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BIFURCATION AND MULTIPLICITY RESULTS FOR CRITICAL MAGNETIC FRACTIONAL PROBLEMS

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Abstract. This article concerns the bifurcation phenomena and the existence of multiple solutions for a non-local boundary value problem driven by the magnetic fractional Laplacian $(-\Delta)^s_A$. In particular, we consider

$$(-\Delta)^s_A u = \lambda u + |u|^{2^*_s-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,$$

where λ is a real parameter and $\Omega \subset \mathbb{R}^n$ is an open and bounded set with Lipschitz boundary.

1. Introduction

The aim of this article is to study a critical non-local boundary value problem that could be considered as the magnetic fractional version of the famous Brézis-Nirenberg problem [9]. More precisely, we deal with the following problem

$$\begin{gathered} (-\Delta)^s_A u = \lambda u + |u|^{2^*_s-2}u, \quad \text{in } \Omega, \\ u = 0, \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open and bounded set with Lipschitz boundary $\partial\Omega$, dimension $n > 2s$ with $s \in (0,1)$, $2^*_s := 2n/(n-2s)$ is the fractional critical Sobolev exponent and $\lambda \in \mathbb{R}$ is a parameter. In particular, we are interested in the extension to the magnetic setting of a classical result due to Cerami, Fortunato and Struwe [11], that has recently been generalized to the diamagnetic fractional case in [14].

A first motivation for the study of (1.1) is the increasing interest in the non-local magnetic PDEs, driven by the so called *magnetic fractional Laplacian* $(-\Delta)^s_A$; see e.g. [1, 2, 5, 12, 15, 18, 21, 22, 36]. This operator has been introduced in [12] through the following representation, when acting on smooth complex-valued functions $u \in C^\infty_0(\mathbb{R}^n, \mathbb{C})$,

$$(-\Delta)^s_A u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)}{|x-y|^{n+2s}} dy, \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $B(x,\varepsilon)$ denotes the ball of center x and radius ε. We can consider $(-\Delta)^s_A$ as a fractional counterpart of the magnetic Laplacian $(\nabla - \mathrm{i}A)^2$, with $A : \mathbb{R}^n \to \mathbb{R}^n$ being a L^∞_{loc}-vector potential; see e.g. [19, Chapter 7]. In this context, when $n = 3$, the curl of A represents a magnetic field acting on a charged particle.

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While, it is clear that for $A = 0$ and u smooth real-valued function, $(-\Delta)^s_A$ coincides with the standard fractional Laplacian defined as principal value integral

$$(-\Delta)^s u(x) = \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n\setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}dy, \quad x \in \mathbb{R}^n.$$

We refer to [13, 23] and the references therein for further details on the fractional Laplacian. Furthermore, $(-\Delta)^s_A$ appears quite naturally in the definition of *magnetic fractional Sobolev spaces* and some non–local functionals that has been recently used in [25, 26, 27, 28, 34] to study characterizations of magnetic Sobolev spaces in the spirit of the works by Bourgain, Brézis and Mironescu [6], Maz'ya and Shaposhnikova [20], and Nguyen [24].

We also want to mention that, for $s = 1/2$, the definition of the fractional operator in (1.2) goes back to the '80's, and is related to the proper definition of a quantized operator corresponding to the symbol of the classical relativistic Hamiltonian, namely

$$\sqrt{(\xi - A(x))^2 + m^2} + V(x), \quad (\xi, x) \in \mathbb{R}^n \times \mathbb{R}^n.$$

More precisely, it is related to the kinetic part of the above symbol. We point out that in the literature there are at least three definitions for such a quantized operator. Two of them are given in terms of pseudo-differential operators, and the third one as the square root of a non–negative operator, see e.g. [17, 16]. The survey [17] also shows that these three non–local operators are in general different, but they coincide when the vector potential A is assumed to be linear, an assumption which is equivalent to require a constant magnetic field when we are in the physically relevant situation of $\mathbb{R}^3$.

Another reason to study critical magnetic fractional problems like (1.1) is provided by the rich background of results concerning the diamagnetic fractional version of the Brézis-Nirenberg problem, namely

$$\begin{gathered} (-\Delta)^s u = \lambda u + |u|^{2^*_s-2}u, \quad \text{in } \Omega, \\ u = 0, \quad \text{in } \mathbb{R}^n \setminus \Omega. \end{gathered} \tag{1.3}$$

In [33], they prove the existence of a non-trivial weak solution of (1.3), whenever $n \geq 4s$ and $\lambda \in (0, \lambda_{1,s})$, with $\lambda_{1,s}$ the first Dirichlet eigenvalue of $(-\Delta)^s$. In [29], a generalization of a classical result due to Capozzi, Fortunato and Palmieri [10] is given, stating the existence of a solution of (1.3), for every $\lambda > 0$ which is not a variational Dirichlet eigenvalue of $(-\Delta)^s$. Another result in the same spirit is proved in [32], with the requirement that $n \in (2s, 4s)$. One of the key points of the aforementioned papers is the introduction of a suitable functional space which allows them to encode in a proper way the non–local boundary condition of (1.3). Also, a crucial ingredient is the knowledge of the family of functions attaining the best fractional Sobolev constant. This latter aspect is however not necessary to prove the fractional counterpart of [11] in [14]. We mention here [3, 35] for other results concerning fractional Brézis-Nirenberg-type problems.

Motivated by the above papers, we study a bifurcation phenomena for problem (1.1) by a variational approach. However, because of the presence of a critical term in (1.1) and the lack of compactness at critical level $L^{2^*_s}$, the related Euler–Lagrange functional does not satisfy a global Palais-Smale condition. For this, as in the classical and in the fractional diamagnetic cases, we provide the Palais-Smale condition in a certain range strongly depending on the best *magnetic fractional Sobolev constant* given by

$$\mathcal{S}_A := \inf_{v\in X_{0,A}\setminus\{0\}} \frac{\displaystyle\iint_{\mathbb{R}^{2n}} \frac{|v(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}{\Big(\displaystyle\int_\Omega |v(x)|^{2^*}\,dx\Big)^{2/2^*}}, \tag{1.4}$$

where $X_{0,A}$ denotes the suitable functional space, where finding solutions of (1.1). See Section 2 for a detailed description of $X_{0,A}$. Of course by the continuous embedding $X_{0,A} \hookrightarrow L^{2^*_s}(\Omega,\mathbb{C})$, given in [15, Lemma 2.2], the constant $\mathcal{S}_A$ is well defined and strictly positive.

Considering the presence of a linear term in (1.1), we also study the following auxiliary eigenvalue problem

$$\begin{gathered}(-\Delta)^s_A u = \lambda u, \quad \text{in } \Omega,\\ u = 0, \quad \text{in } \mathbb{R}^n\setminus\Omega.\end{gathered} \tag{1.5}$$

Inspired by the classical case, we call *variational Dirichlet eigenvalues*, the values of the real λ's for which (1.5) admits a non-trivial weak solution $u\in X_{0,A}$, that will be called *eigenfunction*. Among many properties, we will show that there is a sequence of isolated variational Dirichlet eigenvalues whose associated eigenfuctions form an orthogonal basis of the space $X_{0,A}$. To the best of our knowledge, a systematic study and variational characterizations (akin to the classical one) of the variational Dirichlet eigenvalues of the magnetic fractional Laplacian $(-\Delta)^s_A$ are not explicitly stated in the literature. We refer to Section 3 for more details. In the diamagnetic case, we recall that the variational Dirichlet eigenvalues of $(-\Delta)^s$ are introduced in [31].

We are now ready to state our main result.

Theorem 1.1. *Let $s\in(0,1)$, $n>2s$, $\Omega\subset\mathbb{R}^n$ be an open and bounded set with Lipschitz continuous boundary $\partial\Omega$. Let $\lambda\in\mathbb{R}$ and let λ^* be the variational Dirichlet eigenvalue of problem* (1.5) *given by*

$$\lambda^* := \min\{\lambda_k : \lambda < \lambda_k\}. \tag{1.6}$$

Call $m\in\mathbb{N}$ its multiplicity. Assume that

$$\lambda \in \Big(\lambda^* - \frac{\mathcal{S}_A}{|\Omega|^{2s/n}}, \lambda^*\Big), \tag{1.7}$$

with $\mathcal{S}_A$ defined in (1.4). *Then problem* (1.1) *admits at least m pairs of non-trivial weak solutions $\{-u_{\lambda,i}, u_{\lambda,i}\}$, whose functional norm satisfies*

$$\|u_{\lambda,i}\|_{X_{0,A}} \to 0, \quad \text{as } \lambda\to\lambda^*,$$

for every $i=1,\dots,m$.

The proof of Theorem 1.1 relies on an abstract result due to Bartolo, Benci and Fortunato [4] and on the study of the variational Dirichlet eigenvalues performed in Section 3.

This paper is organized as follows. In Section 2, we introduce the necessary functional and variational setup to study the boundary value problem (1.1). In Section 3, we study the variational Dirichlet eigenvalues of $(-\Delta)^s_A$. In Section 4, we provide the Palais–Smale condition on a suitable range. In the last Section 5, we finally prove Theorem 1.1.

2. Functional and variational setup

Throughout this article, we indicate with $|\Omega|$ the n-dimensional Lebesgue measure of a measurable set $\Omega \subset \mathbb{R}^n$. Moreover, for every $z \in \mathbb{C}$ we denote by $\Re z$ its real part, and by $\overline{z}$ its complex conjugate. Let $\Omega \subset \mathbb{R}^n$ be an open set. We denote by $L^2(\Omega, \mathbb{C})$ the space of measurable functions $u : \Omega \to \mathbb{C}$ such that

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u(x)|^2 \, dx \right)^{1/2} < \infty,$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{C}$.

For $s \in (0,1)$, we define the magnetic Gagliardo semi-norm as

$$[u]_{H^s_A(\Omega)} := \Big(\iint_{\Omega \times \Omega} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \Big)^{1/2}.$$

We denote by $H^s_A(\Omega)$ the space of functions $u \in L^2(\Omega, \mathbb{C})$ such that $[u]_{H^s_A(\Omega)} < \infty$, normed with

$$\|u\|_{H^s_A(\Omega)} := \left(\|u\|^2_{L^2(\Omega)} + [u]^2_{H^s_A(\Omega)} \right)^{1/2}.$$

For $A = 0$, this definition is consistent with the usual fractional space $H^s(\Omega)$. We stress out that $C^\infty_0(\mathbb{R}^n, \mathbb{C}) \subseteq H^s_A(\mathbb{R}^n)$, see [12, Proposition 2.2].

However, to encode the boundary condition $u = 0$ in $\mathbb{R}^n \setminus \Omega$, the natural functional space introduced in [15] to deal with weak solutions of problem (1.1) is

$$X_{0,A} := \big\{ u \in H^s_A(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \big\},$$

which generalizes to the magnetic framework the space introduced in [30]. As in [12], we define the following real scalar product on $X_{0,A}$

$$\langle u, v \rangle_{X_{0,A}} := \Re \iint_{\mathbb{R}^{2n}} \frac{\big(u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)\big) \overline{\big(v(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} v(y)\big)}}{|x-y|^{n+2s}} \, dxdy,$$

which induces the norm

$$\|u\|_{X_{0,A}} := \Big(\iint_{\mathbb{R}^{2n}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^{n+2s}} \, dxdy \Big)^{1/2}.$$

By arguing as in [30, Lemma 7], we see that $(X_{0,A}, \langle \cdot, \cdot \rangle_{X_{0,A}})$ is a Hilbert space and hence reflexive.

We can now describe the variational formulation of problem (1.1). We will say that a function $u \in X_{0,A}$ is a weak solution of (1.1) if

$$\langle u, \varphi \rangle_{X_{0,A}} = \lambda \Re \int_\Omega u(x) \overline{\varphi(x)} \, dx + \Re \int_\Omega |u(x)|^{2^*_s - 2} u(x) \overline{\varphi(x)} \, dx,$$

for every $\varphi \in X_{0,A}$. Clearly, the weak solutions of (1.1) are the critical points of the Euler–Lagrange functional $\mathcal{J}_{A,\lambda} : X_{0,A} \to \mathbb{R}$, associated with (1.1), that is

$$\mathcal{J}_{A,\lambda}(u) := \|u\|^2_{X_{0,A}} - \frac{\lambda}{2} \|u\|^2_{L^2(\Omega)} - \frac{1}{2^*_s} \|u\|^{2^*_s}_{L^{2^*_s}(\Omega)}. \tag{2.1}$$

It is easy to see that $\mathcal{J}_{A,\lambda}$ is well-defined and of class $C^1(X_{0,A}, \mathbb{R})$.

Hence, to prove Theorem 1.1, we apply the following abstract critical point theorem to our functional $\mathcal{J}_{A,\lambda}$.

Theorem 2.1 ([4, Theorem 2.4]). *Let $(H, \|\cdot\|_H)$ be a real Hilbert space. Let $\mathcal{J} : H \to \mathbb{R}$ be a C^1 functional satisfying the following conditions:*

(A1) $\mathcal{J}(u) = \mathcal{J}(-u)$ *and* $\mathcal{J}(0) = 0$;
(A2) *there exists a positive real constant* $c > 0$ *such that* $\mathcal{J}$ *satisfies the Palais-Smale condition at the level c;*
(A3) *there exist two closed subspaces* V, $W \subset H$, *and there exist positive real constants* ρ, δ, $\beta > 0$, *with* $\delta < \beta < c$, *such that*
 (i) $\mathcal{J}(w) \le \beta$, *for every* $w \in W$;
 (ii) $\mathcal{J}(v) \ge \delta$, *for every* $v \in V$ *with* $\|v\|_H = \rho$;
 (iii) $\operatorname{codim} V < \infty$ *and* $\dim W \ge \operatorname{codim} V$.

Then, there exist at least $\dim W - \operatorname{codim} V$ *pairs of critical points of the functional* $\mathcal{J}$, *with critical values contained in the interval* $[\delta, \beta]$.

3. Variational Dirichlet eigenvalues of the fractional magnetic Laplacian

In this section we define the variational Dirichlet eigenvalues of the non-local operator $(-\Delta)^s_A$ and we provide a few results concerning them. Our guideline will be the content of [31, Proposition 9].

We say that $\lambda \in \mathbb{R}$ is a *variational Dirichlet eigenvalue* of $(-\Delta)^s_A$ with eigenfunction $u \in X_{0,A}$, if there exists a non-trivial weak solution $u \in X_{0,A}$ of (1.5), namely

$$\langle u, \varphi\rangle_{X_{0,A}} = \lambda \Re \int_\Omega u(x)\overline{\varphi(x)}\,dx, \quad \text{for every } \varphi \in X_{0,A}. \tag{3.1}$$

To simplify the readability, from now on we will write just *eigenvalue* in place of variational Dirichlet eigenvalue. The rest of this section is devoted to prove some properties of the eigenvalues of the fractional magnetic Laplacian. For this, let us define

$$\Phi(u) = \frac{1}{2}\|u\|^2_{X_{0,A}}.$$

Then Φ admits a local minimum, as follows.

Lemma 3.1. *Let* X^*_A *be a non-empty weakly closed subspace of* $X_{0,A}$ *and let*

$$M_* := \{u \in X^*_A : \|u\|_{L^2(\Omega)} = 1\}.$$

Then there exists $u_* \in M_*$ *such that*

$$\min_{u\in M_*} \Phi(u) = \Phi(u_*), \tag{3.2}$$

and there exists $\lambda_* := 2\Phi(u_*)$ *such that*

$$\langle u, \varphi\rangle_{X_{0,A}} = \lambda_* \Re \int_\Omega u_*(x)\overline{\varphi(x)}\,dx, \quad \textit{for every } \varphi \in X^*_A. \tag{3.3}$$

Proof. Let $\{u_j\}_j \subset M_*$ be a minimizing sequence for the functional Φ, namely

$$\Phi(u_j) \to \inf_{u\in M_*} \Phi(u) \ge 0, \quad \text{as } j \to \infty. \tag{3.4}$$

We recall that the bound from below of $\inf_{u\in M_*} \Phi(u)$ follows from the fact that $\Phi(u) \ge 0$ for every $u \in X_{0,A}$. It follows that $\{\Phi(u_j)\}_j$ is a bounded sequence in $\mathbb{R}$, and therefore $\{u_j\}_j$ is a bounded sequence in $\mathbb{R}$ as well. By the reflexivity of $(X_{0,A}, \langle\cdot,\cdot\rangle_{X_{0,A}})$, there exists a weakly convergent subsequence of $\{u_j\}_j$, still denoted by $\{u_j\}_j$, and since X^*_A is weakly closed, there exists $u_* \in X^*_A$ such that

$$u_j \rightharpoonup u_*, \quad \text{in } X_{0,A}.$$

Since $\{u_j\}_j$ is a bounded sequence in $X_{0,A}$ and $X_{0,A} \hookrightarrow L^2(\Omega,\mathbb{C})$ is compact by [15, Lemma 2.2], we have that

$$u_j \to u_*, \quad \text{in } L^2(\Omega,\mathbb{C}).$$

In particular, this shows that $\|u_*\|_{L^2(\Omega)} = 1$, and therefore $u_* \in M_*$. By Fatou Lemma, we can now conclude that

$$\lim_{j\to\infty} \Phi(u_j) \geq \Phi(u_*) \geq \inf_{u\in M_*} \Phi(u),$$

which, combined with (3.4) proves (3.2).

For the second statement, by (3.2) there exists a Lagrange multiplier λ_* such that (3.3) holds true. By taking $\varphi = u_*$ in (3.3), since $\|u_*\|_{L^2(\Omega)} = 1$ we conclude $\lambda_* = \|u_*\|^2_{X_{0,A}} = 2\Phi(u_*)$. □

The second technical result establishes that eigenfunctions of (1.5) corresponding to different eigenvalues are orthogonal with respect to the real scalar product $\langle\cdot,\cdot\rangle_{X_{0,A}}$.

Lemma 3.2. *Let $\lambda \neq \widetilde{\lambda}$ be two different eigenvalues of* (1.5)*, with eigenfunctions f, $\mathfrak{f} \in X_{0,A}$. Then*

$$\Re \int_\Omega f(x)\overline{\mathfrak{f}(x)}\, dx = \langle f,\mathfrak{f}\rangle_{X_{0,A}} = 0.$$

Proof. Since both $f \not\equiv 0$ and $\mathfrak{f} \not\equiv 0$, we can set $F := f/\|f\|_{L^2(\Omega)}$ and $\mathcal{F} := \mathfrak{f}/\|\mathfrak{f}\|_{L^2(\Omega)}$ two eigenfunctions respectively of λ and $\widetilde{\lambda}$. Now, recalling that for every z, $w \in \mathbb{C}$

$$\Re(z\bar{w}) = \Re(w\bar{z}),$$

and combining it with (3.1), we obtain

$$\begin{aligned}
&\lambda\Re\int_\Omega F(x)\overline{\mathcal{F}(x)}\,dx\\
&= \Re\iint_{\mathbb{R}^{2n}} \frac{\Big(F(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}F(y)\Big)\overline{\Big(\mathcal{F}(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}\mathcal{F}(y)\Big)}}{|x-y|^{n+2s}}\,dx\,dy\\
&= \Re\iint_{\mathbb{R}^{2n}} \frac{\Big(\mathcal{F}(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}\mathcal{F}(y)\Big)\overline{\Big(F(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}F(y)\Big)}}{|x-y|^{n+2s}}\,dx\,dy\\
&= \widetilde{\lambda}\,\Re\int_\Omega \mathcal{F}(x)\overline{F(x)}\,dx;
\end{aligned} \tag{3.5}$$

that is,

$$(\lambda-\widetilde{\lambda})\Re\int_\Omega F(x)\overline{\mathcal{F}(x)}\,dx = 0.$$

Since $\lambda \neq \widetilde{\lambda}$, we have

$$\Re\int_\Omega F(x)\overline{\mathcal{F}(x)}\,dx = 0,$$

and by (3.5) this completes the proof. □

We want to provide a variational characterization of the eigenvalues by means of Rayleigh-type quotients. To this aim, we start defining the following family of subspaces, for any $k \in \mathbb{N}$,

$$\mathbb{E}_{k+1} := \big\{u \in X_{0,A} : \langle u, f_j\rangle_{X_{0,A}} = 0, \text{ for every } j = 1,\dots,k\big\}, \tag{3.6}$$

where $f_j \in X_{0,A}$ are given in Proposition 3.3. It is easy to see that

$$\mathbb{E}_{k+2} \subseteq \mathbb{E}_{k+1} \subseteq X_{0,A}, \quad \text{for any } k \in \mathbb{N}. \tag{3.7}$$

Starting from this, we can state the following results for eigenvalues and eigenfunctions of (1.5).

Proposition 3.3. *Let us define*

$$\lambda_1 := \min_{u \in X_{0,A} \setminus \{0\}} \frac{\|u\|^2_{X_{0,A}}}{\|u\|^2_{L^2(\Omega)}}, \tag{3.8}$$

and by induction

$$\lambda_{k+1} := \min_{u \in \mathbb{E}_{k+1} \setminus \{0\}} \frac{\|u\|^2_{X_{0,A}}}{\|u\|^2_{L^2(\Omega)}}, \quad \textit{for any } k \in \mathbb{N}. \tag{3.9}$$

Then λ_1 and λ_{k+1} are eigenvalues of (1.5)*.*

Also, there exist $f_1 \in X_{0,A}$ and $f_{k+1} \in \mathbb{E}_{k+1}$ which are eigenfunctions respectively of λ_1 and λ_{k+1}, and attain the minimums in (3.8) *and* (3.9)*.*

Proof. We start studying the case of λ_1. Lemma 3.1 ensures that the minimum in (3.8) is well defined, due to (3.2) with $X_A^* = X_{0,A}$, and λ_1 is an eigenvalue of $(-\Delta)^s_A$ because of (3.3). By using again (3.2), we find a function $f_1 \in X_{0,A}$, with $\|f_1\|_{L^2(\Omega)} = 1$, which attains the minimum in (3.8). In particular, by (3.3) with $X_A^* = X_{0,A}$, we see that f_1 is an eigenfunction related to λ_1.

We pass to the case of λ_{k+1} with $k \in \mathbb{N}$. First we observe that $X_A^* = \mathbb{E}_{k+1}$ is weakly closed by construction, hence Lemma 3.1 yields that the minimum in (3.9) is well defined and it is achieved by a certain function $f_{k+1} \in \mathbb{E}_{k+1}$. It remains to prove that λ_{k+1} in (3.9) is an eigenvalue with corresponding eigenfunction given by f_{k+1}. By (3.3) with $X_A^* = \mathbb{E}_{k+1}$, we have

$$\langle f_{k+1}, \varphi \rangle_{X_{0,A}} = \lambda_{k+1} \Re \int_\Omega f_{k+1}(x) \overline{\varphi(x)}\, dx, \quad \text{for every } \varphi \in \mathbb{E}_{k+1}. \tag{3.10}$$

We must prove that (3.10) holds for every $\varphi \in X_{0,A}$. Arguing by induction, we assume that the claim holds for $1, \dots, k$. The base of induction has been already proved, since λ_1 is an eigenvalue of (1.5). Now, we can decompose the space $X_{0,A}$ as

$$X_{0,A} = \operatorname{span}_\mathbb{R}\{f_1, \dots, f_k\} \oplus (\operatorname{span}_\mathbb{R}\{f_1, \dots, f_k\})^\perp = \operatorname{span}_\mathbb{R}\{f_1, \dots, f_k\} \oplus \mathbb{E}_{k+1},$$

where the orthogonal complement has to be intended with respect to the real scalar product $\langle \cdot, \cdot \rangle_{X_{0,A}}$ defined on $X_{0,A}$. The former decomposition implies that we can write any function $w \in X_{0,A}$ as $w = w_1 + w_2$, with $w_2 \in \mathbb{E}_{k+1}$, and

$$w_1 = \sum_{i=1}^k c_i f_i,$$

where c_i are real constants. By (3.10) with $\varphi = w_2 = w - w_1$, we obtain

$$\begin{aligned} &\langle f_{k+1}, w \rangle_{X_{0,A}} - \lambda_{k+1} \Re \int_\Omega f_{k+1}(x) \overline{w(x)}\, dx \\ &= \sum_{i=1}^k c_i \langle f_{k+1}, f_i \rangle_{X_{0,A}} - \lambda_{k+1} \Re \int_\Omega f_{k+1}(x) \overline{f_i(x)}\, dx. \end{aligned} \tag{3.11}$$

By inductive assumption, we know that f_i is an eigenfunction corresponding to λ_i for every $i = 1, \dots, k$, therefore we can plug it into (3.1) finding

$$0 = \langle f_{k+1}, f_i \rangle_{X_{0,A}} = \lambda_i \Re \int_\Omega f_{k+1}(x) \overline{f_i(x)}\, dx,$$

which implies

$$\langle f_{k+1}, f_i \rangle_{X_{0,A}} = 0 = \Re \int_\Omega f_{k+1}(x) \overline{f_i(x)}\, dx,$$

for every $i = 1, \dots, k$. Plugging the former relation into (3.11), we complete the proof. □

The next three technical results are essential for proving that the eigenvalues constitute an orthonormal basis of $L^2(\Omega, \mathbb{C})$ and an orthogonal one of $X_{0,A}$, see Proposition 3.7.

Proposition 3.4. *The eigenfunctions $\{f_k\}_k$ are orthogonal with respect to both the real L^2-scalar product and the real scalar product $\langle \cdot, \cdot \rangle_{X_{0,A}}$.*

Proof. Let j, $k \in \mathbb{N}$ with $j \neq k$. Without loss of generality, we can assume that $j > k$. Therefore,

$$f_j \in \mathbb{E}_j = (\operatorname{span}_{\mathbb{R}}\{f_1, \dots, f_{j-1}\})^{\perp} \subseteq (\operatorname{span}_{\mathbb{R}}\{f_k\})^{\perp},$$

which implies

$$\langle f_j, f_k \rangle_{X_{0,A}} = 0.$$

Now, since f_j is an eigenfunction, by (3.1) we obtain

$$0 = \langle f_j, f_k \rangle_{X_{0,A}} = \lambda_j \Re \int_\Omega f_j(x) \overline{f_k(x)}\, dx,$$

as desired. □

Proposition 3.5. *The eigenvalues of problem* (1.5) *form a sequence of real positive numbers $\{\lambda_k\}_k$ with*

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots, \tag{3.12}$$

and

$$\lambda_k \to \infty, \quad \text{as } k \to \infty. \tag{3.13}$$

Proof. By Proposition 3.3 we know that $\lambda_1 > 0$, while by (3.7) follows (3.12). To prove (3.13), assume by contradiction that there exists a positive real constant $l > 0$ such that $\lambda_k \to l$ as $k \to \infty$. This implies that the sequence $\{\lambda_k\}_k$ is bounded in $\mathbb{R}$. Moreover, since by Proposition 3.3 f_k attains the minimum in (3.9), we have $\|f_k\|^2_{X_{0,A}} = \lambda_k$. Thus, by [15, Lemma 2.2] there exists a subsequence $\{f_{k_j}\}_j \subset X_{0,A}$ such that

$$f_{k_j} \to f, \quad \text{in } L^2(\Omega, \mathbb{C}),$$

for some $f \in L^2(\Omega, \mathbb{C})$. This implies that $\{f_{k_j}\}_j$ is a Cauchy sequence in $L^2(\Omega, \mathbb{C})$. On the other hand, when $i \neq j$ by Proposition 3.4 we have that f_{k_j} and f_{k_i} are orthogonal in $L^2(\Omega, \mathbb{C})$, hence

$$\|f_{k_j} - f_{k_i}\|^2_{L^2(\Omega)} = \|f_{k_j}\|^2_{L^2(\Omega)} + \|f_{k_i}\|^2_{L^2(\Omega)} = 2,$$

which yields a contradiction.

We have to prove that the sequence $\{\lambda_k\}_k$ provides all the eigenvalues of (1.5). Suppose that there exists at least one eigenvalue $\lambda \notin \{\lambda_k\}_k$. Let us denote by

$f \in X_{0,A}$ its corresponding eigenfunction. Without loss of generality, we can also assume that $\|f\|_{L^2(\Omega)} = 1$. Now, by (3.1) with $\varphi = f$, we obtain

$$2\Phi(f) = \iint_{\mathbb{R}^{2n}} \frac{|f(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} f(y)|}{|x-y|^{n+2s}}\, dx\, dy = \lambda. \tag{3.14}$$

Since λ_1 is minimal in (3.8), it holds that

$$\lambda_1 = \Phi(f_1) \le \Phi(f) = \lambda.$$

For this and (3.12), we have that there exists $j \in \mathbb{N}$ such that

$$\lambda_j < \lambda < \lambda_{j+1}.$$

We claim now that $f \notin \mathbb{E}_{j+1}$. Indeed, if $f \in \mathbb{E}_{j+1}$, then by (3.9) and (3.14),

$$\lambda_{j+1} \le 2\,\Phi(f) = \lambda,$$

which yields a contradiction. Thus, since $f \notin \mathbb{E}_{j+1}$, there exists $i \in \{1, \dots, j\}$ such that $\langle f, f_i \rangle_{X_{0,A}} \ne 0$, which is in contrast with Lemma 3.2. This concludes the proof. $\square$

Lemma 3.6. *If $v \in X_{0,A}$ is such that $\langle v, f_k \rangle_{X_{0,A}} = 0$ for every $k \in \mathbb{N}$, then $v \equiv 0$.*

Proof. By contradiction, suppose that there exists $v \in X_{0,A}$ with $v \not\equiv 0$ and such that $\langle v, f_k \rangle_{X_{0,A}} = 0$ for every $k \in \mathbb{N}$. Without loss of generality, we can assume that $\|v\|_{L^2(\Omega)} = 1$. By (3.13), we know that there exists $k \in \mathbb{N}$ such that

$$2\Phi(v) < \lambda_{k+1}.$$

Therefore $v \notin \mathbb{E}_{k+1}$, and this implies that there exists $j \in \mathbb{N}$ such that $\langle v, f_j \rangle_{X_{0,A}} \ne 0$, which is impossible. This completes the proof. $\square$

Proposition 3.7. *The sequence of eigenfunctions $\{f_k\}_k$ corresponding to $\{\lambda_k\}_k$ is an orthonormal basis of $L^2(\Omega, \mathbb{C})$ and an orthogonal basis of $X_{0,A}$.*

Proof. The orthogonality follows from Proposition 3.4. Let us prove that $\{f_k\}_k$ is a basis of $X_{0,A}$. To this aim, let us define $F_k := f_k / \|f_k\|_{X_{0,A}}$. Moreover, for any $g \in X_{0,A}$, we consider

$$g_j := \sum_{k=1}^{j} \langle g, F_k \rangle_{X_{0,A}} F_k,$$

which yields that $g_j \in \mathrm{span}_{\mathbb{R}}\{f_1, \dots, f_j\}$ for any $j \in \mathbb{N}$. We want to show that $g_j \to g$ in $X_{0,A}$ as $j \to \infty$. We define $v_j := g - g_j$ and, since by Proposition 3.4, the $\{f_k\}_k$ are orthogonal with respect to $\langle \cdot, \cdot \rangle_{X_{0,A}}$, we have

$$\begin{aligned} 0 \le \|v_j\|^2_{X_{0,A}} &= \|g\|^2_{X_{0,A}} + \|g_j\|^2_{X_{0,A}} - 2\langle g, g_j \rangle_{X_{0,A}} \\ &= \|g\|^2_{X_{0,A}} + \langle g_j, g_j \rangle_{X_{0,A}} - 2\sum_{k=1}^{j} \langle g, F_k \rangle^2_{X_{0,A}} \\ &= \|g\|^2_{X_{0,A}} - \sum_{k=1}^{j} \langle g, F_k \rangle^2_{X_{0,A}}, \end{aligned}$$

which implies that

$$\sum_{k=1}^{j} \langle g, F_k \rangle^2_{X_{0,A}} \le \|g\|^2_{X_{0,A}},$$

and hence $\sum_{k=1}^{\infty}\langle g,F_k\rangle^2_{X_{0,A}}$ is a convergent series. We can then consider the partial sum

$$\tau_j:=\sum_{k=1}^{j}\langle g,F_k\rangle^2_{X_{0,A}},$$

which is a Cauchy sequence in $\mathbb{R}$. Moreover, due to the orthogonality of $\{f_k\}_k$ with respect to $\langle\cdot,\cdot\rangle_{X_{0,A}}$, we have

$$\|v_h-v_j\|^2_{X_{0,A}}=\Big\|\sum_{k=j+1}^{h}\langle g,F_k\rangle_{X_{0,A}}F_k\Big\|^2_{X_{0,A}}=\tau_h-\tau_j, \tag{3.15}$$

for $h>j$. Since $\{\tau_j\}_j$ is a Cauchy sequence in $\mathbb{R}$, (3.15) implies that $\{v_j\}_j$ is a Cauchy sequence in $X_{0,A}$. Hence, there exists $v\in X_{0,A}$ such that $v_j\to v$ in $X_{0,A}$ as $j\to\infty$. We now note that, if $j\geq k$, then $\langle g_j,F_k\rangle_{X_{0,A}}=\langle g,F_k\rangle_{X_{0,A}}$. Therefore, for $j\geq k$,

$$\langle v_j,F_k\rangle_{X_{0,A}}=\langle g,F_k\rangle_{X_{0,A}}-\langle g_j,F_k\rangle_{X_{0,A}}=0.$$

On the other hand, since $v_j\to v$ in $X_{0,A}$, we obtain

$$\langle v,F_k\rangle_{X_{0,A}}=0,\quad \text{for every } k\in\mathbb{N}.$$

By Lemma 3.6, we know that $v\equiv 0$, hence

$$g_j=g-v_j\;\to\; g-v=g\in X_{0,A},\quad \text{as } j\to\infty,$$

and this shows that $\{f_k\}_k$ is a basis in $X_{0,A}$, since $g_j\in\operatorname{span}_{\mathbb{R}}\{f_1,\dots,f_j\}$ for any $j\in\mathbb{N}$.

Let us conclude by showing that $\{f_k\}_k$ is a basis in $L^2(\Omega,\mathbb{C})$. For this, let $v\in L^2(\Omega,\mathbb{C})$ and $v_j\in C_0^\infty(\Omega,\mathbb{C})$ be such that $\|v_j-v\|_{L^2(\Omega)}\leq 1/j$. By combining [12, Proposition 2.2] and [15, Lemma 2.1], we have $C_0^\infty(\Omega,\mathbb{C})\subset X_{0,A}$, implying $v_j\in X_{0,A}$. We also know that $\{f_k\}_k$ is a basis of $X_{0,A}$, therefore there exists $k_j\in\mathbb{N}$ and a function $w_j\in\operatorname{span}_{\mathbb{R}}\{f_1,\dots,f_{k_j}\}$ such that

$$\|v_j-w_j\|_{X_{0,A}}\leq\frac{1}{j}.$$

Now, by [15, Lemma 2.1] we have

$$\|v_j-w_j\|_{L^2(\Omega)}\leq\|v_j-w_j\|_{H^s_A(\mathbb{R}^n)}\leq C\|v_j-w_j\|_{X_{0,A}}\leq\frac{C}{j}.$$

By triangle inequality, we obtain that

$$\|v-w_j\|_{L^2(\Omega)}\leq\|v-v_j\|_{L^2(\Omega)}+\|v_j-w_j\|_{L^2(\Omega)}\leq\frac{C+1}{j},$$

which shows that $\{f_k\}_k$ is a basis of $L^2(\Omega,\mathbb{C})$. $\square$

We conclude this section with a proposition showing that for any $k\in\mathbb{N}$ each eigenvalue λ_k has finite multiplicity.

Proposition 3.8. *Let $h\geq 0$ and $k\in\mathbb{N}$. If λ_k has multiplicity $h+1$, namely*

$$\lambda_{k-1}<\lambda_k=\ldots=\lambda_{k+h}<\lambda_{k+h+1},$$

then the set of the eigenfunctions corresponding to λ_k is given by

$$\operatorname{span}_{\mathbb{R}}\{f_k,\dots,f_{k+h}\}.$$

Proof. By Proposition 3.3, we know that every element $g \in \operatorname{span}_{\mathbb{R}}\{f_k, \dots, f_{k+h}\}$ is an eigenfunction of (1.5) corresponding to the eigenvalue $\lambda_k = \dots = \lambda_{k+h}$.

Conversely, we need to show that any eigenfunction $\psi \in X_{0,A}$ corresponding to λ_k belongs to $\operatorname{span}_{\mathbb{R}}\{f_k, \dots, f_{k+h}\}$. First, we consider the orthogonal decomposition

$$X_{0,A} = \operatorname{span}_{\mathbb{R}}\{f_k, \dots, f_{k+h}\} \oplus \left(\operatorname{span}_{\mathbb{R}}\{f_k, \dots, f_{k+h}\}\right)^{\perp}.$$

Thus, we can write $\psi = \psi_1 + \psi_2$, where

$$\psi_1 \in \operatorname{span}_{\mathbb{R}}\{f_k, \dots, f_{k+h}\} \quad \text{and} \quad \psi_2 \in \left(\operatorname{span}_{\mathbb{R}}\{f_k, \dots, f_{k+h}\}\right)^{\perp}, \tag{3.16}$$

which implies by Proposition 3.4,

$$\langle \psi_1, \psi_2 \rangle_{X_{0,A}} = 0. \tag{3.17}$$

Being ψ an eigenfunction corresponding to λ_k, by (3.1) and (3.17) we obtain

$$\lambda_k \|\psi\|^2_{L^2(\Omega)} = \|\psi\|^2_{X_{0,A}} = \|\psi_1\|^2_{X_{0,A}} + \|\psi_2\|^2_{X_{0,A}}. \tag{3.18}$$

Since $\{f_k, \dots, f_{k+h}\}$ are eigenfunctions corresponding to λ_k by Proposition 3.3, ψ_1 is an eigenfunction corresponding to λ_k as well. Therefore, by also (3.17) we obtain

$$\lambda_k \Re \int_\Omega \psi_1(x)\overline{\psi_2(x)}\, dx = \langle \psi_1, \psi_2 \rangle_{X_{0,A}} = 0,$$

which implies

$$\|\psi\|^2_{L^2(\Omega)} = \|\psi_1 + \psi_2\|^2_{L^2(\Omega)} = \|\psi_1\|^2_{L^2(\Omega)} + \|\psi_2\|^2_{L^2(\Omega)}. \tag{3.19}$$

Now, by definition of ψ_1, there exist $c_k, \dots, c_{k+h}$ real constants, such that

$$\psi_1 = \sum_{i=k}^{k+h} c_i\, f_i.$$

Byo the orthogonality of $\{f_i\}_i$ in Proposition 3.4 and considering each f_i attains the minimum in (3.9) by Proposition 3.3, we have

$$\|\psi_1\|^2_{X_{0,A}} = \sum_{i=k}^{k+h} c_i^2 \|f_i\|^2_{X_{0,A}} = \sum_{i=k}^{k+h} c_i^2 \lambda_i = \lambda_k \sum_{i=k}^{k+h} c_i^2 = \lambda_k \|\psi_1\|^2_{L^2(\Omega)}. \tag{3.20}$$

Now, since ψ and ψ_1 are eigenfunctions corresponding to λ_k, and $\psi_2 = \psi - \psi_1$, it follows that ψ_2 must be an eigenfunction corresponding to λ_k as well. Therefore, by Lemma 3.2

$$\langle \psi_2, f_1 \rangle_{X_{0,A}} = \dots = \langle \psi_2, f_{k-1} \rangle_{X_{0,A}} = 0,$$

which, together with (3.16), implies

$$\psi_2 \in \left(\operatorname{span}_{\mathbb{R}}\{f_1, \dots, f_{k+h}\}\right)^{\perp} = \mathbb{E}_{k+h+1}. \tag{3.21}$$

Now, we claim that $\psi_2 \equiv 0$. Assume by contradiction that this is not the case. Then, by (3.9) and (3.21)

$$\lambda_k < \lambda_{k+h+1} = \min_{u \in \mathbb{E}_{k+h+1}} \frac{\|u\|^2_{X_{0,A}}}{\|u\|^2_{L^2(\Omega)}} \le \frac{\|\psi_2\|^2_{X_{0,A}}}{\|\psi_2\|^2_{L^2(\Omega)}}. \tag{3.22}$$

We can now compute by (3.18), (3.19), (3.20) and (3.22)

$$\lambda_k \|\psi\|^2_{L^2(\Omega)} = \|\psi_1\|^2_{X_{0,A}} + \|\psi_2\|^2_{X_{0,A}} > \lambda_k \|\psi_1\|^2_{L^2(\Omega)} + \lambda_k \|\psi_2\|^2_{L^2(\Omega)} = \lambda_k \|\psi\|^2_{L^2(\Omega)},$$

which yields a contradiction, hence proving that $\psi_2 \equiv 0$. In particular, this and (3.16) imply that

$$\psi = \psi_1 \in \operatorname{span}_{\mathbb{R}}\{f_k, \dots, f_{k+h}\},$$

as desired. □

4. Palais-Smale condition

In this section we verify that the functional $\mathcal{J}_{A,\lambda}$ satisfies the $(PS)_c$ condition under a suitable level, depending on the magnetic Sobolev constant given in (1.4). For this, we recall that $\{u_j\}_j \subset X_{0,A}$ is a *Palais-Smale sequence for* $\mathcal{J}_{A,\lambda}$ *at level* $c \in \mathbb{R}$ (in short $(PS)_c$ sequence) if

$$\mathcal{J}_{A,\lambda}(u_j) \to c \quad \text{and} \quad \mathcal{J}'_{A,\lambda}(u_j) \to 0 \quad \text{as } j \to \infty. \tag{4.1}$$

We say that $\mathcal{J}_{A,\lambda}$ *satisfies the Palais-Smale condition at level* c if any Palais–Smale sequence $\{u_j\}_j$ at level c admits a convergent subsequence in $X_{0,A}$.

Proposition 4.1. *Let $c \in (-\infty, s\,\mathcal{S}_A^{n/(2s)}/n)$. Then the functional $\mathcal{J}_{A,\lambda}$ satisfies the Palais-Smale condition at any level c.*

Proof. The proof follows its diamagnetic counterpart in [14]. Let $\{u_j\}_j \subset X_{0,A}$ be any sequence verifying (4.1). As usual, we first need to prove the boundedness of $\{u_j\}_j$ in $X_{0,A}$. To this aim, since $u_j \not\equiv 0$ and by (4.1), there exists a positive constant $k > 0$, independent of j, such that

$$|\mathcal{J}_{A,\lambda}(u_j)| \leq k \quad \text{and} \quad \Big|\Big\langle \mathcal{J}'_{A,\lambda}(u_j), \frac{u_j}{\|u_j\|_{X_{0,A}}}\Big\rangle\Big| \leq k, \tag{4.2}$$

for every $j \in \mathbb{N}$. Now, a direct computation combined with (4.2), shows that

$$k\,(1 + \|u_j\|_{X_0,A}) \geq \mathcal{J}_{A,\lambda}(u_j) - \frac{1}{2}\langle \mathcal{J}'_{A,\lambda}(u_j), u_j\rangle = \frac{s}{n}\|u_j\|^{2^*_s}_{L^{2^*_s}(\Omega)}, \quad \text{for any } j \in \mathbb{N}. \tag{4.3}$$

From this, by Hölder inequality and considering that $2/2^*_s < 1$, we have

$$\|u_j\|^2_{L^2(\Omega)} \leq k_*\big(1 + \|u_j\|_{X_{0,A}}\big), \quad \text{for any } j \in \mathbb{N}, \tag{4.4}$$

with $k_* > 0$ another suitable constant independent of j. By (4.2), (4.3) and (4.4), it follows that there exists a constant $\widetilde{k} > 0$, independent of j, such that

$$k \geq \mathcal{J}_{A,\lambda}(u_j) \geq \frac{1}{2}\|u_j\|^2_{X_{0,A}} - \widetilde{k}\left(1 + \|u_j\|_{X_{0,A}}\right), \quad \text{for any } j \in \mathbb{N},$$

and this is sufficient to conclude that the sequence $\{u_j\}_j \subset X_{0,A}$ is bounded.

Since the space $(X_{0,A}, \langle\cdot,\cdot\rangle_{X_{0,A}})$ is a Hilbert space, there exists $u_\infty \in X_{0,A}$ such that $u_j \rightharpoonup u_\infty$ in $X_{0,A}$ as $j \to \infty$, that is

$$\begin{aligned}
&\lim_{j\to\infty} \Re \iint_{\mathbb{R}^{2n}} \frac{\big(u_j(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u_j(y)\big)\overline{\big(\varphi(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}\varphi(y)\big)}}{|x-y|^{n+2s}}\,dx\,dy \\
&= \Re \iint_{\mathbb{R}^{2n}} \frac{\big(u_\infty(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u_\infty(y)\big)\overline{\big(\varphi(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}\varphi(y)\big)}}{|x-y|^{n+2s}}\,dx\,dy,
\end{aligned} \tag{4.5}$$

for any $\varphi \in X_{0,A}$. Also, by [15, Lemma 2.2] and [7, Theorem 4.9], up to a subsequence still labeled by $\{u_j\}_j$, we have

$$u_j \rightharpoonup u_\infty \text{ in } L^{2^*_s}(\Omega,\mathbb{C}), \quad u_j \to u_\infty \text{ in } L^2(\Omega,\mathbb{C}), \quad u_j \to u_\infty \text{ a.e in } \Omega, \tag{4.6}$$

as $j\to\infty$. By (4.3) and the boundedness of $\{u_j\}_j$ in X_0, we have that $\|u_j\|_{L^{2^*_s}(\Omega)}$ is uniformly bounded in j, and therefore the sequence $\{|u_j|^{2^*_s-2}u_j\}_j$ is uniformly bounded in $L^{2^*_s/(2^*_s-1)}(\Omega)$. Thus, by (4.6)

$$|u_j|^{2^*_s-2}u_j \rightharpoonup |u_\infty|^{2^*_s-2}u_\infty, \quad \text{in } L^{2^*_s/(2^*_s-1)}(\Omega,\mathbb{C}), \tag{4.7}$$

as $j\to\infty$.

Now, by passing to the limit as $j\to\infty$ on (4.1) and considering (4.5), (4.6) and (4.7), we have

$$\begin{aligned}
&\Re\iint_{\mathbb{R}^{2n}}\frac{\big(u_\infty(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u_\infty(y)\big)\overline{\big(\varphi(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}\varphi(y)\big)}}{|x-y|^{n+2s}}\,dxdy\\
&=\lambda\Re\int_\Omega u(x)\overline{\varphi(x)}\,dx+\Re\int_\Omega|u(x)|^{2^*_s-2}u(x)\overline{\varphi(x)}\,dx,
\end{aligned}\tag{4.8}$$

for any $\varphi\in X_{0,A}$. This proves that $u_\infty\in X_{0,A}$ is a weak solution of (1.1).

By (4.5) and since $(X_{0,A},\langle\cdot,\cdot\rangle_{X_{0,A}})$ is a Hilbert space, we have

$$\|u_j\|^2_{X_{0,A}}=\|u_j-u_\infty\|^2_{X_{0,A}}+\|u_\infty\|^2_{X_{0,A}}+o(1), \quad \text{as } j\to\infty. \tag{4.9}$$

While, by (4.6) and [8, Theorem 1] we have

$$\|u_j\|^{2^*_s}_{L^{2^*_s}(\Omega)}=\|u_j-u_\infty\|^{2^*_s}_{L^{2^*_s}(\Omega)}+\|u_\infty\|^{2^*_s}_{L^{2^*_s}(\Omega)}+o(1), \quad \text{as } j\to\infty. \tag{4.10}$$

Therefore, by (4.6), (4.9) and (4.10), we obtain

$$\begin{aligned}
\mathcal{J}_{A,\lambda}(u_j)&=\frac12\|u_j-u_\infty\|^2_{X_{0,A}}+\frac12\|u_\infty\|^2_{X_{0,A}}-\frac{\lambda}{2}\|u_\infty\|^2_{L^2(\Omega)}\\
&\quad-\frac{1}{2^*_s}\|u_j-u_\infty\|^{2^*_s}_{L^{2^*_s}(\Omega)}-\frac{1}{2^*_s}\|u_\infty\|^{2^*_s}_{L^{2^*_s}(\Omega)}+o(1)\\
&=\mathcal{J}_{A,\lambda}(u_\infty)+\frac12\|u_j-u_\infty\|^2_{X_{0,A}}-\frac{1}{2^*_s}\|u_j-u_\infty\|^{2^*_s}_{L^{2^*_s}(\Omega)}+o(1),
\end{aligned}\tag{4.11}$$

as $j\to\infty$.

Furthermore, by (4.1), (4.5), (4.6), (4.7) and considering u_∞ as solution of (1.1), we obtain

$$\begin{aligned}
o(1)&=\langle\mathcal{J}'_{A,\lambda}(u_j)-\mathcal{J}'_{A,\lambda}(u_\infty),u_j-u_\infty\rangle\\
&=\iint_{\mathbb{R}^{2n}}\frac{|u_j(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u_j(y)|^2}{|x-y|^{n+2s}}dxdy\\
&\quad-2\Re\iint_{\mathbb{R}^{2n}}\frac{\big(u_j(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u_j(y)\big)\overline{\big(u_\infty(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u_\infty(y)\big)}}{|x-y|^{n+2s}}dx\,dy\\
&\quad+\iint_{\mathbb{R}^{2n}}\frac{|u_\infty(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u_\infty(y)|^2}{|x-y|^{n+2s}}dxdy\\
&\quad-\Re\int_\Omega\Big(|u_j(x)|^{2^*_s-2}u_j(x)-|u_\infty(x)|^{2^*_s-2}u_\infty(x)\Big)\Big(\overline{u_j(x)}-\overline{u_\infty(x)}\Big)\,dx+o(1)\\
&=\iint_{\mathbb{R}^{2n}}\frac{|u_j(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u_j(y)|^2}{|x-y|^{n+2s}}dxdy\\
&\quad-\iint_{\mathbb{R}^{2n}}\frac{|u_\infty(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u_\infty(y)|^2}{|x-y|^{n+2s}}dxdy
\end{aligned}$$

$$- \int_\Omega |u_j(x)|^{2^*_s} dx + \int_\Omega |u_\infty(x)|^{2^*_s} dx + o(1) \quad \text{as } j \to \infty,$$

from which, by (4.9) and (4.10), we obtain the formula

$$\|u_j - u_\infty\|^2_{X_{0,A}} = \|u_j - u_\infty\|^{2^*_s}_{L^{2^*_s}(\Omega)} + o(1), \tag{4.12}$$

as $j \to \infty$. Hence, combining (4.11) and (4.12), we obtain

$$\mathcal{J}_{A,\lambda}(u_j) = \mathcal{J}_{A,\lambda}(u_\infty) + \frac{s}{n}\|u_j - u_\infty\|^2_{X_{0,A}} + o(1), \quad \text{as } j \to \infty.$$

From this, recalling (4.1), combining (2.1) and (4.8) with $\varphi = u_\infty$, we obtain as $j \to \infty$

$$\begin{aligned} c &= \mathcal{J}_{A,\lambda}(u_\infty) + \frac{s}{n}\|u_j - u_\infty\|^2_{X_{0,A}} + o(1) \\ &= \frac{s}{n}\Big(\|u_\infty\|^2_{X_{0,A}} + \|u_j - u_\infty\|^2_{X_{0,A}}\Big) + o(1). \end{aligned} \tag{4.13}$$

By the boundedness of $\{u_j\}_j$ in X_0, up to a subsequence, we also have

$$\|u_j - u_\infty\|^2_{X_{0,A}} \to L \in [0,\infty), \quad \text{as } j \to \infty. \tag{4.14}$$

Combining formula (4.12) with (4.14), and recalling (1.4), we obtain

$$L^{2/2^*_s}\mathcal{S}_A \le L$$

which implies that either $L = 0$ or $L \ge \mathcal{S}_A^{n/(2s)}$. Let us assume by contradiction $L \ge \mathcal{S}_A^{n/(2s)}$. Then, by (4.13) and (4.14) we obtain

$$c = \frac{s}{n}(\|u_\infty\|^2_{X_{0,A}} + L) \ge \frac{s}{n}L \ge \frac{s}{n}\mathcal{S}_A^{n/(2s)},$$

in contrast with our standing assumption on c. Therefore $L = 0$ and from (4.14) we conclude the proof. □

5. Proof of Theorem 1.1

In this section we prove our main result given in Theorem 1.1. To state the number of solutions of problem (1.1), we must explicitly describe λ^* with its multiplicity, as in (1.6). Since, $\lambda^* = \lambda_k$ for some $k \in \mathbb{N}$, with multiplicity $m \in \mathbb{N}$ by assumption, we have that

$$\begin{gathered} \lambda^* = \lambda_1 < \lambda_2 \quad \text{if } k = 1 \\ \lambda_{k-1} < \lambda^* = \lambda_k = \dots = \lambda_{k+m-1} < \lambda_{k+m} \quad \text{if } k \ge 2\,. \end{gathered} \tag{5.1}$$

Also, we have to observe that, under condition (1.7), the parameter $\lambda > 0$. Indeed, by definition of λ^* and taking into account Proposition 3.5, it is easily seen that

$$\lambda^* \ge \lambda_1. \tag{5.2}$$

Now, by Hölder inequality,

$$\|u\|^2_{L^2(\Omega)} \le |\Omega|^{2s/n}\|u\|^2_{L^{2^*_s}(\Omega)}.$$

From this, (1.4) and (3.8), we obtain

$$\lambda_1 \ge \mathcal{S}_A|\Omega|^{-2s/n}\,,$$

which combined with (5.2) yields

$$\lambda^\star \ge \mathcal{S}_A|\Omega|^{-2s/n}\,.$$

Hence, as a consequence of this and of (1.7), we obtain $\lambda^\star > 0$.

Lemma 5.1. *Let $\lambda \in \mathbb{R}$ satisfy* (1.7)*, with $\lambda^\star$ in* (1.6) *counted with multiplicity $m \in \mathbb{N}$. Then, for any $w \in \operatorname{span}_{\mathbb{R}}\{f_1, \dots, f_{k+m-1}\}$, we have*

$$\mathcal{J}_{A,\lambda}(w) \leq \frac{s}{n}(\lambda^\star - \lambda)^{n/(2s)}|\Omega|.$$

Proof. Let $u \in \operatorname{span}_{\mathbb{R}}\{f_1, \dots, f_{k+m-1}\}$. Then, we have

$$u(x) = \sum_{i=1}^{k+m-1} c_i f_i(x)$$

with $c_i \in \mathbb{R}$, for $i = 1, \dots, k+m-1$.

By Proposition 3.7, and taking into account (5.1), we obtain

$$\begin{aligned} \|u\|^2_{X_{0,A}} &= \sum_{i=1}^{k+m-1} c_i^2 \|f_i\|^2_{X_{0,A}} = \sum_{i=1}^{k+m-1} \lambda_i c_i^2 \\ &\leq \lambda_k \sum_{i=1}^{k+m-1} c_i^2 = \lambda_k \|u\|^2_{L^2(\Omega)} \\ &= \lambda^\star \|u\|^2_{L^2(\Omega)}, \end{aligned}$$

so that, by using Hölder inequality, we have

$$\begin{aligned} \mathcal{J}_{A,\lambda}(u) &= \frac{1}{2}\|u\|^2_{X_{0,A}} - \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)} - \frac{1}{2_s^*}\|u\|^{2_s^*}_{L^{2_s^*}(\Omega)} \\ &\leq \frac{1}{2}(\lambda^\star - \lambda)\|u\|^2_{L^2(\Omega)} - \frac{1}{2_s^*}\|u\|^{2_s^*}_{L^{2_s^*}(\Omega)} \\ &\leq \frac{1}{2}(\lambda^\star - \lambda)|\Omega|^{2s/n}\|u\|^2_{L^{2_s^*}(\Omega)} - \frac{1}{2_s^*}\|u\|^{2_s^*}_{L^{2_s^*}(\Omega)}. \end{aligned} \tag{5.3}$$

Now, for $t \geq 0$ we define the function

$$g(t) := \frac{1}{2}(\lambda^\star - \lambda)|\Omega|^{2s/n} t^2 - \frac{1}{2_s^*} t^{2_s^*}.$$

Note that the function g is differentiable in $(0, \infty)$ and

$$g'(t) = (\lambda^\star - \lambda)|\Omega|^{2s/n} t - t^{2_s^* - 1},$$

with $(\lambda^\star - \lambda)|\Omega|^{2s/n} > 0$ since $\lambda < \lambda^\star$ by (1.7). Thus, we have $g'(t) \geq 0$ if and only if

$$t \leq \widetilde{t} := \left[(\lambda^\star - \lambda)|\Omega|^{2s/n}\right]^{1/(2_s^* - 2)}.$$

As a consequence of this, $\widetilde{t}$ is a maximum point for g and so for any $t \geq 0$

$$g(t) \leq \max_{t \geq 0} g(t) = g(\widetilde{t}) = \frac{s}{n}(\lambda^\star - \lambda)^{n/(2s)}|\Omega|. \tag{5.4}$$

By (5.3) and (5.4) we obtain

$$\sup_{u \in \operatorname{span}_{\mathbb{R}}\{f_1, \dots, f_{k+m-1}\}} \mathcal{J}_{A,\lambda}(u) \leq \max_{t \geq 0} g(t) = \frac{s}{n}(\lambda^\star - \lambda)^{n/(2s)}|\Omega|,$$

concluding the proof. □

Lemma 5.2. *Let $\lambda \in \mathbb{R}$ satisfy* (1.7)*, with $\lambda^\star$ in* (1.6) *counted with multiplicity $m \in \mathbb{N}$. Then, there exists $\delta > 0$, with*

$$\delta < \frac{s}{n}(\lambda^\star - \lambda)^{n/(2s)}|\Omega|,$$

and $\rho>0$ such that $\mathcal{J}_{A,\lambda}(v)\geq\delta$, for any $v\in V$ with $\|v\|_{X_{0,A}}=\rho$, where

$$V=\begin{cases}X_0 & \text{if } k=1\\ \mathbb{E}_k & \text{if } k\geq 2\,,\end{cases} \tag{5.5}$$

with $\mathbb{E}_k$ given in (3.6).

Proof. Let $u\in V$. Then

$$\|u\|^2_{X_{0,A}}\geq\lambda^\star\|u\|^2_{L^2(\Omega)} \tag{5.6}$$

Indeed, if $u\equiv 0$, then the assertion is trivial, while if $u\in V\setminus\{0\}$ it follows from the variational characterization of $\lambda^\star=\lambda_k$, given by (3.8) and (3.9).

Thus, by (1.4), (5.6) and taking into account that $\lambda>0$, as seen above, it follows that

$$\begin{aligned}\mathcal{J}_{A,\lambda}(u)&\geq\frac{1}{2}\Big(1-\frac{\lambda}{\lambda^\star}\Big)\|u\|^2_{X_{0,A}}-\frac{1}{2^*_s\mathcal{S}_A^{1/2}}\|u\|^{2^*_s}_{X_{0,A}}\\ &=\|u\|^2_{X_{0,A}}\Big(\frac{1}{2}\Big(1-\frac{\lambda}{\lambda^\star}\Big)-\frac{1}{2^*_s\mathcal{S}_A^{1/2}}\|u\|^{2^*_s-2}_{X_{0,A}}\Big)\,.\end{aligned} \tag{5.7}$$

Thus, let $u\in V$ be such that $\|u\|_{X_{0,A}}=\rho>0$. Since $2^*_s>2$, to conclude the proof it is enough to choose ρ sufficiently small so that

$$\frac{1}{2}\Big(1-\frac{\lambda}{\lambda^\star}\Big)-\frac{1}{2^*_s\mathcal{S}_A^{1/2}}\rho^{2^*_s-2}>0, \tag{5.8}$$

and

$$\rho^2\Big(\frac{1}{2}\Big(1-\frac{\lambda}{\lambda^\star}\Big)-\frac{1}{2^*_s\mathcal{S}_A^{1/2}}\rho^{2^*_s-2}\Big)\Big)<\frac{\rho^2}{2}\Big(1-\frac{\lambda}{\lambda^\star}\Big)<\frac{s}{n}(\lambda^\star-\lambda)^{n/(2s)}|\Omega|. \tag{5.9}$$

□

Proof of Theorem 1.1. Let $\lambda\in\mathbb{R}$ satisfy (1.7), with $\lambda^\star$ in (1.6) counted with multiplicity $m\in\mathbb{N}$.

By (2.1) it is immediate to see that $\mathcal{J}_{A,\lambda}$ satisfies (A1) of Theorem 2.1. While (A2) and (A3) hold true thanks to Proposition 4.1, Lemmas 5.1 and 5.2, considering $W=\operatorname{span}_{\mathbb{R}}\{e_1,\dots,e_{k+m-1}\}$, V given in (5.5) and that, by (1.7),

$$\delta<\frac{s}{n}(\lambda^\star-\lambda)^{n/(2s)}|\Omega|<\frac{s}{n}\mathcal{S}_A^{n/(2s)}.$$

Thus, since $\dim W=k+m-1$ and $\operatorname{codim}V=k-1$, by Theorem 2.1 the functional $\mathcal{J}_{A,\lambda}$ has m pairs $\{-u_{\lambda,i},\,u_{\lambda,i}\}$ of critical points whose critical values $\mathcal{J}_{A,\lambda}(\pm u_{\lambda,i})$ are such that

$$0<\delta\leq\mathcal{J}_{A,\lambda}(\pm u_{\lambda,i})\leq\frac{s}{n}(\lambda^\star-\lambda)^{n/(2s)}|\Omega| \tag{5.10}$$

for any $i=1,\dots,m$.

Since $\mathcal{J}_{A,\lambda}(0)=0$ and by (5.10), it is immediate to see that these critical points are non-trivial. Hence, problem (1.1) admits m pairs of non-trivial weak solutions. Now, fix $i\in\{1,\dots,m\}$. By (5.10) we obtain

$$\begin{aligned}\frac{s}{n}(\lambda^*-\lambda)^{n/(2s)}|\Omega|&\geq\mathcal{J}_{A,\lambda}(u_{\lambda,i})=\mathcal{J}_{A,\lambda}(u_{\lambda,i})-\frac{1}{2}\langle\mathcal{J}'_{A,\lambda}(u_{\lambda,i}),u_{\lambda,i}\rangle\\ &=\frac{s}{n}\,\|u_{\lambda,i}\|^{2^*_s}_{L^{2^*_s}(\Omega)},\end{aligned} \tag{5.11}$$

so that, passing to the limit as $\lambda\to\lambda^*$ in (5.11), it follows that

$$\|u_{\lambda,i}\|_{L^{2^*_s}(\Omega)}\to 0\quad\text{as }\lambda\to\lambda^*. \tag{5.12}$$

Also, by (5.12) and [15, Lemma 2.2], we also get

$$\|u_{\lambda,i}\|_{L^2(\Omega)} \to 0 \quad \text{as } \lambda \to \lambda^*. \tag{5.13}$$

Thus, arguing as in (5.11), we have

$$\frac{s}{n}(\lambda^\star - \lambda)^{n/(2s)}|\Omega| \geq \mathcal{J}_{A,\lambda}(u_{\lambda,i}) = \frac{1}{2}\|u_{\lambda,i}\|^2_{X_{0,A}} - \frac{\lambda}{2}\|u_{\lambda,i}\|^2_{L^2(\Omega)} - \frac{1}{2^*_s}\|u_{\lambda,i}\|^{2^*_s}_{L^{2^*_s}(\Omega)},$$

which combined with (5.12) and (5.13) gives

$$\|u_{\lambda,i}\|_{X_{0,A}} \to 0 \quad \text{as } \lambda \to \lambda^*.$$

This completes the proof. □

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