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ORIGIN OF THE p-LAPLACIAN AND A. MISSBACH

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Abstract. We describe the historical process of derivation of the p-Laplace operator from a nonlinear Darcy law and the continuity equation. The story begins with nonlinear flows in channels and ditches. As the nonlinear Darcy law we use the power law discovered by Smreker and verified in experiments by Missbach for flows through porous media in one space dimension. These results were generalized by Christianovitch and Leibenson to porous media in higher space dimensions. We provide a brief description of Missbach's experiments.

1. Introduction

The authors of this article have often been confronted with the question on the origin of the p-Laplace operator. The main goal of the present work is to answer this question at satisfactory technical and historical levels. We do not attempt to provide or claim complete answers to many questions that arise in our investigation of the available resources. In particular, we leave the question of *competitiveness* of mathematical models with the p-Laplacian to alternative mathematical models still widely open in practical applications [51, 58].

2. The Filtration Problem and the Equation

An important task of hydrodynamics engineering throughout the 18^{th} century was to build reliable water supplies for fast growing urban centers. The need for water sparked a number new directions in theoretical research on hydrodynamics and hydrology. Numerous interesting mathematical problems in this area are derived and formulated in the monograph by Jacob Bear [3]. Among them we are interested in *filtration of fluids through porous media* and *unsaturated flow*; see [3, Sect. 5.2, 5.10, 5.11] and [3, Sect. 9.4], respectively. A mathematical model for such phenomena is presented in J. I. Díaz and F. de Thélin [14]. It is described by the following *nonlinear* initial-boundary value problem of parabolic type for the

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unknown function $u = u(x,t)$ of the space and time variables, x and t, respectively:

$$\begin{gathered}\frac{\partial}{\partial t}\, b(u) - \operatorname{div}\phi\big(\nabla u - K(b(u))\mathbf{e}\big) + g(x,u) = f(x,t) \quad \text{in } \Omega\times(0,\infty)\,,\\ u(x,t) = 0 \quad \text{on } \partial\Omega\times(0,\infty)\,,\\ b(u(x,0)) = b(u_0(x)) \quad \text{in } \Omega\,.\end{gathered} \tag{2.1}$$

Here, $\Omega \subset \mathbb{R}^N$ is a bounded open subset of the N-dimensional Euclidean space $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$, $b : \mathbb{R}\to\mathbb{R}$, $K : \mathbb{R}\to\mathbb{R}$, and $g(x,\,\cdot\,) : \mathbb{R}\to\mathbb{R}$ are continuous functions satisfying some additional hypotheses ([14, Sect. 1]), such as b being monotone increasing with $b(0) = 0$, $\mathbf{e}$ denotes a given unit vector in $\mathbb{R}^N$, and for some $1 < p < \infty$,

$$\phi(\boldsymbol{\zeta}) = |\boldsymbol{\zeta}|^{p-2}\boldsymbol{\zeta} \quad \text{for every } \boldsymbol{\zeta}\in\mathbb{R}^N\,. \tag{2.2}$$

As usual, $t \in \mathbb{R}_+ := [0,\infty)$. Finally, $f : \Omega\times(0,\infty)\to\mathbb{R}$ is (typically) a Lebesgue--integrable function standing for sources (if $f(x,t) > 0$) and sinks (if $f(x,t) < 0$), whereas $u_0 : \Omega\to\mathbb{R}$ stands for the prescribed initial data, usually assumed to be Lebesgue-measurable and (essentially) bounded.

For filtration of fluids through porous media in laminar regime one begins with

$$\text{the continuity equation} \qquad \frac{\partial\theta}{\partial t} + \operatorname{div}\mathbf{v} = 0 \tag{2.3}$$

$$\text{and the Darcy law} \qquad \mathbf{v} = -K(\theta)\,\nabla\Phi(\theta)\,, \tag{2.4}$$

where $\theta = \theta(x,t)$ is the volumetric moisture content, $K = K(\theta)$ is the hydraulic conductivity, and the potential Φ is given by $\Phi(\theta) = \psi(\theta) + z$ with $\psi(\theta)$ being the hydrostatic potential and z the gravitational potential. For instance, if $N = 3$ then we fix the unit vector $\mathbf{e} = (0,0,-1) \in \mathbb{R}^N$ in the direction opposite (but parallel) to the gravitational force, perpendicular to the horizontal plane $(x_1,x_2,0)$, so that the gravitational potential $z = z(x) = gx_3 + \text{const}$ at the point $x = (x_1,x_2,x_3) \in \mathbb{R}^3$ yields the gravitational force

$$\mathbf{G} = \mathbf{G}(x) = -\nabla z = \big(0,0,\ -\tfrac{\partial z}{\partial x_3}\big) = (0,0,-g) = -g\mathbf{e} \in \mathbb{R}^3\,.$$

To simplify our notation, we normalize the gravitational constant to one, $g = 1$; hence, $\mathbf{G} = -\nabla z = -\mathbf{e} \in \mathbb{R}^3$. Thus, we obtain

$$\nabla\Phi(\theta) = \psi'(\theta)\,\nabla\theta - \mathbf{e}$$

which, after being inserted into Darcy's law (2.4), yields

$$\mathbf{v} = -K(\theta)\,\psi'(\theta)\,\nabla\theta + K(\theta)\mathbf{e} = -\nabla\varphi(\theta) + K(\theta)\mathbf{e} \in \mathbb{R}^3 \tag{2.5}$$

where

$$\varphi(\theta) := \int_0^\theta K(\vartheta)\,\psi'(\vartheta)\,\mathrm{d}\vartheta \quad \text{for } \theta\in\mathbb{R}\,.$$

In general, the vector field $\mathbf{v}$ stands for the seepage flow which, in our applications, will be proportional to the fluid velocity, thus denoted by $\mathbf{v}$. It is reasonable to assume $K(\vartheta) > 0$ and $\psi'(\vartheta) > 0$ (see Bear [3]), so that also $\varphi'(\vartheta) > 0$ holds. As a consequence, $\varphi : \mathbb{R}\to\mathbb{R}$ is a strictly monotone increasing, continuously differentiable function.

Beginning in the 1870s, many engineers concerned with fluid dynamics (including the works in [20, 21, 22, 23, 24, 32, 36, 37, 38, 43, 44, 45, 46, 52, 54, 55, 56, 57, 62, 64, 66, 67]) have discovered that, if the fluid flow is in *turbulent regime*, the linear

Darcy law (2.4) does not provide the correct relationship between the *pressure slope* (force),

$$\mathbf{F} = -\nabla\varphi(\theta) + K(\theta)\mathbf{e}\,,$$

on the right-hand side and the *velocity*, $\mathbf{v}$, on the left-hand side of Darcy's law (2.4). Oscar Smreker [55, Eqs. (5)–(7), pp. 361–362] shows by rigorous calculations how linear Darcy's law leads to a contradiction in a practical problem (dug well, "Schachtbrunnen" in German). Among several "correction" alternatives to Darcy's law, O. Smreker [54, 55, 56] suggested the following *power law*:

$$\mathbf{F} = -K(\theta)\,\nabla\Phi(\theta) = -\nabla\varphi(\theta) + K(\theta)\mathbf{e} \tag{2.6}$$

is given by

$$\mathbf{F} = \mathrm{const}\cdot|\mathbf{v}|^{p'-2}\mathbf{v} \quad \text{with some } p' > 2\,, \tag{2.7}$$

with the power $s = (p'-2)+1 = p'-1$, where the multiplicative constant is set to one, const $= 1$, for simplicity. Smreker's work [54] suggests $p'-1 = 3/2$, i.e., $p' = 2.5$, whereas Reynolds's measurements [52] show $p'-1 = 1.723$. A. M. White [62] proposed an analogous relation with $p'-1 = 1.8$. All such corrections to Darcy's law allow only the power range $1 \le p'-1 \le 2$. Denoting by $p = p'/(p'-1)$ the conjugate exponent, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$, we thus have to deal with the range $3/2 \le p \le 2$ and the velocity

$$\mathbf{v} = |\mathbf{F}|^{p-2}\mathbf{F}\,. \tag{2.8}$$

Inserting (2.6) and (2.8) into the continuity equation (2.3) we finally arrive at problem (2.1), where $b = \varphi_{-1}$ denotes the inverse function to φ and $f \equiv g \equiv 0$.

We refer an interested reader to J. I. Díaz and F. de Thélin [14] for how to obtain problem (2.1) in a model dealing with *unsaturated flow* (gas flow, typically). There, $p = 3/2$.

It is now evident, that the *p-Laplace operator* Δ_p,

$$\Delta_p u \equiv \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)\,, \quad \text{for } 1 < p < \infty\,, \tag{2.9}$$

is created by the nonlinear power law (2.7) or, equivalently, by (2.8). The continuity equation (2.3) is standard for both, linear and nonlinear Darcy's laws. This means that the origin of the p-Laplacian Δ_p is closely tied to <u>*who was the first*</u> to plug the power law (2.8) into the continuity equation (2.3) or at least into its stationary special case div $\mathbf{v} = 0$. There seems to be a wide-spread agreement in the literature that the power law (2.8) with $p = 5/3$ was suggested first by Oscar Smreker [54] [1] in 1878 in the equivalent form (2.7) with $p' = 2.5$. A number of "power laws" (with a more general exponent $s = p'-1$) by various authors followed afterwards. We will discuss the most important ones in the following two sections.

In this context ("*Who was the first?*"), we should mention the articles by Smreker [56] from 1881 (used also in his doctoral dissertation [57] in 1914) and by N. E. Zhukovskii [64] from 1889 (reprinted in his collected works [65] in 1937), in which they give the explicit formula for the radially symmetric solution, $u(x) \equiv u(|x|)$, of the so-called *p-harmonic equation*, $\Delta_p u = 0$, for any $1 < p < \infty$, $p \ne N$,

$$u(r) = C_0 + \mathrm{const}\cdot r^{1-\mu} \quad \text{for every } r = |x| > 0\,, \tag{2.10}$$

$$\mu = \frac{N-1}{p-1} \ge 0, \quad \mu \ne 1,$$

[1] Ref. [6] by M. Brenčič provides a "Short description of life and work of Oskar Smreker".

see [57, Eq. (I), p. 36] and [65, Eq. (13), p. 19], respectively. Here, $C_0 \in \mathbb{R}$ is a constant; $C_0 = u(0)$ if $\mu < 1$ and $C_0 = u(+\infty) := \lim_{r\to+\infty} u(r)$ if $\mu > 1$. Although this formula is valid in any dimension $N \geq 1$, both engineers, in [56, 64], treat only the planar case ($N = 2$) given by the hydroengineering model. They had *never* written down the p-harmonic equation ($\Delta_p u = 0$) explicitly throughout their entire articles [56, 64]; rather, they preferred to refer to Smreker's work in [54] for the power law. In fact, since every radially symmetric solution $u(x) \equiv u(|x|)$ to the p-harmonic equation $\Delta_p u = 0$ in the plane ($N = 2$) satisfies the stationary case of the continuity equation (2.3), div $\mathbf{v} = 0$, which is equivalent to

$$\Delta_p u(x) = r^{1-N} \cdot \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{N-1}\,|u'(r)|^{p-2}u'(r)\right) = 0 \quad \text{for every } r = |x| > 0\,,$$

both, Smreker [56] and Zhukovskii [64], may have very easily used an alternative way (e.g., the surface integral over a sphere) to obtain in the plane ($N = 2$),

$$r\,|u'(r)|^{p-2}u'(r) = \text{const} \qquad \text{for all } \ r = |x| > 0\,,\ x \in \mathbb{R}^2\,,$$

whence (2.10) follows with $N = 2$ and $\mu = 1/(p-1) > 0$ (recall that $p \neq N = 2$).

3. Flow in a Channel or Porous Media

The rapid development of hydrology in the late 18$^{\text{th}}$ and early 19$^{\text{th}}$ centuries required new theoretical background and related new measurement techniques. Much of this research, particularly by French engineers closely connected with the famous Parisian engineering school *École des ponts et chaussées*, was published in 1804 in the monograph by one of its former directors, baron Gaspard Riche de Prony [50]. This book is a very comprehensive description of French research on water flow through channels and large pipes. Some studies treat also smaller (thinner) pipes and hoses which, towards the end of the 19$^{\text{th}}$ century, developed into research on filtration through soil, sand, and other similar porous materials. Mathematically, all models in this research are set in space dimension one. The spectrum of specialists involved in the 18$^{\text{th}}$ century research begins with *civil engineers* (count Pierre Louis George du Buat [8] and Pierre-Simon Girard [26]), continues with *theoretical engineers* and *applied mathematicians* like de Prony himself and Antoine de Chézy [9], and ends up with *mathematicians* (marquess Pierre-Simon de Laplace [33]). The author, de Prony [50], describes and further develops the research findings of his former teacher, Antoine de Chézy [9], published in 1775 which contains also his famous mathematical formula on the average flow velocity. De Prony's book [50] was further influenced by the work of P. L. G. du Buat [8] and P.-S. Girard [26]. One of their most important discoveries was the formula for the resistance force due to adhesion of the fluid to the contact surface, cf. G. R. de Prony [50, pp. 44, 58]:

If u stands for the average flow velocity, then this resistance force, $\chi\delta s\,\phi(u)$, is proportional to a polynomial function $\phi = \phi(u)$ of degree one to three, where χ, δ, and s are some positive constants that describe the adhesion to the contact surface, and

$$\phi(u) = c + \alpha u + \beta u^2 + \gamma u^3 \tag{3.1}$$

with some nonnegative constants c, α, β, and γ. We refer to pages 44 and 58 of de Prony's book [50]. Calculation of these constants from available measurements was a subject of strong theoretical and practical interest to civil engineers working on the constructions of channels and water pipelines throughout entire France ([50, pp. 65–90]).

The transformation of the research interests in water flow through channels and large pipes into research on filtration through porous materials began in mid-19$^{\text{th}}$ century in the work by Henry Darcy [11] in 1856, a French hydroengineer working in Dijon, with his famous (linear) *Darcy law*, and by Jules Dupuit [18], another French engineer and economist, published in 1863, who, in contrast, works with de Prony's quadratic law (3.1) (where $\gamma = 0$) for the dependence of the resistance force or pressure loss (difference) on the average flow velocity, u. While *Darcy's law* became quickly a very popular, simple tool for calculating the dependence of force or pressure on the velocity u for small absolute values of u, de Prony's quadratic law has turned out to fit the filtration problems much more accurately also with higher velocities required to filter a sufficient amount of liquid (water) needed by a large urban community. Towards the end of the 19$^{\text{th}}$ century, several civil engineers throughout Western Europe have adopted de Prony's polynomial formula (3.1) (typically quadratic or cubic) in their investigation of fluid filtration phenomena; Oscar Smreker [54, 55][2] (an Austrian-born engineer based in the city of Mannheim, Germany, and active in several neighboring countries) seems to be the first of them in 1878–1879 (with another work [56] in 1881), followed by C. Kröber [32] in 1884 and Philipp Forchheimer [20] in 1886 and [21] in 1901 (another Austrian engineer active also in Germany). Especially Forchheimer's latter article, [21], became a landmark in nonlinear fluid dynamics. Owing to Forchheimer's tremendous theoretical and practical activity in filtration problems, which includes several lecture notes and comprehensive textbooks [22, 23, 24], de Prony's and Smreker's quadratic law (3.1), $\gamma = 0$, in filtration theory is called *Forchheimer's equation*. We will stick to this terminology in the rest of this article while keeping in mind earlier contributions by de Prony and Smreker. Smreker's main merit is an early application of Forchheimer's quadratic formula (3.1) in civil engineering, particularly in the construction of a water supply system to the Alsatian city of Strasbourg (France) ([54], see the sketches following p. 128). This engineering project plays the key role in Smreker's works [54, 55, 56] mentioned above (in 1878–1881). This work (and from his other articles to follow it) is collected in his doctoral dissertation [57] (Dr.–Ing.) from 1914 at the age of sixty. By then he had designed and/or built numerous water supply systems in various European cities: Belgrade, Ljubljana, Lvow (Lemberg), Mannheim, Prague, Trieste, Vilnius, etc. Greater details on his achievements can be found in M. Brenčič's survey [6].

Nevertheless, it was Oscar Smreker [54] again who has discovered that, at "low" velocity levels v, neither the linear *Darcy law* nor the quadratic (or cubic) *de Prony-Forchheimer* law (3.1) describes the relation between the pressure loss and the velocity v accurately. He suggested the following correction for the *(pressure) slope* h/ℓ,

$$\frac{h}{\ell} = \frac{v^2}{2g} \cdot \xi \quad \text{where} \quad \xi = f(v) \quad \text{for} \;\; v > 0\,, \tag{3.2}$$

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[2] In fact, the latter article, [55], was intended to be an introduction to the former one, [54]. The temporal order of publication is publisher's mistake; see publisher's remark at the end of the latter [55].

with the gravitational constant (acceleration) g given by $g = 9.81$ (m/s^2) and the function $\xi = f(v)$ taking the "hyperbolic" form

$$f(v) = \alpha + \frac{\beta}{\sqrt{v}} \quad \text{for } v > 0 \tag{3.3}$$

with some positive constants α and β. The (positive) quantities h and ℓ, respectively, stand for the difference h of water levels before and after the (horizontal) filter of length ℓ; cf. Forchheimer [21, Fig. 1, p. 1736] and Smreker [55, pp. 358–360]. Formulas (3.2) and (3.3) yield a very special, but important case of the famous *power law*,

$$\frac{h}{\ell} = \frac{v^{3/2}}{2g} \cdot \left(\alpha \sqrt{v} + \beta\right) \approx \frac{\beta}{2g} \cdot v^{3/2} \quad \text{for } v > 0\,, \tag{3.4}$$

with the approximation by the power $(\beta/2g) \cdot v^{3/2}$ being valid for small velocities $v > 0$. In his work [54, p. 127], Smreker suggests also a much more general relation, namely,

$$\xi = f(v) = \alpha + \sum_{n=1}^{\infty} \beta_n \, v^{-1/n} \quad \text{for } v > 0 \tag{3.5}$$

with some nonnegative constants α and β_n. This is how the *power law*

$$\frac{h}{\ell} = \text{const} \cdot v^s \quad (1 < s < 2) \quad \text{for } v > 0 \tag{3.6}$$

was discovered for the (pressure) slope h/ℓ. Starting with the articles [21, 54], the precise value of the constant $s \in (1, 2)$ was the subject of numerous measurements and theoretical investigations; $s > 1$ shows the tendency to approach one ($s \searrow 1$). Of course, the case $s = 1$ renders (linear) *Darcy's law*. The power law (3.6) for soil permeability and *high* water velocity v was confirmed in the experiments performed by F. Zunker [66] in 1920 with $s = 3/2$; see also Zunker's survey article [67]. He claims that Darcy's law is applicable to *medium* water velocities v. In Great Britain, the two nonlinear Darcy laws, the quadratic law (3.1) (where $\gamma = 0$) and the power law (3.6) (where $s = 1.723$), appear for the first time in 1883 in the work by Osborne Reynolds [52, Sect. III, §37, pp. 973–976]. He considers very briefly also Smreker's general problem (3.2) (cf. [54, p. 119]). However, the relation of his research findings to those of O. Smreker [54] is unclear[3].

It was not until mid-1930s when Smreker's power law (3.6) was verified by Alois Anton Missbach [43] – [46] in many laboratory experiments with sugar juice and water penetrating a medium consisting of tiny glass balls of constant diameter. The final comparison of Smreker's power law with A. Missbach's laboratory experiments were published in the (now) famous article [46]. His experiments are so well-documented in the series of articles [43] – [46][4] that many researchers in nonlinear fluid dynamics, especially in the "West" (Americas, Australia, Europe, and New Zeeland), consider A. Missbach's article [46] as the *verification* of Smreker's power law (3.6). For this reason, this power law is often called *Missbach's equation* in Western literature (or the *Darcy-Missbach equation* in [51]). We will use this terminology in the rest of this article, although many authors from Russia, the mainland China, and Taiwan prefer to attribute the power law to, e.g., the prolific Russian engineer S. V. Izbash [29, 30]; see also S. V. Izbash and Kh. Yu. Khaldre

[3] Reynolds [52] seems to be unaware of Smreker's results in [54] published five years earlier.

[4] Part VI (Ref. [45]) of Missbach's work appeared before Part V (Ref. [44]).

[31] and H. Watanabe [61]. A. Missbach's work [46] summarizes the results of a large research program sponsored by several sugar refineries in Czechoslovakia in the early 1930s on efficient sugar juice filtration. It is the final part (Part VII) of the series of seven articles on *Filtration ability of separated and saturated juices* inspired by the scientific and industrial activities of Missbach's doctoral adviser, Jaroslav Dědek, who himself also contributed to this article series (Part III), cf. J. Dědek and D. Ivančenko [12]. The findings of the research reported in Missbach's article [46], albeit obtained with penetrating water rather than sugar juice, were immediately incorporated into industrial sugar production. This article is written in two parallel originals, Czech and German. Further details on his professional involvement with the Czechoslovakian sugar producing industry will be provided in Section 9. A very practical application of *Missbach's equation* to nonlinear Darcy flow (also called *non-Darcian flow*) is provided in P. M. Quinn, J. A. Cherry, and B. L. Parker [51]. This flow occurs in high-precision straddle packer tests conducted in boreholes in a fractured dolostone aquifer using constant rate injection step tests to identify the conditions of change from Darcian to non-Darcian flow. An interesting comparison of *Forchheimer's* and *Missbach's equations*, (3.1) and (3.6), respectively, is available in the survey article by K. P. Stark and R. E. Volker [58] who, unfortunately, seem to be unaware of O. Smreker's pioneering work [54, 55, 56, 57].

4. The Russian School

Significant contributions to the filtration problem in porous materials by Russian (or Soviet) engineers and scientists began in early 1920s by N. N. Pavlovskii [48] in a hand-written monograph of 753 pages. It provides a very well-written, up--to-date introduction to hydraulics from a (mostly) theoretical point of view, with plenty of valuable references to the literature. In Russia, this time is characterized by massive industrialization (1920s and 1930s). In the first chapter, Pavlovskii surveys constitutive laws (Darcy's law, Forchheimer's quadratic and cubic laws, and the power law). In the second chapter, he suggests a criterion based on the Reynolds number to establish the validity range of the linear Darcy law and the range where a nonlinear law must be used instead. According to V. I. Aravin and S. N. Numerov [2], p. 4 and also p. 33 with a detailed explanation, Pavlovskii's work [48] is the first one to use Reynolds number for this purpose. Despite of the fact that the monograph [48] thoroughly discusses various constitutive laws in its first two chapters, the partial differential equations used throughout the book to study the seepage are only linear.

Serious interests in nonlinear (and non-Newtonian) fluid dynamics in the former Soviet Union began in early 1930s with the works by S. V. Izbash [29, 30], who has published the power law (3.6) already in 1931 in a monograph available only in Russian. Decisive contributions to fluid dynamics were made by N. E. Zhukovskii (see his collected works [65] from 1937), the most relevant for us being [64] from 1889. As we have already mentioned in Section 2, he gives the explicit formula for the radially symmetric solution, $u(x) \equiv u(|x|)$, of the p-harmonic equation, $\Delta_p u = 0$, see [64, Eq. (13), p. 19]. In the same article, [64], Zhukovskii discusses applicability of various constitutive laws to filtration of water through sandy soil known to that date, i.e., Darcy's, Kröber's, and Smreker's power-type laws [11, 32, 54], and compares them to scores of available experimental results. For instance, he derives Laplace's equation by inserting the (linear) Darcy law into the differential equation of continuity. Using the Laplace equation he studies several configurations of water wells scattered in the field (standalone well, wells in a row, and wells on a circle). For the standalone case, he finds out that the discrepancy between theoretical predictions from the formula based on the solution of Laplace's equation and the reality (measured data) is too large. To fix this problem, he suggests to use the velocity v given by Kröber's and Smreker's power law [32, 54] $-\frac{\mathrm{d}u}{\mathrm{d}x} = i = \mathrm{const} \cdot v^{p'-1}$, $2 < p' < \infty$, cf. eq. (2.8), to be plugged into the stationary case of the continuity equation (2.3) as described above. In particular, eq. (2.7) plays the role of the constitutive law.

To the best of our knowledge, all work on nonlinear (and non-Newtonian) fluid dynamics until 1940, throughout the entire world, treated only spatially one-dimensional problems. (Smreker's and Zhukovskii's radially symmetric planar solution in [56, 64] mentioned above is essentially one-dimensional.) It was the Russian scientist S. A. Christianovitch [10] who employed nonlinear constitutive laws (Forchheimer's quadratic and cubic laws and Missbach's power law) to derive nonlinear partial differential equations for the seepage movement of underground water. He restricts himself to the spatially two-dimensional case. In the case of the power law, he obtains the following equation (re-written in contemporary notation):

$$\Delta_p u \equiv \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = 0\,,$$

for the unknown function $u = u(x, y)$. Since he works in two space dimensions, he can use methods of complex analysis and suggest analytical techniques to obtain approximations of the solution to this equation with the so-called *p-Laplace operator* Δ_p, $1 < p < \infty$. The common (linear) Laplace operator Δ is obtained for the (linear) Darcy law ($p = 2$).

Another notable person in the Russian hydraulic engineering school was L. S. Leibenson who investigated seepage of oil and gas in the oil and gas fields near the city of Baku (now Azerbaijan, formerly Soviet Union). Much of his research from the 1920s and early 1930s was published not only in brief article form, but also as a survey monograph [35]. His most important findings concern turbulent filtration of gas in porous medium [36, 37] (see also [40]). It was his article [36] where the *doubly nonlinear parabolic equation*,

$$\frac{\partial u^m}{\partial t} = c\, \Delta_p u \quad \text{for } (x, y, z, t) \in \mathbb{R}^3 \times (0, T)\,, \tag{4.1}$$

with $m + 1 = p = 3/2$, appeared for the first time. Here, $u = u(x, y, z, t)$ is the unknown function of space and time, and $c > 0$ is some constant. Thanks to $m = p - 1$, eq. (4.1) is called $(p-1)$-*homogeneous*. He used the separation of space and time variables,

$$u(x, y, z, t) = v(t)\, w(x, y, z)\,,$$

in order to obtain the following equation with the so-called 1-*Laplacian*,

$$\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right) + A\sqrt{w} = 0\,, \tag{4.2}$$

where $w = w(x, y, z)$ is the unknown function of space and $A > 0$ is a constant. This article, [36], published in 1945 seems to be the first one to derive and consider a quasilinear *parabolic* (time-dependent) problem, eq. (4.1), with the *p-Laplace operator* Δ_p in space dimension three (defined in eq. (2.9)), albeit for $p = 3/2$

only. For the p-harmonic equation, $\Delta_p u = 0$ with $p = 3/2$, Leibenson [36] finds solutions in the spatially one-dimensional and radially symmetric cases. In contrast, S. A. Christianovitch [10] (in 1940) treated only a quasilinear *elliptic* (stationary) problem, $\Delta_p u = 0$, in two space dimensions, but for any $1 < p < \infty$.

In his next work [37], immediately following [36], L. S. Leibenson allows for a wider range of values of p, $3/2 \le p \le 2$. Also his doubly nonlinear parabolic equation (4.1) becomes more general,

$$\frac{\partial}{\partial t}\Big(u^{\frac{1}{m+1}}\Big) = c\,\Delta_p u \quad \text{for } \ (x,y,z,t) \in \mathbb{R}^3 \times (0,T)\,, \tag{4.3}$$

with $m > 0$, which is no longer $(p-1)$-homogeneous. This equation results from Leibenson's studies [37] of filtration of turbulent polytropic gas flow through porous medium; $m > 0$ is called the *polytropic index* of the gas. It is a direct generalization of an earlier work by L. S. Leibenson [34] which still uses the linear Darcy law, whereas [37] uses Smreker's power law[5] (3.6). Practically all Leibenson's results we have mentioned above are very carefully collected and explained in his monograph [39] published in 1947; his scientific articles [34, 36, 37, 38] are reprinted in [40].

An important member of the Russian school was also P. Ya. Polubarinova-Kochina. Her Russian monograph [49] from 1952 (translated into English in 1962) became quickly a widely used textbook by hydrogeologists all over the world.

5. From Darcy's law to Forchheimer's equation (from linear to nonlinear diffusion)

Although fluid flow through channels, large pipes, and hoses had occupied theoretical hydrologists since the 18^{th} century (see de Prony's equation (3.1)), fluid flow through porous media attracted major attention much later, in mid-19^{th} century. We recall from Section 3 the research on filtration through porous materials by Henry Darcy [11] in 1856 (the linear *Darcy law*) and by Jules Dupuit [18] in 1863 (working with de Prony's quadratic law). The idea of the quadratic law (3.1) was picked up by Ph. Forchheimer who, in his groundbreaking work [21], developed applications of de Prony's quadratic law to filtration through porous materials (soil, in particular),

$$i = av + bv^2\,. \tag{5.1}$$

Here, the quantity i is the (negative) total piezometric head gradient, $i = -\frac{\mathrm{d}u}{\mathrm{d}x}$, v stands for the average seepage velocity, and a and b are nonnegative constants determined by the properties of the fluid and medium; typically, $a > 0$ and $b > 0$. His article [21], published in 1901, meant also the introduction of nonlinear diffusion after several decades of intensive studies of linear diffusion prompted by Darcy's law. A number of workers have inferred that Forchheimer's equation has sound physical backing apart from its attraction as a relatively simple nonlinear expression. We refer the reader to J. Bear, D. Zaslavsky, and S. Irmay [4], for example, who have derived the Forchheimer relation by inferred arguments from the fundamental Navier-Stokes equations for the general case when inertia terms are considered; see also Irmay [28]. A few decades later, in 1930, Ph. Forchheimer [24] extended his nonlinear Darcy law to

$$i = av + bv^m\,, \tag{5.2}$$

[5] Leibenson [37] was apparently not aware of Missbach's work [46].

where m is a constant typically taking values in the interval $(1,2]$, i.e., $1 < m \le 2$.

Remark 5.1. From the point of view of Mathematical Physics, relation (**??**) means that if $a > 0$, then the head gradient i has nearly linear, nontrivial growth

$$i(v) - i(0) = i = av\Big(1 + \frac{b}{a}v\Big) \approx av \tag{5.3}$$

for low velocity v. On one hand, this phenomenon was confirmed for certain types of fluids and media from both theoretical and experimental viewpoints, e.g., in the work of V. I. Aravin and S. N. Numerov [2], E. Lindquist [41], and J. C. Ward [60]. On the other hand, the nontrivial growth (5.1) ($a > 0$), which yields

$$v = v(i) = -\frac{a}{2b} + \sqrt{\big(\frac{a}{2b}\big)^2 + \frac{i}{b}} = \frac{a}{2b}\Big(-1 + \sqrt{1 + \big(\frac{2b}{a}\big)^2 \frac{i}{b}}\Big) > 0 \quad \text{if also } b > 0\,,$$

whence $v \approx i/a$ for $i \ge 0$ small, does not occur for other types of fluids and media studied in M. Anandakrishnan and G. H. Varadarajulu [1], C. R. Dudgeon [16], C. R. Dudgeon and C. N. Yuen [17], L. Escande [19], A. Missbach [43, 44, 45, 46], A. K. Parkin [47], A. M. White [62], and J. K. Wilkins [63].

6. Missbach's power law (nonlinear, power-type diffusion)

In contrast with Forchheimer's approach to generalizing Darcy's law, Alois Missbach [46] based his approach to the porous medium problem on numerous *experimental results* that became available in the 1930s in various rapidly developing industries, such as sugar and petroleum (oil) production, where certain types of fluids are filtered through special porous media. Missbach's experiments were prompted by theoretical and experimental results obtained much earlier by C. Kröber [32], O. Reynolds [52], O. Smreker [54], and F. Zunker [66]. The experimental results obtained during the sugar beet campaign of 1935 in Czechoslovakia led A. Missbach [46] to verifying the *power law relation*

$$i = c\,v^m \tag{6.1}$$

between the head gradient and the velocity, i and v, respectively, published in 1937. The power m typically takes values in the interval $(1,2)$. A couple of years before Missbach's article appeared, in 1935, A. M. White [62] proposed an analogous relation with $m = 1.8$. As a porous medium, Missbach used gravels, sands, and packings of uniform spheres (e.g., tiny glass balls), while in his starting experiments [43] – [45] the fluid was represented by sugar juice of various sugar contents. However, in his most important work for us, [46], he used water as the penetrating fluid (Figure 1 below). He found out that the power m stays in $(1,2)$ and tends to 1 with the decreasing diameter of the spheres. C. R. Dudgeon [16] carried out tests on coarse materials serving as porous medium (gravels, sands, and packings of uniform spheres) and confirmed that while the results followed closely an expression of Missbach's form (6.1) the values of c and m were not constant for the particular material for all fluid flow conditions. These and other experimental results have confirmed Missbach's equation (6.1). A theoretical derivation of the special case of Missbach's equation (6.1) for $m = 3/2$ has been given in E. Skjetne and J.-L. Auriault [53]. The authors of the present article have not been able to find any reference concerned with a theoretical derivation of Missbach's equation (6.1) for an arbitrary power $m \in (1,2)$. The article by A. Brieghel-Müller [7] thoroughly surveys almost all results concerning constitutive laws for filtration known up to 1940 and discusses their applicability to filtration processes in sugar production.

Since experiments and measurements play a decisive role in A. Missbach's work [43] – [46], we provide a brief description of his apparatus. A. Missbach [46] calls his experimental laboratory equipment "Apparatus for testing the hydraulic conductivity (permeability, porosity) through a layer of glass balls".

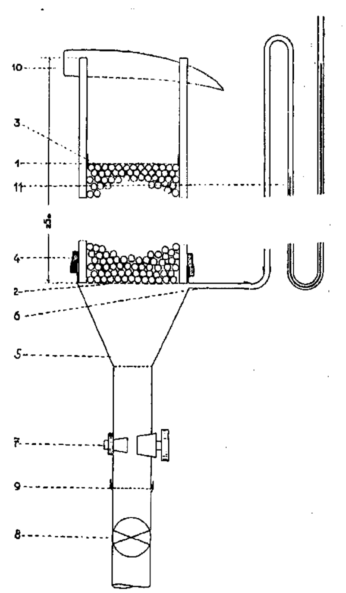

FIGURE 1. Apparatus for testing the hydraulic conductivity through a layer of glass balls.

Figure 1 is a scanned copy of the original figure from Missbach's work [46], p. 294, Obr. 1 (in the Czech edition) and p. 424, Abb. 1 (in the German edition). Missbach [46] credits the use of tiny glass balls to Zunker [66].

Figure 1 **description**:

(1) Glass tube with strong walls of internal diameter 45 mm, slightly longer than 200 mm.
(2) Lower sieve.
(3) Upper sieve with a steel spring.
(4) Connecting rubber hose with strong walls.
(5) Tin funnel with a sieve insole.
(6) Thin connection pipe for the differential water manometer.
(7) Faucet for flow regulation.
(8) Outlet for flow regulation.
(9) Screw thread with an inserted filter cloth.
(10) Trench for draining overflowing liquid.
(11) Manometer.

In contrast with earlier filtration experiments (e.g., F. Zunker [66, 67]) which used a system of parallel capillary tubes having undesirable side effects, A. Missbach [46] decided to construct an apparatus of a relatively large diameter (45 mm) whose walls do not influence (obstruct, slow down) the fluid flow through the layer of tiny glass balls. He used glass balls of four (4) different sizes (A, B, C, D; specified in [46, Table I]) and varied both, the thickness (height) of the layer of glass balls and the pressure of the fluid penetrating through the layer. The fluid used in this experiment was tap water, carefully filtered, with no air bubbles and other "pollutants". The filtered water was pumped through the outlet for flow regulation (8) from the bottom, under the atmospheric pressure of up to 0.5 atm, then led to penetrate through the layer of glass balls upwards. In order to guarantee a constant fluid flow velocity, v, throughout the horizontal cross section of the glass tube, a sieve insole (2) is inserted into the glass tube. The upper sieve with a steel spring (3) prevents the glass balls from being moved upwards by the penetrating fluid. Finally, the overflowing liquid is drained into the trench (10) and its volume is measured in a cylindrical vessel.

The thickness of the layer of glass balls, the size of the balls (A, B, C, D), the vertical pressure difference in the layer, the flow velocity, and many other important measurements are carefully recorded in [46, Tables II through V]. These experiments provide evidence for Missbach's power law relation (6.1).

7. Comparison of the Forchheimer and Missbach equations (two different types of nonlinear diffusion)

Both, Forchheimer's and Missbach's models have been very useful in a number of various situations. Which of the two nonlinear models is better (i.e., more accurate) depends strongly on the fluid properties and the velocity v. A brief comparison of the two models has been carried out e.g. in P. M. Quinn, J. A. Cherry, and B. L. Parker [51], K. P. Stark and R. E. Volker [58], and numerically in R. E. Volker [59]. The experimental conditions in [51] seem to be slightly more favorable for Missbach's model. We refer to Figure 5 in [51, Chapt. 9, pp. 9–12] for a detailed comparison of the two models. It is interesting to observe that the authors in [58, Chapt. 5, pp. 131–196] slightly favor Forchheimer's model for water penetrating a porous medium *between two horizontal plates* (see [58, pp. 144, 185–186, and 196]), whereas A. Missbach [46] obtains highly favorable results for filtration of water through a porous medium *in a vertical cylinder* described in the previous section (with applications to filtration of saturated sugar juice). Although the laminar flow regime often obeys the linear Darcy law, it is always nonlinear in character. Thus, Missbach's equation applies also to the laminar flow regime and in the transition to a turbulent regime.

8. Some basic analytic and numerical results for the p-Laplacian

A comprehensive survey on only basic analytic and numerical results for the p-Laplacian would have to contain literally hundreds of references. As this is not the purpose of our present article, we have decided to mention only a few ones. Perhaps the very basic monograph on modern (nonlinear) functional-analytic methods for the p-Laplacian and similar quasilinear partial differential operators is the classical book by J.-L. Lions [42]. Besides methods of Nonlinear Analysis it contains also many applications to various mathematical models. Among important topics are the *global climate modelling* treated in J.-I. Díaz, G. Hetzer, and L. Tello [13] and *nonlinear fluid dynamics* in J. I. Díaz and F. de Thélin [14].

The spectrum of the (positive) p-Laplace operator $-\Delta_p$ on the Sobolev space $W_0^{1,p}(\Omega)$ (that is, a monotone nonlinear operator with the zero Dirichlet boundary conditions) has been an interesting open problem for decades, with the exception of the first eigenvalue; see the monograph by S. Fučík, J. Nečas, J. Souček, and V. Souček [25]. The Fredholm alternative at the first eigenvalue is studied in P. Drábek, P. Girg, P. Takáč, and M. Ulm [15] in a bounded domain $\Omega \subset \mathbb{R}^N$ and in J. Benedikt, P. Girg and P. Takáč [5] in a bounded open interval $\Omega \subset \mathbb{R}^1$. Bifurcations at the first eigenvalue are treated in P. Girg and P. Takáč [27].

9. A short sketch of Missbach's biography

A. Missbach (full name Alois Anton Missbach) was born on the 11$^{\text{th}}$ of June, 1897 in Plenkovice near Znojmo, Moravia (present Czech Republic), and baptized on June 13$^{\text{th}}$, 1897. According to the population statistics office ("matrika") in the town of Libáň in Eastern Bohemia (Czech Republic), A. Missbach had moved to Libáň in 1923 and stayed there until July 26$^{\text{th}}$, 1945. He was employed as a technical engineer from 1923 through 1945 in the sugar refinery in Libáň where he performed his research reported in Refs. [43] – [46]. While working full time as an engineer (the second technical adjunct), he defended his doctoral thesis on June 26$^{\text{th}}$, 1936 at the Czech Technical University in Brno, Moravia. He received the degree of Doctor of Technical Sciences (Dr. techn.). His thesis advisor was the well-known expert in Chemistry and sugar production, prof. Ing. Dr. techn. et Dr. agr. h.c. Jaroslav Dědek.

A. Missbach got married in 1928 in the famous Old Town Hall in the historic center of Prague, then the capital of Czechoslovakia. According to the statistics office in Libáň, he moved out to Havraň near the town of Most in Northwestern Bohemia (Czech Republic). As far as we know from the municipal office of Havraň, several months later he moved to the nearby village of Lenešice, also near the town of Most. He was the director of the sugar refinery in Havraň at least during his stay there. His last residence known to us was the town of Most starting on August 12$^{\text{th}}$, 1953. Both sugar refineries, in Libáň and Havraň, have been closed down several decades ago.

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local museum library, particularly Refs. [12] and [43] – [46] in both Czech and German.

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