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BLOWUP OF SOLUTIONS TO DEGENERATE KIRCHHOFF-TYPE DIFFUSION PROBLEMS INVOLVING THE FRACTIONAL p-LAPLACIAN

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Abstract. We study an initial boundary value problem for Kirchhoff-type parabolic equation with the fractional p-Laplacian. We first discuss the blow up of solutions in finite time with three initial energy levels: subcritical, critical and supercritical initial energy levels. Then we estimate an upper bound of the blowup time for low and for high initial energies.

1. Introduction

In this article we consider the parabolic initial boundary value problem involving the fractional p-Laplacian

$$\begin{gathered} \partial_t u + [u]_{s,p}^{(\lambda-1)p} \mathcal{L}_K^p u = |u|^{q-2}u, \quad \text{in } \Omega \times \mathbb{R}^+, \partial_t u = \partial u/\partial t, \\ u(x,0) = u_0(x), \quad \text{in } \Omega, \\ u(x,t) = 0, \quad \text{in } (\mathbb{R}^N \backslash \Omega) \times \mathbb{R}_0^+, \end{gathered} \tag{1.1}$$

where $[u]_{s,p} = \big(\iint_Q |u(x,t) - u(y,t)|^p K(x-y)\,dx\,dy \big)^{1/p}$, p and q satisfy $2 < p\lambda < q < p_s^*$ with $\lambda \in [1, p_s^*/p)$ and $p_s^* := Np/(N-sp)$, $s \in (0,1)$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$. The initial function is $u_0 \geq 0$ on Ω, $\mathcal{L}_K^p$ is a nonlocal integro-differential operator, which is defined by

$$\mathcal{L}_K^p \varphi(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \backslash B_\varepsilon(x)} |\varphi(x) - \varphi(y)|^{p-2} [\varphi(x) - \varphi(y)] K(x-y) dy,$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, where $B_\varepsilon(x)$ denotes the ball in $\mathbb{R}^N$ with radius $\varepsilon > 0$ centered at $x \in \mathbb{R}^N$. The kernel $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ satisfies the following assumptions

- (A1) $m(x)K \in L^1(\mathbb{R}^N)$, where $m(x) = \min\{|x|^p, 1\}$; there exists $K_0 > 0$, such that $K(x) \geq K_0 |x|^{-(N+ps)}$ for a.e. $x \in \mathbb{R}^N \setminus \{0\}$.

A typical example for K is the singular kernel $K(x) = |x|^{-(N+ps)}$. In this way, $\mathcal{L}_K^p \varphi(x) = (-\Delta)_p^s \varphi(x)$ for all $\varphi(x) \in C_0^\infty(\mathbb{R}^N)$. We refer the reader to [7, 14, 22, 39] for further details on fractional Laplacian and the fractional Sobolev spaces. In this

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case, $[u]_{s,p}$ becomes the celebrated Gagliardo semi-norm. As well known, problem (1.1) has been used to model some physical phenomena occurring in nonlocal reaction-diffusion problems, non-Newtonian fluid, non-Newtonian filtration and turbulent flows of a gas in a porous medium, and so on. In the non-Newtonian fluid theory, the quantity p is characteristic of the medium. Media with $p > 2$ are called dilatant fluid and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids.

To explain the motivation for problem (1.1), let us introduce a prototype of nonlocal problem (1.1) in $\mathbb{R}^N \times \mathbb{R}_0^+$. Nonlocal evolutions of the form

$$\partial_t u(x,t) = \int_{\mathbb{R}^N} [u(y,t) - u(x,t)]\mathcal{K}(x-y)dy, \tag{1.2}$$

and its variants, have been recently used to model diffusion processes. More precisely, as stated, if $u(x,t)$ is thought of as a density of population at the point x and time t and $\mathcal{K}(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int_{\mathbb{R}^N} u(y,t)\mathcal{K}(x-y)dy$ is the rate at which individuals are arriving at position x from all other places and $\int_{\mathbb{R}^N} u(x,t)\mathcal{K}(x-y)dy$ is the rate at which they are leaving location x to travel to all other sites. If we consider the effects of total population, then problem (1.2) becomes

$$\begin{aligned} \partial_t u(x,t) = M\Big(\iint_{\mathbb{R}^{2N}} |u(x,t)-u(y,t)|^2\mathcal{K}(x-y)\,dx\,dy\Big) \\ \times \int_{\mathbb{R}^N} [u(y,t) - u(x,t)]\mathcal{K}(x-y)dy, \end{aligned} \tag{1.3}$$

where the coefficient $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ accounts for the possible changes of total population in $\mathbb{R}^N$. This signifies that the behavior of individuals is subject to total population, such as the diffusion process of bacteria. As a matter of fact, model (1.3) is meaningful, since the way of measurements are usually taken in average sense. It is worthy pointing out that there are some papers dedicated to the study of Kirchhoff-type parabolic problems. For example, Gobbino in [11] investigated the properties of solutions for the degenerate parabolic equations of Kirchhoff type

$$u_t - M\Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)\Delta u = 0, \tag{1.4}$$

where the Kirchhoff function $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous, which have been studied by many authors, see [11] and the references therein for more details; see also [3, 26] for wave equations of Kirchhoff type.

In the classical case, let us sketch the recent advances concerning the equation

$$u_t - \Delta u = f(u). \tag{1.5}$$

Liu and Zhao [16] considered the initial-boundary value problem with initial data $J(u_0) < d$ for $I(u_0) < 0$ and $I(u_0) \geq 0$, and initial data $J(u_0) = d$ for $I(u_0) \geq 0$. In [30] Xu studied the same problem with critical initial data $J(u_0) = d, I(u_0) < 0$, and initial data $J(u_0) > d, I(u_0) > 0$. A powerful technique for treating the above problem is the so-called potential well method, which was established by Payne and Sattinger [25]. Gazzola and Weth [12] studied the initial-boundary value problem of (1.5), where $f(u) = |u|^{p-1}u$. They proved finite time blow-up of solutions with high initial energy $J(u_0) > d$ by the comparison principle and variational methods. Xu and Su [32] studied the initial boundary value problem of $u_t - \Delta u_t - \Delta u = u^p$.

More precisely, they used the family of the potential wells to prove the nonexistence of solutions with initial energy $J(u_0) \le d$, and obtained finite time blowup with high initial energy $J(u_0) > d$ by comparison principle. Very recently, Xu et al. in [33] discussed the same problem and established a new finite time blowup theorem for the solution of problem for arbitrary high initial energy.

In the fractional case, Caffarelli and Silvestre [4] introduced the s-harmonic extension to define the fractional Laplacian operator. Nezza et al. [22] established the corresponding Sobolev inequality and Poincaré inequality on the cone Sobolev spaces. Fu and Pucci in [8] proved the existence of global solutions with exponential decay and showed the blow-up in finite time of solutions to the space-fractional diffusion problem

$$\begin{gathered} u_t + (-\Delta)^s u = |u|^{p-1}u, \quad x \in \Omega, t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \mathbb{R}^n \setminus \Omega, t \ge 0, \end{gathered} \tag{1.6}$$

provided that $M \equiv 1$ and p satisfies $1 < p \le 2_s^* - 1 = \frac{n+2s}{n-2s}$. More works on fractional equations can be found in [1, 13, 18, 29] and the references therein.

In recent years, a lot of interest has grown about Kirchhoff-type problems, see for example [3, 10, 27, 35]. In these papers, to obtain the existence of weak solutions, the authors always assume that the Kirchhoff function $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuous and nondecreasing function and satisfies the following conditions:

$$\text{there exists } m_0 > 0 \text{ such that } M(t) \ge m_0 \text{ for all } t \in \mathbb{R}_0^+. \tag{1.7}$$

A typical example is $M(t) = m_0 + bt^m$ with $m_0 > 0$, $b \ge 0$ for all $t \in \mathbb{R}_0^+$. Naturally, we distinguish the problem into non-degenerate and degenerate cases in accordance with $M(0) > 0$ and $M(0) = 0$ respectively. It is worthwhile pointing out that the degenerate case is rather interesting and is treated in well-known papers in Kirchhoff theory, see for example [5]. From a physical point of view, the fact that $M(0) = 0$ means that the base tension of the string is zero. For some recent results in the degenerate case, see for instance [2, 6, 20, 28, 31, 36] and the references therein. In these papers, the Kirchhoff function M was assumed to fulfill more general conditions which cover the degenerate case. In this paper, we assume that M is the simple power function $M(t) = t^{\lambda-1}$ with $\lambda \in [1, p_s^*/p)$ for all $t \in \mathbb{R}_0^+$, which implies problem (1.1) is degenerate, see [34, 37, 38] for more results about this type. Pan et al. [24] first studied the global solutions for degenerate Kirchhoff-type wave problem in the setting of fractional Laplacian by combing the Galerkin method with potential well theory. Pan et al. [23] investigated for the first time the existence of a global solution for degenerate Kirchhoff-type diffusion problems involving fractional p-Laplacian by combing the Galerkin method with potential well theory. Recently, Xiang et al. [19] studied a diffusion model of Kirchhoff-type driven by a nonlocal integro-differential operator, and obtained the existence of nonnegative local solutions. Also, they showed that the nonnegative local solutions blow up in finite time with arbitrary negative initial energy. In particular, the authors gave an estimate for the lower and upper bounds of the blow-up time under certain hypotheses on M which cover the degenerate case $M(0) = 0$.

Zhou and Yang [40] studied an evolution m-Laplace equation involving variable source in which the upper bound of the blowup time for the blow-up solutions with positive initial energy was estimated. Xu et al. [33] discussed the initial boundary value problem of $u_t - \Delta u_t - \Delta u = u^p$, and estimated the upper bound of the blowup time for arbitrary high initial energy.

Motivated by the above works, we complete the picture of weak solutions for problem (1.1) in the setting of fractional p-Laplacian by potential well theory and concave function method. More precisely, we shall prove the finite time blow-up of solutions for problem (1.1) at three different energy levels: $J(u_0) < d$, $J(u_0) = d$, $J(u_0) > d$. Furthermore, we will estimate the upper bound of the blowup time at low initial energy and arbitrary high initial energy.

The outline of this paper is as follows. In Section 2, we recall some necessary definitions and properties of the fractional Sobolev spaces and introduce the family of potential wells. In Section 3, we prove the finite time blow-up for problem (1.1) with low initial energy $J(u_0) < d$ and estimate the upper bound of the blowup time. In Section 4, we show the finite time blow-up for problem (1.1) with critical energy $J(u_0) = d$. In Section 5, we establish a new finite time blowup theorem for the solution of problem (1.1) for arbitrary high initial energy and estimate the upper bound of the blowup time.

2. Preliminaries

2.1. **Functional spaces.** In this section, we first recall some definitions and properties of the fractional Sobolev spaces, see [9, 22, 35] for further details.

Let $0 < s < 1 < p < \infty$ be real numbers and the fractional critical exponent p_s^* be defined as

$$p_s^* = \begin{cases} \frac{Np}{N-sp}, & \text{if } sp < N, \\ \infty, & \text{if } sp \geq N. \end{cases} \tag{2.1}$$

In the following, we denote $Q = \mathbb{R}^{2N} \setminus \mathcal{G}$, where

$$\mathcal{G} = \mathcal{C}(\Omega) \times \mathcal{C}(\Omega) \subset \mathbb{R}^{2N},$$

and $\mathcal{G} = \mathbb{R}^N \setminus \Omega$. W is a linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function u in W belongs to $L^p(\Omega)$ and

$$\iint_Q |u(x) - u(y)|^p K(x-y)\,dx\,dy < \infty.$$

The space W is equipped with the norm

$$\|u\|_W = \Big(\|u\|_{L^p(\Omega)} + \iint_Q |u(x) - u(y)|^p K(x-y)\,dx\,dy\Big)^{1/p}.$$

It is easy to get that $\|\cdot\|_W$ is a norm on W, see [35]. We shall work in the closed linear subspace

$$W_0 = \{u \in W : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}. \tag{2.2}$$

For any $p \in [1, +\infty)$, we define the fractional Sobolev space $W^{s,p}(\Omega)$ as follows

$$W^{s,p}(\Omega) = \big\{u \in L^p(\Omega) : \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \in L^p(\Omega \times \Omega)\big\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \Big(\|u\|_{L^p(\Omega)} + \iint_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{1/p}.$$

Lemma 2.1 ([35, Lemma 2.3]). *Let $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ satisfy assumption(A1). Then there exists a positive constant $C_0 = C_0(N, p, s)$ such that for any $v \in W_0$ and $q \in [1, p_s^*]$,*

$$\begin{aligned}\|v\|^p_{L^q(\Omega)} &\le C_0 \iint_{\Omega\times\Omega} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \\ &\le \frac{C_0}{K_0} \iint_Q |v(x)-v(y)|^p K(x-y)\,dx\,dy.\end{aligned}$$

Definition 2.2. Let $p \ge 1$ and W be a reflexive Banach space. A function f defined and measurable in Q belongs to the space $L^p(0, T; W)$, if

$$\|f\|_{L^p}(0,T;W) = \Big(\int_0^T \|f(x,t)\|^p_W dt\Big)^{1/p} < \infty,$$

Using [8], we can get an equivalent norm on W_0 defined as

$$\|v\|_{W_0(\Omega)} = \Big(\iint_Q |v(x)-v(y)|^p K(x-y)\,dx\,dy\Big)^{1/p}.$$

Definition 2.3. A function $u \in L^\infty(0,\infty; W_0)$ is said to be a (weak) solution of problem (1.1), if $u_t \in L^2(0,\infty; L^2(\Omega))$ and for a.e. $t > 0$,

$$\int_\Omega \partial_t u(x,t)\phi dx + \langle u, \phi\rangle_{W_0} = \int_\Omega |u|^{q-2}u\phi dx,$$

where

$$\begin{aligned}\langle u, \phi\rangle_{W_0} = M(\|u\|^p_{W_0}) \iint_Q &|u(x,t)-u(y,t)|^{p-2}[u(x,t)-u(y,t)] \\ &\times [\phi(x)-\phi(y)]K(x-y)\,dx\,dy,\end{aligned}$$

for any $\phi \in W_0$.

Then we introduce some functionals

$$J(u) = \frac{1}{p\lambda}\|u\|^{p\lambda}_{W_0} - \frac{1}{q}\|u\|^q_q, \tag{2.3}$$

$$I(u) = \|u\|^{p\lambda}_{W_0} - \|u\|^q_q, \tag{2.4}$$

and the potential well

$$\mathcal{W} = \{u \in W_0 \mid I(u) > 0, J(u) < d\} \cup \{0\},$$
$$\mathcal{V} = \{u \in W_0 \mid I(u) < 0, J(u) < d\}, \quad d = \inf_{u\in\mathcal{N}} J(u).$$

The Nehari manifold

$$\mathcal{N} = \{u \in W_0 : I(u) = 0, \|u\|_{W_0} \ne 0\},$$

separates the two unbounded sets

$$\mathcal{N}_+ = \{u \in W_0 \mid I(u) > 0\}, \quad \mathcal{N}_- = \{u \in W_0 \mid I(u) < 0\}.$$

2.2. **Family of potential wells.** In this section, we introduce a family of potential wells $\mathcal{W}_\delta$ and its corresponding sets $\mathcal{V}_\delta$, and give a series of their properties for problem (1.1). Firstly, let the definitions of functionals $J(u), I(u)$ and the potential well $\mathcal{W}$ with its depth d given above hold. Next, we give some properties of above sets and functionals.

For $\delta > 0$, we define

$$I_\delta(u) = \delta\|u\|_{W_0}^{p\lambda} - \|u\|_q^q, \quad d(\delta) = \inf_{u\in\mathcal{N}_\delta} J(u),$$
$$\mathcal{N}_\delta = \{u \in W_0 \mid I_\delta(u) = 0, \|u\|_{W_0} \neq 0\}, \quad r(\delta) = \Big(\frac{\delta}{C_*^q}\Big)^{\frac{1}{q-p\lambda}},$$

where C_* is the embedding constant from W_0 into $L^q(\Omega)$.

For $0 < \delta < q/(p\lambda)$, we define

$$\begin{gathered} \mathcal{W}_\delta = \{u \in W_0 \mid I_\delta(u) > 0, J(u) < d(\delta)\} \cup \{0\}, \\ \mathcal{V}_\delta = \{u \in W_0 \mid I_\delta(u) < 0, J(u) < d(\delta)\}, \\ \int_0^t \|u_\tau\|_2^2 d\tau + J(u) \le J(u_0)\,. \end{gathered} \tag{2.5}$$

Lemma 2.4. *Let $u \in W_0$. Then we have*

- (i) *If $I_\delta(u) < 0$, then $\|u\|_{W_0} > r(\delta)$. In particular, if $I(u) < 0$, then $\|u\|_{W_0} > r(1)$.*
- (ii) *If $I_\delta(u) = 0$, then $\|u\|_{W_0} \ge r(\delta)$ or $\|u\|_{W_0} = 0$. In particular, if $I(u) = 0$, then $\|u\|_{W_0} \ge r(1)$ or $\|u\|_{W_0} = 0$.*
- (iii) *If $I_\delta u = 0$ and $\|u\|_{W_0} \neq 0$, then $J(u) > 0$ for $0 < \delta < q/(p\lambda)$, $J(u) = 0$ for $\delta = q/(p\lambda)$, $J(u) < 0$ for $\delta > q/(p\lambda)$.*

Proof. (i) It is easy to see that $\|u\|_{W_0} \neq 0$ thanks to $I_\delta(u) < 0$. Thus from

$$\delta\|u\|_{W_0}^{p\lambda} < \|u\|_q^q \le C_*^q\|u\|_{W_0}^q = C_*^q\|u\|_{W_0}^{p\lambda}\|u\|_{W_0}^{q-p\lambda},$$

we obtain $\|u\|_{W_0} > r(\delta)$.

(ii) On the one hand, if $\|u\|_{W_0} = 0$, then $I_\delta(u) = 0$. On the other hand, if $\|u\|_{W_0} \neq 0$ and $I_\delta(u) = 0$, then by

$$\delta\|u\|_{W_0}^{p\lambda} = \|u\|_q^q \le C_*^q\|u\|_{W_0}^{p\lambda}\|u\|_{W_0}^{q-p\lambda},$$

we obtain $\|u\|_{W_0} \ge r(\delta)$.

(iii) The conclusion follows from Lemma 2.4(ii) and by $I_\delta(u) = 0$, we have

$$\begin{aligned} J(u) &= \Big(\frac{1}{p\lambda} - \frac{\delta}{q}\Big)\|u\|_{W_0}^{p\lambda} + \frac{\delta}{q}\|u\|_{W_0}^{p\lambda} - \frac{1}{q}\|u\|_q^q \\ &= \Big(\frac{1}{p\lambda} - \frac{\delta}{q}\Big)\|u\|_{W_0}^{p\lambda} + \frac{1}{q} I_\delta u\,, \end{aligned}$$

which implies (iii). □

Lemma 2.5. *$d(\delta)$ satisfies the following properties:*

- (i) *$d(\delta) \ge a(\delta)r^{p\lambda}(\delta)$ for $a(\delta) = 1/(p\lambda) - \delta/q, 0 < \delta < q/(p\lambda)$.*
- (ii) *$\lim_{\delta\to 0} d(\delta) = 0, d(q/(p\lambda)) = 0$ and $d(\delta) < 0$ for $\delta > q/(p\lambda)$.*
- (iii) *$d(\delta)$ is increasing on $0 < \delta \le 1$, decreasing on $1 \le \delta \le q/(p\lambda)$ and takes the maximum $d = d(1)$ at $\delta = 1$.*

Proof. (i) If $u \in \mathcal{N}$, then by lemma 2.4(ii) we have $\|u\|_{W_0} \geq r(\delta)$. Hence from

$$J(u) = \Big(\frac{1}{p\lambda} - \frac{\delta}{q}\Big)\|u\|_{W_0}^{p\lambda} + \frac{1}{q}I_\delta(u) = a(\delta)\|u\|_{W_0}^{p\lambda} \geq a(\delta)r^{p\lambda}(\delta),$$

it follows that $d(\delta) \geq a(\delta)r^{p\lambda}(\delta)$.

(ii) For any $u \in W_0, \|u\|_{W_0} \neq 0$, we define $\theta = \theta(\delta)$ by

$$\delta\|\theta u\|_{W_0}^{p\lambda} = \|\theta u\|_q^q, \tag{2.6}$$

i.e. $\delta\|u\|_{W_0}^{p\lambda} = \theta^{q-p\lambda}\|u\|_q^q$. Hence, for any $\delta > 0$, there exists a unique

$$\theta(\delta) = \Big(\frac{\delta\|u\|_{W_0}^{p\lambda}}{\|u\|_q^q}\Big)^{\frac{1}{q-p\lambda}},$$

satisfying (2.6), which implies that $\theta u \in \mathcal{N}_\delta$, we have $\lim_{\delta\to 0}\theta(\delta) = 0$. It is easy to see that

$$\lim_{\delta\to 0} J(\theta u) = \lim_{\theta\to 0} J(\theta u) = 0$$

and $\lim_{\delta\to 0} d(\delta) = 0$. From lemma 2.4 (iii), we can complete this proof.

(iii) It is enough to prove that for any $0 < \delta' < \delta'' < 1$ or $1 < \delta'' < \delta' < q/(p\lambda)$ and for any $u \in \mathcal{N}_{\delta''}$, there exist a $v \in \mathcal{N}_{\delta'}$ and a constant $\varepsilon(\delta', \delta'')$ such that $J(v) < J(u) - \varepsilon(\delta', \delta'')$. In fact, for above u we can define $\theta(\delta)$, then $I_\delta(\theta(\delta)u) = 0$ and $\theta(\delta'') = 1$. Let $g(\theta) = J(\theta u)$, we obtain

$$\frac{d}{d\theta}g(\theta) = \frac{1}{\theta}\Big((1-\delta)\|\theta u\|_{W_0}^{p\lambda} + I_\delta(\theta u)\Big) = \theta^{p\lambda-1}(1-\delta)\|u\|_{W_0}^{p\lambda}.$$

Taking $v = \theta(\delta')u$, then $v \in \mathcal{N}_{\delta'}$. For $0 < \delta' < \delta'' < 1$, we have

$$\begin{aligned} J(v) - J(u) &= g(1) - g(\theta(\delta')) \\ &= \int_{\theta(\delta')}^1 \frac{d}{d\theta}(g(\theta))d\theta \\ &= \int_{\theta(\delta')}^1 (1-\delta)\theta^{p\lambda-1}\|u\|_{W_0}^{p\lambda}d\theta \\ &> (1-\delta'')r^{p\lambda}(\delta'')\theta^{p\lambda-1}(\delta')\,(1-\theta(\delta')) \equiv \varepsilon(\delta', \delta''). \end{aligned}$$

For $1 < \delta'' < \delta' < q/(p\lambda)$, we have

$$\begin{aligned} J(u) - J(v) &= g(1) - g(\theta(\delta')) \\ &> (\delta''-1)r^{p\lambda}(\delta'')\theta^{p\lambda-1}(\delta'')\,(\theta(\delta')-1) \equiv \varepsilon(\delta', \delta''). \end{aligned}$$

Therefore, the conclusion of (iii) is proved. □

Lemma 2.6. *Assume $0 < J(u) < d$ for some $u \in W_0$, and $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = J(u)$. Then the sign of $I_\delta(u)$ doesn't change for $\delta_1 < \delta < \delta_2$.*

Proof. $J(u) > 0$ implies $\|u\|_{W_0} \neq 0$. If the sign of $I_\delta(u)$ is changeable for $\delta_1 < \delta < \delta_2$, then we choose $\overline{\delta} \in (\delta_1, \delta_2)$ and $I_{\overline{\delta}}(u) = 0$. Therefore, we can get $J(u) \geq d(\overline{\delta})$, which contradicts $J(u) = d(\delta_1) = d(\delta_2) < d(\overline{\delta})$. □

3. Blow up with low initial energy $J(u_0) < d$

Definition 3.1. Let $u(t)$ be a weak solution of problem (1.1). We define the maximal time existence $T_{\max}$ of $u(t)$ as follows:

(i) If $u(t)$ exists for $0 \le t < \infty$, then $T_{\max} = \infty$.
(ii) If there exists a $t_0 \in (0, \infty)$ such that $u(t)$ exists for $0 \le t < t_0$, but does not exists at $t = t_0$, then $T_{\max} = t_0$.

Lemma 3.2 (Invariant set for $J(u_0) < d$)**.** *Let $u_0 \in W_0$, $0 < e < d$, $\delta_1 < \delta_2$ be the two roots of equation $d(\delta) = e$. Then All weak solutions u of problem* (1.1) *with $J(u_0) = e$ belong to $\mathcal{V}_\delta$ for $\delta_1 < \delta < \delta_2$, $0 \le t < T_{\max}$, provided $I(u_0) < 0$, where $T_{\max}$ is the maximal existence time of $u(t)$.*

Proof. Let $u(t)$ be any weak solution of problem (1.1) with $J(u_0) = e$, $I(u_0) < 0$. From $J(u_0) = e$, $I(u_0) < 0$ and Lemma 2.6, it follows $I_\delta(u_0) < 0$ and $J(u_0) < d(\delta)$. Then $u_0(x) \in \mathcal{V}_\delta$ for $\delta_1 < \delta < \delta_2$.

We prove $u(t) \in V_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T_{\max}$. Arguing by contradiction, by time continuity of $I(u)$, we suppose that there exists a $\delta_0 \in (\delta_1, \delta_2)$ and $t_0 \in (0, T_{\max})$ such that $u(t_0) \in \partial\mathcal{V}_{\delta_0}$, $I_{\delta_0}(u(t_0)) = 0$ or $J(u(t_0)) = d(\delta_0)$. From

$$\int_0^t \|u(\tau)\|_2^2 d\tau + J(u) \le J(u_0) < d(\delta), \quad \delta_1 < \delta < \delta_2,\ 0 \le t < T_{\max}, \tag{3.1}$$

we can see that $J(u(t_0)) \ne d(\delta_0)$. Assume $I_{\delta_0}(u(t_0)) = 0$ and t_0 is the first time such that $I_{\delta_0}(u(t_0)) = 0$, then $I_{\delta_0}(u(t)) < 0$ for $0 \le t < t_0$. By Lemma 2.4(i) we have $\|u(t_0)\|_{W_0} > r(\delta_0)$ for $0 \le t < t_0$. Hence $\|u(t_0)\|_{W_0} > r(\delta_0)$, then $\|u(t_0)\|_{W_0} \ne 0$. From $u(t_0) \in \mathcal{N}_{\delta_0}$ and $J(u(t_0)) \ne d(\delta_0)$, we have $J(u(t_0)) > d(\delta_0)$, which contradicts (3.1). □

Remark 3.3. If the assumption $J(u_0) = e$ is replaced by $0 < J(u_0) \le e$ in Lemma 3.2, then the conclusion of Lemma 3.2 still holds.

3.1. Finite time blow-up at low initial energy.

In this section, we establish the finite time blow-up of solutions of problem (1.1). By Lemma 2.1 we know that W_0 is continuously embedding in $L^2(\Omega)$, let S be the best embedding constant. Then the main result of this section is stated as follows.

Theorem 3.4 (Blow-up for $J(u_0) < d$)**.** *Suppose that $u_0 \in W_0, J(u_0) < d$ and $I(u_0) < 0$. then any nontrivial solution of problem* (1.1) *must blowup in finite time. There exists a $T > 0$ such that*

$$\lim_{t \to T} \int_0^t \|u\|_2^2 d\tau = +\infty. \tag{3.2}$$

Proof. Let $u(t)$ be any weak solution of problem (1.1) with $J(u_0) < d$ and $I(u_0) < 0$. We define

$$M(t) = \int_0^t \|u\|_2^2 d\tau,$$

then $M'(t) = \|u\|_2^2$, and

$$M''(t) = 2(u, u_t) = 2\int_\Omega u_t u dx = 2\|u\|_q^q - 2\|u\|_{W_0}^{p\lambda} = -2I(u). \tag{3.3}$$

Notice that

$$J(u) = \frac{1}{p\lambda}\|u\|_{W_0}^{p\lambda} - \frac{1}{q}\|u\|_q^q = \Big(\frac{1}{p\lambda} - \frac{1}{q}\Big)\|u\|_{W_0}^{p\lambda} + \frac{1}{q}I(u);$$

thus

$$I(u) = qJ(u) - \frac{q-p\lambda}{p\lambda}\|u\|_{W_0}^{p\lambda}.$$

Applying the basic inequality $s \le s^\alpha + 1$ for any $s \ge 0$ and $\alpha \ge 1$, we can get

$$\begin{aligned}
M''(t) =& \frac{2(q-p\lambda)}{p\lambda}\|u\|_{W_0}^{p\lambda} - 2qJ(u)\\
\ge& \frac{2(q-p\lambda)}{p\lambda}(\|u\|_{W_0}^2 - 1) + 2q\int_0^t \|u_\tau\|_2^2 d\tau - 2qJ(u_0)\\
\ge& \frac{2C(q-p\lambda)}{p\lambda}\|u\|_2^2 + 2q\int_0^t \|u_\tau\|_2^2 - \Big(2qJ(u_0) + \frac{2(q-p\lambda)}{p\lambda}\Big)\\
=& \frac{2C(q-p\lambda)}{p\lambda}M'(t) + 2q\int_0^t \|u_\tau\|_2^2 d\tau - \Big(2qJ(u_0) + \frac{2(q-p\lambda)}{p\lambda}\Big),
\end{aligned}$$

where $C = S^2$. Note that

$$\begin{aligned}
\Big(\int_0^t (u_\tau, u)d\tau\Big)^2 =& \Big(\frac{1}{2}\int_0^t \frac{d}{d\tau}\|u\|_2^2\Big)^2\\
=& \Big(\frac{1}{2}\|u\|_2^2 - \frac{1}{2}\|u_0\|_2^2\Big)^2\\
=& \frac{1}{4}\big(\|u\|_2^4 - 2\|u\|_2^2\|u_0\|_2^2 + \|u_0\|_2^4\big)\\
=& \frac{1}{4}\big((M'(t))^2 - 2M'(t)\|u_0\|_2^2 + \|u_0\|_2^4\big)\,.
\end{aligned}$$

It follows that

$$(M'(t))^2 = 4\Big(\int_0^t\int_\Omega u_\tau u\,dx\,d\tau\Big)^2 + 2M'(t)\|u_0\|_2^2 - \|u_0\|_2^4\,. \tag{3.4}$$

Using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
& M''(t)M(t) - \frac{q}{2}\,(M'(t))^2\\
&\ge 2q\int_0^t \|u_\tau\|_2^2 d\tau \int_0^t \|u\|_2^2 d\tau - 2q\Big(\int_0^t\int_\Omega u_\tau u\,dx\,d\tau\Big)^2 + \frac{q}{2}\|u_0\|_2^4\\
&\quad - \Big(2qJ(u_0) + \frac{2(q-p\lambda)}{p\lambda}\Big)M(t) + \frac{2C(q-p\lambda)}{p\lambda}M'(t)M(t) - q\|u_0\|_2^2 M'(t)\\
&\ge \frac{2C(q-p\lambda)}{p\lambda}M'(t)M(t) - \Big(2qJ(u_0) + \frac{2(q-p\lambda)}{p\lambda}\Big)M(t) - q\|u_0\|_2^2 M'(t).
\end{aligned}$$

We discuss the following two cases:

(i) If $J(u_0) \le 0$, then

$$\begin{aligned}
& M(t)M''(t) - \frac{q}{2}(M'(t))^2\\
&\ge \frac{2C(q-p\lambda)}{p\lambda}M(t)M'(t) - q\|u_0\|_2^2 M'(t) - \frac{2(q-p\lambda)}{p\lambda}M(t).
\end{aligned}$$

Now we prove $I(u) < 0$ for $t > 0$. If it is false, we must be allowed to choose a $t_0 > 0$ such that $I(u(t_0)) = 0$ and $I(u) < 0$ for $0 \le t < t_0$. From Lemma 2.4(i), we have $\|u\|_{W_0} > r(1)$ for $0 \le t < t_0$, $\|u(t_0)\|_{W_0} \ge r(1)$ and $J(u(t_0)) \ge d$, which contradicts

(2.5). From (3.3), we can get $M''(t) > 0$ for $t \geq 0$. From $M'(0) = \|u_0\|_2^2 \geq 0$, we can see that there exists a $t_0 \geq 0$ such that $M'(t_0) > 0$. For $t \geq t_0$ we have

$$M(t) \geq M'(t_0)(t - t_0) + M(t_0) > M'(0)(t - t_0).$$

Therefore, for sufficiently large t, we obtain

$$\frac{C(q - p\lambda)}{p\lambda} M(t) > q\|u_0\|_2^2,$$
$$\frac{C(q - p\lambda)}{p\lambda} M'(t) > \frac{2(q - p\lambda)}{p\lambda},$$

then

$$M(t)M''(t) - \frac{q}{2}(M'(t))^2 > 0.$$

(ii) If $0 < J(u_0) < d$, then by Lemma 3.2 we have $u(t) \in \mathcal{V}_\delta$ for $1 < \delta < \delta_2$, $t \geq 0$ and $I_\delta(u) < 0$, $\|u\|_{W_0} > r(\delta)$ for $1 < \delta < \delta_2$, $t \geq 0$, where δ_2 is the larger root of equation $d(\delta) = J(u_0)$. Hence, $I_{\delta_2}(u) \leq 0$ and $\|u\|_{W_0} > r(\delta_2)$ for $t \geq 0$. By (3.3) we have

$$\begin{gathered} M''(t) = -2I(u) = 2(\delta_2 - 1)\|u\|_{W_0}^{p\lambda} - 2I_{\delta_2}(u) \\ \geq 2(\delta_2 - 1)\|u\|_{W_0}^{p\lambda} \geq 2(\delta_2 - 1)r^{p\lambda}(\delta_2), \quad t \geq 0, \\ M'(t) \geq 2(\delta_2 - 1)r^{p\lambda}(\delta_2)t + M'(0) \geq 2(\delta_2 - 1)r^{p\lambda}(\delta_2)t, \quad t \geq 0, \\ M(t) \geq 2(\delta_2 - 1)r^{p\lambda}(\delta_2)t^2, \quad t \geq 0. \end{gathered}$$

Therefore, for sufficiently large t, we have

$$\frac{C(q - p\lambda)}{p\lambda} M(t) > q\|u_0\|_2^2,$$
$$\frac{C(q - p\lambda)}{p\lambda} M'(t) > 2qJ(u_0) + \frac{2(q - p\lambda)}{p\lambda}.$$

Consequently,

$$\begin{aligned} &M(t)M''(t) - \frac{q}{2}(M'(t))^2 \\ &\geq \frac{2C(q - p\lambda)}{p\lambda} M'(t)M(t) - \Big(2qJ(u_0) + \frac{2(q - p\lambda)}{p\lambda}\Big)M(t) - q\|u_0\|_2^2 M'(t) \\ &= \Big(\frac{C(q - p\lambda)}{p\lambda} M(t) - q\|u_0\|_2^2\Big)M'(t) \\ &\quad + \Big(\frac{C(q - p\lambda)}{p\lambda} M'(t) - 2qJ(u_0) - \frac{2(q - p\lambda)}{p\lambda}\Big)M(t) > 0. \end{aligned}$$

The remainder of the proof is the same as that in [32]. □

3.2. **Blow up time with low initial energy.** We give an upper bound for the blow up time. By Lemma 2.1, we know that the Sobolev space $W_0 \hookrightarrow L^q(\Omega)$ continuously. Let C_* be the optimal constant of the embedding then

$$\|u\|_q \leq C_* \|u\|_{W_0}, \tag{3.5}$$

$$\alpha_1 := C_*^{-\frac{q}{q - p\lambda}}, \tag{3.6}$$

$$J_1 = \frac{q - p\lambda}{p\lambda q} C_*^{-\frac{p\lambda q}{q - p\lambda}} = \frac{q - p\lambda}{p\lambda q}\alpha_1^{p\lambda}. \tag{3.7}$$

By [23, Lemma 3.4], we know that

$$J_1 = \frac{q - p\lambda}{p\lambda q} \frac{1}{C_*^{\frac{p\lambda q}{q-p\lambda}}} = d.$$

Then the main result of this article reads as follows.

Theorem 3.5. *Suppose $q > p\lambda$, $q > 2$. Then the solution of problem (1.1) will blow up in finite time if the initial value u_0 is chosen to ensure that $J(u_0) < d$ and $\|u_0\|_{W_0} > \alpha_1$. Moreover, the blow-up time T can be estimated from above by T^*, where*

$$T^* = \frac{q\big(⨍_\Omega u_0^2(x)\big)^{\frac{2-q}{2}}}{(q-2)(q-p\lambda)\Big(1 - \big(\big(\frac{1}{p\lambda} - J(u_0)\alpha_1^{-p\lambda}\big)q\big)^{-\frac{q}{q-p\lambda}}\Big)} \tag{3.8}$$

and

$$⨍_\Omega f(x)dx = \frac{1}{|\Omega|}\int_\Omega f(x)dx$$

where $|\Omega|$ is the Lebesgue measure of Ω.

Lemma 3.6. *The energy defined in (2.3) is nonincreasing with*

$$J(u(t)) = J(u_0) - \int_0^t \|u_\tau\|_2^2 d\tau. \tag{3.9}$$

Proof. From (2.3), we have

$$\begin{aligned} J'(u(t)) =& \frac{d}{dt}\Big(\frac{1}{p\lambda}\|u\|_{W_0}^{p\lambda} - \frac{1}{q}\|u\|_q^q\Big) \\ =& -\int_\Omega |u|^{q-2}uu_t dx + \int_\Omega [u]_{s,p}^{(\lambda-1)p}(-\Delta)_p^s uu_t dx \\ =& -\int_\Omega \Big(|u|^{q-2}u - [u]_{s,p}^{(\lambda-1)p}(-\Delta)_p^s u\Big) u_t dx \\ =& -\int_\Omega u_t^2 dx, \end{aligned}$$

which yields (3.9). □

We deduce from (2.3) and (3.5) that

$$J(u(t)) = \frac{1}{p\lambda}\|u\|_{W_0}^{p\lambda} - \frac{1}{q}\|u\|_q^q \geq \frac{1}{p\lambda}\alpha^{p\lambda} - \frac{1}{q}(C_*\alpha)^q, \tag{3.10}$$

where $\alpha(t) = \|u(\cdot,t)\|_{W_0}$.

Lemma 3.7. *Let $g : [0,\infty) \mapsto \mathbb{R}$ be defined by*

$$g(\alpha) = \frac{1}{p\lambda}\alpha^{p\lambda} - \frac{1}{q}C_*^q\alpha^q.$$

Then the following properties hold under the assumptions of Theorem 3.5:

(i) *g is increasing for $0 < \alpha < \alpha_1$ and decreasing for $\alpha \geq \alpha_1$;*
(ii) *$\lim_{\alpha\to\infty} g(\alpha) = -\infty$ and $g(\alpha_1) = J_1$.*

Proof. (i) The first derivative of $g(\alpha)$ is

$$g'(\alpha) = \alpha^{p\lambda-1} - C_*^q \alpha^{q-1}.$$

Note that $g'(\alpha) = 0$ implied that $\alpha_1 = C_*^{-\frac{q}{q-p\lambda}}$, hence (i) follows.

(ii) Since $p\lambda < q$, we have that $\lim_{\alpha\to\infty} g(\alpha) = -\infty$. α_1 is the extreme point and a routine computation gives rise to $g(\alpha_1) = J_1$. Then (ii) holds. □

Lemma 3.8. *Under the assumptions of Theorem 3.5, there exists a positive constant $\alpha_2 > \alpha_1$ such that*

$$\|u(\cdot,t)\|_{W_0} \ge \alpha_2, \quad t \ge 0, \tag{3.11}$$

$$\int_\Omega |u|^q dx \ge (C_*\alpha_2)^q, \tag{3.12}$$

$$\frac{\alpha_2}{\alpha_1} \ge \Big(\Big(\frac{1}{p\lambda} - J(0)\alpha_1^{-p\lambda}\Big)q\Big)^{\frac{1}{q-p\lambda}} > 1. \tag{3.13}$$

Proof. Since $J(u_0) < J_1$, it follows from Lemma 3.7 that there exists a positive constant $\alpha_2 > \alpha_1$ such that $J(u_0) = g(\alpha_2)$. Let $\alpha_0 = \|u_0\|_{W_0}$, by (3.10), we have $g(\alpha_0) \le J(u_0) = g(\alpha_2)$. Since $\alpha_0, \alpha_2 \ge \alpha_1$, it follows from Lemma 3.7(i) that $\alpha_0 \ge \alpha_2$ so (3.11) holds for $t = 0$.

Now we prove (3.11) by contradiction. Suppose that $\|u(\cdot,t_0)\|_{W_0} < \alpha_2$ for some $t_0 > 0$. By the continuity of $\|u(\cdot,t)\|_{W_0}$ and $\alpha_1 < \alpha_2$, we may choose t_0 such that $\|u(\cdot,t_0)\|_{W_0} > \alpha_1$. Then it follows from (3.10) that

$$J(u_0) = g(\alpha_2) < g(\|u(\cdot,t_0)\|_{W_0}) \le J(u(t_0)),$$

which contradicts Lemma 3.6, and (3.11) follows.

By (2.3) and Lemma 3.6, we obtain

$$\int_\Omega \frac{1}{q}|u|^q dx \ge \frac{1}{p\lambda}\|u\|_{W_0}^{p\lambda} - J(u_0) \ge \frac{1}{p\lambda}\alpha_2^{p\lambda} - J(u_0) = \frac{1}{q}(C_*\alpha_2)^q,$$

and (3.12) follows.

Since $J(u_0) < J_1$, by a straightforward computation, we can check

$$\Big(\frac{1}{p\lambda} - J(u_0)\alpha_1^{-p\lambda}\Big)q > 1.$$

Denote $\beta = \alpha_2/\alpha_1$, then $\beta > 1$ by the fact that $\alpha_2 > \alpha_1$. So it follows from $J(u_0) = g(\alpha_2)$ and (3.6) that

$$\begin{aligned} J(u_0) &= g(\beta\alpha_1) \\ &= \frac{1}{p\lambda}(\beta\alpha_1)^{p\lambda} - \frac{1}{q}C_*^q(\beta\alpha_1)^q \\ &\ge \alpha_1^{p\lambda}\Big(\frac{1}{p\lambda} - \frac{\beta^{q-p\lambda}}{q}C_*^q\alpha_1^{q-p\lambda}\Big) \\ &= \alpha_1^{p\lambda}\Big(\frac{1}{p\lambda} - \frac{\beta^{q-p\lambda}}{q}\Big), \end{aligned} \tag{3.14}$$

which implies that the inequality in (3.13). □

Lemma 3.9. *Under the assumptions of Theorem 3.5, we have the estimate*

$$0 < H(0) \le H(t) \le \frac{1}{q}\int_\Omega |u|^q dx, \tag{3.15}$$

where $H(t) = J_1 - J(u(t))$ *for* $t \geq 0$.

Proof. From Lemma 3.6, we know that $H(t)$ is nondecreasing in t. Thus

$$H(t) \geq H(0) = J_1 - J(u_0) > 0, \ t \geq 0. \tag{3.16}$$

Combining (2.3), (3.7) and (3.11), $J(u(t)) > 0$ and $\alpha_2 > \alpha_1$, we have

$$H(t) = J_1 - J(u(t)) \leq J_1 - \frac{1}{p\lambda}\alpha_1^{p\lambda} + \frac{1}{q}\int_\Omega |u|^q dx \leq \frac{1}{q}\int_\Omega |u|^q dx.$$

This completes the proof. ∎

Proof of Theorem 3.5. Let

$$M(t) = \frac{1}{2}\int_\Omega u^2(x,t)dx.$$

Then by the definition of $J(u(t))$ and $H(t)$, the derivative of $M(t)$ satisfies

$$\begin{aligned} M'(t) &= \int_\Omega u u_t dx \\ &= - \|u\|_{W_0}^{p\lambda} + \|u\|_q^q \\ &= \|u\|_q^q - p\lambda J(u(t)) - \frac{p\lambda}{q}\|u\|_q^q \\ &= \frac{q-p\lambda}{q}\|u\|_q^q - p\lambda J_1 + p\lambda H(t). \end{aligned} \tag{3.17}$$

From (3.6), (3.7) and (3.12), we obtain

$$\begin{aligned} p\lambda J_1 = \frac{q-p\lambda}{q} C_*^{-\frac{p\lambda q}{q-p\lambda}} &= \frac{q-p\lambda}{q}(C_*\alpha_1)^q \\ &= \frac{q-p\lambda}{q}\Big(\frac{\alpha_1}{\alpha_2}\Big)^q (C_*\alpha_2)^q \\ &\leq \frac{q-p\lambda}{q}\Big(\frac{\alpha_1}{\alpha_2}\Big)^q \int_\Omega |u|^q dx. \end{aligned} \tag{3.18}$$

So, we have

$$M'(t) \geq \tilde{C}\|u\|_q^q, \tag{3.19}$$

where

$$\tilde{C} = \Big(1 - (\frac{\alpha_1}{\alpha_2})^q\Big)\frac{q-p\lambda}{q}.$$

By Hölder's inequality, we have

$$M^{q/2}(t) \leq \bar{C}\int_\Omega |u|^q dx, \tag{3.20}$$

where

$$\bar{C} = 2^{-q/2}|\Omega|^{\frac{q-2}{2}},$$

and $|\Omega|$ is the Lebesgue measure of Ω. Then it follows from (3.19) and (3.20) that

$$M'(t) \geq \frac{\tilde{C}}{\bar{C}} M^{q/2}(t),$$

which means that

$$M(t) = \Big(\Big(\frac{1}{2}\int_\Omega |u_0|^2 dx\Big)^{\frac{2-q}{2}} - \frac{(q-2)\tilde{C}}{2\bar{C}}t\Big)^{-\frac{2}{q-2}}. \tag{3.21}$$

Let

$$\tilde{T} := \frac{2^{q/2}\bar{C}}{(q-2)\tilde{C}}\Big(\int_\Omega |u_0|^2dx\Big)^{\frac{2-q}{2}} \in (0,\infty)\,. \tag{3.22}$$

Then $M(t)$ blows up at time $\tilde{T}$. Therefore, $u(x,t)$ ceases to exist at some finite time $T \le \tilde{T}$, that is to say, $u(x,t)$ blows up at a finite time T.

Next, we estimate T. By (3.13) and the values of $\tilde{C}, \bar{C}$, we have

$$\frac{2^{q/2}\bar{C}}{(q-2)\tilde{C}} \le \frac{|\Omega|^{\frac{q-2}{2}}}{(q-2)\Big(1-\big((\frac{1}{p\lambda}-J(u_0)\alpha_1^{-p\lambda})q\big)^{\frac{q-p\lambda}{q}}\Big)^{\frac{q-p\lambda}{q}}}.$$

The above inequalities combined with (3.22) give $T \le \tilde{T} \le T^*$, where T^* is defined in (3.8). The remainder of the proof is the same as that in [40]. $\square$

4. Blow up with critical initial energy $J(u_0) = d$

In this section, we prove the finite time blow-up of solution for problem (1.1) with the critical initial condition $J(u_0) = d$.

Theorem 4.1. *Suppose that $u_0 \in W_0$, $J(u_0) = d$ and $I(u_0) < 0$. Then any nontrivial solution of problem* (1.1) *must blow up in finite time.*

Proof. Let $u(t)$ be any weak solution of problem (1.1) with $J(u_0) = d$ and $I(u_0) < 0$, T being the existence time of $u(t)$. We prove that $T < \infty$. Arguing by contradiction, we assume that $T = \infty$. Now we define

$$M(t) = \int_0^t \|u\|_2^2 d\tau.$$

By Theorem 3.4 and $J(u_0) = d$ we have

$$\begin{aligned}
M''(t) =& \frac{2(q-p\lambda)}{p\lambda}\|u\|_{W_0}^{p\lambda} - 2qJ(u)\\
=& \frac{2(q-p\lambda)}{p\lambda}\|u\|_{W_0}^{p\lambda} + 2q\int_0^t \|u_\tau\| d\tau - 2qJ(u_0)\\
\ge& \frac{2(q-p\lambda)}{p\lambda}(\|u\|_{W_0}^{2} - 1) + 2q\int_0^t \|u_\tau\|_2^2 d\tau - 2qJ(u_0)\\
\ge& \frac{2C(q-p\lambda)}{p\lambda}\|u\|_2^2 + 2q\int_0^t \|u_\tau\|_2^2 d\tau - \Big(2qJ(u_0) + \frac{2(q-p\lambda)}{p\lambda}\Big)\\
=& \frac{2C(q-p\lambda)}{p\lambda}M'(t) + 2q\int_0^t \|u_\tau\|_2^2 d\tau - \Big(2qJ(u_0) + \frac{2(q-p\lambda)}{p\lambda}\Big).
\end{aligned}$$

According to the estimate of the $(M'(t))^2$ in Theorem 3.4 which is (3.4), we obtain

$$\begin{aligned}
&M''(t)M(t) - \frac{q}{2}\left(M'(t)\right)^2\\
&\ge 2q\int_0^t \|u_\tau\|_2^2 d\tau \int_0^t \|u\|_2^2 d\tau - 2q\Big(\int_0^t\int_\Omega u_\tau u\,dx\,d\tau\Big)^2\\
&\quad - \Big(2qJ(u_0) + \frac{2(q-p\lambda)}{p\lambda}\Big)M(t) + \frac{2C(q-p\lambda)}{p\lambda}M'(t)M(t)\\
&\quad - q\|u_0\|_2^2 M'(t) + \frac{q}{2}\|u_0\|_2^4
\end{aligned}$$

$$\geq \frac{2C(q-p\lambda)}{p\lambda}M'(t)M(t) - \Big(2qJ(u_0) + \frac{2(q-p\lambda)}{p\lambda}\Big)M(t) - q\|u_0\|_2^2 M'(t).$$

By using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} &M(t)M''(t) - \frac{q}{2}(M'(t))^2 \\ &\geq \frac{2C(q-p\lambda)}{p\lambda}M'(t)M(t) - \Big(2qJ(u_0) + \frac{2(q-p\lambda)}{p\lambda}\Big)M(t) - q\|u_0\|_2^2 M'(t) \\ &= \Big(\frac{C(q-p\lambda)}{p\lambda}M(t) - q\|u_0\|_2^2\Big)M'(t) \\ &\quad + \Big(\frac{C(q-p\lambda)}{p\lambda}M'(t) - 2qJ(u_0) - \frac{2(q-p\lambda)}{p\lambda}\Big)M(t). \end{aligned} \tag{4.1}$$

On the other hand, from $J(u_0) = d > 0$, $I(u_0) < 0$ and the continuity of $J(u)$ and $I(u)$ with respect to t, it follows that there exists a sufficiently small $t_1 > 0$ such that $J(u(t_1)) > 0$ and $I(u) < 0$ for $0 \leq t \leq t_1$. Hence $(u_t, u) = -I(u) > 0$, $u_t \neq 0, \|u_t\| > 0$ for $0 \leq t \leq t_1$. From this and the continuity of $\int_0^t \|u_\tau\|_2^2 d\tau$, we can choose a t_1 such that

$$0 < J(u(t_1)) = d_1 = d - \int_0^{t_1} \|u_\tau\|_2^2 d\tau < d.$$

Thus we take $t = t_1$ as the initial time, then we know that $u(t) \in \mathcal{V}_\delta$ for $\delta \in (\delta_1, \delta_2)$, $t_1 \leq t < \infty$, where (δ_1, δ_2) is the maximal interval including $\delta = 1$ such that $d(\delta) > d_1$ for $\delta \in (\delta_1, \delta_2)$. Hence we have $I_\delta(u) < 0$ and $\|u\|_{W_0} > r(\delta)$ for $\delta \in (1, \delta_2)$, $t_1 \leq t < \infty$, and $I_{\delta_2}(u) \leq 0, \|u\|_{W_0} \geq r(\delta_2)$ for $t_1 \leq t < \infty$. Thus from (3.3) we obtain

$$\begin{aligned} M''(t) = -2I(u) &= 2(\delta_2 - 1)\|u\|_{W_0}^{p\lambda} - 2I_{\delta_2}(u) \\ &\geq 2(\delta_2 - 1)\|u\|_{W_0}^{p\lambda} \\ &\geq 2(\delta_2 - 1)r^{p\lambda}(\delta_2) \equiv C(\delta_2), \quad t_1 \leq t < \infty, \end{aligned} \tag{4.2}$$

$$M'(t) \geq C(\delta_2)(t - t_1) + M'(t_1) \geq C(\delta_2)(t - t_1), \quad t_1 \leq t < \infty, \tag{4.3}$$

$$M(t) \geq \frac{1}{2}C(\delta_2)(t-t_1)^2 + M(t_1) > \frac{1}{2}C(\delta_2)(t-t_1)^2, \quad t_1 \leq t < \infty. \tag{4.4}$$

From (4.3) and (4.4) it follows that for sufficiently large t we have

$$\frac{C(q-p\lambda)}{p\lambda}M(t) > q\|u_0\|_2^2,$$

and

$$\frac{C(q-p\lambda)}{p\lambda}M'(t) > 2qd + \frac{2(q-p\lambda)}{p\lambda}, \quad t_1 \leq t < \infty.$$

Thus (4.1) yields

$$M(t)M'(t) - \frac{q}{2}(M'(t))^2 > 0,$$

which gives

$$(M^{-\alpha}(t))'' = \frac{-\alpha}{M^{\alpha+2}}(t)\left(M(t)M'(t) - (\alpha+1)(M'(t))^2\right) \leq 0, \quad \alpha = \frac{q-2}{2}.$$

From this it follows that there exists a $T_1 > 0$ such that

$$\lim_{t\to T_1} M^{-\alpha}(t) = 0, \quad \text{and} \quad \lim_{t\to T_1} M(t) = +\infty,$$

which contradicts that $T = +\infty$. $\square$

5. Blow up time with high initial energy $J(u_0) > 0$

In this section, we establish a finite time blowup theorem for the solution of problem (1.1) with arbitrary high initial energy. At the same time, we estimate the upper bound of the blowup time.

Theorem 5.1. *Let $u(x,t)$ be a weak solution to problem* (1.1)*, $u_0 \in W_0$. Suppose that $J(u_0) > 0$ and*

$$\frac{p\lambda q}{q-p\lambda} J(u_0) < B\|u_0\|_2^{p\lambda} \tag{5.1}$$

hold. Then the solution $u(x,t)$ blows up in finite time, where B is best constant of inequality $\|u\|_{W_0}^{p\lambda} \geq B\|u\|_2^{p\lambda}$ with $B = S^{p\lambda}$. In addition there exists a t_1 as

$$0 < t_1 \leq \frac{2\varphi(0)}{(\alpha-1)\varphi'(0)},$$

such that

$$\lim_{t\to t_1} \int_0^t \|u\|_2^2 d\tau = +\infty, \tag{5.2}$$

where

$$\varphi(t) = \Big(\int_0^t \|u\|_2^2 d\tau\Big) + \varepsilon^{-1}\|u_0\|_2^2 \int_0^t \|u\|_2^2 d\tau + c, \tag{5.3}$$

$$1 < \alpha < \frac{B(q-p\lambda)\|u_0\|_2^{p\lambda}}{p\lambda q J(u_0)}, \tag{5.4}$$

$$0 < \varepsilon < \frac{1}{p\lambda\alpha\|u_0\|_2^2}\Big(\frac{2B(q-p\lambda)}{q}\|u_0\|_2^2 - 2p\lambda\alpha J(u_0) - \frac{2(q-p\lambda)}{q}\Big), \tag{5.5}$$

$$c > \frac{1}{4}\varepsilon^{-2}\|u\|_2^4. \tag{5.6}$$

Lemma 5.2 ([15])**.** *Suppose that a positive, twice-differentiable function $\psi(t)$ satisfy the inequality*

$$\psi''(t)\psi(t) - (1+\theta)(\psi'(t))^2 \geq 0, \quad t > 0,$$

where $\theta > 0$ is a constant. If $\psi(0) > 0$ and $\psi'(0) > 0$, then there exists $0 < t_1 \leq \frac{\psi(0)}{\theta\psi'(0)}$ such that $\psi(t)$ tends to ∞ as $t \to t_1$.

To prove the high energy blowup, we first establish the following lemma.

Lemma 5.3. *Assume that $u_0 \in W_0$ satisfies* (5.1)*. Then $u \in \mathcal{N}_- = \{u \in W_0 | I(u) < 0\}$.*

Proof. Let $u(t)$ be any weak solution of problem (1.1). Multiplying (1.1) by $u_t(t)$ and integrating on Ω, then we have

$$\|u_t(t)\|_2^2 = -\frac{1}{p\lambda}\frac{d}{dt}\|u\|_{W_0}^{p\lambda} + \frac{1}{q}\frac{d}{dt}\|u\|_q^q;$$

that is,

$$-\|u_t(t)\|_2^2 = \frac{d}{dt}\Big(\frac{1}{p\lambda}\|u\|_{W_0}^{p\lambda} - \frac{1}{q}\|u\|_q^q\Big).$$

Then, we could obtain

$$\frac{d}{dt}J(u) = -\|u_t(t)\|_2^2 \le 0. \tag{5.7}$$

Multiplying (1.1) by u and integrate on $\Omega \times (0,t)$, we have

$$\frac{1}{2}\|u\|_2^2 - \frac{1}{2}\|u_0\|_2^2 + \int_0^t (\|u\|_{W_0}^{p\lambda} - \|u\|_q^q) d\tau = 0\,;$$

that is,

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 = -I(u). \tag{5.8}$$

Note that

$$\begin{aligned} J(u_0) =& \frac{q-p\lambda}{p\lambda q}\|u_0\|_{W_0}^{p\lambda} + \frac{1}{q}I(u_0) \\ \ge& \frac{B(q-p\lambda)}{p\lambda q}\|u_0\|_2^{p\lambda} + \frac{1}{q}I(u_0)\,. \end{aligned}$$

Then (5.1) indicates that $I(u_0) < 0$.

Next, we prove $u(t) \in \mathcal{N}_-$ for all $t \in [0,T)$. Arguing by contradiction, by the continuity of $I(t)$ in t, we assume that there exists an $s \in (0,T)$ such that $u(t) \in \mathcal{N}_-$ for $0 \le t < s$ and $u(s) \in \mathcal{N}$, then by (5.8) we have

$$\frac{d}{dt}\|u(t)\|_2^2 = -2I(u) > 0, \quad \text{for all } t \in [0,s), \tag{5.9}$$

which implies that $\|u_0\|_2^2 < \|u(s)\|_2^2$. Then, we have

$$\|u_0\|_2^{p\lambda} < \|u(s)\|_2^{p\lambda}. \tag{5.10}$$

From (5.7) it follows that

$$J(u(s)) \le J(u_0) \quad \text{for all } t \in [0,s). \tag{5.11}$$

By the definition of $J(u)$ and $u(s) \in \mathcal{N}$, we arrive to

$$J(u(s)) = \frac{q-p\lambda}{p\lambda q}\|u\|_{W_0}^{p\lambda} + \frac{1}{q}I(u(s)) \ge \frac{B(q-p\lambda)}{p\lambda q}\|u\|_2^{p\lambda}.$$

Combining (5.1) and (5.11), we obtain

$$\frac{B(q-p\lambda)}{p\lambda q}\|u\|_2^{p\lambda} \le J(u_0) < \frac{B(q-p\lambda)}{p\lambda q}\|u_0\|_2^{p\lambda};$$

that is

$$\|u(s)\|_2^{p\lambda} < \|u_0\|_2^{p\lambda}.$$

This contradicts (5.10). □

Now we show high energy blowup and estimate the upper bound of the blowup time of solutions for problem(1.1).

Proof. Arguing by contradiction, we assume the existence time of solutions $T = +\infty$. Integrating of (5.7) with from 0 to t,

$$J(u) + \int_0^t \|u_\tau\|_2^2 d\tau = J(u_0). \tag{5.12}$$

From (5.8) we have

$$
\begin{aligned}
\frac{d}{dt}\|u\|_2^2 &= -2I(u) \\
&= -2(\|u\|_{W_0}^{p\lambda} - \|u\|_q^q) \\
&= -2p\lambda\Big(\frac{1}{p\lambda}\|u\|_{W_0}^{p\lambda} - \frac{1}{q}\|u\|_q^q\Big) + \Big(2 - \frac{2p\lambda}{q}\Big)\|u\|_q^q \\
&= -2p\lambda J(u) + \frac{2q-2p\lambda}{q}\|u\|_q^q.
\end{aligned} \tag{5.13}
$$

In the rest of the proof, we consider the following two cases.

(i) $J(u) \ge 0$, for all $t > 0$. From (5.1), we choose α satisfying (5.4). Substituting (5.12) into (5.13), as $J(u) \ge 0$ in this case we obtain

$$
\begin{aligned}
\frac{d}{dt}\|u\|_2^2 &= 2p\lambda(\alpha-1)J(u) - 2p\lambda\alpha J(u) + \frac{2(q-p\lambda)}{q}\|u\|_q^q \\
&\ge -2p\lambda\alpha J(u_0) + 2p\lambda\alpha\int_0^t \|u_\tau\|_2^2 d\tau + \frac{2(q-p\lambda)}{q}\|u\|_q^q.
\end{aligned} \tag{5.14}
$$

From Lemma 5.3, we know that $\|u\|_{W_0}^{p\lambda} < \|u\|_q^q$. Therefore, applying the basic inequality $s \le s^\alpha + 1$ for any $s \ge 0$ and $\alpha \ge 1$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|u\|_2^2 \\
&\ge -2p\lambda\alpha J(u_0) + 2p\lambda\alpha\int_0^t \|u_\tau\|_2^2 d\tau + \frac{2(q-p\lambda)}{q}\|u\|_q^q \\
&> -2p\lambda\alpha J(u_0) + 2p\lambda\alpha\int_0^t \|u_\tau\|_2^2 d\tau + \frac{2(q-p\lambda)}{q}\|u\|_{W_0}^{p\lambda} \\
&> -2p\lambda\alpha J(u_0) + 2p\lambda\alpha\int_0^t \|u_\tau\|_2^2 d\tau + \frac{2(q-p\lambda)}{q}(\|u\|_{W_0}^2 - 1) \\
&> -2p\lambda\alpha J(u_0) + 2p\lambda\alpha\int_0^t \|u_\tau\|_2^2 d\tau + \frac{2B(q-p\lambda)}{q}\|u\|_2^2 - \frac{2(q-p\lambda)}{q}.
\end{aligned} \tag{5.15}
$$

Then

$$
\frac{d}{dt}\|u\|_2^2 - \frac{2B(q-p\lambda)}{q}\|u\|_2^2 > -2p\lambda\alpha J(u_0) - \frac{2(q-p\lambda)}{q}, \tag{5.16}
$$

which yields

$$
\begin{aligned}
\|u\|_2^2 &> \|u_0\|_2^2 e^{\frac{2B(q-p\lambda)}{q}t} \\
&\quad + \frac{q}{B(q-p\lambda)}\Big(p\lambda\alpha J(u_0) + \frac{q-p\lambda}{q}\Big)\Big(1 - e^{\frac{2B(q-p\lambda)}{q}t}\Big).
\end{aligned} \tag{5.17}
$$

Next, we define $y(t) = \int_0^t \|u(\tau)\|_2^2 d\tau$. Since the solution $u(x,t)$ is global, thus the function $y(t)$ is bounded for all $t \ge 0$. Then we have

$$
y'(t) = \|u(t)\|_2^2, \quad y''(t) = \frac{d}{dt}\|u\|_2^2.
$$

Substituting (5.17) into (5.15), we obtain

$$
\begin{aligned}
y''(t) &> \Big(\frac{2B(q-p\lambda)}{q}\|u_0\|_2^2 - 2p\lambda\alpha J(u_0) - \frac{2(q-p\lambda)}{q}\Big)e^{\frac{2B(q-p\lambda)}{q}t} \\
&\quad + 2p\lambda\alpha\int_0^t \|u_\tau\|_2^2 d\tau \\
&> p\lambda\alpha\varepsilon\|u_0\|_2^2 + 2p\lambda\alpha\int_0^t \|u_\tau\|_2^2 d\tau \\
&= A(t).
\end{aligned} \tag{5.18}
$$

By (5.4), we can take $\varepsilon > 0$ small enough such that

$$
\varepsilon < \frac{1}{p\lambda\alpha\|u_0\|_2^2}\Big(\frac{2B(q-p\lambda)}{q}\|u_0\|_2^2 - 2p\lambda\alpha J(u_0) - \frac{2(q-p\lambda)}{q}\Big), \tag{5.19}
$$

then we pick $c > 0$ large enough such that

$$
c > \frac{1}{4}\varepsilon^{-2}\|u\|_2^4. \tag{5.20}
$$

We now define the auxiliary function $\varphi(t) = y^2(t) + \varepsilon^{-1}\|u_0\|_2^2 y(t) + c$. Hence

$$
\varphi'(t) = \left(2y(t) + \varepsilon^{-1}\|u_0\|_2^2\right) y'(t), \tag{5.21}
$$

$$
\varphi''(t) = \left(2y(t) + \varepsilon^{-1}\|u_0\|_2^2\right) y''(t) + 2(y'(t))^2. \tag{5.22}
$$

Set $\delta = 4c - \varepsilon^{-2}\|u_0\|_2^4$, because of (5.6), $\delta > 0$. Now, from (5.21) we can write

$$
\begin{aligned}
(\varphi'(t))^2 &= \left(2y(t) + \varepsilon^{-1}\|u_0\|_2^2\right)^2 (y'(t))^2 \\
&= \left(4y^2(t) + 4\varepsilon^{-1}\|u_0\|_2^2 y(t) + \varepsilon^{-2}\|u_0\|_2^4\right)(y'(t))^2 \\
&= \left(4y^2(t) + 4\varepsilon^{-1}\|u_0\|_2^2 y(t) + 4c - \delta\right)(y'(t))^2 \\
&= (4\varphi(t) - \delta)(y'(t))^2.
\end{aligned} \tag{5.23}
$$

The above equality yields

$$
4\varphi(t)(y'(t))^2 = (\varphi'(t))^2 + \delta(y'(t))^2. \tag{5.24}
$$

By integrating

$$
\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 = (u, u_t) \tag{5.25}
$$

from 0 to t, we obtain

$$
\frac{1}{2}\left(\|u(t)\|_2^2 - \|u_0\|_2^2\right) = \int_0^t (u, u_\tau)d\tau.
$$

Hence

$$
\|u(t)\|_2^2 = \|u_0\|_2^2 + 2\int_0^t (u, u_\tau)d\tau.
$$

This equality along with the Hölder and Young's inequality give

$$
\begin{aligned}
&(y'(t))^2\\
&= \|u(t)\|_2^4\\
&= \Big(\|u_0\|_2^2 + 2\int_0^t (u, u_\tau)d\tau\Big)^2\\
&\le \Big(\|u_0\|_2^2 + 2\Big(\int_0^t \|u\|_2^2 d\tau\Big)^{1/2}\Big(\int_0^t \|u_\tau\|_2^2 d\tau\Big)^{1/2}\Big)^2\\
&\le \|u_0\|_2^4 + 4y(t)\int_0^t \|u_\tau\|_2^2 d\tau + 2\varepsilon\|u_0\|_2^2 y(t) + 2\varepsilon^{-1}\|u_0\|_2^2\int_0^t \|u_\tau\|d\tau\\
&= B(t).
\end{aligned} \tag{5.26}
$$

From (5.22) and (5.24), we obtain

$$
\begin{aligned}
2\varphi(t)\varphi''(t) &= 2\left(\left(2y(t) + \varepsilon^{-1}\|u_0\|_2^2\right) y''(t) + 2(y'(t))^2\right)\varphi(t)\\
&= 2\left(2y(t) + \varepsilon^{-1}\|u_0\|_2^2\right) y''(t)\varphi(t) + 4(y'(t))^2\varphi(t)\\
&= 2\left(2y(t) + \varepsilon^{-1}\|u_0\|_2^2\right) y''(t)\varphi(t) + (\varphi'(t))^2 + \delta(y'(t))^2.
\end{aligned} \tag{5.27}
$$

By (5.19) and the fact that $e^{\frac{2C(q-p\lambda)}{q}} > 1$ and $\varphi > 0$, we obtain

$$
\begin{aligned}
&2\varphi(t)\varphi''(t) - (1+\alpha)(\varphi'(t))^2\\
&> 2\varphi(t)\left(2y(t) + \varepsilon^{-1}\|u_0\|_2^2\right)\Big(2p\lambda\alpha\int_0^t \|u_\tau\|_2^2 d\tau + p\lambda\alpha\varepsilon\|u_0\|_2^2\Big) - 4\alpha\varphi(t)B(t)\\
&> 2p\lambda\alpha\varphi(t)\big(2y(t) + \varepsilon^{-1}\|u_0\|_2^2\big)\Big(2\int_0^t \|u_\tau\|_2^2 d\tau + \varepsilon\|u_0\|_2^2\Big) - 4\alpha\varphi(t)B(t)\\
&= 2p\lambda\alpha B(t)\varphi(t) - 4\alpha B(t)\varphi(t) > 0;
\end{aligned}
$$

that is,

$$
\varphi(t)\varphi''(t) - \frac{1+\alpha}{2}(\varphi'(t))^2 > 0, \quad t \in [0, T],
$$

which implies that

$$
(\varphi^{-\beta}(t))'' = -\frac{\beta}{\varphi^{\beta+2}}(\varphi''(t)\varphi(t) - (\beta+1)(\varphi'(t))^2) < 0, \quad \beta = \frac{\alpha-1}{2} > 0.
$$

Since $\varphi(0) > 0$ and $\varphi'(0) > 0$, by Lemma 5.2, there exists t_* such that

$$
0 < t_* \le \frac{2\varphi(0)}{(\alpha-1)\varphi'(0)},
$$

such that

$$
\lim_{t\to t_*} \varphi^{-\beta}(t) = 0, \quad \text{and} \quad \lim_{t\to t_*} \varphi(t) = +\infty,
$$

which contradicts $T = +\infty$. Now, by considering the continuity of φ with respect to y, we can conclude that $y(t)$ tends to ∞ at some finite time which is a contradiction.

(ii) There exist some $\tilde{t}$ such that $J(u(\tilde{t})) < 0$. Since $J(u_0) > 0$, by the continuity of $J(u)$ in t, we can assume that there exists a first time $t_0 > 0$ such that $J(u(t_0)) = 0$ and $J(u(\hat{t})) < 0$ for some $\hat{t} > t_0$. We take $u(\hat{t})$ as a new initial datum, then from Lemma 5.3, we have $u(t) \in \mathcal{N}_-$ for $t > \hat{t}$. Then similar to the proof of Theorem 3.4, we can prove the finite time blowup of the solution.

Combining the above two cases, we conclude that $u(x,t)$ blows up in finite time. $\square$

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