Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 162, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXPONENTIAL ESTIMATES FOR QUANTUM GRAPHS

SETENAY AKDUMAN, ALEXANDER PANKOV

Communicated by Vicentiu D. Radulescu

Abstract. The article studies the exponential localization of eigenfunctions associated with isolated eigenvalues of Schrödinger operators on infinite metric graphs. We strengthen the result obtained in [3] providing a bound for the rate of exponential localization in terms of the distance between the eigenvalue and the essential spectrum. In particular, if the spectrum is purely discrete, then the eigenfunctions decay super-exponentially.

1. Introduction

A quantum graph is a metric graph equipped with a self-adjoint Hamiltonian. For a comprehensive introduction to quantum graphs we refer to [4, 6, 11, 12] and references therein). Other aspects of differential equations on graphs and networks are available in [13, 14, 15] and references therein.

Typically, Hamiltonians are operators of Schrödinger type generated by the second-order differential expression

$$-\frac{d^2}{dx^2} + V(x)$$

on the edges of graph and certain conditions at the vertices. In this paper we use the Kirchhoff vertex conditions and impose sufficiently weak assumptions on the potential V under which the operator is self-adjoint and bounded below.

Our main concern in this paper is the exponential localization of eigenfunctions associated with isolated eigenvalues. In the case of classical Schrödinger operators this topic goes back to Schnol's paper [20] (see also [7]). For one-dimensional operators similar results were obtained in [16, 21]. The current state of the art of the topic is reviewed in [10, 18]. The first localization result for operators on metric graphs is obtained in [3]. The approach in that paper relies upon an elementary perturbation theory for linear operators and provides the exponential decay of eigenfunctions with sufficiently small rate. Papers [8, 9] are devoted to an extension of Agmon's geometric approach to quantum graphs.

In this article we obtain a stronger result on exponential decay of eigenfunction than in [3]. We provide a bound for the rate of decay in terms of the distance between the associated eigenvalue and the essential spectrum. Though not optimal,

2010 *Mathematics Subject Classification.* 34B45, 34L40, 81Q35.
Key words and phrases. Infinite metric graph; Schrödinger operator; eigenfunction; exponential decay.

Submitted July 14, 2018. Published September 10, 2018.

the bound is strong enough to imply that all eigenfunctions decay superexponentiall fast provided that the spectrum is purely discrete. The techniques relies upon estimates of the derivative of solution to the Schrödinger equation in terms of the solution itself on certain special domains. Such domains, called quasi-balls and quasi-annuli, are defined in terms of properly regularized distance function introduced in [3]. This approach can be considered as a suitable variant of the original Schnol's method [20]. Also it permits us to obtain an extension to quantum graphs for another Schnol's result [20] that provides an estimate for the distance from a real number to the spectrum in terms of exponential growth of solution to the Schrödinger equation, and known as Schnol's theorem [18] in the classical setting. As consequence, we have a condition for a point to belong to the spectrum in terms of such solutions. Under some stronger assumptions the last result is obtained in [12]. As an application, we give sufficient conditions under which eigenfunctions belong to all L^p spaces.

This article is organized as follows. In Section 2 we recall basic information about metric graphs and Schrödinger operators on them. Section 3 is the technical core of the paper. In Section 4 we prove the main results while Section 5 is dealing with some consequences of the main results.

2. Metric graphs and Schrödinger operators

Let us consider a graph $\Gamma = (E, V)$ with countably infinite sets of edges E and vertices V. We allow loops and multiple edges, and assume that the graph is connected, i.e., any two vertices are terminal vertices of a path of edges. Recall that the degree $\deg(v)$ of a vertex $v \in V$ is the number of edges emanating from v. We assume that all vertices of Γ have finite degrees which are positive due to the connectedness of Γ. For any vertex $v \in V$ we denote by E_v the set of edges adjacent to v.

The graph Γ is said to be a *metric graph* if each edge e is identified with an interval $[0, l_e]$ of real line. We always assume that there exist two positive constants $\underline{l}$ and $\bar{l}$ such that

$$\underline{l} \le l_e \le \bar{l} \tag{2.1}$$

for all $e \in E$. If $e \in E$, we denote by x_e the induced coordinate of e (we often skip the index e in this notation). The same symbol x is often used for a point on Γ.

The distance $d(x, y)$ between two points x and y in Γ is defined as the length of a shortest path that connects these points. Furthermore, there is a natural measure, dx, on Γ which coincides with the Lebesgue measure on each edge. Thus, Γ is a non-compact metric measure space. We fix an arbitrary vertex $o \in V$ considered as an origin and set

$$d(x) = d(x, o)\,. \tag{2.2}$$

We utilize the standard notation $L^p(\Gamma)$, $1 \le p \le \infty$, for the Lebesgue spaces on Γ with respect to the measure dx. The norm in a Banach space E is denoted by $\|\cdot\|_E$, and we set $\|\cdot\| = \|\cdot\|_{L^2}$. The space $L^p_{\rm loc}(\Gamma)$, $1 \le p \le \infty$, consists of all measurable functions f on Γ such that $f|_e \in L^p(e)$ for all $e \in E$.

The Sobolev space $H^1(\Gamma)$ consists of all *continuous* complex valued functions f on Γ such that $f|_e \in H^1(e)$ for all edges $e \in E$ and

$$\|f\|^2_{H^1} = \sum_{e\in E} \|f\|^2_{H^1(e)} < \infty\,.$$

For every function $f \in H^1(\Gamma)$ we have $f(x) \to 0$ as $x \to \infty$ in the sense that $d(x) \to \infty$. Furthermore, there is a continuous, dense embedding $H^1(\Gamma) \subset L^p(\Gamma)$ if $p \geq 2$.

The space $BS(\Gamma)$ of Stepanov bounded functions (known also under the name uniform L^1 space [18]) consists of all functions $f \in L^1_{\rm loc}(\Gamma)$ such that

$$\|f\|_{BS} = \sup_{e\in E} \|f\|_{L^1(e)} < \infty \,.$$

We need the following inequality (see [2, Lemma 2.1]). For every $\varepsilon > 0$,

$$\int_\Gamma |f(x)||u(x)|^2 dx \leq \|f\|_{BS} \left(\varepsilon \|u'\|^2 + (\varepsilon^{-1} + \underline{l}^{-1})\|u\|^2\right) , \tag{2.3}$$

whenever $f \in BS(\Gamma)$ and $u \in H^1(\Gamma)$.

Let $V(x)$ be a real function on Γ. Throughout this paper we accept the following assumption

(A1) The function V is locally integrable on Γ and $V_- \in BS(\Gamma)$.

Here and thereafter we use the notation $a_+ = \max[a, 0]$ and $a_- = -\min[a, 0]$.

We consider the Schrödinger operator L associated with the differential expression,

$$\mathcal{L} = -\frac{d^2}{dx^2} + V(x)$$

together with certain vertex conditions. The domain $D(L)$ of L consists of all $u \in L^2(\Gamma)$ such that u and u' are absolutely continuous on each edge of Γ (hence, $u'' \in L^1_{\rm loc}(\Gamma)$),

$$u \text{ is continuous at all vertices of } \Gamma\,, \tag{2.4}$$

$$\sum_{e\in E_v} \frac{du}{dn_e}(v) = 0 \tag{2.5}$$

for all vertices $v \in V$, where $\frac{d}{dn_e}$ stands for the outward derivatives at the endpoints of the edge e, and $\mathcal{L}u \in L^2(\Gamma)$. Then the action of L is defined by $Lu = \mathcal{L}u$ for all $u \in D(L)$. As shown in [2], L is a densely defined, self-adjoint operator in $L^2(\Gamma)$. Furthermore, L is bounded below and $D(L) \subset H^1(\Gamma)$. Notice that conditions (2.4) and (2.5) are called *Kirchhoff vertex conditions*. Alternatively, the operator L can be defined in terms of quadratic forms [2].

Note that the distance function d is not smooth and does not satisfy the Kirchhoff vertex conditions. To overcome this difficulty, we need the following lemma (see [3, Lemma 4.1]).

Lemma 2.1. *There exists a function $\eta \in C(\Gamma \times \Gamma)$ such that for every $y \in \Gamma$ the function $\eta(\cdot, y)$ belongs to $C^2(e)$ on each edge e, its first and second derivatives with respect to the first variable are bounded on Γ uniformly with respect to y, η satisfies the Kirchhoff vertex conditions with respect to the first variable, and*

$$d(x, y) - c_0 \leq \eta(x, y) \leq d(x, y) + c_0\,, \quad (x, y) \in \Gamma \times \Gamma\,, \tag{2.6}$$

with $c_0 > 0$ independent of (x, y).

To abbreviate we set $\eta(x) = \eta(x, o)$.

3. Preliminary results

First we provide a corrected version of [7, Theorem 10, Section 3] (no proof is given there).

Lemma 3.1. *Let A be a self-adjoint operator in a Hilbert space H, with the domain $D(A)$, $\lambda_0 \in \mathbb{R}$, and $\delta > 0$. The spectral subspace of A that corresponds to the interval $[\lambda_0 - \delta, \lambda_0 + \delta]$ is infinite dimensional if and only if there exists a sequence $u_n \in D(A)$ such that $\|u_n\| = 1$, $u_n \to 0$ weakly in H and $\|Au_n - \lambda_0 u_n\| \le \delta$ for all n.*

Proof. Without loss, we may assume that $\lambda_0 = 0$.

(a) Sufficiency. Let $\Delta = [-\delta, \delta]$. Assume that $\dim E(\Delta)H = \infty$. Then there exists an orthonormal sequence $u_n \in D(A) \cap E(\Delta)H$ such that $u_n \to 0$ weakly and

$$\|Au_n\|^2 = \int_\Delta \lambda^2 d(E(\lambda)u_n, u_n) \le \int_\Delta \delta^2 d(E(\lambda)u_n, u_n) = \delta^2 \|u_n\|^2 = \delta^2.$$

(b) Necessity. Suppose the contrary. Then $\sigma(A) \cap \Delta$ consists of finite number of isolated eigenvalues of finite multiplicity. Therefore, there exists $\delta_1 > \delta$ such that

$$\sigma(A) \cap \Delta_1 = \sigma(A) \cap \Delta\,,$$

where $\Delta_1 = [-\delta_1, \delta_1]$. Then

$$\begin{aligned}
\delta_1^2 &= \delta_1^2 \|u_n\|^2 \\
&= \int_{\Delta_1} \delta_1^2 d(E(\lambda)u_n, u_n) + \int_{\mathbb{R}\setminus\Delta_1} \delta_1^2 d(E(\lambda)u_n, u_n) \\
&\le \int_{\Delta_1} \delta_1^2 d(E(\lambda)u_n, u_n) + \int_{\mathbb{R}\setminus\Delta_1} \lambda^2 d(E(\lambda)u_n, u_n) \\
&\le \int_{\Delta_1} \delta_1^2 d(E(\lambda)u_n, u_n) + \int_{\mathbb{R}} \lambda^2 d(E(\lambda)u_n, u_n) \\
&= \delta_1^2 \|E(\Delta_1)u_n\|^2 + \|Au_n\|^2 \\
&= \delta_1^2 \|E(\Delta)u_n\|^2 + \|Au_n\|^2 \\
&\le \delta_1^2 \|E(\Delta)u_n\|^2 + \delta^2.
\end{aligned}$$

Since $u_n \to 0$ weakly and $E(\Delta)H$ is finite dimensional, then $\|E(\Delta)u_n\| \to 0$. Passing to the limit, we obtain that $\delta_1 \le \delta$, a contradiction. $\square$

Remark 3.2. We recall an easy consequence of the spectral theorem. If A is a self-adjoint operator and $\sigma(A) \cap [\lambda_0 - \delta, \lambda_0 + \delta] = \emptyset$, then

$$\|Au - \lambda_0 u\| > \delta \|u\|$$

for all $u \in D(A)$, $u \neq 0$.

Given $x_0 \in \Gamma$ and $R > 0$, we introduce balls

$$B(x_0, R) = \{x \in \Gamma : d(x, x_0) \le R\}$$

and quasi-balls

$$\Omega(x_0, R) = \{x \in \Gamma : \eta(x, x_0) \le R\}.$$

If $x_0 = 0$, we use the abbreviations $B(R)$ and $\Omega(R)$, respectively. For $R > c_0$, inequality (2.6) implies that

$$B(x_0, R - c_0) \subset \Omega(x_0, R) \subset B(x_0, R + c_0). \tag{3.1}$$

A solution of equation

$$-u'' + V(x)u - \lambda u = 0, \tag{3.2}$$

is a function u on Γ such that u and u' are absolutely continuous on each edge of Γ and (3.2) holds almost everywhere.

Lemma 3.3. *Let $R_1 < R$ and $x_0 \in \Gamma$. If u is a solution of (3.2) that satisfies the Kirchhoff vertex conditions, then*

$$\int_{\Omega(x_0,R_1)} |u'(x)|^2 dx \le C(\|(V-\lambda)^-\|_{BS} + 1)^2 \int_{\Omega(x_0,R)} |u(x)|^2 dx, \tag{3.3}$$

where $C > 0$ depends on $R - R_1$ but not on x_0 and λ.

Proof. Without loss of generality, we assume that $\lambda = 0$. Let $\psi(r)$, $r \in \mathbb{R}$, be a smooth function such that $0 \le \psi(r) \le 1$ for all $r \in \mathbb{R}$, $\psi(r) = 1$ for $r \le R_1$, $\psi(r) = 0$ for $r \ge R$, and $|\psi'(r)|$ and $|\psi''(r)|$ are bounded by a constant that depends only on $R - R_1$. By Lemma 2.1, the function

$$\varphi(x) = \psi(\eta(x, x_0))$$

is smooth on every edge of Γ and satisfies Kirchhoff vertex conditions (2.4) and (2.5).

Since $u(x)$ and $\varphi^2(x)u(x)$ satisfy the Kirchhoff condition, and $\operatorname{supp}\varphi^2 u \subset \Omega(x_0, R)$, integration by parts implies

$$\begin{aligned} 0 = \int_\Gamma (\mathcal{L}u)(\varphi^2 u)dx &= \int_\Gamma \big\{u'(\varphi^2 u)' + V(x)(\varphi u)^2\big\}dx \\ &= \int_\Gamma \big\{(\varphi^2)'uu' + \varphi^2(u')^2 + V(x)(\varphi u)^2\big\}dx \end{aligned} \tag{3.4}$$

and

$$\int_\Gamma (\varphi^2)'uu'dx = \frac{1}{2}\int_\Gamma (\varphi^2)'(u^2)'dx = -\frac{1}{2}\int_\Gamma (\varphi^2)''u^2 dx\,. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\int_\Gamma \varphi^2(u')^2 dx = \frac{1}{2}\int_\Gamma (\varphi^2)''u^2 dx - \int_\Gamma V(x)(\varphi u)^2 dx\,.$$

Hence,

$$\|\varphi u'\|^2 \le \frac{1}{2}\|\,|(\varphi^2)''|^{1/2}u\|^2 + \int_\Gamma V^-(x)(\varphi u)^2 dx\,. \tag{3.6}$$

By inequality (2.3), for any $\varepsilon > 0$ the integral in the right-hand side of (3.6) is bounded above by

$$\|V^-\|_{BS}(\varepsilon\|\varphi u' + \varphi' u\|^2 + (\varepsilon^{-1} + \underline{l}^{-1})\|\varphi u\|^2)\,.$$

Taking $\varepsilon = 1/(4\|V^-\|_{BS})$, using the inequality

$$\|\varphi u' + \varphi' u\|^2 \le 2(\|\varphi u'\|^2 + \|\varphi' u\|^2)\,,$$

and estimating $\|\varphi' u\|$ and $\||(\varphi^2)''|^{1/2}u\|$ in terms of

$$\int_{\Omega(x_0,R)} |u(x)|^2 dx\,,$$

from (3.6), we obtain

$$\|\varphi u'\|^2 \le C(\|V^-\|_{BS} + 1)^2 \int_{\Omega(x_0,R)} |u(x)|^2 dx\,.$$

Since the left-hand side of (3.3) does not exceed $\|\varphi u'\|^2$, the result follows. ☐

Also we need an estimate of type (3.3) on quasi-annuli

$$\Omega(x_0, R', R) = \{x \in \Gamma : R' \le \eta(x, x_0) \le R\}\,.$$

Lemma 3.4. *Let $R' < R_1' < R_1 < R$ and $x_0 \in \Gamma$. If u is a solution of (3.2) that satisfies the Kirchhoff vertex conditions, then*

$$\int_{\Omega(x_0,R_1',R_1)} |u'(x)|^2 dx \le C(\|(V-\lambda)^-\|_{BS} + 1)^2 \int_{\Omega(x_0,R',R)} |u(x)|^2 dx,$$

where $C > 0$ depends on $R - R_1$ and $R_1' - R'$ but not on x_0 and λ.

Proof. We follow the same arguments as in the proof of Lemma 3.3. The main difference is that now we choose a smooth function $\psi(r)$, $r \in \mathbb{R}$, such that $0 \le \psi(r) \le 1$ for all $r \in \mathbb{R}$, $\psi(r) = 1$ if $R_1' \le r \le R_1$ and $\psi(r) = 0$ if either $r \le R'$ or $r \ge R$. The function ψ can be chosen in such a way that its first and second derivatives are bounded by a constant that depends only on $R - R_1$ and $R_1' - R'$. Then we use the test function $\varphi^2(x)u(x)$, where $\varphi(x) = \psi(\eta(x))$. ☐

4. Exponential estimates

We begin with the exponential decay of eigenfunctions.

Theorem 4.1. *There exists a constant $c > 0$, independent of V, with the following property. If $u \in L^2(\Gamma)$ is an eigenfunction of L associated with an isolated eigenvalue λ of finite multiplicity, and κ is the distance from λ to $\sigma_{\rm ess}(L)$, then for any $\alpha > 0$ such that*

$$\alpha < \ln\Big(1 + \frac{\kappa^2}{c(\|(V-\lambda)^-\|_{BS} + 1)^2}\Big)$$

then

$$|u(x)| \le C_\alpha e^{-\frac{\alpha}{2} d(x)}\,, \quad x \in \Gamma\,, \tag{4.1}$$

for some $C_\alpha > 0$. If, in addition, $\sigma(L)$ is purely discrete, then (4.1) holds for all $\alpha > 0$.

Proof. Without loss of generality, we assume that $\lambda = 0$. Consider the function

$$J(r) = \int_{\Omega^c(r)} |u(x)|^2 dx\,,$$

where the superscript ^{c} stands for the complement of a subset in Γ. Let A be the set of all α such that

$$J(r) \le Ce^{-\alpha r}\,, \quad r > 0\,, \tag{4.2}$$

with some $C = C(\alpha) > 0$, and let $\alpha_0 = \sup A$.

Assume that $\alpha_0 \ne +\infty$. Then for every $\delta > 0$ and every $C > 0$ there exists a sequence $r_n \to \infty$ satisfying

$$J(r_n) > Ce^{-(\alpha_0+\delta)r_n}\,. \tag{4.3}$$

As a consequence, given $\delta > 0$, there exists a sequence $\rho_n \to \infty$ such that

$$J(\rho_n) \le e^{\alpha_0+\delta} J(\rho_n + 1). \tag{4.4}$$

Indeed, if this is not so, then

$$J(r) < e^{-(\alpha_0+\delta)} J(r-1)$$

for all $r > r_0$. Iterating this inequality, we obtain that

$$J(r) < Ce^{-(\alpha_0+\delta)r}$$

for some $C > 0$ which contradicts (4.3).

Now we choose a smooth function φ on $\mathbb{R}$ such that $0 \le \varphi(r) \le 1$ for all $r \in \mathbb{R}$ and

$$\varphi(r) = \begin{cases} 0 & \text{for } r \le 1/4 \\ 1 & \text{for } r \ge 3/4 \end{cases}$$

and set $\varphi_n(x) = \varphi(\eta(x) - \rho_n)$. Then we define the functions $u_n(x) = \varphi_n(x)u(x)$ and $v_n(x) = \|u_n\|^{-1}u_n(x)$. By Lemma 2.1, both functions u_n and v_n satisfy the Kirchhoff vertex conditions. Notice that $\|v_n\| = 1$ and $\operatorname{supp} v_n \subset \Omega^c(\rho_n + 1/4)$. Hence, $v_n \to 0$ weakly in $L^2(\Gamma)$. It is easily seen that

$$\|u_n(x)\|^2 \ge J(\rho_n + 1) \ge e^{-(\alpha_0+\delta)} J(\rho_n)\,. \tag{4.5}$$

On the other hand,

$$Lu_n = -\varphi'' u_n - 2\varphi' u_n'\,.$$

Therefore,

$$\|Lu_n\| \le C_1\Big\{ \int_{\Omega(\rho_n+\frac{1}{4},\rho_n+\frac{3}{4})} |u'|^2 dx \Big\}^{1/2} + C_2\Big\{ \int_{\Omega(\rho_n,\rho_n+1)} |u|^2 dx \Big\}^{1/2}.$$

By Lemma 3.4 and inequalities (4.4) and (4.5),

$$\begin{aligned}
\|Lu_n\|^2 &\le a \int\limits_{\Omega(\rho_n,\rho_n+1)} |u|^2 dx \\
&= a[J(\rho_n) - J(\rho_{n+1})] \le a[J(\rho_n) - e^{-(\alpha_0+\delta)} J(\rho_n)] \\
&= aJ(\rho_n)\{1 - e^{-(\alpha_0+\delta)}\} \le ae^{(\alpha_0+\delta)}\{1 - e^{-(\alpha_0+\delta)}\}\|u_n\|^2 \\
&\le a\{e^{\alpha_0+\delta} - 1\}\|u_n\|^2\,,
\end{aligned}$$

where $a = c(\|V^-\|_{BS} + 1)^2$. Hence,

$$\|Lv_n\| \le a(e^{\alpha_0+\delta} - 1)\,.$$

Using Lemma 3.1, we conclude that

$$\kappa^2 \le a(e^{\alpha_0+\delta} - 1)\,.$$

Since $\delta > 0$ is arbitrary, it follows that $\kappa^2 \le a(e^{\alpha_0} - 1)$ and

$$\alpha_0 \ge \ln\Big(1 + \frac{\kappa^2}{a}\Big)\,.$$

As consequence, for any $\alpha < \ln\big(1+\frac{\kappa^2}{a}\big)$ there exists a constant $C = C(\alpha)$ such that

$$\int_{\Omega^c(r)} |u(x)|^2 dx \le Ce^{-\alpha r} \tag{4.6}$$

for all $r \ge 0$ provided that $\alpha_0 < \infty$. If $\alpha_0 = \infty$, inequality (4.6) holds trivially. Finally, the previous argument shows that if $\kappa = \infty$, then $\alpha_0 = \infty$, and inequality (4.6) holds in all possible cases.

Now we show that integral estimate (4.6) implies uniform decay of (4.1). The inclusion $\Omega(r) \subset B(r+c_0)$ implies that $B^c(r+c_0) \subset \Omega^c(r)$. As consequence,

$$\int_{B^c(r+c_0)} |u(x)|^2 dx \le C e^{-\alpha r} \tag{4.7}$$

for all $r \ge 0$. Since, by (3.1),

$$B(x_0, R-c_0) \subset \Omega(x_0,R) \subset B(x_0, R+c_0)\,,$$

we have that, for any $y \in \Gamma$,

$$B(y,\bar{l}) \subset \Omega(y,\bar{l}+c_0) \subset \Omega(y,\bar{l}+c_0+1) \subset B(y,\bar{l}+2c_0+1)\,,$$

where $\bar{l}$ is defined by (2.1). If

$$d(y) = d(y,o) > (r+c_0) + (\bar{l}+2c_0+1) = r+\bar{l}+3c_0+1\,,$$

then $B(y,\bar{l}+2c_0+1) \cap B(r+c_0) = \emptyset$ and, hence,

$$\Omega(y,\bar{l}+c_0+1) \subset B(y,\bar{l}+2c_0+1) \subset B^c(r+c_0)\,.$$

By (4.7),

$$\int_{\Omega(y,\bar{l}+c_0+1)} |u(x)|^2 dx \le C e^{-\alpha r}\,, \tag{4.8}$$

and, by Lemma 3.3,

$$\begin{aligned}
\int_{B(y,\bar{l})} |u'(x)|^2 dx &\le \int_{\Omega(y,\bar{l}+c_0)} |u'(x)|^2 dx \\
&\le C_1 \int_{\Omega(y,\bar{l}+c_0+1)} |u(x)|^2 dx \\
&\le C_2 e^{-\alpha r}\,.
\end{aligned} \tag{4.9}$$

Since all edges have length less than or equal to $\bar{l}$, there is an edge $e \subset B(y,\bar{l})$ that contains y. By (4.9),

$$\int_e |u'(x)|^2 dx \le C_2 e^{-\alpha r}\,,$$

while (4.6) yields

$$\int_e |u(x)|^2 dx \le C e^{-\alpha r}\,.$$

Hence,

$$\|u\|^2_{H^1(e)} \le C_3 e^{-\alpha r}\,.$$

Since the length l_e satisfies $l_e \ge \underline{l} > 0$, then the embedding constant of $H^1(e) \subset L^\infty(e)$ is independent of l_e. As consequence,

$$|u(y)| \le C_4 e^{-\alpha r/2}\,.$$

Now, we take $y \in \Gamma$ such that

$$\rho = d(y) = r + \bar{l} + 3c_0 + 2\,.$$

Then $r = \rho - \bar{\lambda} - 3c_0 - 2$, and

$$|u(y)| \le \widetilde{C} e^{-\alpha\rho/2}\,,$$

where

$$\widetilde{C} = C_4 e^{\frac{\alpha}{2}(\bar{l}+3c_0+2)}\,.$$

This completes the proof. $\square$

Corollary 4.2. *Assume that the spectrum of L is purely discrete. If u is an eigenfunction of L, then for any $\alpha > 0$ there exists $C_\alpha > 0$ such that*

$$|u(x)| \le C_\alpha e^{-\alpha d(x)}, \quad x \in \Gamma .$$

Now we provide an estimate for the distance between $\lambda \in \mathbb{R}$ and the spectrum $\sigma(L)$ in terms of solutions to equation (3.2).

Theorem 4.3. *There exists a constant $c > 0$, independent of V, with the following property. Suppose that $u \ne 0$ is a solution of equation (3.2) on Γ that satisfies the Kirchhoff vertex conditions and*

$$\int_{B(r)} |u(x)|^2 dx \le Ce^{\alpha r} \tag{4.10}$$

for some $\alpha > 0$ and $C > 0$, then the distance of the point λ from $\sigma(L)$ does not exceed

$$c(\|(V-\lambda)^-\|_{BS} + 1)(e^\alpha - 1)^{1/2} .$$

In particular, if (4.10) holds for all $\alpha > 0$ with $C = C_\alpha > 0$, then $\lambda \in \sigma(L)$.

Proof. Without lost of generality assume that $\lambda = 0$. By (3.1),

$$\int_{B(r-c_0)} |u(x)|^2 dx \le \int_{\Omega(r)} |u(x)|^2 dx \le \int_{B(r+c_0)} |u(x)|^2 dx .$$

Hence, the function

$$J(r) = \int_{\Omega(r)} |u(x)|^2 dx$$

satisfies $J(r) \le Ce^{\alpha r}$ for some $C > 0$.

For any given $\delta > 0$, there exists a sequence $\rho_n \to \infty$ such that

$$J(\rho_{n+1}) < e^{\alpha+\delta} J(\rho_n) . \tag{4.11}$$

If not, then

$$J(r) > e^{\alpha+\delta} J(r-1)$$

for all sufficiently large r. Iterating this inequality, we obtain that

$$J(r) \ge Ce^{(\alpha+\delta)r} ,$$

with $C > 0$, which is incompatible with (4.10).

As in the proof of Theorem 4.1, we choose a smooth function φ on $\mathbb{R}$ such that $0 \le \varphi(r) \le 1$ for all $r \in \mathbb{R}$ and

$$\varphi(r) = \begin{cases} 0 & \text{for } r \le 1/4 \\ 1 & \text{for } r \ge 3/4 , \end{cases}$$

and set $\varphi_n(x) = \varphi(\eta(x) - \rho_n)$. Then we define the function

$$u_n(x) = \{1 - \varphi_n(x)\} u(x) .$$

Note that $\operatorname{supp} u_n \subset \Omega^c(\rho_n + 1/4)$, and u_n satisfies the Kirchhoff conditions. It is easily seen that

$$\|u_n(x)\|^2 \ge J(\rho_n) . \tag{4.12}$$

On the other hand,

$$Lu_n = -\varphi_n'' u - 2\varphi_n' u'$$

and, hence,

$$\|Lu_n\| \le C_1\Big\{\int_{\Omega(\rho_n+\frac14,\rho_n+\frac34)} |u'|^2 dx\Big\}^{1/2} + C_2\Big\{\int_{\Omega(\rho_n,\rho_n+1)} |u|^2 dx\Big\}^{1/2}.$$

Using Lemma 3.4 and inequalities (4.11) and (4.12), we obtain

$$\begin{aligned}\|Lu_n\|^2 &\le C(\|V^-\|_{BS}+1)^2 \int_{\Omega(\rho_n,\rho_{n+1})} |u(x)|^2 dx,\\ &= C(\|V^-\|_{BS}+1)^2 \{J(\rho_{n+1}) - J(\rho_n)\}\\ &\le C(\|V^-\|_{BS}+1)^2 (e^{\alpha+\delta}-1) J(\rho_n)\\ &\le C(\|V^-\|_{BS}+1)^2 (e^{\alpha+\delta}-1)\|u_n\|^2.\end{aligned}$$

Thus,

$$\|Lu_n\| \le c(\|V^-\|_{BS}+1)(e^{\alpha+\delta}-1)^{1/2}\|u_n\|,$$

where $c=\sqrt{C}$. From this inequality and Remark 3.2, it follows that the distance to the point λ from $\sigma(L)$ does not exceed

$$c(\|V^-\|_{BS}+1)(e^{\alpha+\delta}-1)^{1/2},$$

and since the number $\delta > 0$ is arbitrary, the result follows. □

5. Applications

In this section we make an additional assumption. Namely, we assume that there exist $\mu > 0$ and $C_\mu > 0$ such that for all $r > 0$,

$$|B(r)| \le C_\mu e^{\mu r}, \tag{5.1}$$

where $|S|$ is the measure of $S \subset \Gamma$. The infimum of all such μ is denoted by μ_0. If $\mu_0 > 0$, the graph Γ is of *exponential growth*. Otherwise, if $\mu_0 = 0$, then Γ is of *sub-exponential growth*.

Let u be a continuous function on Γ such that

$$|u(x)| \le C_\alpha e^{-\alpha d(x)}$$

with positive constants α and C_α. If $p \in [0,\infty)$ and $\mu < \alpha p$, with μ from inequality (5.1), then

$$\int_\Gamma |u(x)|^p dx \le \sum_{n=1}^\infty \int_{B(n)\setminus B(n-1)} |u(x)|^p dx \le C\sum_{n=1}^\infty e^{-(\alpha p-\mu)n} < \infty,$$

and, hence, $u \in L^p(\Gamma)$. Together with Theorem 4.1, this implies the following results.

Corollary 5.1. *Assume that Γ is of sub-exponential growth. If $u \in L^2(\Gamma)$ is an eigenfunction associated with an isolated eigenvalue of finite multiplicity, then $u \in L^p(\Gamma)$ for all $p \in [1,\infty]$.*

Corollary 5.2. *Assume that Γ is of exponential growth and the spectrum of L is purely discrete. If $u \in L^2(\Gamma)$ is any eigenfunction of L, then $u \in L^p(\Gamma)$ for all $p \in [1,\infty]$.*

Note that these statements are non-trivial only in the case when $p \in [1,2)$ because $u \in H^1(\Gamma) \subset L^p(\Gamma)$ if $p \in [2,\infty]$. The following statement is an easy consequence of Theorem 4.3.

Corollary 5.3. *Let $u \neq 0$ be a solution of* (3.2) *on* Γ *that satisfies the Kirchhoff vertex conditions and, for some $\beta > 0$ and $C_b > 0$,*

$$|u(x)| \leq C_\beta e^{\beta d(x)}\,,$$

and β_0 is the infimum of all such β. Then the distance of the point λ from $\sigma(L)$ does not exceed

$$c(\|(V-\lambda)^-\|_{BS}+1)(e^{2\beta_0+\mu_0}-1)^{1/2}\,,$$

where the constant $c > 0$ is independent of V. In particular, if $\beta_0 = \mu_0 = 0$, then $\lambda \in \sigma(L)$.

In Corollary 5.3, $\beta_0 = \mu_0 = 0$ means that both the graph Γ and the solution u are of sub-exponential growth. Also we point out that if u is bounded, then $\beta_0 = 0$.

Acknowledgements. A. Pankov was supported by Simons Foundation, Award 410289.

References

[1] S. Agmon; *Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators*, Mathematical Notes 29, Princeton University Press, Princeton, NJ, 1982.

[2] S. Akduman, A. Pankov; *Schrödinger operators with locally integrable potentials on infinite metric graphs*, Applicable Analysis, **96** (2017), 2149-2161 .

[3] S. Akduman, A. Pankov; *Exponential decay of eigenfunctions of Schrödinger operators on infinite metric graphs*, Complex Variables and Elliptic Equations, **62** (2017), 957-966.

[4] G. Berkolaiko, P. Kuchment; *Introduction to Quantum Graphs*, Amer. Math. Soc., Providence RI, 2013.

[5] H. Brezis; *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.

[6] J. Blank, P. Exner, M. Havliček; *Hilbert Space Operators in Quantum Physics*, Springer, New York, 2008.

[7] I. Glazman; *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators*, Israel Progr. Sci. Translations, Jerusalem, 1965.

[8] M. E. Harrell, A. V. Maltsev; *On Agmon metrics and exponential localization for quantum graphs*, Commun. Math. Phys., (2018), doi: 10.1007/s00220-018-3124-x .

[9] E. M. Harrell, A. V. Maltsev; *Localization and landscape functions on quantum graphs*, arXiv: 1803.01186.

[10] P. D. Hislop; *Exponential decay of two-body eigenfunctions: A review*, Electr. J. Differ. Equat., Conf. 04 (2000), 265–288.

[11] P. Kuchment; *Quantum graphs: I, Some basic structures*, Waves Random Media, **14** (2004), S107–S128.

[12] P. Kuchment; *Quantum grephs: II, Some spectral properties for infinite and combinatorial graphs*, J. Phys. A: Math. Gen., **38** (2005), 4887–4900.

[13] D. Mugnolo; *Semigroup Methods for Evolution Equations on Networks*, Springer, Cham, 2015.

[14] Y. V. Pokornyi, V. L. Pryadiev; *Some problems in the qualitative Sturm-Liouville theory an a spatial network* (Russian), Uspekhi Mat. Nauk, **59**, No 3, 115–150 (2004); English translation: Russian Math. Surveys, **59** (2004), 515–552.

[15] Y. V. Pokornyi, O. M. Penkin, V. L. Pryadiev, A. V. Borovkikh, K. P. Lazarev, S. A. Shabrov; *Differential Equations on Geometric Graphs* (Russian), Fizmatlit, Moscow, 2005.

[16] C. R. Putnam; *On isolated eigenfunctions associated with bounded potentials*, Amer. J. Math., **72** (1950), 135–147.

[17] M. Reed, B. Simon; *Methods of Modern Mathematical Physics: II, Fourier Analysis, Self-Adjointness*, Academic Press, San Diego, 1975.

[18] B. Simon; *Schrödinger semigroupes*, Bull. Amer. Math. Soc., **7** (1982), 447–526.

[19] K. Schmüdgen; *Unbounded Self-adjoint Operators on Hilbert Space*, Springer, Dordrecht, 2012.

[20] E. Schnol; *On the behaavior of eigenfunctions of the Schrödinger equations*, Mat. Sbornik, **42** (1957), 273–285 (in Russian).
[21] A. Wintner; *On the smallness of isolated eigenfunctions*, Amer. J. Math., **71** (1949), 603–611.

Setenay Akduman
Department of Mathematics, Izmir Democracy University, Izmir, 35140, Turkey
E-mail address: `setenay.akduman@idu.edu.tr`

Alexander Pankov
Department of Mathematics, Morgan State University, Baltimore, MD 21251, USA
E-mail address: `alexander.pankov@morgan.edu`