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ASYMPTOTIC BEHAVIOR OF THE SIXTH-ORDER BOUSSINESQ EQUATION WITH FOURTH-ORDER DISPERSION TERM

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Abstract. In this article, we investigate the initial-value problem for the sixth-order Boussinesq equation with fourth order dispersion term. Existence of a a global solution and asymptotic behavior in Morrey spaces are established under suitable conditions. The proof is mainly based on the decay properties of the solutions operator in Morrey spaces and the contraction mapping principle.

1. Introduction

In this article, we investigate the initial-value problem for the sixth-order Boussinesq equation with fourth-order dispersion term

$$u_{tt} - a\Delta u_{tt} - 2b\Delta u_t - \alpha\Delta^3 u + \beta\Delta^2 u - \Delta u = \Delta f(u) \tag{1.1}$$

with initial value

$$u(x,0) = u_0(x), \quad u_t(x,0) = \Lambda U_1(x), \quad x \in \mathbb{R}^n. \tag{1.2}$$

Here $u = u(x,t)$ is the unknown function of $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ and $t > 0$, Δu_{tt} denotes the dispersion term, a, b, α, β are positive constants. The nonlinear term $f(u) = O(u^2)$ and $\Lambda = (-\Delta)^{1/2}$. The initial value $u_0(x)$ and $U_1(x)$ are given functions.

Zhang et al. [25] investigated the first initial boundary value problem for (1.1) with $f(u) = u^2$ in a unit circle. The existence and the uniqueness of strong solution were established and the solution was constructed in the form of series in the small parameter present in the initial conditions. The long-time asymptotics was also obtained in the explicit form. The author considered the initial-boundary value problem for (1.1) in the unit ball $B \subset \mathbb{R}^3$, similar results were established in [8]. Wang and Wang [18] proved the global existence and asymptotic decay estimates of solutions to problem (1.1), (1.2) with L^1 initial data. Their proof is based on the contraction mapping principle and makes use of the sharp decay estimates for the linearized problem. When $n \geq 2$, global existence and optimal decay estimate of solutions to (1.1), (1.2) with L^2 data were established by Wang and Zhang [19]. By constructing a class of special initial value, Wang [16] proved that global existence

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and faster decay estimate of solutions to (1.1), (1.2). [12] proved that the Cauchy problem for (1.1) is globally well-posed. Under certain conditions, they also proved that the global solution decays exponentially to zero in the infinite time limit. Very recently, the well-posedness of global solutions and blow-up of solutions were obtained by Wang [17]. Moreover, the asymptotic behavior of the solution was established by the multiplier method.

If the fourth-order dispersion term Δu_{tt} is neglected, (1.1) is reduced to the sixth-order Boussinesq equation with damped term. Guo and Fang [6] established global existence and pointwise estimates of classical solutions by virtue of the Fourier analysis and Greens function. If the damped term Δu_t is also neglected, then it is reduced to the classical sixth-order Boussinesq equation that models the nonlinear lattice dynamics in elastic crystals [10]. The study of the classical sixth-order Boussinesq equation has a long history and lots of interesting results have been established, we may refer to [1, 2, 3, 4, 13, 14] for local well-posedness, global well-posedness, stability of solitary waves and blow-up and so on. For other type of higher order hyperbolic equation, we may refer to [15, 20, 21, 22, 23, 24] and references therein.

It is well known that the Morrey space $\mathcal{M}_{p,q}$ generalizes the Lebesgue space L^q. For (1.1), there are few results about global existence and asymptotic behavior in Morrey space $\mathcal{M}_{p,q}$. Our main purpose is to investigate global existence and decay estimates of solutions to problem (1.1), (1.2) in Morrey spaces. More precisely, we prove that problem (1.1), (1.2) has a unique global solution in Morrey spaces under suitable conditions. Moreover, decay estimates of this solution are also established. We state our main results as follows:

Theorem 1.1. *Let $1 \le p \le q_1 \le n < mq_1$, $q_1 \le q_2$ and m is a positive integer. Assume that for $0 \le k \le m$, $\nabla^k u_0, \nabla^{(k-1)_+} U_1 \in M_{p,q_1} \cap M_{p,q_2}$. Put*

$$\mathcal{E}_0 = \sum_{k=0}^{m} \Big[\|\nabla^k u_0\|_{\mathcal{M}_{p,q_1} \cap M_{p,q_2}} + \|\nabla^{(k-1)_+} U_1\|_{\mathcal{M}_{p,q_1} \cap M_{p,q_2}} \Big].$$

Then there exists $\epsilon_0 > 0$, such that if $\mathcal{E}_0 \le \epsilon_0$, problem (1.1), (1.2) has a unique global solution u such that

$$\nabla^k u \in C([0,\infty); \mathcal{M}_{p,q_1} \cap \mathcal{M}_{p,q_2}), \quad \nabla^l \partial_t u \in C([0,\infty); \mathcal{M}_{p,q_1} \cap \mathcal{M}_{p,q_2}).$$

Moreover, we have the following decay estimates:

$$\|\nabla^k u(t)\|_{\mathcal{M}_{p,q_1}} \le C\mathcal{E}_0 (1+t)^{-k/2}, \tag{1.3}$$

$$\|\nabla^k u(t)\|_{\mathcal{M}_{p,q_2}} \le C\mathcal{E}_0 (1+t)^{-\frac{k}{2} - \frac{n}{2}(\frac{1}{q_1} - \frac{1}{q_2})}, \tag{1.4}$$

$$\|\nabla^l \partial_t u(t)\|_{\mathcal{M}_{p,q_1}} \le C\mathcal{E}_0 (1+t)^{-\frac{l+1}{2}}, \tag{1.5}$$

$$\|\nabla^l \partial_t u(t)\|_{\mathcal{M}_{p,q_2}} \le C\mathcal{E}_0 (1+t)^{-\frac{l+1}{2} - \frac{n}{2}(\frac{1}{q_1} - \frac{1}{q_2})}, \tag{1.6}$$

where $0 \le l \le m-2$ in (1.5) and (1.6).

Remark 1.2. If $p = q_1 = q_2 = 2$, then $\mathcal{M}_{p,q_1} = \mathcal{M}_{p,q_2} = L^2$, therefore Theorem 1.1 reduces to [19, Theorem 1]. If $p = q_1 = 1$, then Theorem 1.1 gives the existence and decay estimate of global solutions to problem (1.1), (1.2) in L^1 space. In a word, the results obtained in this paper generalize the results in [18] and [19].

There are two goals in this paper. Firstly, study the wave equation with damped term in Morrey spaces. To the best of our knowledge, there are only a few results in this setting, since the classical energy method used in Sobolev space can not be applied in Morrey spaces. Secondly, we hope that the method used here provide an idea for studying hyperbolic equations with damping terms and related models in Morrey spaces.

The plan of the paper is as follows. Firstly, we recall the definition of Morrey spaces and state some useful lemmas in Section 2. Section 3 is devoted to establish the decay estimate of the solutions operator in Morrey spaces. Finally, global existence and decay estimates are proved by Banach fixed point theorem in Section 4.

Notation: The Fourier transform of a function u is defined by

$$\widehat{u}(\xi) = \mathcal{F}[u](\xi) := \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)dx.$$

We denote its inverse transform by $\mathcal{F}^{-1}$.

The Morrey space $\mathcal{M}_{p,q}(1 \le p \le q \le \infty(q = \infty, p < q))$ is defined as the set of functions $f \in L^p(\mathbb{R}^n)$ such that

$$\begin{aligned} \|f\|_{\mathcal{M}_{p,q}} &= \sup_{z\in\mathbb{R}^n}\sup_{r>0} r^{n(\frac{1}{q}-\frac{1}{p})}\Big(\int_{B(z,r)} |f(y)|^p dy\Big)^{1/p} \\ &= \sup_{z\in\mathbb{R}^n}\sup_{r>0} r^{n(\frac{1}{q}-\frac{1}{p})}\|f\|_{L^p(B(z,r))} < \infty. \end{aligned} \tag{1.7}$$

Here $B(z, r)$ denotes the open ball in $\mathbb{R}^n$ with radius r centered at z.

2. Some lemmas

In this section, we state some useful results, such as interpolation inequalities in Morrey spaces, which may be found in [5, 7, 9, 11].

Lemma 2.1. *Let $1 \le p \le q \le \infty$ $(q = \infty, p < q)$. Then*
1. *$\mathcal{M}_{p,q}$ is a Banach space,*
2. *$\mathcal{M}_{p,p} \cong L^p$,*
3. *$\mathcal{M}_{p,\infty} \cong L^\infty$.*

Lemma 2.2. *Let $1 \le p \le q < \infty$, $\lambda > 0$ and let*

$$\vartheta_\lambda(x) = \vartheta(\frac{x}{\lambda}). \tag{2.1}$$

Then

$$\|\vartheta_\lambda\|_{\mathcal{M}_{p,q}} = \lambda^{n/q}\|\vartheta\|_{\mathcal{M}_{p,q}}. \tag{2.2}$$

Proof. Owing to the definition of Morrey spaces and direct computation, we obtain (2.2). Here we omit the details. □

Remark 2.3. Let X be a homogeneous Banach space. The smoothness degree of X is defined as

$$\deg(X) := \log_{\lambda^{-1}} \Lambda(\lambda), \quad \forall\lambda > 0,$$

where

$$\Lambda(\lambda) = \frac{\|\vartheta_\lambda(\cdot)\|_X}{\|\vartheta(\cdot)\|_X}, \quad \forall\lambda > 0,$$

and ϑ is a nonzero function in X. Then $\deg(\mathcal{M}_{p,q}) = -n/q$.

From the definition of Morrey spaces we have the following interpolation result.

Lemma 2.4. *Let $\frac{1}{q}=\frac{1-\theta}{q_1}+\frac{\theta}{q_2}$ and $0<\theta<1$. If $f\in\mathcal{M}_{p,q_1}\cap\mathcal{M}_{p,q_2}$, then $f\in\mathcal{M}_{p,q}$ and*

$$\|f\|_{\mathcal{M}_{p,q}}\le C\|f\|^{1-\theta}_{\mathcal{M}_{p,q_1}}\|f\|^{\theta}_{\mathcal{M}_{p,q_2}}. \tag{2.3}$$

Lemma 2.5. *Let $1\le p\le q<\infty$ and $n<mq$. If $f\in\mathcal{M}_{p,q}$ and $\nabla^m f\in\mathcal{M}_{p,q}$, then $f\in\mathcal{M}_{p,\infty}$ and*

$$\|f\|_{\mathcal{M}_{p,\infty}}\le C\|f\|^{1-\theta}_{\mathcal{M}_{p,q}}\|\nabla^m f\|^{\theta}_{\mathcal{M}_{p,q}} \tag{2.4}$$

with $\theta=n/(mq)$.

Proof. By Lemma 2.2, the interpolation result (2.4) may be established. For the details, we may refer to [5]. Here we omit the details. □

Let $\beta\in[0,n)$. Assume that $\varpi(\xi)$ is smooth on $\mathbb{R}^n$ and homogeneous of degree $-\beta$ in ξ. We call $\varpi(\xi)\in\sum_1^{-\beta}(\mathbb{R}^n)$ if $\varpi(\xi)$ satisfies

$$|D^\alpha\varpi(\xi)|\le C_\alpha|\xi|^{-\beta-\alpha}.$$

Lemma 2.6 ([11]). *If $\varpi(\xi)\in\sum_1^{0}(\mathbb{R}^n)$, and $1<p\le q<\infty$, then*

$$T=\varpi(D):\mathcal{M}_{p,q}\longrightarrow\mathcal{M}_{p,q}$$

is a bounded operator.

3. Decay properties of the solution operator

We investigate the linearized equation of (1.1),

$$u_{tt}-a\Delta u_{tt}-2b\Delta u_t-\alpha\Delta^3u+\beta\Delta^2u-\Delta u=0. \tag{3.1}$$

Taking the Fourier transform of (3.1), (1.2) yields

$$(1+a|\xi|^2)\widehat{u}_{tt}+2b|\xi|^2\widehat{u}_t+(|\xi|^2+\beta|\xi|^4+\alpha|\xi|^6)\widehat{u}=0, \tag{3.2}$$

$$\widehat{u}(\xi,0)=\widehat{u}_0(\xi),\qquad \widehat{u}_t(\xi,0)=|\xi|\widehat{U}_1(\xi). \tag{3.3}$$

The characteristic equation is

$$(1+a|\xi|^2)\lambda^2+2b|\xi|^2\lambda+(|\xi|^2+\beta|\xi|^4+\alpha|\xi|^6)=0.$$

Let $\lambda=\lambda_\pm(\xi)$ be the corresponding eigenvalues. Solving the characteristic equation, we arrive at

$$\lambda_\pm(\xi)=\frac{-b|\xi|^2\pm|\xi|\sqrt{-1-(a+\beta-b^2)|\xi|^2-(\alpha+a\beta)|\xi|^4-a\alpha|\xi|^6}}{1+a|\xi|^2}. \tag{3.4}$$

Then the solution to problem (3.2), (3.3) is

$$\widehat{u}(\xi,t)=\widehat{G}(\xi,t)|\xi|\widehat{U}_1(\xi)+\widehat{\mathcal{G}}(\xi,t)\widehat{u}_0(\xi), \tag{3.5}$$

where

$$\begin{gathered}\widehat{G}(\xi,t)=\frac{1}{\lambda_+(\xi)-\lambda_-(\xi)}\Big(e^{\lambda_+(\xi)t}-e^{\lambda_-(\xi)t}\Big),\\ \widehat{\mathcal{G}}(\xi,t)=\frac{1}{\lambda_+(\xi)-\lambda_-(\xi)}\Big(\lambda_+(\xi)e^{\lambda_-(\xi)t}-\lambda_-(\xi)e^{\lambda_+(\xi)t}\Big).\end{gathered} \tag{3.6}$$

Let $G(x,t)=\mathcal{F}^{-1}[\widehat{G}(\cdot,t)](x)$ and $\mathcal{G}(x,t)=\mathcal{F}^{-1}[\widehat{\mathcal{G}}(\cdot,t)](x)$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Then, we apply $\mathcal{F}^{-1}$ to (3.5) and obtain the solution formula to problem (3.1), (1.2):

$$u(t)=G(t)*\Lambda U_1+\mathcal{G}(t)*u_0. \tag{3.7}$$

Owing to Duhamel principle, we obtain the solution formula to problem (1.1), (1.2):

$$u(t) = G(t) * \Lambda U_1 + \mathcal{G}(t) * u_0 + \int_0^t G(t-\tau) * (I - a\Delta)^{-1}\Delta f(u)(\tau)d\tau. \tag{3.8}$$

To establish decay properties of solutions operator in Morrey spaces, we shall make analysis for G and $\mathcal{G}$ by the Fourier splitting frequency technique. To this end, let

$$\chi(\xi) = \begin{cases} 1, & |\xi| < r, \\ 0, & |\xi| > 2r. \end{cases}$$

be smooth cut-off functions, where $0 < r < 1$ is constant. Define

$$\widehat{G}_l(\xi,t) = \chi(\xi)\widehat{G}(\xi,t), \quad \widehat{G}_h(\xi,t) = (1-\chi(\xi))\widehat{G}(\xi,t),$$
$$\widehat{\mathcal{G}}_l(\xi,t) = \chi(\xi)\widehat{\mathcal{G}}(\xi,t), \quad \widehat{\mathcal{G}}_h(\xi) = (1-\chi(\xi))\widehat{\mathcal{G}}(\xi,t)$$

Then

$$G_l(x,t) = \chi(D)G(x,t), \quad G_h(x,t) = (1-\chi(D))G(x,t),$$
$$\mathcal{G}_l(x,t) = \chi(D)\mathcal{G}(x,t), \quad \mathcal{G}_h(x,t) = (1-\chi(D))\mathcal{G}(x,t),$$
$$G(x,t) = G_l(x,t) + G_h(x,t), \quad \mathcal{G}(x,t) = \mathcal{G}_l(x,t) + \mathcal{G}_h(x,t),$$

where the operator $\chi(D)$ is defined by

$$\chi(D) = \mathcal{F}^{-1}[\chi(\xi)].$$

The following energy estimate in the Fourier space has been obtained in [18, 16], which may be derived by the energy method in the Fourier space.

Lemma 3.1. *The solution of problem* (3.2), (3.3) *satisfies*

$$\begin{aligned} &|\xi|^2(1+|\xi|^2)|\widehat{u}(\xi,t)|^2 + |\widehat{u}_t(\xi,t)|^2 \\ &\le Ce^{-c\omega(\xi)t}(|\xi|^2(1+|\xi|^2)|\widehat{u}_0(\xi)|^2 + |\xi|^2|\widehat{U}_1(\xi)|^2), \end{aligned} \tag{3.9}$$

for $\xi \in \mathbb{R}^n$ *and* $t \ge 0$, *where* $\omega(\xi) = \frac{|\xi|^2}{1+|\xi|^2}$.

The above lemma and the solution formula (3.5) imply that the decay estimates of solution operators $G(t)$ and $\mathcal{G}(t)$ hold.

Lemma 3.2. *Let* $\widehat{G}$ *and* $\widehat{\mathcal{G}}$ *be the fundamental solutions of* (3.1) *in the Fourier space, which are given explicitly in* (3.6). *Then we have*

$$|\xi|^2(1+|\xi|^2)|\widehat{G}(\xi,t)|^2 + |\widehat{G}_t(\xi,t)|^2 \le Ce^{-c\omega(\xi)t}, \tag{3.10}$$

$$|\xi|^2(1+|\xi|^2)|\widehat{\mathcal{G}}(\xi,t)|^2 + |\widehat{\mathcal{G}}_t(\xi,t)|^2 \le C|\xi|^2(1+|\xi|^2)e^{-c\omega(\xi)t} \tag{3.11}$$

for $\xi \in \mathbb{R}^n$ *and* $t \ge 0$, *where* $\omega(\xi) = \frac{|\xi|^2}{1+|\xi|^2}$.

Lemma 3.2 implies that

$$|\xi|\widehat{G}_l(\xi,t) \sim e^{-c|\xi|^2 t}, \quad \widehat{\mathcal{G}}_l(\xi,t) \sim e^{-c|\xi|^2 t}, \tag{3.12}$$

$$|\xi|\widehat{G}_h(\xi,t) \sim e^{-ct}, \quad \widehat{\mathcal{G}}_h(\xi,t) \sim e^{-ct}. \tag{3.13}$$

Lemma 3.3. *Let $1 \le p \le q_1 \le q_2 \le \infty$, and let k be nonnegative integer. The following decay properties hold for solution operator:*

$$\begin{aligned}\|\nabla^k G(t) * \Lambda\phi\|_{\mathcal{M}_{p,q_2}} \le C(1+t)^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2})}\|\phi\|_{\mathcal{M}_{p,q_1}} \\ + Ce^{-ct}\|\nabla^{k-1+l}\phi\|_{\mathcal{M}_{p,q_2}},\end{aligned} \tag{3.14}$$

$$\begin{aligned}\|\nabla^k \mathcal{G}(t) * \psi\|_{\mathcal{M}_{p,q_2}} \le C(1+t)^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2})}\|\psi\|_{\mathcal{M}_{p,q_1}} \\ + Ce^{-ct}\|\nabla^{k+l}\psi\|_{\mathcal{M}_{p,q_2}},\end{aligned} \tag{3.15}$$

$$\begin{aligned}\|\nabla^k \partial_t G(t) * \Lambda\phi\|_{\mathcal{M}_{p,q_2}} \le C(1+t)^{-\frac{k+1}{2}-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2})}\|\phi\|_{\mathcal{M}_{p,q_1}} \\ + Ce^{-ct}\|\nabla^{k+1+l}\phi\|_{\mathcal{M}_{p,q_2}}\end{aligned} \tag{3.16}$$

$$\begin{aligned}\|\nabla^k \partial_t \mathcal{G}(t) * \psi\|_{\mathcal{M}_{p,q_2}} \le C(1+t)^{-\frac{k+1}{2}-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2})}\|\psi\|_{\mathcal{M}_{p,q_1}} \\ + Ce^{-ct}\|\nabla^{k+2+l}\psi\|_{\mathcal{M}_{p,q_2}}.\end{aligned} \tag{3.17}$$

Proof. The proofs of (3.14)–(3.17) is similar, we only prove (3.14). Obviously, it holds that

$$\begin{aligned}&\|\nabla^k G(t) * \Lambda\phi\|_{\mathcal{M}_{p,q}} \\ &\le \|\nabla^k G_l(t) * \Lambda\phi\|_{\mathcal{M}_{p,q}} + \|\nabla^k G_h(t) * \Lambda\phi\|_{\mathcal{M}_{p,q}} =: \mathscr{I}_1 + \mathscr{I}_2.\end{aligned} \tag{3.18}$$

By using (3.12), we have

$$|\nabla^{k+1} G_l(t)| \le C(1+t)^{-\frac{n+k}{2}} e^{-c_k\frac{|x|^2}{t}},$$

which implies

$$\begin{aligned}&\|\nabla^k G_l(t) * \Lambda\phi\|_{L^p(B(z,R))} \\ &\le C(1+t)^{-\frac{n+k}{2}p}\int_{\mathbb{R}^n}\chi_{Z,R}(x)|e^{-c_k\frac{|x|^2}{t}} * \phi|^p(x)dx \\ &\le C(1+t)^{-\frac{k}{2}p}\|\phi\|^p_{L^p(B(z,R))}.\end{aligned}$$

Then the above inequality and the definition of $\mathcal{M}_{p,q_1}$ implies

$$\|\nabla^k G_l(t) * \Lambda\phi\|_{\mathcal{M}_{p,q_1}} \le C(1+t)^{-k/2}\|\phi\|_{\mathcal{M}_{p,q_1}} \tag{3.19}$$

Thanks to (3.12) and Höld inequality, we arrive at

$$\begin{aligned}|\nabla^k G_l(t) * \Lambda\phi| &\le C(1+t)^{-\frac{n+k}{2}}\int_{\mathbb{R}^n} e^{-c\frac{|x-y|^2}{t}}|\phi(y)|dy \\ &\le C(1+t)^{-\frac{n+k}{2}}\int_0^1 ds\int_{B(x,|ct\log s|^{1/2})}|\phi(y)|dy \\ &\le C(1+t)^{-\frac{n+k}{2}}\int_0^1 |t\log s|^{\frac{n}{2}(1-\frac{1}{p})}\|\phi\|_{L^p(B(x,|ct\log s|^{1/2}))}ds \\ &\le C(1+t)^{-\frac{n+k}{2}}\int_0^1 |t\log s|^{\frac{n}{2}(1-\frac{1}{q_1})}ds\|\phi\|_{\mathcal{M}_{p,q_1}} \\ &\le C(1+t)^{-\frac{k}{2}-\frac{n}{2q_1}}\|\phi\|_{\mathcal{M}_{p,q_1}},\end{aligned}$$

which implies

$$\|\nabla^k G_l(t) * \Lambda\phi\|_{\mathcal{M}_{p,\infty}} \le C(1+t)^{-\frac{k}{2}-\frac{n}{2q_1}}\|\phi\|_{\mathcal{M}_{p,q_1}}. \tag{3.20}$$

Lemma 2.4, (3.19) and (3.20) yield

$$\begin{aligned}\mathscr{I}_1 &\le \|\nabla^k G_l(t) * \Lambda\phi\|_{\mathcal{M}_{p,\infty}}^{1-\frac{q_1}{q_2}} \|\nabla^k G_l(t) * \Lambda\phi\|_{\mathcal{M}_{p,q_2}}^{\frac{q_1}{q_2}} \\ &\le C(1+t)^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2})}\|\phi\|_{\mathcal{M}_{p,q_1}}.\end{aligned} \tag{3.21}$$

Set $\varpi(\xi)=1$, (3.15) gives

$$|\xi|^2|\widehat{G}_h(\xi,t)| \le Ce^{-ct}\varpi(\xi),$$

which together with Lemma 2.6 with $\varpi(\xi)=1$ yields

$$\mathscr{I}_2 \le Ce^{-ct}\|\nabla^k\phi\|_{\mathcal{M}_{p,q_2}}. \tag{3.22}$$

Inserting (3.21) and (3.22) into (3.18) immediately yields (3.14). The proof is complete. □

Noting that the boundness of the operator $(I-a\Delta)^{-1}$ in Morrey spaces, the following lemma immediately follows from (3.14) and (3.16).

Lemma 3.4. *Let $1\le p\le q_1\le q_2\le\infty$, and let k be nonnegative integer. The following decay properties of solution operator hold:*

$$\begin{aligned}&\|\nabla^k G(t) * (I-a\Delta)^{-1}\Delta f\|_{\mathcal{M}_{p,q_2}} \\ &\le C(1+t)^{-\frac{k+1}{2}-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2})}\|f\|_{\mathcal{M}_{p,q_1}} + Ce^{-ct}\|\nabla^{k-2+l}f\|_{\mathcal{M}_{p,q_2}},\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}&\|\nabla^k \partial_t G(t) * (I-a\Delta)^{-1}\Delta f\|_{\mathcal{M}_{p,q_2}} \\ &\le C(1+t)^{-\frac{k+2}{2}-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2})}\|f\|_{\mathcal{M}_{p,q_1}} + Ce^{-ct}\|\nabla^{k+l}f\|_{\mathcal{M}_{p,q_2}}.\end{aligned} \tag{3.24}$$

4. Proof of main results

In this section, our main purpose is to prove Theorem 1.1. For this purpose, we need the following lemma(see [26]).

Lemma 4.1. *Assume that $f=f(v)$ is a smooth function satisfying $f(v)=O(v^{1+\sigma})$ for $v\to 0$, where $\sigma\ge 1$ is an integer. Let $v\in L^\infty$ and $\|v\|_{L^\infty}\le M_0$ for a positive constant M_0. Let $1\le p,q,r\le+\infty$ and $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$, and let $k\ge 0$ be an integer. Then we have*

$$\|\partial_x^k f(v)\|_{L^p} \le C\|v\|_{L^\infty}^{\sigma-1}\|v\|_{L^q}\|\partial_x^k v\|_{L^r},$$

Furthermore,

$$\begin{aligned}\|\partial_x^\alpha(f(v_1)-f(v_2))\|_{L^p} \le C\Big\{&(\|\partial_x^\alpha v_1\|_{L^q}+\|\partial_x^\alpha v_2\|_{L^q})\|v_1-v_2\|_{L^r} + (\|v_1\|_{L^r} \\ &+\|v_2\|_{L^r})\|\partial_x^\alpha(v_1-v_2)\|_{L^q}\Big\}(\|v_1\|_{L^\infty}+\|v_2\|_{L^\infty})^{\sigma-1}.\end{aligned}$$

where $C=C(M_0)$ is a constant depending on M_0.

Proof of Theorem 1.1. To prove existence and decay estimate of global solutions to problem (1.1), (1.2), we define the mapping by (3.8)

$$\mathscr{T}(u) = G(t)*\Lambda U_1 + \mathcal{G}(t)*u_0 + \int_0^t G(t-\tau)*(I-a\Delta)^{-1}\Delta f(u)(\tau)d\tau. \tag{4.1}$$

Based on the decay properties of solutions operator, we define the function space

$$X = \big\{\nabla^k u \in C([0,\infty);\mathcal{M}_{p,q_1}\cap\mathcal{M}_{p,q_2}), k=0,1,\dots,m \,\big|\, \|u\|_X<\infty\big\},$$

where

$$\begin{aligned}\|u\|_X = \sup_{t\geq 0}\sum_{k=0}^{m}\Big\{&(1+t)^{\frac{k}{2}}\|\nabla^k u(t)\|_{\mathcal{M}_{p,q_1}}\\ &+(1+t)^{\frac{k}{2}+\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2})}\|\nabla^k u(t)\|_{\mathcal{M}_{p,q_2}}\Big\}.\end{aligned}\tag{4.2}$$

For $R>0$, let

$$Y=\{u\in X: \|u\|_X\leq R\}.$$

Then Y is a closed set of X. Hence, Y is also a Banach space. To prove existence and decay estimate of global solutions to problem (1.1), (1.2), it is suffice to prove that the mapping $\mathscr{T}$ has a unique fixed point in the function space Y.

By (4.1), Minkowski inequaity, (3.14), (3.15) and (3.23), we have

$$\begin{aligned}&\|\nabla^k\mathscr{T}(u)(t)\|_{\mathcal{M}_{p,q_1}}\\ &\leq \|\nabla^k G(t)*\Lambda U_1\|_{\mathcal{M}_{p,q_1}}+\|\nabla^k\mathcal{G}(t)*u_0\|_{\mathcal{M}_{p,q_1}}\\ &\quad+\int_0^t\|\nabla^k G(t-\tau)*(I-a\Delta)^{-1}\Delta f(u)(\tau)\|_{\mathcal{M}_{p,q_1}}d\tau\\ &\leq C(1+t)^{-k/2}(\|U_1\|_{\mathcal{M}_{p,q_1}}+\|u_0\|_{\mathcal{M}_{p,q_1}})\\ &\quad+Ce^{-ct}(\|\nabla^{(k-1)_+}U_1\|_{\mathcal{M}_{p,q_1}}+\|\nabla^k u_0\|_{\mathcal{M}_{p,q_1}})\\ &\quad+C\int_0^{t/2}(1+t-\tau)^{-\frac{k+1}{2}}\|f(u)(\tau)\|_{\mathcal{M}_{p,q_1}}d\tau\\ &\quad+C\int_{t/2}^t(1+t-\tau)^{-1/2}\|\nabla^k f(u)(\tau)\|_{\mathcal{M}_{p,q_1}}d\tau\\ &\quad+C\int_0^t e^{-c(t-\tau)}\|\nabla^k f(u)(\tau)\|_{\mathcal{M}_{p,q_1}}d\tau.\end{aligned}\tag{4.3}$$

Lemma 2.5 implies

$$\|\nabla^j u\|_{L^\infty}\leq C\|\nabla^j u\|^{1-\theta}_{\mathcal{M}_{p,q_1}}\|\nabla^{j+m}u\|^{\theta}_{\mathcal{M}_{p,q_1}},$$

where $\theta=n/(mq_1)$, which together with (4.2) gives

$$\|\nabla^j u(\tau)\|_{L^\infty}\leq C\|u\|_X(1+\tau)^{-\frac{j}{2}-\frac{n}{2q_1}}.\tag{4.4}$$

It follows from the definition of Morrey spaces, Lemma 4.1, (4.4) and (4.2) that

$$\|f(u)(\tau)\|_{\mathcal{M}_{p,q_1}}\leq C\|u(\tau)\|_{L^\infty}\|u(\tau)\|_{\mathcal{M}_{p,q_1}}\leq C\|u\|_X^2(1+\tau)^{-\frac{n}{2q_1}},\tag{4.5}$$

$$\|\nabla^k f(u)(\tau)\|_{\mathcal{M}_{p,q_1}}\leq \|u(\tau)\|_{L^\infty}\|\nabla^k u(\tau)\|_{\mathcal{M}_{p,q_1}}\leq C\|u\|_X^2(1+\tau)^{-\frac{k}{2}-\frac{n}{2q_1}}.\tag{4.6}$$

We insert (4.5) and (4.6) into (4.3) and arrive at

$$
\begin{aligned}
&\|\nabla^k \mathscr{T}(u)(t)\|_{\mathcal{M}_{p,q_1}} \\
&\le C(1+t)^{-k/2}(\|U_1\|_{\mathcal{M}_{p,q_1}} + \|u_0\|_{\mathcal{M}_{p,q_1}} \\
&\quad + \|\nabla^{(k-1)_+}U_1\|_{\mathcal{M}_{p,q_1}} + \|\nabla^k u_0\|_{\mathcal{M}_{p,q_1}}) \\
&\quad + C\|u\|_X^2 \int_0^{t/2} (1+t-\tau)^{-\frac{k+1}{2}}(1+\tau)^{-\frac{n}{2q_1}}\,d\tau \\
&\quad + C\|u\|_X^2 \int_{t/2}^{t} (1+t-\tau)^{-1/2}(1+\tau)^{-\frac{k}{2}-\frac{n}{2q_1}}\,d\tau \\
&\quad + C\|u\|_X^2 \int_0^{t} e^{-c(t-\tau)}(1+\tau)^{-\frac{k}{2}-\frac{n}{2q_1}}\,d\tau \\
&\le C(1+t)^{-k/2}(\|U_1\|_{\mathcal{M}_{p,q_1}} + \|u_0\|_{\mathcal{M}_{p,q_1}} \\
&\quad + \|\nabla^{(k-1)_+}U_1\|_{\mathcal{M}_{p,q_1}} + \|\nabla^k u_0\|_{\mathcal{M}_{p,q_1}}) \\
&\quad + C\|u\|_X^2(1+t)^{-k/2}\varrho(t) + C\|u\|_X^2(1+t)^{-\frac{k}{2}-\frac{n}{2q_1}+\frac{1}{2}},
\end{aligned}
\tag{4.7}
$$

where

$$
\varrho(t) = \begin{cases} (1+t)^{-1/2}, & n > 2q_1, \\ (1+t)^{-1/2}\log(2+t), & n = 2q_1, \\ (1+t)^{\frac{1}{2}-\frac{n}{2q_1}}, & n < 2q_1. \end{cases}
$$

Similarly, we have

$$
\begin{aligned}
&\|\nabla^k \mathscr{T}(u)(t)\|_{\mathcal{M}_{p,q_2}} \\
&\le \|\nabla^k G(t) * \Lambda U_1\|_{\mathcal{M}_{p,q_2}} + \|\nabla^k \mathcal{G}(t) * u_0\|_{\mathcal{M}_{p,q_2}} \\
&\quad + \int_0^t \|\nabla^k G(t-\tau) * (I-a\Delta)^{-1}\Delta f(u)(\tau)\|_{\mathcal{M}_{p,q_2}}\,d\tau \\
&\le C(1+t)^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2})}(\|U_1\|_{\mathcal{M}_{p,q_1}} + \|u_0\|_{\mathcal{M}_{p,q_1}}) \\
&\quad + Ce^{-ct}(\|\nabla^{(k-1)_+}U_1\|_{\mathcal{M}_{p,q_2}} + \|\nabla^k u_0\|_{\mathcal{M}_{p,q_2}}) \\
&\quad + C\int_0^{t/2} (1+t-\tau)^{-\frac{k+1}{2}-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2})}\|f(u)(\tau)\|_{\mathcal{M}_{p,q_1}}\,d\tau \\
&\quad + C\int_{t/2}^{t} (1+t-\tau)^{-1/2}\|\nabla^k f(u)(\tau)\|_{\mathcal{M}_{p,q_2}}\,d\tau \\
&\quad + C\int_0^{t} e^{-c(t-\tau)}\|\nabla^k f(u)(\tau)\|_{\mathcal{M}_{p,q_2}}\,d\tau \\
&\le C(1+t)^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2})}\mathcal{E}_0 + C\|u\|_X^2(1+t)^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2})}\varrho(t) \\
&\quad + C\|u\|_X^2(1+t)^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2})-\frac{n}{2q_1}+\frac{1}{2}} \\
&\le C(1+t)^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2})}\mathcal{E}_0 + C\|u\|_X^2(1+t)^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2})}.
\end{aligned}
\tag{4.8}
$$

where we have used

$$
\begin{aligned}
\|\nabla^k f(u)(\tau)\|_{\mathcal{M}_{p,q_2}} &\le C\|u\|_{L^\infty}\|\nabla^k u(\tau)\|_{\mathcal{M}_{p,q_2}} \\
&\le C(1+\tau)^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2})-\frac{n}{2q_1}}\|u\|_X^2,
\end{aligned}
$$

which may be derived from Lemma 4.1 and (4.2), (4.4).

Combining (4.7) and (4.8) yields

$$\|\mathscr{T}(u)\|_X \le C\mathcal{E}_0 + C\|u\|_X^2$$

Thus, we arrive at

$$\|\mathscr{T}(u)\|_X \le C\mathcal{E}_0 \le R, \tag{4.9}$$

provided that taking $R = 2C\mathcal{E}_0$ and $\mathcal{E}_0$ suitably small.

The definition of Morrey spaces, Lemma 4.1, Lemma 2.5 and (4.2) yield

$$\begin{aligned}
&\|\nabla^k (f(\bar{u}) - f(\tilde{u}))(\tau)\|_{\mathcal{M}_{p,q_1}} \\
&\le C\big\{\|\bar{u}-\tilde{u}\|_{L^\infty}(\|\nabla^k \bar{u}(\tau)\|_{\mathcal{M}_{p,q_1}} + \|\nabla^k \tilde{u}(\tau)\|_{\mathcal{M}_{p,q_1}}) \\
&\quad + (\|\bar{u}\|_{L^\infty} + \|\tilde{u}\|_{L^\infty})\|\nabla^k(\bar{u}-\tilde{u})(\tau)\|_{\mathcal{M}_{p,q_1}}\big\} \\
&\le C(1+\tau)^{-\frac{k}{2}-\frac{n}{2q_1}} R\|\bar{u}-\tilde{u}\|_X.
\end{aligned} \tag{4.10}$$

Using (4.1), Minkowski inequality and (4.10), we obtain

$$\begin{aligned}
&\|\nabla^k(\mathscr{T}(\bar{u}) - \mathscr{T}(\tilde{u}))(t)\|_{\mathcal{M}_{p,q_1}} \\
&\le \int_0^t \|\nabla^k G(t-\tau) * (I - a\Delta)^{-1}\Delta(f(\bar{u}) - f(\tilde{u}))(\tau)\|_{\mathcal{M}_{p,q_1}} d\tau \\
&\le C\int_0^{t/2} (1+t-\tau)^{-\frac{k+1}{2}} \|(f(\bar{u}) - f(\tilde{u}))(\tau)\|_{\mathcal{M}_{p,q_1}} d\tau \\
&\quad + C\int_{t/2}^t (1+t-\tau)^{-1/2}\|\nabla^k(f(\bar{u}) - f(\tilde{u}))(\tau)\|_{\mathcal{M}_{p,q_1}} d\tau \\
&\quad + C\int_0^t e^{-c(t-\tau)}\|\nabla^k(f(\bar{u}) - f(\tilde{u}))(\tau)\|_{\mathcal{M}_{p,q_1}} d\tau \\
&\le CR\|\bar{u}-\tilde{u}\|_X \int_0^{t/2} (1+t-\tau)^{-\frac{k+1}{2}}(1+\tau)^{-\frac{n}{2q_1}} d\tau \\
&\quad + CR\|\bar{u}-\tilde{u}\|_X \int_{t/2}^t (1+t-\tau)^{-1/2}(1+\tau)^{-\frac{k}{2}-\frac{n}{2q_1}} d\tau \\
&\quad + CR\|\bar{u}-\tilde{u}\|_X \int_0^t e^{-c(t-\tau)}(1+\tau)^{-\frac{k}{2}-\frac{n}{2q_1}} d\tau \\
&\le CR\|\bar{u}-\tilde{u}\|_X (1+t)^{-k/2}\varrho(t) + CR\|u_1 - u_2\|_X (1+t)^{-\frac{k}{2}-\frac{n}{2q_1}+\frac{1}{2}} \\
&\le CR\|\bar{u}-\tilde{u}\|_X (1+t)^{-k/2}.
\end{aligned} \tag{4.11}$$

Similarly,

$$\begin{aligned}
&\|\nabla^k(\mathscr{T}(\bar{u})-\mathscr{T}(\tilde{u}))(t)\|_{\mathcal{M}_{p,q_2}} \\
&\le \int_0^t \|\nabla^k G(t-\tau)*(I-a\Delta)^{-1}\Delta(f(\bar{u})-f(\tilde{u}))(\tau)\|_{\mathcal{M}_{p,q_2}}\,d\tau \\
&\le C\int_0^{t/2}(1+t-\tau)^{-\frac{k+1}{2}-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2})}\|(f(\bar{u})-f(\tilde{u}))(\tau)\|_{\mathcal{M}_{p,q_1}}\,d\tau \\
&\quad + C\int_{t/2}^t (1+t-\tau)^{-1/2}\|\nabla^k(f(\bar{u})-f(\tilde{u}))(\tau)\|_{\mathcal{M}_{p,q_2}}\,d\tau \\
&\quad + C\int_0^t e^{-c(t-\tau)}\|\nabla^k(f(\bar{u})-f(\tilde{u}))(\tau)\|_{\mathcal{M}_{p,q_2}}\,d\tau \qquad (4.12)\\
&\le CR\|\bar{u}-\tilde{u}\|_X\int_0^{t/2}(1+t-\tau)^{-\frac{k+1}{2}-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2})}(1+\tau)^{-\frac{n}{2q_1}}\,d\tau \\
&\quad + CR\|\bar{u}-\tilde{u}\|_X\int_{t/2}^t (1+t-\tau)^{-1/2}(1+\tau)^{-\frac{k}{2}-\frac{n}{q_1}+\frac{n}{2q_2}}\,d\tau \\
&\quad + CR\|\bar{u}-\tilde{u}\|_X\int_0^t e^{-c(t-\tau)}(1+\tau)^{-\frac{k}{2}-\frac{n}{q_1}+\frac{n}{2q_2}}\,d\tau \\
&\le CR\|\bar{u}-\tilde{u}\|_X(1+t)^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{q_2})},
\end{aligned}$$

where we have used

$$\begin{aligned}
&\|\nabla^k(f(\bar{u})-f(\tilde{u}))(\tau)\|_{\mathcal{M}_{p,q_2}} \\
&\le C\Big\{\|\bar{u}-\tilde{u}\|_{L^\infty}(\|\nabla^k\bar{u}(\tau)\|_{\mathcal{M}_{p,q_2}}+\|\nabla^k\tilde{u}(\tau)\|_{\mathcal{M}_{p,q_2}}) \\
&\quad + (\|\bar{u}\|_{L^\infty}+\|\tilde{u}\|_{L^\infty})\|\nabla^k(\bar{u}-\tilde{u})(\tau)\|_{\mathcal{M}_{p,q_2}}\Big\} \\
&\le C(1+\tau)^{-\frac{k}{2}-\frac{n}{q_1}+\frac{n}{2q_2}}R\|\bar{u}-\tilde{u}\|_X.
\end{aligned}$$

Combining (4.11) and (4.12) yields

$$\|\mathscr{T}(\bar{u})-\mathscr{T}(\tilde{u})\|_X \le CR\|\bar{u}-\tilde{u}\|_X$$

Thus, we arrive at

$$\|\mathscr{T}(\bar{u})-\mathscr{T}(\tilde{u})\|_X \le \frac{1}{2}\|\bar{u}-\tilde{u}\|_X. \qquad (4.13)$$

Inequalities (4.9) and (4.13) imply that $\mathscr{T}$ is a strictly contracting mapping. The contraction mapping principle implies that the mapping $\mathscr{T}$ has a unique fixed point $u \in Y$, which is a global solution to problem (1.1), (1.2). Moreover, u verifies decay estimates (1.3) and (1.4).

In what follows, we prove the decay estimates (1.5) and (1.6). Owing to (4.1), Minkowski inequality, (3.16), (3.17), (3.24), (4.5), (4.6) and (1.3), we arrive at

$$
\begin{aligned}
&\|\nabla^l \partial_t u(t)\|_{\mathcal{M}_{p,q_1}} \\
&\le \|\nabla^l \partial_t G(t) * \Lambda U_1\|_{\mathcal{M}_{p,q_1}} + \|\nabla^l \partial_t \mathcal{G}(t) * u_0\|_{\mathcal{M}_{p,q_1}} \\
&\quad + \int_0^t \|\nabla^l \partial_t G(t-\tau) * (1-a\Delta)^{-1}\Delta f(u)(\tau)\|_{\mathcal{M}_{p,q_1}}\,d\tau \\
&\le C(1+t)^{-\frac{l+1}{2}}(\|U_1\|_{M_{p,q_1}} + \|u_0\|_{\mathcal{M}_{p,q_1}}) \\
&\quad + Ce^{-ct}(\|\nabla^{l+1}U_1\|_{M_{p,q_1}} + \|\nabla^{l+2}u_0\|_{\mathcal{M}_{p,q_1}}) \\
&\quad + C\int_0^{t/2}(1+t-\tau)^{-\frac{l+2}{2}}\|f(u)(\tau)\|_{\mathcal{M}_{p,q_1}}\,d\tau \\
&\quad + C\int_{t/2}^t (1+t-\tau)^{-1/2}\|\nabla^{l+1}f(u)(\tau)\|_{\mathcal{M}_{p,q_1}}\,d\tau \\
&\quad + C\int_0^t e^{-c(t-\tau)}\|\nabla^l f(u)(\tau)\|_{M_{p,q_1}}\,d\tau \\
&\le C\mathcal{E}_0(1+t)^{-\frac{l+1}{2}} + C\mathcal{E}_0^2\int_0^{t/2}(1+t-\tau)^{-\frac{l+2}{2}}(1+\tau)^{-\frac{n}{2q_1}}\,d\tau \\
&\quad + C\mathcal{E}_0^2\int_{t/2}^t (1+t-\tau)^{-1/2}(1+\tau)^{-\frac{l+1}{2}-\frac{n}{2q_1}}\,d\tau \\
&\quad + C\mathcal{E}_0^2\int_0^t e^{-c(t-\tau)}(1+\tau)^{-\frac{l}{2}-\frac{n}{2q_1}}\,d\tau \\
&\le C\mathcal{E}_0(1+t)^{-\frac{l+1}{2}}.
\end{aligned} \tag{4.14}
$$

Similarly,

$$\begin{aligned}
&\|\nabla^l \partial_t u(t)\|_{\mathcal{M}_{p,q_2}} \\
&\le \|\nabla^l \partial_t G(t) * \Lambda U_1\|_{\mathcal{M}_{p,q_2}} + \|\nabla^l \partial_t \mathcal{G}(t) * u_0\|_{\mathcal{M}_{p,q_2}} \\
&\quad + \int_0^t \|\nabla^l \partial_t G(t-\tau) * (I - a\Delta)^{-1} \Delta f(u)(\tau)\|_{\mathcal{M}_{p,q_2}} d\tau \\
&\le C(1+t)^{-\frac{l+1}{2} - \frac{n}{2}(\frac{1}{q_1} - \frac{1}{q_2})} (\|U_1\|_{\mathcal{M}_{p,q_1}} + \|u_0\|_{\mathcal{M}_{p,q_1}}) \\
&\quad + Ce^{-ct} (\|\nabla^{l+1} U_1\|_{\mathcal{M}_{p,q_2}} + \|\nabla^{l+2} u_0\|_{\mathcal{M}_{p,q_2}}) \\
&\quad + C\int_0^{t/2} (1+t-\tau)^{-\frac{l+2}{2} - \frac{n}{2}(\frac{1}{q_1} - \frac{1}{q_2})} \|f(u)(\tau)\|_{\mathcal{M}_{p,q_1}} d\tau \\
&\quad + C\int_{t/2}^t (1+t-\tau)^{-1/2} \|\nabla^{l+1} f(u)(\tau)\|_{\mathcal{M}_{p,q_2}} d\tau \\
&\quad + C\int_0^t e^{-c(t-\tau)} \|\nabla^l f(u)(\tau)\|_{\mathcal{M}_{p,q_2}} d\tau \\
&\le C\mathcal{E}_0 (1+t)^{-\frac{l+1}{2} - \frac{n}{2}(\frac{1}{q_1} - \frac{1}{q_2})} \\
&\quad + C\mathcal{E}_0^2 \int_0^{t/2} (1+t-\tau)^{-\frac{l+2}{2} - \frac{n}{2}(\frac{1}{q_1} - \frac{1}{q_2})} (1+\tau)^{-\frac{n}{2q_1}} d\tau \\
&\quad + C\mathcal{E}_0^2 \int_{t/2}^t (1+t-\tau)^{-1/2} (1+\tau)^{-\frac{l+1}{2} - \frac{n}{2}(\frac{1}{q_1} - \frac{1}{q_2}) - \frac{n}{2q_1}} d\tau \\
&\quad + C\mathcal{E}_0^2 \int_0^t e^{-c(t-\tau)} (1+\tau)^{-\frac{l}{2} - \frac{n}{2}(\frac{1}{q_1} - \frac{1}{q_2}) - \frac{n}{2q_1}} d\tau \\
&\le C\mathcal{E}_0 (1+t)^{-\frac{l+1}{2} - \frac{n}{2}(\frac{1}{q_1} - \frac{1}{q_2})}.
\end{aligned} \tag{4.15}$$

Inequalities (4.14) and (4.15) imply that (1.5) and (1.6) hold. Moreover, (4.14) and (4.15) also imply that $\nabla^l \partial_t u \in C([0,\infty); \mathcal{M}_{p,q_1} \cap \mathcal{M}_{p,q_2})$. The proof is complete. □

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References

[1] A. Dé Godefroy; *Existence, decay and blow-up for solutions to the sixth-order generalized Boussinesq equation*, Discrete Contin. Dyn. Syst., **35** (2015), 117–137.

[2] A. Esfahani, L. Farah, H. Wang; *Global existence and blow-up for the generalized sixth-order Boussinesq equation* , Nonlinear Anal., **75** (2012), 4325–4338.

[3] A. Esfahani, L.G. Farah; *Local well-posedness for the sixth-order Boussinesq equation*, J. Math. Anal. Appl., **385** (2012), 230–242.

[4] A. Esfahani, S. Levandosky; *Stability of solitary waves for the generalized higher-order Boussinesq equation*, J. Dynam. Diff. Eqs., **24** (2012), 391–425.

[5] Y. Giga, T. Miyakawa; *Navier-Stokes flow in $\mathbb{R}^3$ with measures as initial vorticity and Morrey spaces*, Commun. Partial Differential Equations, **14** (1989), 577–618.

[6] C. H. Guo, S. M. Fang; *Global existence and pointwise estimates of solutions for the generalized sixth-order Boussinesq equation*, Commun. Math. Sci., **15** (2017), 1457–1487.

[7] H. Kozono, M. Yamazaki; *Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data*, Commun.Partial Differential Equations, **19** (1994), 959–1014.

[8] S. Lai S, Y. Wang, Y. Wu, Q. Lin; *An initial-boundary value problem for a generalized Boussinesq water system in a ball*, Int J Appl. Math. Sci., **3** (2006), 117–33.
[9] T. Miyakawa; *On Morrey spaces of measures: basic properties and potential estimates*, Hiroshiama Math. J., **20** (1990), 213–222.
[10] G. Maugin; *Nonlinear Waves in Elastic Crystals*, Oxford Mathematical Monographs Series, Oxford University Press, Oxford, 2000.
[11] M. E. Taylor; *Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations*, Commun. Partial Differential Equations, **17** (1992), 1407–1456.
[12] N. Polat, E. P!sk!n; *Existence and asymptotic behavior of solution of Cauchy problem for the damped sixth-order Boussinesq equation*, Acta Mathematicae Applicatae Sinica, English Series **31** (2015), 735–746.
[13] H. Wang, A. Esfahani; *Well-posedness for the Cauchy problem associated to a periodic Boussinesq equation*, Nonlinear Anal., **89** (2013), 267–275.
[14] H. Wang, A. Esfahani; *Global rough solutions to the sixth-order Boussinesq equation*, Nonlinear Anal., **102** (2014), 97–104.
[15] S. Wang, X. Su; *Global existence and asymptotic behavior of solution for the sixth order Boussinesq equation with the damped term*, Nonlinear Analysis, **120** (2015), 171–185.
[16] Y. X. Wang; *Existence and asymptotic behaviour of solutions to the generalized damped Boussinesq equation*, Electronic Journal of Differential Equations, **2012** No. 96 (2012), 1-11.
[17] Y. Wang; *Cauchy problem for the sixth-order damped multidiemnsional Boussinesq equation*, Electronic Journal of Differential Equations, **2016** no. 64 (2016) 1-16.
[18] Y. Z. Wang, K. Y. Wang; *Decay estimate of solutions to the sixth order damped Boussinesq equation*, Appl. Math. Comput., **239** (2014), 171–179.
[19] Y.-Z. Wang, Q. N. Zhang; *Asymptotic behavior of global solutions to the Boussinesq equation in multidimensions*, Abstr. Appl. Anal., 2013, Art. ID 154102, 5 pp.
[20] Y.-Z. Wang, S. Chen; *Asymptotic profile of solutions to the double dispersion equation*, Nonlinear Anal., **134** (2016), 236–254.
[21] Y.-Z. Wang, H. Zhao; *Pointwise estimates of global small solutions to the generalized double dispersion equation*, J. Math. Anal. Appl., **448** (2017), 672-690.
[22] Y. Wang, Y. X. Wang; *Asymptotic behavior of solutions to a class of nonlinear wave equations of sixth order with damping*, Math. Methods Appl. Sci., **40** (2017), 1922–1936.
[23] R. Xu, Y. Yang, B. Liu, J. Shen, S. Huang; *Global existence and blowup of solutions for the multidimensional sixth-order "good" Boussinesq equation*, Z. Angew. Math. Phys., **66** (2015), 955–976.
[24] S. Xia, J. Yuan; *Existence and scattering of small solutions to a Boussinesq type equation of sixth order*, Nonlinear Anal., **73** (2010), 1015–1027.
[25] Y. Zhang, Q. Lin, S. Lai; *Long time asymptotic for the damped Boussinesq equation in a circle*, J. Partial Diff. Eqs., **18** (2005), 97–113.
[26] S. M. Zheng; *Nonlinear Evolution Equations*, Monographs and Surveys in Pure and Applied Mathematics, 133, Chapan Hall/CRC, 2004.

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