be such that $U_1(R) = U_2(R)$. Consider*

$$\phi(x) = \begin{cases} U_1(r) & \text{if } 0 < |x| \leq R, \\ U_2(r) & \text{if } |x| \geq R. \end{cases}$$

*We have*

- *If $U_1'(R) \geq U_2'(R)$, then $\phi(x)$ is a continuous weak super-solution of* (1.1).
- *If $U_1'(R) \leq U_2'(R)$, then $\phi(x)$ is a continuous weak sub-solution of* (1.1).

**Lemma 3.5.** *Assume* (A1), (A2)*:*

(i) *If the initial value $\phi$ in* (1.3) *is a continuous weak super-(sub-) solution of* (1.1)*, then the solution $u(t, x; \phi)$ of* (1.2)-(1.3) *is non-increasing (non-decreasing) in $t$ as long as it exists, for any $x$; strictly if $\phi$ is not a solution.*

(ii) *If $\phi$ is radial, then $u(t, x; \phi)$ is radial in the $x$ variable for any $t > 0$.*

To prove Theorem 3.1 we adapt the main ideas developed in [21, 10, 38].

As a consequence of the proof of Proposition 2.12 we obtain the following result (see Lemma 3.6 below) which will be useful to prove the stability of the solutions, and replaces a longer elliptic estimate performed in [10, Lemma 4.3] and adapted in [38, 8]. We stress that in fact the proof in the critical case, considered in [8], suffers from a flaw.

**Lemma 3.6.** *Assume* (A1)–(A5)*. Assume $\beta > \alpha$ then $\mathcal{A}(\beta) > \mathcal{A}(\alpha)$.*

*Proof.* Since $U(r, \beta) > U(r, \alpha)$ for any $r > 0$ (see Lemma 2.12), we already know that $\mathcal{A}(\beta) \geq \mathcal{A}(\alpha)$, so we just need to prove that the inequality is strict. Set $h(s) = [U(e^s, \alpha_2) - U(e^s, \alpha_1)]e^{(m(l_s) - \lambda_1)s}$, and, following the notation of Proposition 2.12, $Q(s) = e^{\lambda_1 s}$. Following the main line in the proof of Proposition 2.12 we see that $\dot{h}(s) = Z(s)/Q^2(s)$. In particular, from (2.25) and (2.26), $\dot{h}(s) > 0$ for any $s \in \mathbb{R}$. Since $\lim_{s\to-\infty} h(s) = 0$ we see that $h(s) > 0$ for any $s \in \mathbb{R}$, and $\lim_{s\to+\infty} h(s) > 0$.

If $l_s > \sigma^*$, then $\lim_{s\to+\infty} h(s) = \mathcal{A}(\beta) - \mathcal{A}(\alpha) > 0$, and the proof is complete.

Assume now $l_s = \sigma^*$, and also assume by contradiction that $\mathcal{A}(\beta) = \mathcal{A}(\alpha)$. In this case we see that $\lim_{s\to+\infty} h(s) = \mathcal{B}(\beta) - \mathcal{B}(\alpha) \in (0, +\infty)$. However, from (2.25), since $A = -2\lambda_1$, for any $\bar{s} \in \mathbb{R}$ we find

$$\dot{h}(\bar{s}) = \int_{-\infty}^{\bar{s}} \mathrm{e}^{As} Q(s)[\partial_{y_1} g(P_1^+, +\infty; l_s)W(s) - D(s)] > 0\,.$$

Therefore $\liminf_{s\to+\infty} \dot{h}(s) \geq \dot{h}(0) > 0$, hence $\mathcal{B}(\beta) - \mathcal{B}(\alpha) = \lim_{s\to+\infty} h(s) = +\infty$, but this is a contradiction. Hence $\mathcal{A}(\beta) > \mathcal{A}(\alpha)$. □

**Lemma 3.7.** *Assume* (A1)–(A5) *and* $l_u \geq l_s$. *If* $l_s > \sigma^*$.
*Then* $\|U(r,\beta) - U(r,\alpha)\|_{m+|\lambda_1|} \to 0$ *as* $\beta \to \alpha$,
*while if* $l_s = \sigma^*$, *then* $|||U(r,\beta) - U(r,\alpha)|||_{m+|\lambda_1|} \to 0$ *as* $\beta \to \alpha$.

*Proof.* We develop the proof assuming $l_s > \sigma^*$, the case $l_s = \sigma^*$ is completely analogous. It is well known that, for any fixed $R > 0$ and any $\varepsilon > 0$, there is $\delta_1(\varepsilon, \alpha, R) > 0$ such that

$$\sup\{|U(r,\beta) - U(r,\alpha)| \mid 0 \leq r \leq R\} < \varepsilon \tag{3.1}$$

whenever $|\beta - \alpha| < \delta_1$ (this is a continuous dependence on initial data argument for the singular equation (1.4)). Further from (2.40) we see that for $r$ large enough we have

$$\big|\big(U(r,\beta) - U(r,\alpha)\big)\big|(1 + r^{m-\lambda_1}) \cong |\mathcal{A}(\beta) - \mathcal{A}(\alpha)| + o(r^{|\lambda_2 - \lambda_1|/2}) \tag{3.2}$$

Thus, for any $\varepsilon > 0$ there exists $M(\varepsilon)$ such that $|o(r^{|\lambda_2-\lambda_1|/2})| \leq C\varepsilon$, when $r \geq M(\varepsilon)$. Further from Lemma 2.20 we see that for any $\varepsilon > 0$ we can find $\delta_2(\varepsilon, \alpha) > 0$ such that $|\mathcal{A}(\beta) - \mathcal{A}(\alpha)| \leq \varepsilon$ if $|\beta - \alpha| < \delta_2$. Therefore

$$\big|\big(U(r,\beta) - U(r,\alpha)\big)\big|\,(1 + r^{m-\lambda_1}) \leq \varepsilon\,, \quad \text{for } r \geq M\,. \tag{3.3}$$

The proof follows from (3.2)–(3.3), choosing $M = R$ and $\delta(R, \alpha, \varepsilon) = \min\{\delta_1, \delta_2\}$. □

We are now ready to prove Theorem 3.1.

*Proof of Theorem 3.1.* We give the proof just in the case $l_s > \sigma^*$, when $l_s = \sigma^*$ the calculations are completely analogous and we omit them. Fix $\varepsilon > 0$ (small) and $\alpha > 0$; let $\phi(x)$ be such that $\|U(|x|, \alpha) - \phi(x)\|_{m+|\lambda_1|} = \delta$, where $\delta > 0$ will be chosen below.

Let $|\eta| < \alpha$ and set

$$z(r,\eta) = [U(r, \alpha + \eta) - U(r,\alpha)](1 + r^{m-\lambda_1}) \tag{3.4}$$

Observe that $z(0,\eta) = \eta$ and $\lim_{r\to+\infty} z(r,\eta) = \mathcal{A}(\alpha + \eta) - \mathcal{A}(\alpha)$. So we can set

$$\underline{z}(\eta) = \min\{|z(r,\eta)| \mid r > 0\} \quad \text{and} \quad \overline{z}(\eta) = \max\{|z(r,\eta)| \mid r > 0\}. \tag{3.5}$$

Moreover $z(r,\eta)$ is uniformly positive (respectively negative) for any $r > 0$ if $\eta > 0$ (resp. $\eta < 0$), so $\underline{z}(\eta) > 0$ if $\eta \neq 0$: This follows from Lemmas 2.12 and 3.6.

Finally, from Lemma 3.7, we know that $\lim_{\eta\to 0} \underline{z}(\eta) = \lim_{\eta\to 0} \overline{z}(\eta) = 0$. Then, for any $\varepsilon > 0$ we can find $d = d(\varepsilon) > 0$ such that $\overline{z}(-d) < \varepsilon$, and $\overline{z}(d) < \varepsilon$. Set $\alpha_1 = \alpha - d$, $\alpha_2 = \alpha + d$, and choose $\delta = \min\{\underline{z}(-d)\,, \underline{z}(d)\}$. Then

$$\begin{gathered} U(|x|, \alpha_1) < \phi(x) < U(|x|, \alpha_2), \\ \|U(|x|, \alpha_i) - U(|x|, \alpha)\|_{m-\lambda_1} \leq \varepsilon \quad \text{for } i = 1, 2 \end{gathered} \tag{3.6}$$

Therefore, from the comparison principle (see, e.g., [20, Appendix]), we have that

$$U(|x|,\alpha_1) < u(t,x;\phi) < U(|x|,\alpha_2), \quad \text{for any } t \geq 0,\ x \in \mathbb{R}^n, \tag{3.7}$$

and the proof is complete. □

3.2. **Weak asymptotic stability.** To prove weak asymptotic stability we follow the outline of the proof of [21, Theorem 4.1] and adapted in [10, 38].

**Proposition 3.8.** *Assume the hypotheses of Theorem 3.1 and consider the stationary problem* (1.4)*. Then, for any radial GS* $U(\cdot,d)$ *of* (1.4)*, there is a sequence of radial strict super-solutions* $\overline{U}^{(1)}(\cdot,e^1) > \overline{U}^{(2)}(\cdot,e^2) > \dots > U(\cdot,d)$ *of* (1.1) *and a sequence of radial strict sub-solutions* $\underline{U}^{(1)}(\cdot,c^1) < \underline{U}^{(2)}(\cdot,c^2) < \dots < U(\cdot,d)$ *such that* $U(\cdot,d)$ *is the only solution of* (1.1) *satisfying* $\underline{U}^{(k)}(\cdot,c^k) < U(\cdot,d) < \overline{U}^{(k)}(\cdot,e^k)$*, for every* $k$*. Moreover*

$$\lim_{k\to\infty} \underline{U}^{(k)}(\cdot,c^k) = U(\cdot,d) = \lim_{k\to\infty} \overline{U}^{(k)}(\cdot,e^k) \tag{3.8}$$

*Proof.* Let $h:[0,+\infty)\to[0,1]$ be a monotone decreasing $C^\infty$ function such that $h(0)=1$ and $h(r)\equiv 0$ for $r \geq 1$. Let $\mathcal{G}(y_1,s;l_s) = g(y_1,s;l_s) - g(y_1,+\infty;l_s)$ and observe that $\mathcal{G}(y_1,s;l_s) \geq 0$ and it is decreasing in $s$ for any $y_1, s$.

Assume first $\mathcal{G}(y_1,s) \not\equiv 0$, i.e. consider the generic case, and denote

$$\begin{aligned}
\overline{g}^{(k)}(y_1,s) &= g(y_1,s;l_s) + \frac{h(\mathrm{e}^s)}{2k}\mathcal{G}(y_1,s;l_s)\\
\underline{g}^{(k)}(y_1,s) &= g(y_1,s;l_s) - \frac{h(\mathrm{e}^s)}{2k}\mathcal{G}(y_1,s;l_s)
\end{aligned}$$

and let $\overline{f}^{(k)}$, $\underline{f}^{(k)}$ be the corresponding functions obtained via (2.2). Notice that by construction $\overline{g}^{(k)}(y_1,s)$, and $\underline{g}^{(k)}(y_1,s)$ are both decreasing in $s$ for any $k \geq 1$; Hence $\overline{f}^{(k)} \geq f \geq \underline{f}^{(k)}$ satisfy (A1)–(A5) so that Lemma 2.10, and Proposition 2.12 hold. In particular all the regular solutions of the respective problem (1.4), say $\overline{U}^{(k)}(r,\alpha)$, $U(r,\alpha)$, $\underline{U}^{(k)}(r,\alpha)$, are GSs. Further the corresponding trajectories of (2.3), say $\overline{\mathbf{y}}^{(\mathbf{k})}(s,\alpha)$, $\mathbf{y}(s,\alpha)$, $\underline{\mathbf{y}}^{(\mathbf{k})}(s,\alpha)$ are monotone increasing in their first component and converge to $\mathbf{P}^+$, and have the asymptotic expansion as described in Proposition 2.16. More precisely they both have either the expansion (2.40) or (2.41), where the function $\Psi(\ln(r))$ coincide for $r \geq 1$, while the coefficients $a = \overline{\mathcal{A}}^{(k)}(\alpha)$, $a = \underline{\mathcal{A}}^{(k)}(\alpha)$ and $b = \overline{\mathcal{B}}^{(k)}(\alpha)$, $b = \underline{\mathcal{B}}^{k}(\alpha)$ are different, see Lemma 2.20. Further by construction, $\overline{U}^{(k)}(r,\alpha)$, $\underline{U}^{(k)}(r,\alpha)$ are respectively super- and sub-solutions for the original problem (1.4).

We divide our argument in several steps.

**Step 1.** If there is $R>0$ such that $U(R,d) = \overline{U}^{(k)}(R,c)$ (respectively $U(R,d) = \underline{U}^{(k)}(R,e)$), then $U(r,d) \geq \overline{U}^{(k)}(r,c)$ (respectively $U(r,d) \leq \underline{U}^{(k)}(r,e)$) for any $r \geq R$.

Let $\tau(\xi)\colon (0,2) \to \mathbb{R}$ be the inverse of the function $\xi(\tau)$ defined in (2.11). We consider

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ B(l_*) & -A(l_*) & 0 \\ 0 & 0 & C \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \xi \end{pmatrix} - \begin{pmatrix} 0 \\ g(y_1,\tau(\xi);l_*) \\ 0 \end{pmatrix}, \tag{3.9}$$

where $A(l_*)$, $B(l_*)$, $C$ coincide with $A(l_u)$, $B(l_u)$, $\varpi$ for $s \le 0$ and with $A(l_s)$, $B(l_s)$, $\gamma$ for $s \ge 0$, and similarly $g(y_1, \tau(\xi); l_*)$ equals $g(y_1, \frac{\ln(\xi)}{\varpi}; l_u)$ for $\xi \le 1$ (i.e. $s \le 0$) and $g(y_1, \frac{\ln(2-\xi)}{\varpi}; l_s)$ for $\xi \ge 1$ (i.e. $s \ge 0$). Notice that (3.9) coincides with (2.8) when $\xi \le 1$ (i.e. $s \le 0$) and it is equivalent to (2.9) when $\xi \ge 1$ (i.e. $s \le 0$ and $\zeta \le 1$, it differs from (2.9) just in the fact that $\xi = 2 - \zeta$). Further we recall that the unstable manifold $\mathbf{W}^u(l_*)$ defined in (2.11) has dimension 2 and connects the $\xi$-axis and the graph of $\mathbf{y}^*(s, l_*)$; Further it is a graph on the $y_2 = 0$ plane, see Remark 2.14. Let us denote

$$E = \{(y_1, y_2, \xi) : 0 < y_1 < y_1^u(\tau(\xi), l_*)\,,\ 0 < \xi < 2\}$$

and by $E_0 := \mathbf{W}^u(l_*)$. It follows that $E_0 \subset E$ and $E_0$ splits $E$ in 2 open components, say $E^+$ and $E^-$ (the one with larger and smaller $y_2$).

By construction the flow of the modified system (3.9) where $g$ is replaced respectively by $\overline{g}^k$ and by $\underline{g}^k$ on $\mathbf{W}^u(l_*)$ points towards $E^-$ and $E^+$ respectively for $s \le 0$, and it is tangent to $E^0$ for $s \ge 0$. So the corresponding manifolds $\overline{\mathbf{W}}^{\mathbf{u},(\mathbf{k})}(l_*)$ and $\underline{\mathbf{W}}^{\mathbf{u},(\mathbf{k})}(l_*)$ lie respectively in $E^-$ and $E^+$.

Now assume $U(R, d) = \overline{U}^{(k)}(R, c)$ and consider the corresponding trajectories $\mathbf{y}(s; l_*)$, and $\overline{\mathbf{y}}^{(\mathbf{k})}(s; l_*)$: Then $y_1(\ln(R); l_*) = \overline{y}_1^{(k)}(\ln(R); l_*)$ and $y_2(\ln(R); l_*) \ge \overline{y}_2^{(k)}(\ln(R); l_*)$. Hence $y_1(s; l_*) \ge \overline{y}_1^{(k)}(s; l_*)$ for $s$ in a right neighborhood of $\ln(R)$. Then the claim in Step 1 concerning $\overline{U}^{(k)}(r, c)$ follows. The claim concerning $\underline{U}^{(k)}(r, e)$ is analogous.

We continue the discussion for later purposes. We know that $y_2(\ln(R); l_*) \ge \overline{y}_2^{(k)}(\ln(R); l_*)$, assume first $y_2(\ln(R); l_*) > \overline{y}_2^{(k)}(\ln(R); l_*)$. Then $y_1(s; l_*) > \overline{y}_1^{(k)}(s; l_*)$ for $s$ in a right neighborhood of $\ln(R)$.

Assume now $y_2(\ln(R); l_*) = \overline{y}_2^{(k)}(\ln(R); l_*)$: Then $R \ge 1$. In fact assume for contradiction that $0 < R < 1$, then $\overline{\mathbf{y}}(\ln(R); l_*) = \mathbf{Q} = \overline{\mathbf{y}}^{(\mathbf{k})}(\ln(R); l_*)$ is such that $(\mathbf{Q}, \xi(\ln(R)) \in E^0$, but from (2.3) we obtain $\dot{\overline{y}}_2(\ln(R); l_*) < \dot{y}_2^{(k)}(\ln(R); l_*)$. Hence $\overline{\mathbf{y}}^{(\mathbf{k})}(r; l_*)$ crosses transversally $E^0$ at $s = \ln(R)$, going from $E^+$ to $E^-$, in particular it is in $E^+$ when $s$ is in a sufficiently small left neighborhood of $\ln(R)$. But $(\overline{\mathbf{y}}^{(\mathbf{k})}(s; l_*), \xi(s)) \in \overline{\mathbf{W}}^{\mathbf{u},(\mathbf{k})}(l_*) \subset E^-$, and this is a contradiction, so $R > 1$.

Observe that if $R \ge 1$ then $\overline{\mathbf{y}}^{(\mathbf{k})}(s; l_*)$ and $\mathbf{y}(s; l_*)$ are solutions of the same equation (2.3) for $s \ge 0$ which coincide for $s = \ln(R)$, so they coincide for $s \ge 0$.

We have already proved the following result, i.e.

**Step 2.** For any $0 < r < 1$ we have that

$$\overline{U}^{(k)}(r, d) < U(r, d) < \underline{U}^{(k)}(r, d) \tag{3.10}$$

and either (3.10) holds for any $r > 0$ or the functions coincide for any $r \ge 1$. Moreover $\overline{\mathcal{A}}^{(k)}(d) \le \mathcal{A}(d) \le \underline{\mathcal{A}}^{(k)}(d)$.

**Step 3.** Fix $d$ and the corresponding coefficient $\mathcal{A}(d)$. It is possible to choose $c^k \le d \le e^k$ so that $\underline{\mathcal{A}}^{(k)}(e^k) = \overline{\mathcal{A}}^{(k)}(c^k) = \mathcal{A}(d)$.

Fix $\tau > 0$ and $0 < c < d < e$; let $\mathbf{y}(s, \tau, \mathbf{P}; l_s)$, $\mathbf{y}(s, \tau, \mathbf{Q}; l_s)$, $\mathbf{y}(s, \tau, \mathbf{R}; l_s)$ be the trajectories of (2.3) corresponding to the solutions $U(r, c)$, $U(r, d)$, $U(r, e)$ of (1.4). It follows that $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ are points in $W^u(\tau, l_s)$ and $\mathbf{P}$, $\mathbf{R}$ are respectively the closest to and the farthest from the origin. Let us consider the lines $\ell^l, \ell^r$ parallel to the $y_2$-axis and passing through $\mathbf{P}$ and $\mathbf{R}$ respectively: We denote by

$\overline{\mathbf{P}^{(k)}}$ and $\overline{\mathbf{R}^{(k)}}$, the intersections of $\overline{W}^{u,(k)}(\tau, l_s)$ respectively with $\ell^l$ and with $\ell^r$. Using continuous dependence on initial data of ODE we see that $\overline{\mathbf{P}^{(k)}} \to \mathbf{P}$ and $\overline{\mathbf{R}^{(k)}} \to \mathbf{R}$ as $k \to \infty$. Since $a(\mathbf{Q})$ is continuous, see Remark 2.18 and (2.42), we see that $a(\overline{\mathbf{P}^{(k)}}) \to a(\mathbf{P}) = \mathcal{A}(c) < \mathcal{A}(d)$, while $a(\overline{\mathbf{R}^{(k)}}) \to a(\mathbf{R}) = \mathcal{A}(e) > \mathcal{A}(d)$. Therefore we can choose $N$ large enough so that $a(\overline{\mathbf{P}^{(k)}}) < \mathcal{A}(d) < a(\overline{\mathbf{R}^{(k)}})$ for any $k \geq N$. Hence we can find $\overline{\mathbf{Q}}^{(\mathbf{k})} \in \overline{W}^{u,(k)}(\tau, l_s)$ between $\overline{\mathbf{P}^{(k)}}$ and $\overline{\mathbf{R}^{(k)}}$ such that $a(\overline{\mathbf{Q}}^{(\mathbf{k})}) = \mathcal{A}(d)$. Correspondingly we find $e^k$ such that $\overline{\mathcal{A}}^{(k)}(e^k) = a(\overline{\mathbf{Q}}^{(\mathbf{k})}) = \mathcal{A}(d)$. Note that in view of *Step 2* we have $e^k \geq d$. The proof for $\underline{\mathcal{A}}^{(k)}(c^k)$ is analogous.

Then, putting together Step 1 and Step 3, we see that $U(r,d)$ is the unique solution of the original equation (1.4) such that

$$\underline{U}^{(k)}(r, c^k) \leq U(r,d) \leq \overline{U}^{(k)}(r, e^k), \text{ for any } r \geq 0 \tag{3.11}$$

**Step 4.** Formula (3.8) and the following Remark hold.

**Remark 3.9.** $\overline{\mathcal{B}}^{(k)}(e^k)$ and $\underline{\mathcal{B}}^{(k)}(c^k)$ are respectively strictly decreasing and increasing in $k$ and they both converge to $\mathcal{B}(d)$.

*Proof.* To prove (3.8) it is sufficient to observe that, by construction, the functions $\underline{U}^k(r, c^k)$ and $\overline{U}^k(r, e^k)$ are bounded and monotonically respectively increasing and decreasing in $k$. Then, from standard elliptic estimates we see that they both converge to solutions of the original problem (1.4) as $k \to +\infty$. Then, from *Step 3* we see that the limit of both $\underline{U}^k(r, c^k)$ and $\overline{U}^k(r, e^k)$ is the same solution $U(r,d)$ of the original problem (1.4).

Now, we consider Remark 3.9. From Step 3 we know that $\underline{\mathcal{A}}^{(k)}(e^k) = \mathcal{A}(d) = \overline{\mathcal{A}}^{(k)}(c^k)$. Further from the previous argument we also infer that $\overline{\mathcal{B}}^{(k)}(e^k)$ and $\underline{\mathcal{B}}^{(k)}(c^k)$ are respectively decreasing and increasing and converge to $\mathcal{B}(d)$. As next step, we show that $\underline{\mathcal{B}}^{(k)}(c^k) < \underline{\mathcal{B}}^{(k-1)}(c^{k-1}) < \mathcal{B}(d) < \overline{\mathcal{B}}^{(k-1)}(e^{k-1}) < \overline{\mathcal{B}}^{(k)}(e^k)$, i.e. $\underline{\mathcal{B}}^{(k)}(c^k)$ and $\overline{\mathcal{B}}^{(k)}(e^k)$ are strictly increasing and decreasing. As usual we just prove the last inequality, the others being analogous. Let $\overline{u}^j(x)$ be the radial function defined by $\overline{u}^j(x) = \overline{U}^j(|x|, e^j)$. Observe that $\Delta[\overline{u}^k(x)) - \overline{u}^{k-1}(x)] \leq 0$, hence from standard arguments (see [29, Theorem 3.8]), we see that there is $C > 0$ such that $\overline{U}^k(r, e^k) - \overline{U}^{k-1}(r, e^{k-1}) > Cr^{-(n-2)}$. Assume $\overline{\mathcal{B}}^{(k)}(e^k) = \overline{\mathcal{B}}^{(k-1)}(e^{k-1})$ for contradiction. Since $\overline{\mathcal{A}}^{(k)}(e^k) = \mathcal{A}(d)$ for any $k$, from the construction in Lemma 2.20 it follows that $\overline{\mathbf{y}}^{(\mathbf{k})}(s, e^k; l_s) \equiv \overline{\mathbf{y}}^{(\mathbf{k-1})}(s, e^{k-1}; l_s)$ for any $s \geq 0$, i.e. $\overline{U}^k(r, e^k) = \overline{U}^{k-1}(r, e^{k-1})$ for $r \geq 1$, but this is a contradiction and the Remark is proved. $\square$

From Remark 3.9 we see that the inequalities in (3.11) are strict for $r$ large. Then, from *Step 1* we conclude the proof of Proposition 3.8 in the case $G(y_1, s) \not\equiv 0$.

Assume now $G(y_1, s) \equiv 0$, this is the case, e.g., when $f(u,r) = cu|u|^{q-2}$. Following [21] we denote by $\underline{f}^{(k)}(u,r) := [1 - \mu h(r)/k] f(u,r)$ and $\overline{f}^{(k)}(u,r) := [1 + \mu h(r)/k] f(u,r)$, for $k \in \mathbb{N}$ and where $\mu > 0$ is chosen small enough so that $\underline{f}^{(1)}(u,r)$ satisfies (A6); then it is easy to check that $\underline{f}^{(k)}(u,r)$ and $\overline{f}^{(k)}(u,r)$ satisfy (A6) for any $k \in \mathbb{N}$. So Proposition 2.8 holds, and in all the 3 cases all the regular solutions of (1.4), denoted respectively by $\underline{U}^{(k)}(r, \alpha)$, $U(r, \alpha)$, $\overline{U}^{(k)}(r, \alpha)$, are GSs, but a a priori they might not be ordered. However repeating the argument of *Step 1* in [21, Theorem 4.1], it it easy to prove that

$$\overline{U}^{(k)}(r,\alpha) \le U(r,\alpha) \le \underline{U}^{(k)}(r,\alpha) \tag{3.12}$$

for any $r > 0$ and any $\alpha > 0$. The proof might be concluded arguing as in [21, Theorem 4.1]. However notice that we can also repeat the argument at the end of Step 1 of this proof to get (3.10) for any $r > 0$, and then carry on through Step 2,3,4, of this proof and conclude also in this case, with no further changes. □

### 3.3. **Proof of the weak asymptotic stability.**

Now we consider $d > 0$ fixed, and we use the shorthand notation $\overline{U}^{(1)}(r, e^1) = \overline{U}(r)$, $\underline{U}^{(1)}(r, c^1) = \underline{U}(r)$, $\overline{u}(t,x) = u(t,x;\overline{U}(|x|))$, $\underline{u}(t,x) = u(t,x;\underline{U}(|x|))$.

**Lemma 3.10.** *Under the hypotheses of Theorem 3.1, we have* $\overline{u}(t,x) \searrow U(|x|,d)$ *and* $\underline{u}(t,x) \nearrow U(|x|,d)$ *as* $t \to +\infty$*, with the norm* $\|\cdot\|_l$*, for any* $0 \le l < m + |\lambda_2|$.

Notice that if $l_s = \sigma^*$ then $\|\cdot\|_{m+|\lambda_1|} = \|\cdot\|_{m+|\lambda_2|}$.

*Proof.* Let us set $B := \lim_{|x|\to+\infty}[\overline{U}(|x|) - \underline{U}(|x|)]|x|^{m+|\lambda_2|}$ and notice that $B > 0$ is finite, see Proposition 3.8 and Remark 3.9. Fix $0 \le l < m + |\lambda_2|$ and observe that for any $\varepsilon > 0$ we can find $\rho > 0$ such that

$$[\overline{U}(|x|) - \underline{U}(|x|)]|x|^l < 2B|x|^{l-m-|\lambda_2|} < \varepsilon/2 \tag{3.13}$$

for $\|x\| \ge \rho$.

Since $\overline{U}(|x|)$ and $\underline{U}(|x|)$ are respectively a radial super- and sub-solution of (1.1), then $\overline{u}(t,x)$ and $\underline{u}(t,x)$ are respectively radially symmetric super- and sub-solution of (1.2). Further they are resp. monotone decreasing and increasing in $t$, so they converge to a radial solution of (1.1), see Lemma 3.5. From Lemma 3.8 we know that $U(r,d)$ is the unique solution of (1.4) between $\overline{U}(r)$ and $\underline{U}(r)$, so $\overline{u}(t,x)$ and $\underline{u}(t,x)$ converge monotonically to $U(|x|,d)$ as $t \to +\infty$, for any fixed $x \in \mathbb{R}^n$. Then, from the equiboundedness of the functions involved and of their derivatives we see that the convergence is uniform in any ball of radius $R > 0$ fixed. Hence setting $R = \rho > 0$, for any $\varepsilon > 0$ we find $T(\varepsilon) > 0$ such that

$$[\overline{u}(x,t) - \underline{u}(x,t)]|x|^l \le \varepsilon/2 \tag{3.14}$$

for any $|x| \le \rho$. Further from (3.13) and the comparison principle we easily find that

$$[\overline{u}(x,t) - \underline{u}(x,t)]|x|^l \le [\overline{U}(|x|) - \underline{U}(|x|)]|x|^l \le \varepsilon/2 \tag{3.15}$$

for $|x| \ge \rho$. Hence the Lemma follows from (3.14) and (3.15). □

*Proof of Theorem 3.2.* Assume for definiteness $l_s > \sigma^*$, the case $l_s \ge \sigma^*$ being analogous. Fix $d > 0$ and denote

$$\begin{aligned} \overline{W}(r,d) &= [\overline{U}(r) - U(r,d)](1 + r^{m+|\lambda_2|}), \quad \overline{\delta} = \inf_{r>0} \overline{W}(r,d) \\ \underline{W}(r,d) &= [U(r,d) - \underline{U}(r)](1 + r^{m+|\lambda_2|}), \quad \underline{\delta} = \inf_{r>0} \underline{W}(r,d) \end{aligned} \tag{3.16}$$

Observe that $\overline{W}(r,d)$, $\underline{W}(r,d)$ are both positive for any $r > 0$, see Proposition 3.8. Further $\overline{W}(0,d) = e^1 - d > 0$, $\underline{W}(0,d) = d - c^1 > 0$, $\lim_{r\to+\infty} \overline{W}(r,d) = \overline{B}^{(1)}(e^1) - B(d) > 0$, $\lim_{r\to+\infty} \underline{W}(r,d) = B(d) - \underline{B}^{(1)}(c^1) > 0$, see Remark 3.9. It follows that $\delta = \min\{\overline{\delta}, \underline{\delta}\} > 0$.

Now let us consider $\phi$ such that $\|\phi - U(|\cdot|, d)\|_{m+|\lambda_2|} < \delta$: by construction we have $\underline{U}(|x|) \le \phi(x) \le \overline{U}(|x|)$, for any $x \in \mathbb{R}^n$. Therefore

$$\underline{u}(t,x) \le u(t,x;\phi) \le \overline{u}(t,x)$$

for any $t > 0$ and any $x \in \mathbb{R}^n$. So from Lemma 3.10 we easily complete the proof. □

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Luca Bisconti

Dipartimento di Matematica e Informatica "U. Dini", Università degli Studi di Firenze, Via S. Marta 3, I-50139 Firenze, Italy

*E-mail address*: `luca.bisconti@unifi.it`

Matteo Franca

Dipartimento di Ingegneria Industriale e Scienze Matematiche, Università Politecnica delle Marche, Via Brecce Bianche, I-60131 Ancona, Italy

*E-mail address*: `franca@dipmat.univpm.it`