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MONOTONE AND OSCILLATION SOLUTIONS TO SECOND-ORDER DIFFERENTIAL EQUATIONS WITH ASYMPTOTIC CONDITIONS MODELING OCEAN FLOWS

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Abstract. In this article, we study the existence of monotone bounded solutions and of oscillatory solutions to a second-order differential equation with asymptotic conditions. Such asymptotic conditions arise in the study of the ocean flow in arctic gyres. Our approach relies on functional-analytic techniques.

1. Introduction

In this article, we study the existence of monotone bounded solutions and of oscillatory solutions for the second-order differential equation

$$x'' + a(t)f(x) = h(t), \quad t \geq t_0, \tag{1.1}$$

where the real-valued function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, $a\colon [t_0, +\infty) \to [0, \infty)$ and $h\colon [t_0, +\infty) \to \mathbb{R}$ are continuous. From the view of physics, it is interesting to consider the asymptotic conditions

$$\lim_{t\to\infty} x(t) = \psi_0 \quad \text{and} \quad \lim_{t\to\infty} \{x'(t)\exp(t)\} = 0, \tag{1.2}$$

where $\psi_0 \in \mathbb{R}$ is a constant.

As a special form of equation (1.1), the equation

$$x'' = \frac{F(x)}{\cosh^2(t)} - \frac{2\omega \sinh(t)}{\cosh^3(t)}\,, \quad t \geq t_0, \tag{1.3}$$

with the asymptotic conditions

$$\lim_{t\to\infty} x(t) = \psi_0 \quad \text{and} \quad \lim_{t\to\infty} \{x'(t)\cosh(t)\} = 0\,, \tag{1.4}$$

is a recently derived model for arctic gyres with a vanishing azimuthal velocity (see the discussions in [10] and the discussions in [1]). Recently, Chu has studied (1.3)-(1.4) in a systematic way in the recent papers [1, 2, 3, 4]. Note that the second condition in (1.4) is equivalent to the second one in (1.2). We point out that the specific form of (1.4) and of the associated differential equation is due to physically relevant considerations (see the discussion [5]).

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To prove the existence of monotone solutions and oscillatory solutions of (1.1)-(1.2), we will apply Schauder fixed point theorem. To do this, we transform the problem (1.1)-(1.2) into an integral equation. In fact, if $x(t)$ is a solution of the problem (1.1)-(1.2), integrating the equation (1.1) on $[t,\infty)$, we have

$$x'(t) = -\int_t^\infty h(s)\mathrm{d}s + \int_t^\infty a(s)f(x(s))\mathrm{d}s, \quad t \ge t_0, \tag{1.5}$$

then integrating (1.5) on $[t,\infty)$, we obtain

$$x(t) = \psi_0 + \int_t^\infty (s-t)h(s)\mathrm{d}s - \int_t^\infty (s-t)a(s)f(x(s))\mathrm{d}s, \quad t \ge t_0. \tag{1.6}$$

To make the integral equation (1.6) equivalent to problem (1.1)-(1.2), we assume that

$$\lim_{t\to\infty}\{\exp(t)a(t)\} = 0, \quad \lim_{t\to\infty}\{\exp(t)h(t)\} = 0. \tag{1.7}$$

Indeed, suppose that $x\colon [t_0,\infty) \to \mathbb{R}$ is a continuous function satisfying (1.6), and $\lim_{t\to\infty} x(t) = \psi_0$. It is easy to show that x satisfies (1.5) and the second condition in (1.2), since

$$\lim_{t\to\infty}\Big\{\exp(t)\int_t^\infty h(s)\mathrm{d}s\Big\} = \lim_{t\to\infty}\{\exp(t)h(t)\} = 0,$$

$$\lim_{t\to\infty}\Big\{\exp(t)\int_t^\infty a(s)f(x(s))\mathrm{d}s\Big\} = \lim_{t\to\infty}\{\exp(t)a(t)f(x(t))\} = 0.$$

Therefore, in this paper, we shall study the equivalent integral equation (1.6) of the problem (1.1)-(1.2) under condition (1.7).

2. Monotone solutions

In this section, we study the existence of monotone bounded solutions for the integral equation (1.6) under suitable conditions.

Theorem 2.1. *Assume that $a, h\colon [t_0,+\infty) \to [0,\infty)$ are continuous with*

$$\int_{t_0}^\infty h(s)\mathrm{d}s > 0. \tag{2.1}$$

Suppose further that the limit

$$J = \lim_{t\to\infty}\frac{a(t)}{h(t)} \tag{2.2}$$

exists and $J \ne 0$, and there exists a constant $\gamma > 0$ such that

$$\max_{x\in[\psi_0-\gamma,\psi_0+\gamma]} f(x) < \frac{1}{J}. \tag{2.3}$$

Then there exists some $T_\gamma \ge t_0$ such that (1.6) has at least one decreasing bounded continuous solution $x\colon [T_\gamma,\infty) \to \mathbb{R}$ satisfying $\lim_{t\to\infty} x(t) = \psi_0$. More precisely, we have that

$$x(t) > \psi_0, \quad x'(t) < 0, \quad \text{for all } t > T_\gamma. \tag{2.4}$$

Proof. Set

$$M_\gamma = \max_{x\in[\psi_0-\gamma,\psi_0+\gamma]} |f(x)|.$$

Obviously, $0 \le M_\gamma < \infty$ since f is continuous. From (1.7), we have

$$\int_{t_0}^{\infty} sa(s)\mathrm{d}s < \infty \quad \text{and} \quad \int_{t_0}^{\infty} sh(s)\mathrm{d}s < \infty. \tag{2.5}$$

By (2.5), we can choose $T_0 \ge \max\{t_0, 0\}$ large enough such that

$$M_\gamma \int_{T_0}^{\infty} sa(s)\mathrm{d}s < \frac{\gamma}{2} \quad \text{and} \quad \int_{T_0}^{\infty} sh(s)\mathrm{d}s < \frac{\gamma}{2}.$$

Define the closed and convex subset

$$X_0 = \big\{x \in C([T_0,\infty),\mathbb{R}) : \lim_{t\to\infty} x(t) = \psi_0\big\}$$

of the Banach space X of all bounded functions $x \in C([T_0,\infty),\mathbb{R})$, endowed with the supremum norm $\|x\| = \sup_{t\ge T_0}\{|x(t)|\}$. Set

$$\Omega = \big\{x \in X_0 : \psi_0 - \gamma \le x(t) \le \psi_0 + \gamma, \quad t \ge T_0\big\}.$$

Let $\mathcal{T}\colon \Omega \to X_0$ be the operator defined as

$$[\mathcal{T}(x)](t) = \psi_0 + \int_t^{\infty} (s-t)h(s)\mathrm{d}s - \int_t^{\infty} (s-t)a(s)f(x(s))\mathrm{d}s, \quad t \ge T_0. \tag{2.6}$$

Note that

$$\Big|\int_t^{\infty} (s-t)h(s)\,ds\Big| \le \int_t^{\infty} sh(s)\,ds, \quad t \ge T_0\,,$$
$$\Big|\int_t^{\infty} (s-t)a(s)f(x(s))\,ds\Big| \le M_\gamma \int_t^{\infty} sa(s)\,ds, \quad t \ge T_0\,,$$

which confirms that $\mathcal{T}\colon \Omega \to X_0$. Also, for any $x \in \Omega$, we have $\lim_{t\to\infty}[\mathcal{T}(x)](t) = \psi_0$ since

$$\lim_{t\to\infty} \int_t^{\infty} sh(s)\,ds = 0, \quad \lim_{t\to\infty} \int_t^{\infty} sa(s)\,ds = 0.$$

We shall apply the Schauder fixed point theorem [20] to prove that there exists a fixed point for the operator $\mathcal{T}$ in the nonempty closed bounded convex set Ω, and then we prove that (2.4) holds. It is divided into four steps.

Step 1. We prove that $\mathcal{T}(\Omega) \subset \Omega$. For any $x \in \Omega$ and $t \ge T_0$, we have

$$\begin{aligned}
|[\mathcal{T}(x)](t) - \psi_0| &= \Big|\int_t^{\infty} (s-t)h(s)\mathrm{d}s - \int_t^{\infty} (s-t)a(s)f(x(s))\mathrm{d}s\Big| \\
&\le \int_t^{\infty} (s-t)h(s)\mathrm{d}s + \int_t^{\infty} (s-t)|a(s)f(x(s))|\mathrm{d}s \\
&\le \int_t^{\infty} sh(s)\mathrm{d}s + \int_t^{\infty} M_\gamma sa(s)\mathrm{d}s \\
&\le \int_{T_0}^{\infty} sh(s)\mathrm{d}s + M_\gamma \int_{T_0}^{\infty} sa(s)\mathrm{d}s \le \gamma,
\end{aligned}$$

which shows that $\mathcal{T}\colon \Omega \to \Omega$ is well-defined.

Step 2. We prove that $\mathcal{T}\colon \Omega \to \Omega$ is continuous. For a given $\varepsilon > 0$, there exists a $T_* \ge T_0$ such that

$$M_\gamma \int_{T_*}^{\infty} sa(s)ds < \frac{\varepsilon}{3}.$$

By the fact that $f\colon [\psi_0-\gamma,\psi_0+\gamma]\to\mathbb{R}$ is continuous, there exists a constant $\delta>0$ such that for all $x,y\in[\psi_0-\gamma,\psi_0+\gamma]$ with $|x-y|<\delta$, we have

$$|f(x)-f(y)|<\frac{2\varepsilon}{3T_*^2a_*},\quad \text{for all } t\in[t_0,T_*],$$

where $a_*=\max_{t\in[T_0,T_*]}a(t)$. Therefore, for all $x_1,x_2\in\Omega$ with $\|x_1-x_2\|<\delta$, we obtain

$$\begin{aligned}
|[\mathcal{T}(x_1)](t)-[\mathcal{T}(x_2)](t)| &= \Big|\int_t^\infty (s-t)a(s)[f(x_2(s))-f(x_1(s))]\Big|\\
&\le \int_t^\infty (s-t)a(s)|f(x_2(s))-f(x_1(s))|ds\\
&\le \int_{T_0}^{T_*} (s-T_0)a(s)|f(x_2(s))-f(x_1(s))|ds\\
&\quad + \int_{T_*}^\infty (s-T_*)a(s)|f(x_2(s))-f(x_1(s))|ds\\
&= I_1+I_2.
\end{aligned}$$

Since

$$\begin{gathered}
I_1\le \frac{2\varepsilon}{3T_*^2a_*}a_*\int_{T_0}^{T_*}(s-T_0)ds=\frac{2\varepsilon}{3T_*^2}\frac{(T_*-T_0)^2}{2}<\frac{\varepsilon}{3},\\
\begin{aligned}
I_2 &\le \int_{T_*}^\infty sa(s)\Big\{|f(x_1(s))|+|f(x_2(s))|\Big\}ds\\
&\le 2M_\gamma\int_{T_*}^\infty sa(s)ds<\frac{2\varepsilon}{3},
\end{aligned}
\end{gathered}$$

we have

$$\|[\mathcal{T}(x_1)]-[\mathcal{T}(x_2)]\|\le\varepsilon.$$

Therefore, $\mathcal{T}:\Omega\to\Omega$ is a continuous.

Step 3. We prove that $\mathcal{T}(\Omega)$ is relatively compact in X. Since $\mathcal{T}(\Omega)\subset\Omega$, we know that $\mathcal{T}(\Omega)$ is uniform bounded. Differentiating two sides of (2.6) with respect to t, we obtain

$$[\mathcal{T}(x)]'(t)=-\int_t^\infty h(s)\mathrm{d}s+\int_t^\infty a(s)f(x(s))\mathrm{d}s,\quad t\ge T_0.$$

For all $t\ge T_0$, we have

$$\begin{aligned}
|[\mathcal{T}(x)]'(t)| &\le \Big|\int_t^\infty h(s)\,ds\Big|+\Big|\int_t^\infty a(s)f(x(s))\,ds\Big|\\
&\le \int_t^\infty h(s)\,ds+M_\gamma\int_t^\infty a(s)\,ds\\
&\le \int_{T_0}^\infty h(s)\,ds+M_\gamma\int_{T_0}^\infty a(s)\,ds,
\end{aligned}$$

which means that for all $x\in\Omega$, we have

$$\Big|[\mathcal{T}(x)]'(t)\Big|\le K,\quad t\ge T_0,$$

where

$$K=\int_{T_0}^\infty h(s)\,ds+M_\gamma\int_{T_0}^\infty a(s)\,ds.$$

Let $\{x_n\}$ be an arbitrary sequence in Ω. Then we have

$$|[\mathcal{T}(x_n)]'(t)| \le K, \quad t \ge T_0, \quad n \ge 1.$$

Applying the mean value theorem, we obtain

$$|[\mathcal{T}(x_n)](t_1) - [\mathcal{T}(x_n)](t_2)| \le K|t_1 - t_2|, \quad t_1, t_2 \ge T_0, \quad n \ge 1,$$

which implies that $\{[\mathcal{T}(x_n)]\}$ is equicontinuous in X.

Furthermore, since

$$\lim_{t\to\infty} \Big[\psi_0 + \int_t^\infty (s-t)h(s)\mathrm{d}s - \int_t^\infty (s-t)a(s)f(x(s))\mathrm{d}s\Big] = \psi_0,$$

so for every $\epsilon > 0$, there exists $t_\epsilon > T_0$ such that

$$|[\mathcal{T}(x_n)](t) - \psi_0| \le \epsilon, \quad t \ge t_\epsilon, \quad n \ge 1.$$

Therefore, $\{[\mathcal{T}(x_n)]\}$ is equiconvergent in X.

By using the Arzela-Ascoli theorem [20], we obtain that $\{[\mathcal{T}(x_n)]\}$ is relatively compact in X.

We have proved that all assumptions of the Schauder fixed point theorem are satisfied. Therefore, the operator $\mathcal{T}$ has a fixed point x in Ω, and this fixed point corresponds to a bounded solution of (1.6) on $[T_0, \infty)$.

Step 4. We show that the fixed point is decreasing. Let x be the fixed point of $\mathcal{T}$. Define

$$H(t) = \frac{\int_t^\infty (s-t)a(s)f(x(s))\mathrm{d}s}{\int_t^\infty (s-t)h(s)\mathrm{d}s}, \quad t > T_0.$$

Then

$$H(t) \le \max_{x\in[\psi_0-\gamma, \psi_0+\gamma]} f(x) \cdot \frac{\int_t^\infty (s-t)a(s)\mathrm{d}s}{\int_t^\infty (s-t)h(s)\mathrm{d}s}.$$

Since

$$\lim_{t\to\infty} \frac{\int_t^\infty (s-t)a(s)\mathrm{d}s}{\int_t^\infty (s-t)h(s)\mathrm{d}s} = \lim_{t\to\infty} \frac{\int_t^\infty a(s)\mathrm{d}s}{\int_t^\infty h(s)\mathrm{d}s} = \lim_{t\to\infty} \frac{a(t)}{h(t)} = J,$$

using the condition (2.3), we know that there exists $T_1 \ge T_0$ such that $H(t) < 1$ for $t > T_1$, which yields

$$\int_t^\infty (s-t)a(s)f(x(s))\mathrm{d}s < \int_t^\infty (s-t)h(s)\mathrm{d}s, \quad t > T_1,$$

and hence for all $t > T_1$, we have

$$x(t) = \psi_0 + \int_t^\infty (s-t)h(s)\mathrm{d}s - \int_t^\infty (s-t)a(s)f(x(s))\mathrm{d}s > \psi_0.$$

Define

$$L(t) = \frac{\int_t^\infty a(s)f(x(s))\mathrm{d}s}{\int_t^\infty h(s)\mathrm{d}s}, \quad t > T_0.$$

Then

$$L(t) \le \max_{x\in[\psi_0-\gamma, \psi_0+\gamma]} f(x) \cdot \frac{\int_t^\infty a(s)\mathrm{d}s}{\int_t^\infty h(s)\mathrm{d}s}.$$

Since (2.3) holds, there exists $T_2 \ge T_0$ such that $L(t) < 1$ for $t > T_2$, which implies

$$x'(t) = -\int_t^\infty h(s)\mathrm{d}s + \int_t^\infty a(s)f(x(s))\mathrm{d}s < 0, \quad t > T_2.$$

Let $T_\gamma = \max\{T_1, T_2\}$, then (2.4) holds. $\square$

Example 2.2. Consider the equation

$$x'' + \frac{1}{\cosh^2(t)} \frac{x}{8\psi_0} = e^{-2t}, \quad t \geq t_0. \tag{2.7}$$

It is easy to see that

$$J = \lim_{t\to\infty} \frac{\frac{1}{\cosh^2(t)}}{e^{-t}} = 4. \tag{2.8}$$

We suppose that $\psi_0 > 0$, choose any $\gamma \in [0, \psi_0)$, then it is easy to check that

$$\max_{x\in[\psi_0-\gamma,\psi_0+\gamma]} \frac{x}{8\psi_0} < \frac{1}{4}.$$

We know that the solution of (2.7) is

$$x(t) = \psi_0 + \int_t^\infty (s-t)e^{-s}\mathrm{d}s - \int_t^\infty (s-t)\frac{x(s)}{8\psi_0}\frac{1}{\cosh^2(s)}\mathrm{d}s, \quad t \geq t_0. \tag{2.9}$$

Obviously, $x(t) > \psi_0$ for $t \geq t_0$. Indeed, $\lim_{t\to\infty}\{x(t)\} = \psi_0$ and

$$x'(t) = -\int_t^\infty e^{-s}\mathrm{d}s + \int_t^\infty \frac{x(s)}{8\psi_0}\frac{1}{\cosh^2(s)}\mathrm{d}s < 0.$$

Therefore, $x(t)$ decreases towards ψ_0 as t decreases towards infinity.

In fact, we can prove another result in a similar way.

Theorem 2.3. *Assume that $a, h\colon [t_0, +\infty) \to [0, \infty)$ are continuous and (2.1), (2.2) hold. Suppose further that there exists a constant $\eta > 0$ such that*

$$\min_{x\in[\psi_0-\eta,\psi_0+\eta]} f(x) > \frac{1}{J}. \tag{2.10}$$

Then there exists some $T_\eta \geq t_0$ such that there exists a increasing bounded continuous solution $x\colon [T_\eta, \infty) \to \mathbb{R}$ to the equation (1.6), and $\lim_{t\to\infty}\{x(t)\} = \psi_0$. More precisely, we have

$$x(t) < \psi_0, \quad x'(t) > 0, \quad \textit{for all } t > T_\eta. \tag{2.11}$$

Proof. Proceeding as in Steps 1–3 in the proof of Theorem 2.1, we know that the equation (1.6) has at least one bounded continuous solution $x\colon [T_\eta, \infty) \to \mathbb{R}$ satisfying $\lim_{t\to\infty}\{x(t)\} = \psi_0$.

We only need to prove the solution above is increasing. Define

$$H(t) = \frac{\int_t^\infty (s-t)a(s)f(x(s))\mathrm{d}s}{\int_t^\infty (s-t)h(s)\mathrm{d}s}, \quad t > T_0.$$

Then

$$H(t) \geq \min_{x\in[\psi_0-\eta,\psi_0+\eta]} f(x)\frac{\int_t^\infty (s-t)a(s)\mathrm{d}s}{\int_t^\infty (s-t)h(s)\mathrm{d}s}.$$

Since

$$\lim_{t\to\infty} \frac{\int_t^\infty (s-t)a(s)\mathrm{d}s}{\int_t^\infty (s-t)h(s)\mathrm{d}s} = \lim_{t\to\infty} \frac{\int_t^\infty a(s)\mathrm{d}s}{\int_t^\infty h(s)\mathrm{d}s} = \lim_{t\to\infty} \frac{a(t)}{h(t)} = J,$$

by (2.10), we know that there exists $T_1 \geq T_0$ such that $H(t) > 1$ for $t > T_1$, which yields that

$$\int_t^\infty (s-t)a(s)f(x(s))\mathrm{d}s > \int_t^\infty (s-t)h(s)\mathrm{d}s, \quad t > T_1,$$

and hence for all $t > T_1$, we have

$$x(t) = \psi_0 + \int_t^\infty (s-t)h(s)\mathrm{d}s - \int_t^\infty (s-t)a(s)f(x(s))\mathrm{d}s < \psi_0.$$

Define

$$L(t) = \frac{\int_t^\infty a(s)f(x(s))\mathrm{d}s}{\int_t^\infty h(s)\mathrm{d}s}, \quad t > T_0.$$

Then

$$L(t) \geq \min_{x\in[\psi_0-\eta,\psi_0+\eta]} f(x)\frac{\int_t^\infty a(s)\mathrm{d}s}{\int_t^\infty h(s)\mathrm{d}s}.$$

Since (2.10) holds, there exists $T_2 \geq T_0$ such that $L(t) > 1$ for $t > T_2$, which implies

$$x'(t) = -\int_t^\infty h(s)\mathrm{d}s + \int_t^\infty a(s)f(x(s))\mathrm{d}s > 0, \quad t > T_2.$$

Let $T_\eta = \max\{T_1, T_2\}$, then (2.11) holds. □

Example 2.4. Consider the equation

$$x'' + \frac{1}{\sinh^2(t)}\frac{x}{4\psi_0} = e^{-2t}, \quad t \geq t_0. \tag{2.12}$$

Then we know that

$$J = \lim_{t\to\infty} \frac{\frac{1}{\sinh^2(t)}}{e^{-2t}} = 4. \tag{2.13}$$

Assume that $\psi_0 > 0$, take any $\gamma > 0$, then we have

$$\max_{x\in[\psi_0-\gamma,\psi_0+\gamma]} \frac{x}{4\psi_0} > \frac{1}{4}.$$

We know that the solution of (2.12) is

$$x(t) = \psi_0 + \int_t^\infty (s-t)e^{-2s}\mathrm{d}s - \int_t^\infty (s-t)\frac{x(s)}{4\psi_0}\frac{1}{\sinh^2(s)}\mathrm{d}s, \quad t \geq t_0. \tag{2.14}$$

Obviously, $x(t) < \psi_0$ for $t \geq t_0$. Indeed, $\lim_{t\to\infty}\{x(t)\} = \psi_0$, and

$$x'(t) = -\int_t^\infty e^{-2s}\mathrm{d}s + \int_t^\infty \frac{x(s)}{4\psi_0}\frac{1}{\sinh^2(s)}\mathrm{d}s > 0.$$

Therefore, $x(t)$ increases towards ψ_0 as t increases towards infinity.

3. Oscillatory solutions

In this section, we study the existence of oscillatory solutions for the integral equation (1.6) under suitable conditions. Define a function $H\colon [t_0,\infty) \to \mathbb{R}$ as

$$H(t) = \int_t^\infty (s-t)h(s)\mathrm{d}s.$$

For a fixed $\lambda > t_0$, we denote the upper bound of H by

$$\|H\| = \sup_{t\geq\lambda>t_0} |H(t)|.$$

Fix a positive real number $R > \|H\|$ and define

$$M_R = \sup_{x\in[-R,R]} |f(x)|, \quad g(t) = M_R\int_t^\infty a(s)\mathrm{d}s, \quad t \geq t_0,$$

$$G(t) = \int_t^\infty g(s)\mathrm{d}s.$$

Now we state and prove the main result of this section.

Theorem 3.1. *Assume that* $G(t_0) < +\infty$ *and*

$$\limsup_{t\to+\infty} \frac{H(t)}{G(t)} > 1, \quad \liminf_{t\to+\infty} \frac{H(t)}{G(t)} < -1. \tag{3.1}$$

Then for every ε *with* $0 < \varepsilon < R - \|H\|$, *there exist a real number* $T(\varepsilon) > 0$, *a positive integer* $N(\varepsilon)$, *and two increasing divergent sequences of positive numbers* $\{t_n\}_{n\geq 1}$, $\{s_n\}_{n\geq 1}$, *such that* (1.6) *has a solution* $x(t)$ *defined on* $[T(\varepsilon), +\infty)$ *satisfying* $\lim_{t\to\infty} x(t) = \psi_0$ *and*

$$x(t_n) > \psi_0 \quad \textit{and} \quad x(s_n) < \psi_0, \quad \textit{for all } n \geq N(\varepsilon).$$

Proof. To prove the above result, by (1.6), we just need to prove that the equation

$$x(t) = \int_t^\infty (s-t)h(s)\mathrm{d}s - \int_t^\infty (s-t)a(s)f(x(s))\mathrm{d}s, \quad t \geq t_0, \tag{3.2}$$

has a solution $x(t)$ such that $\lim_{t\to\infty} x(t) = 0$ and

$$x(t_n) > 0 \quad \text{and} \quad x(s_n) < 0.$$

Given a real number $\lambda > t_0$, choose an ε with $0 < \varepsilon < R - \|H\|$. Since $G(t_0) < +\infty$, there exists a number $T(\varepsilon) > \lambda$ such that $G(t) < \varepsilon$ for all $t \geq T(\varepsilon)$. Define the closed and convex subset

$$X_\varepsilon = \{x \in C([T(\varepsilon), +\infty), \mathbb{R}) : \lim_{t\to\infty} x(t) = 0\}$$

of the Banach space X of all functions $x \in C([T(\varepsilon), +\infty), \mathbb{R})$, endowed with the supremum $\|\cdot\|$. Set

$$\Omega = \{x \in X_\varepsilon : \|x - H\| \leq \varepsilon\}.$$

Define an operator $\mathcal{F}\colon \Omega \to \Omega$ as

$$[\mathcal{F}(x)](t) = H(t) - \int_t^\infty \int_s^\infty a(\tau)f(x(\tau))\mathrm{d}\tau\mathrm{d}s, \quad t \geq T(\varepsilon). \tag{3.3}$$

Note that

$$|[\mathcal{F}(x)](t) - H(t)| \leq \int_t^\infty M_{\|H\|+\varepsilon} \int_s^\infty a(\tau)\mathrm{d}\tau\mathrm{d}s \leq G(t) < \varepsilon, \quad t \geq T(\varepsilon). \tag{3.4}$$

Therefore, the operator $\mathcal{F}\colon \Omega \to \Omega$ is well-defined.

We shall apply the Schauder fixed point theorem to prove that there exists a fixed point for the operator $\mathcal{F}$ in the nonempty closed bounded convex set Ω.

First, we prove that the operator $\mathcal{F}$ is uniformly continuous. For a given constant $\xi > 0$, there exists a $T(\xi) > T(\varepsilon)$ such that

$$G(t) < \frac{\xi}{3}, \quad t \geq T(\xi).$$

Furthermore, there exists a $\delta(\xi) > 0$ such that

$$|a(t)f(x_1) - a(t)f(x_2)| < \frac{\xi}{3(T(\xi))^2},$$

holds for all $t \in [T(\varepsilon), T(\xi)]$ and $x_1, x_2 \in [-\|H\| - \varepsilon, \|H\| + \varepsilon]$ with $\|x_1 - x_2\| < \delta(\xi)$. Now for all $x_1, x_2 \in \Omega$ satisfying $\|x_1 - x_2\| < \delta(\xi)$, we have

$$\begin{aligned}
|[\mathcal{F}(x_1)](t) - [\mathcal{F}(x_2)](t)| &\leq \int_{T(\varepsilon)}^{\infty} \int_s^{\infty} |a(\tau) f(x_2(\tau)) - a(\tau) f(x_1(\tau))| \mathrm{d}\tau \mathrm{d}s \\
&= \int_{T(\varepsilon)}^{\infty} (s - T(\varepsilon)) |a(s) f(x_2(s)) - a(s) f(x_1(s))| \mathrm{d}s \\
&\leq |T(\xi) - T(\varepsilon)| \int_{T(\varepsilon)}^{T(\xi)} |a(s) f(x_2(s)) - a(s) f(x_1(s))| \mathrm{d}s \\
&\quad + \int_{T(\xi)}^{\infty} \int_s^{\infty} |a(\tau) f(x_2(\tau))| \mathrm{d}\tau \mathrm{d}s \\
&\quad + \int_{T(\xi)}^{\infty} \int_s^{\infty} |a(\tau) f(x_1(\tau))| \mathrm{d}\tau \mathrm{d}s \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Note that

$$I_1 < [T(\xi) - T(\varepsilon)]^2 \frac{\xi}{3(T(\xi))^2} < \frac{\xi}{3}, \quad I_2 + I_3 < \frac{2}{3}\xi.$$

Then we conclude that

$$|\mathcal{F}(x_1)](t) - \mathcal{F}(x_2)](t)| < \xi.$$

Therefore $\mathcal{F}$ is uniformly continuous.

Next, we apply the Arzela-Ascoli theorem to prove that the set $\mathcal{F}(\Omega)$ is relatively compact. Since $\mathcal{F}(\Omega) \subset \Omega$, we know that $\mathcal{F}(\Omega)$ is uniformly bounded. For any two real numbers t_1, t_2 with $t_2 \geq t_1 \geq T(\varepsilon)$, we have

$$\begin{aligned}
|[\mathcal{F}(x)](t_2) - [\mathcal{F}(x)](t_1)| &\leq |H(t_2) - H(t_1)| + \int_{t_1}^{t_2} \int_s^{\infty} |a(\tau) f(x(\tau))| \mathrm{d}\tau \mathrm{d}s \\
&\leq \int_{t_1}^{t_2} \int_s^{\infty} |h(\tau)| \mathrm{d}\tau \mathrm{d}s + \int_{t_1}^{t_2} g(s) \mathrm{d}s, \quad x \in \Omega,
\end{aligned}$$

which shows that $\mathcal{F}(\Omega)$ is equicontinuous.

From the definition of $\mathcal{F}$, we have

$$|\mathcal{F}(x)](t)| \leq |H(t)| + G(t), \quad t \geq T(\varepsilon), \quad \text{for all } x \in \Omega. \tag{3.5}$$

By (3.5) and $\lim_{t\to\infty} H(t) = 0$, we know that the set $\mathcal{F}(\Omega)$ is equiconvergent. Therefore $\mathcal{F}(\Omega)$ is relatively compact.

Up to now, all conditions of the Schauder fixed point theorem are established. Therefore, the operator $\mathcal{F}$ has a fixed point in Ω, that is, the equation (3.2) has a solution $x(t)$, which satisfies $\lim_{t\to\infty} x(t) = 0$.

Finally, we prove that the solution $x(t)$ is oscillatory. From (3.4), we have

$$|x(t) - H(t)| = |[\mathcal{F}(x)](t) - H(t)| \leq G(t), \quad t \geq T(\varepsilon),$$

which yields

$$H(t) - G(t) \leq x(t) \leq H(t) + G(t), \quad \text{for all } t \geq T(\varepsilon). \tag{3.6}$$

By (3.1), we know that there exist a positive integer $N(\varepsilon)$ and two sequences of positive numbers $\{t_n\}_{n\geq 1}$, $\{s_n\}_{n\geq 1}$, t_n, $s_n \to \infty$ as $n \to \infty$, such that

$$H(t_n) - G(t_n) > 0 \quad \text{and} \quad H(s_n) + G(s_n) < 0, \quad \text{for all } n \geq N(\varepsilon),$$

it follows from (3.6) that

$$x(t_n) > 0 \quad \text{and} \quad x(s_n) < 0, \quad \text{for all } n \geq N(\varepsilon).$$

The proof is complete. □

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