Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 166, pp. 1–27.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

LINEAR ELLIPTIC AND PARABOLIC PDEs WITH NONLINEAR MIXED BOUNDARY CONDITIONS AND SPATIAL HETEROGENEITIES

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Communicated by Ratnasingham Shivaji

Abstract. This article concerns the positive solutions of a boundary-value problem constituted by a linear elliptic partial differential equation, subject to nonlinear mixed boundary conditions containing spatial heterogeneities with arbitrary sign along the boundary. The results obtained in this work provide us the global bifurcation diagram of positive solutions, the pointing behavior of them when the parameters change and the dynamics of the positive solutions of the associated parabolic problem. The main contribution of this paper is to give general results about existence, uniqueness, stability and pointing behavior of positive solutions, for boundary-value problems with nonlinear boundary conditions of mixed type containing spatial heterogeneities. The main technical tools used to develop the mathematical analysis are local and global bifurcation, monotonicity techniques, the Characterization of the Strong Maximum Principle given by Amann and López-Gómez [5], blow up arguments and some of the techniques used in the previous works [19, 20, 33, 34]. The results obtained in this paper are the natural continuation of the previous ones in [11].

1. Introduction

In this article we consider the boundary-value problem with nonlinear mixed boundary conditions and spatial heterogeneities given by

$$\begin{gathered} -\Delta u = \lambda u \quad \text{in } \Omega\,, \\ u = 0 \quad \text{on } \Gamma_0\,, \\ \partial u + V(x)u = \gamma b(x) u^q \quad \text{on } \Gamma_1\,, q>1\,, \end{gathered} \tag{1.1}$$

where:

(i) Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$ of class $\mathcal{C}^2$, with boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are disjoint open and closed subsets of $\partial\Omega$;
(ii) $-\Delta$ stands for the minus Laplacian operator in $\mathbb{R}^N$ and $\lambda \in \mathbb{R}$ is the bifurcation parameter;

2010 *Mathematics Subject Classification.* 35J65, 35J25,35B09, 35B35, 35B40.
Key words and phrases. Nonlinear mixed boundary conditions; positive solutions; spatial heterogeneities; nonlinear flux with arbitrary sign; blow up in finite time; elliptic and parabolic boundary value problems.

Submitted January 12, 2018. Published September 12, 2018.

(iii) the spatial heterogeneities on the boundary come given by the potentials $V, b \in \mathcal{C}(\Gamma_1)$, where $b > 0$ on Γ_1 and V possesses arbitrary sign in each point $x \in \Gamma_1$;
(iv) $\partial u(x)$ stands for the outer normal derivative of u at $x \in \Gamma_1$, and $\gamma \in \mathbb{R}$.

This work is devoted to analyzing the structure of the set of positive solutions of (1.1) depending on the sign of the parameter $\gamma \in \mathbb{R}$ on the nonlinear mixed boundary conditions, to ascertain the pointing behavior of positive solutions of (1.1) when $\gamma < 0$ and λ changes, and to obtain the dynamics of the positive solutions of the associated parabolic problem to (1.1) depending on the values of the bifurcation parameter $\lambda \in \mathbb{R}$ and on the sign of the parameter $\gamma \in \mathbb{R}$.

By a *positive solution* of (1.1) we will mean any couple $(\lambda, u) \in \mathbb{R} \times W_p^2(\Omega)$ for some $p > N$, with $u > 0$ in Ω satisfying (1.1). It should be noted that $W_p^2(\Omega) \subset \mathcal{C}^{2-\frac{N}{p}}(\bar{\Omega})$ and that any function $u \in W_p^2(\Omega)$, $p > N$ is a.e. twice differentiable (cf. [29, Theorem VIII.1]). We will say that a positive solution (λ, u) of (1.1) is *strongly positive* in Ω, and we will denote it by $u \gg 0$, if $u(x) > 0$ for all $x \in \Omega \cup \Gamma_1$ and $\partial u(x) < 0$ for all $x \in \Gamma_0$.

In the particular case when $\gamma = 0$, (1.1) becomes in a linear boundary-value problem which exhibits vertical bifurcation to positive solutions from the trivial branch $(\lambda, u) = (\lambda, 0)$ at a unique value of λ which will be denoted by σ_1. The results obtained along this work will show that the sign of the parameter γ plays a crucial role in our problem. Indeed, we will see that although the partial differential equation of (1.1) is linear, in the particular case when $\gamma < 0$ the structure of the set of positive solutions of (1.1) is a typical structure of a sublinear problem, whereas in the particular case when $\gamma > 0$ it is the typical structure of a superlinear problem. The same occurs in the study of the stability of the positive solutions of (1.1).

The main techniques used to carry out the mathematical analysis are monotonicity techniques, local and global bifurcation, blow up arguments and some of the techniques used in the previous works [33, 34, 19, 20].

Hereinafter, for each $V \in \mathcal{C}(\Gamma_1)$, $\mathfrak{B}(V(x))$ will stand for the boundary operator defined by

$$\mathfrak{B}(V(x))u := \begin{cases} u & \text{on } \Gamma_0\,, \\ \partial u + V(x)u & \text{on } \Gamma_1\,, \end{cases}$$

and $\mathfrak{D}$ the Dirichlet boundary operator on $\partial\Omega$.

From the results in [4, Theorem 12.1] and [5, Theorem 2.2], it is known that for any $K \in \mathcal{C}(\bar{\Omega})$ and $V \in \mathcal{C}(\Gamma_1)$, the boundary eigenvalue problem

$$\begin{gathered} (-\Delta + K(x))\varphi = \sigma\varphi \quad \text{in } \Omega\,, \\ \mathfrak{B}(V(x))\varphi = \bar{0} \quad \text{on } \partial\Omega\,, \end{gathered} \tag{1.2}$$

admits a unique eigenvalue which possesses a positive eigenfunction, unique up multiplicative constant, named *principal eigenvalue* of (1.2). Hereafter we will denote it by $\sigma_1^{\Omega}[-\Delta + K(x), \mathfrak{B}(V(x))]$. Also the principal eigenvalue of (1.2) is simple and dominant in the sense that any other eigenvalue λ of (1.2) satisfies

$$\Re e(\lambda) > \sigma_1^{\Omega}[-\Delta + K(x), \mathfrak{B}(V(x))]\,,$$

where $\mathfrak{Re}(\lambda)$ stands for the real part of λ. In addition, if φ^* stands for the positive eigenfunction of (1.2) associated to $\sigma_1^\Omega[-\Delta+K(x),\mathfrak{B}(V(x))]$, unique up multiplicative constant, then

$$\varphi^* \gg 0 \quad \text{in } \Omega, \tag{1.3}$$

$$\varphi^* \in W^2(\Omega) := \cap_{p>N} W_p^2(\Omega) \subset \mathcal{C}^{1+\alpha}(\bar{\Omega}) \quad \text{for all} \quad \alpha \in (0,1)\,. \tag{1.4}$$

Hereinafter we will denote

$$\sigma_1 := \sigma_1^\Omega[-\Delta, \mathfrak{B}(V(x))]\,,$$

and by $\varphi_1 > 0$, the principal eigenfunction associated to the principal eigenvalue σ_1, normalized so that $\|\varphi_1\|_{L^\infty(\Omega)} = 1$. By (1.3) and (1.4) we obtain

$$\varphi_1 \gg 0 \quad \text{in } \Omega \quad \text{and} \quad \varphi_1 \in \mathcal{C}^{1+\alpha}(\bar{\Omega}) \quad \forall \alpha \in (0,1)$$

Also we will denote

$$\sigma_0 := \sigma_1^\Omega[-\Delta, \mathfrak{D}]\,,$$

that is, the principal eigenvalue of the $-\Delta$ operator in the domain Ω under Dirichlet boundary conditions. Owing to [7, Proposition 3.1] we know that

$$\sigma_1 < \sigma_0 \tag{1.5}$$

Finally, we will denote

$$\mathcal{C}^1_{\Gamma_0}(\bar{\Omega}) := \{u \in \mathcal{C}^1(\bar{\Omega}) : u|_{\Gamma_0} = 0\}$$

As it was mentioned, in the particular case when $\gamma = 0$, (1.1) becomes in the linear boundary-value problem

$$\begin{gathered} -\Delta u = \lambda u \quad \text{in } \Omega \\ \mathfrak{B}(V(x))u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.6}$$

and owing to [4, Theorem 12.1], we know that (1.6) possesses positive solutions if, and only if $\lambda = \sigma_1$. In this case we obtain vertical bifurcation to positive solutions from the trivial branch $(\lambda, u) = (\lambda, 0)$ at $\lambda = \sigma_1$, because all the positive solutions of (1.6) are positive multiple of φ_1, being φ_1 the principal eigenfunction associated to the principal eigenvalue σ_1 of (1.6), normalized so that $\|\varphi_1\|_{L^\infty(\Omega)} = 1$.

There is a big amount of literature about the topics of existence, uniqueness and stability of solutions of elliptic boundary-value problems with nonlinear boundary conditions, and about the dynamics of the solutions of parabolic problems with nonlinear boundary conditions, among others, [3, 6, 8, 9, 10, 11, 12, 13, 15, 16, 17, 22, 25, 26, 27, 30, 31, 32, 35]. The main contribution of this paper, together with the previous works [8, 9, 10, 11, 22], lies in providing general results about the structure of the set of positive solutions and about the stability of them, for very general nonlinear boundary-value problems with nonlinear mixed boundary conditions, containing spatial heterogeneities with arbitrary sign. In particular, in this work, just as in [11], the outer normal derivative of the solution u depends in a nonlinear way of u, and may be positive, negative and vanish in different regions of Γ_1, depending on the sign of $\gamma b(x)u(x)^q - V(x)u(x)$ in each point $x \in \Gamma_1$. In [11] were analyzed the existence, uniqueness and stability of the positive solutions of the semilinear boundary-value problem with nonlinear mixed boundary conditions

$$\begin{gathered} -\Delta u = \lambda u - a(x)u^p \quad \text{in } \Omega\,,\ p > 1 \\ u = 0 \quad \text{on } \Gamma_0\,, \\ \partial u + V(x)u = b(x)u^q \quad \text{on } \Gamma_1\,,\ q > 1 \end{gathered}$$

where the domain Ω and the potentials on the boundary $V, b \in \mathcal{C}^1(\Gamma_1)$ possess similar properties to the considered in this work, and the spatial heterogeneity $a \in \mathcal{C}(\bar{\Omega})$, with $a > 0$, satisfies that either

$$\begin{gathered}\Omega_0 := \operatorname{int}\{x \in \Omega : a(x) = 0\},\ \Omega_0 \in \mathcal{C}^2,\ \bar{\Omega}_0 \subset \Omega, \text{ and } a(x) \text{ is} \\ \text{bounded away from zero in any compact subsets of } (\Omega \setminus \bar{\Omega}_0) \cup \Gamma_1,\end{gathered} \tag{1.7}$$

or

$$a(x) \text{ is bounded away from zero in any compact subset } \Omega \cup \Gamma_1 \tag{1.8}$$

The results obtained in the current work are, in some sense, the natural continuation of the results in [11], to cover the case therein when $\Omega_0 = \Omega$, that is, when the potential $a = 0$ in Ω. Clearly, the results obtained in this paper can not be obtained substituting in [11] $a = 0$ in Ω, because the case $\Omega_0 = \Omega$ does not satisfy the assumptions (1.7) neither (1.8) required in [11].

The organizations of this article is as follows: Section 1 is the Introduction. Section 2 contains results about the profile and regularity of the positive solutions of (1.1) and the main results about local and global bifurcation to positive solutions of (1.1) from the trivial branch $(\lambda, u) = (\lambda, 0)$. In Section 3 is carried out a very sharp analysis, in the particular case when $\gamma < 0$, about the global structure of the set of positive solutions of (1.1)(Section 3.1), about the pointing behavior of the positive solutions of (1.1) when $\lambda \uparrow \sigma_0$ (Section 3.2) and about the dynamics of the positive solutions of the associated parabolic problem to (1.1) (Section 3.3). Finally, in Section 4 is analyzed the particular case when $\gamma > 0$, obtaining some results about the dynamics of the positive solutions of the parabolic problem associated to (1.1) (Section 4.1), and some results about the structure of the set of positive solutions of (1.1) (Section 4.2).

2. Regularity and bifurcation of positive solutions to (1.1)

This section contains results about the profile and regularity of the positive solutions of (1.1) and the main results about local and global bifurcation to positive solutions of (1.1) from the trivial branch $(\lambda, u) = (\lambda, 0)$. The next result gives the regularity and profile of the positive solutions of (1.1) and a necessary condition for the existence of them.

Theorem 2.1. *If u_λ is a positive solution of* (1.1) *for the value λ of the parameter, then*

$$\lambda = \sigma_1^\Omega[-\Delta, \mathfrak{B}(V(x) - \gamma b(x) u_\lambda^{q-1})]\,, \tag{2.1}$$

$u_\lambda \in \mathcal{C}^{1+\alpha}(\bar{\Omega})$ for all $\alpha \in (0,1)$, and u_λ is strongly positive in Ω.

Proof. Let u_λ be a positive solution of (1.1) for the value λ. Then, u_λ is a positive function in Ω satisfying the problem

$$\begin{gathered}-\Delta u_\lambda = \lambda u_\lambda \quad \text{in } \Omega\,, \\ u_\lambda = 0 \quad \text{on } \Gamma_0\,, \\ \big(\partial + V(x) - \gamma b(x) u_\lambda^{q-1}\big) u_\lambda = 0 \quad \text{on } \Gamma_1\,,\ q > 1\,;\end{gathered}$$

that is, λ is an eigenvalue of the problem

$$\begin{gathered}-\Delta\theta = \lambda\theta \quad \text{in } \Omega\,, \\ \mathfrak{B}\big(V(x) - \gamma b(x) u_\lambda^{q-1}\big)\theta = 0 \quad \text{on } \partial\Omega\,,\ q > 1\,,\end{gathered} \tag{2.2}$$

and $\theta = u_\lambda$ is a positive eigenfunction of (2.2) associated to the eigenvalue λ. Then, owing to the uniqueness of the principal eigenvalue of (2.2) (cf. (1.2), [4]), we obtain (2.1) and that u_λ is its principal eigenfunction. The remaining assertions of the theorem follow from the structure and regularity of the principal eigenfunction of (2.2) (cf. (1.3), (1.4), [4]). This completes the proof. $\square$

Hereafter, by *continuum* we will mean a closed and connected set. The following theorem collects the main results about bifurcation of positive solutions of (1.1) from the trivial branch $(\lambda, u) = (\lambda, 0)$. It is [10, Theorem 1.1] for the particular case therein when $p = 1$, $q > 1$ and $a = 0$ in Ω.

Theorem 2.2. *The following hold:*

(i) The value $\lambda = \sigma_1$ is the unique bifurcation value to positive solutions of (1.1) *from the trivial branch $(\lambda, u) = (\lambda, 0)$.*

(ii) A differentiable continuum $\mathfrak{C}$ of solutions of (1.1) *emanates from the bifurcation point $(\lambda, u) = (\sigma_1, 0)$ and in a small neighborhood $\mathcal{V}$ of $(\sigma_1, 0)$ in $\mathbf{R} \times \mathcal{C}^1_{\Gamma_0}(\bar{\Omega})$, the unique non-trivial solutions of* (1.1) *belong to $\mathfrak{C}$. In addition,*

$$\mathfrak{C} \cap \mathcal{V} = \{(\lambda, u) = (\sigma_1 + \mu_q(s), s(\varphi_1 + v_q(s))) : s \in (-\varepsilon, \varepsilon)\} \tag{2.3}$$

for $\varepsilon > 0$ small enough, with

$$(\mu_q, v_q) \in \mathcal{C}^1((-\varepsilon, \varepsilon), \mathbf{R} \times \mathcal{C}^1_{\Gamma_0}(\bar{\Omega})), \quad (\mu_q(0), v_q(0)) = (0, 0), \tag{2.4}$$

and $\int_\Omega v_q(s)\varphi_1 = 0$ for all $s \in (-\varepsilon, \varepsilon)$. Furthermore,

$$\lim_{s \to 0} \frac{\mu_q(s)}{s^{q-1}} = \mathcal{D}(q, \gamma), \tag{2.5}$$

where

$$\mathcal{D}(q, \gamma) := -\gamma \int_{\Gamma_1} b(x)\varphi_1^{q+1}. \tag{2.6}$$

Remark 2.3. It should be noted that owing to (2.3), (2.4), (2.5) and (2.6) and since $b > 0$ and $\varphi_1 \gg 0$ in Ω, we obtain the bifurcation to positive solutions from the trivial branch at $\lambda = \sigma_1$ produces for $s \in (0, \varepsilon)$ and it is supercritical if $\gamma < 0$ and subcritical if $\gamma > 0$.

Hereinafter we will denote by $\mathfrak{C}^+$ the maximal subcontinuum of $\mathfrak{C}$ composed by the positive solutions of (1.1) emanating from the trivial branch at $\lambda = \sigma_1$.

Remark 2.4. Since $\lambda = \sigma_1$ is a simple eigenvalue of the linearization of (1.1) at $(\lambda, u) = (\sigma_1, 0)$, and owing to the fact that $(\lambda, u) = (\sigma_1, 0)$ is the unique bifurcation point to positive solutions of (1.1) from the trivial branch, it follows from the updated version of the Global Alternative of Rabinowitz [24, Theorem 1.27] given by López-Gómez in [21, Theorem 6.4.3], that either $\mathfrak{C}^+$ is unbounded in $\mathbb{R} \times \mathcal{C}^1_{\Gamma_0}(\bar{\Omega})$, or it contains a pair $(\tilde{\lambda}, \tilde{u})$ with $\tilde{u} \gg 0$ in Ω satisfying

$$\int_\Omega \tilde{u}\varphi_1 = 0,$$

which is impossible since $\varphi_1 \gg 0$ in Ω. Then, we obtain $\mathfrak{C}^+$ is unbounded in $\mathbb{R} \times \mathcal{C}^1_{\Gamma_0}(\bar{\Omega})$ and by the L_p-estimates, unbounded in $\mathbb{R} \times L^\infty(\Omega)$.

3. The case $\gamma < 0$

In this section, in the particular case when $\gamma < 0$, we will analyze the existence, uniqueness and stability of the positive solutions of (1.1), and we will obtain the structure of the global bifurcation diagram of positive solutions of (1.1), the pointing behavior of the positive solutions of (1.1) when $\lambda \uparrow \sigma_0$ and the dynamics of the positive solutions of the parabolic problem associated to (1.1). Along this section we will denote by $\tilde{\gamma} := -\gamma > 0$, and hence, (1.1) will be written in the form

$$\begin{gathered} -\Delta u = \lambda u \quad \text{in } \Omega\,, \\ u = 0 \quad \text{on } \Gamma_0\,, \\ \partial u + V(x)u + \tilde{\gamma} b(x) u^q = 0 \quad \text{on } \Gamma_1\,,\ q > 1 \end{gathered} \tag{3.1}$$

3.1. **Structure of the set of positive solutions to** (1.1)**.** The following is the main result of this section, which gives the structure of the global bifurcation diagram of positive solutions of (3.1).

Theorem 3.1. *If*

$$b(x) \geq \underline{b} > 0 \quad \textit{for all } x \in \Gamma_1\,, \tag{3.2}$$

then the following hold:

(i) (3.1) *possesses a positive solution if, and only if*

$$\sigma_1 < \lambda < \sigma_0\,. \tag{3.3}$$

(ii) The positive solution of (3.1)*, if it exists, is unique, strongly positive in* Ω*, and linearly and globally asymptotically stable as steady-state of the parabolic problem associated to* (3.1)*. Hereafter we will denote it by* u_λ*.*

(iii) For any $\lambda \in (\sigma_1, \sigma_0)$*,* $\dot{u}_\lambda := \frac{d\,u_\lambda}{d\,\lambda}$ *is strongly positive in* Ω*, that is,*

$$\dot{u}_\lambda(x) > 0 \quad \forall x \in \Omega \cup \Gamma_1 \quad \textit{and} \quad \partial \dot{u}_\lambda(x) < 0 \quad \forall x \in \Gamma_0 \tag{3.4}$$

In particular, for each $x \in \Omega \cup \Gamma_1$*, the map* $(\sigma_1, \sigma_0) \to (0, \infty)$ *defined by*

$$\lambda \mapsto u_\lambda(x) \tag{3.5}$$

is strictly increasing.

(iv) There exist uniform $L^\infty(\Omega)$*-bounds for the positive solutions of* (3.1) *in compact intervals of* λ *contained in* $[\sigma_1, \sigma_0)$*.*

(v) The positive solutions of (3.1) *belong to a differentiable continuum* $\mathfrak{C}^+$ *of positive solutions, which emanates supercritically from the trivial branch at the unique bifurcation value to positive solutions of* (3.1) $\lambda = \sigma_1$*, bifurcates from infinity at* $\lambda = \sigma_0$ *and it is increasing in* $\|\cdot\|_{L^\infty(\Omega)}$ *with the* $\lambda-$*parameter. In particular,*

$$\mathcal{P}_\lambda(\mathfrak{C}^+) = [\sigma_1, \sigma_0), \tag{3.6}$$

$$\lim_{\lambda \downarrow \sigma_1} \|u_\lambda\|_{L^\infty(\Omega)} = 0\,, \quad \lim_{\lambda \uparrow \sigma_0} \|u_\lambda\|_{L^\infty(\Omega)} = \infty\,, \tag{3.7}$$

where $\mathcal{P}_\lambda(\mathfrak{C}^+)$ *denotes the* λ*-projection of the continuum* $\mathfrak{C}^+$ *over the* λ*-axis.*

To prove Theorem 3.1 we need some lemmas. Next result gives a sufficient condition for the existence of a positive strict subsolution of (3.1).

Lemma 3.2. *For each* $\lambda > \sigma_1$*,* (3.1) *possesses a positive strict subsolution arbitrarily small, which is strongly positive in* Ω*.*

Proof. Let $\lambda > \sigma_1$ be. Owing to the monotonicity and continuous dependence of the principal eigenvalue with respect to the potential on the boundary (cf. [7, Proposition 3.5], [7, Theorem 8.2, Remark 8.3]), there exists $\varepsilon > 0$ small enough such that

$$\sigma_1 := \sigma_1^\Omega[-\Delta, \mathfrak{B}(V(x))] < \sigma_1^\Omega[-\Delta, \mathfrak{B}(V(x)+\varepsilon)] < \lambda \tag{3.8}$$

Let us fix $\varepsilon > 0$ satisfying (3.8) and let us denote by

$$\sigma_1^\varepsilon := \sigma_1^\Omega[-\Delta, \mathfrak{B}(V(x)+\varepsilon)]$$

and by φ_ε the principal eigenfunction associated to the principal eigenvalue σ_1^ε, normalized so that

$$\|\varphi_\varepsilon\|_{L^\infty(\Omega)} = 1 \tag{3.9}$$

By construction, φ_ε is strongly positive in Ω and it satisfies the problem

$$\begin{gathered} -\Delta\varphi_\varepsilon = \sigma_1^\varepsilon \varphi_\varepsilon \quad \text{in } \Omega\,, \\ \varphi_\varepsilon = 0 \quad \text{on} \quad \Gamma_0\,, \\ \big(\partial + V(x) + \varepsilon\big)\varphi_\varepsilon = 0 \quad \text{on } \Gamma_1 \end{gathered} \tag{3.10}$$

Now, let us consider the function $\underline{u}_\lambda := \alpha\varphi_\varepsilon$ for $\alpha > 0$ satisfying

$$0 < \alpha < \Big(\frac{\varepsilon}{\tilde{\gamma}\|b\|_{L^\infty(\Gamma_1)}}\Big)^{\frac{1}{q-1}} \tag{3.11}$$

By construction, and thanks to (3.8), (3.9), (3.10) and (3.11), it is easy to see that $\underline{u}_\lambda$ is a positive strict subsolution of (3.1) for any fixed $\alpha > 0$ satisfying (3.11). Moreover, since φ_ε is strongly positive in Ω and $\alpha > 0$, we obtain $\underline{u}_\lambda$ is strongly positive in Ω. This completes the proof. □

The next result gives a sufficient condition for the existence of a positive strict supersolution of (3.1).

Lemma 3.3. *If* (3.2) *holds, then for each*

$$\lambda < \sigma_0 \tag{3.12}$$

Equation (3.1) *possesses a positive strict supersolution arbitrarily large and strongly positive in* Ω

Proof. Let $\lambda < \sigma_0$ be. Owing to the dominance of the principal eigenvalue of the operator $-\Delta$ under Dirichlet boundary conditions (cf. [7, Proposition 3.1, Corollary 9.2]) and to the limiting behavior of the principal eigenvalue $\sigma_1^\Omega[-\Delta, \mathfrak{B}(n)]$ when $n \uparrow \infty$ (cf. [7, Theorem 9.1]), the following hold

$$\sigma_1^\Omega[-\Delta, \mathfrak{B}(n)] < \sigma_0\,, \quad \forall n \in \mathbf{N}\,, \quad \lim_{n\uparrow\infty} \sigma_1^\Omega[-\Delta, \mathfrak{B}(n)] = \sigma_0 \tag{3.13}$$

Then, owing to (3.12) and (3.13), there exists $n \in \mathbf{N}$ large enough such that

$$\lambda < \sigma_1^\Omega[-\Delta, \mathfrak{B}(n)] < \sigma_0 \tag{3.14}$$

Let us fix $n \in \mathbf{N}$ satisfying (3.14) and let us denote by

$$\sigma_1^n := \sigma_1^\Omega[-\Delta, \mathfrak{B}(n)]$$

and by φ_1^n the principal eigenfunction associated to the principal eigenvalue σ_1^n, normalized so that

$$\|\varphi_1^n\|_{L^\infty(\Omega)} = 1 \tag{3.15}$$

Set

$$m_n := \min_{x\in\Gamma_1} \varphi_1^n \tag{3.16}$$

Since φ_1^n is strongly positive in Ω, we obtain $\varphi_1^n(x) > 0$ for all $x \in \Gamma_1$ and hence,

$$m_n > 0 \tag{3.17}$$

By definition φ_1^n satisfies the problem

$$\begin{gathered} -\Delta\varphi_1^n = \sigma_1^n\varphi_1^n \quad \text{in } \Omega\,, \\ \varphi_1^n = 0 \quad \text{on } \Gamma_0\,, \\ (\partial + n)\varphi_1^n = 0 \quad \text{on } \Gamma_1 \end{gathered} \tag{3.18}$$

Now, let us consider the function $\overline{u}_\lambda := \kappa\varphi_1^n$ for $\kappa > 0$ satisfying

$$\kappa > \Big(\frac{\|V(x) - n\|_{L^\infty(\Gamma_1)}}{\tilde\gamma \underline{b} m_n^{q-1}}\Big)^{\frac{1}{q-1}} \tag{3.19}$$

By construction and owing to (3.14), (3.2), (3.16), (3.17), (3.18) and (3.19), it is easy to see that $\overline{u}_\lambda$ is a positive strict supersolution of (3.1) for any $\kappa > 0$ satisfying (3.19). Moreover, since φ_1^n is strongly positive in Ω and $\kappa > 0$, we obtain $\overline{u}_\lambda$ is strongly positive in Ω. This completes the proof. □

Proof of Theorem 3.1. (i) To prove the necessary condition for the existence of positive solution of (3.1), let u_λ be a positive solution of (3.1) for the value λ of the parameter. Then, owing to (2.1), to the dominance of the principal eigenvalue of the operator $-\Delta$ in the domain Ω under Dirichlet boundary conditions (cf. [7, Proposition 3.1]), to the facts that u_λ is strongly positive in Ω, $b > 0$ on Γ_1 and $\tilde\gamma > 0$ and to the monotonicity of the principal eigenvalue with respect to the potential on the boundary conditions (cf. [7, Proposition 3.5]), we obtain

$$\sigma_1 < \lambda = \sigma_1^\Omega[-\Delta, \mathfrak{B}(V(x) + \tilde\gamma b(x)u_\lambda^{q-1})] < \sigma_0\,,$$

which proves (3.3) and ends the proof of the necessary condition for the existence of positive solutions of (3.1).

We now prove the sufficient condition (3.3) for the existence of positive solution of (3.1). Indeed, owing to Lemma 3.2 and Lemma 3.3, for each λ satisfying (3.3), there exist a positive strict subsolution $\underline{u}_\lambda$ of (1.1) arbitrarily small, and a positive strict supersolution $\overline{u}_\lambda$ of (1.1) arbitrarily large, both of them strongly positive in Ω. Thanks to the fact that both of them are strongly positive in Ω, taking $\alpha > 0$ small enough in Lemma 3.2 or $\kappa > 0$ large enough in Lemma 3.3, it is possible to obtain $0 < \underline{u}_\lambda < \overline{u}_\lambda$, and hence, the sub-super solutions method (cf. [2]) implies the existence of a positive solution u_λ of (3.1) with $0 < \underline{u}_\lambda < u_\lambda < \overline{u}_\lambda$, for each λ satisfying (3.3). This completes the proof of *i*).

(ii) To prove the uniqueness of positive solution of (3.1), when it exists, we will argue by contradiction. Let λ be satisfying (3.3) and suppose that u_1 and u_2 are two positive solutions of (3.1) for the value λ of the parameter with

$$u_1 \neq u_2 \tag{3.20}$$

Owing to (2.1) we obtain

$$\lambda = \sigma_1^\Omega\big[-\Delta, \mathfrak{B}\big(V(x) + \tilde\gamma b(x)u_i^{q-1}\big)\big]\,, \quad i = 1, 2 \tag{3.21}$$

Arguing as in [1, Theorem 4.1], set

$$J(t) := (tu_2 + (1-t)u_1)^q\,, \quad t \in [0, 1]$$

By construction we obtain

$$u_2^q - u_1^q = J(1) - J(0) = \int_0^1 J'(t)\,dt = q(u_2 - u_1)\int_0^1 (tu_2 + (1-t)u_1)^{q-1}\,dt$$

and hence,

$$\frac{u_2^q - u_1^q}{u_2 - u_1} = q\int_0^1 (tu_2 + (1-t)u_1)^{q-1}\,dt \tag{3.22}$$

Now, since $u_1 \gg 0$,

$$(tu_2 + (1-t)u_1)^{q-1} \gg t^{q-1}u_2^{q-1}, \quad 0 \le t < 1, \tag{3.23}$$

and hence, owing to (3.22) and (3.23) we obtain

$$\frac{u_2^q - u_1^q}{u_2 - u_1} = q\int_0^1 (tu_2 + (1-t)u_1)^{q-1}\,dt > q\int_0^1 t^{q-1}u_2^{q-1}\,dt = u_2^{q-1} \tag{3.24}$$

Now, let us consider the function $\Theta := u_2 - u_1$. By construction it satisfies

$$\begin{gathered}(-\Delta - \lambda)\Theta = 0 \quad \text{in } \Omega, \\ \Theta = 0 \quad \text{on } \Gamma_0, \\ \partial\Theta + \Big(V(x) + \tilde{\gamma}b(x)\frac{u_2^q(x) - u_1^q(x)}{u_2(x) - u_1(x)}\Big)\Theta = 0 \quad \text{on } \Gamma_1\end{gathered} \tag{3.25}$$

Owing to (3.24) and the facts that $b > 0$ and $\tilde{\gamma} > 0$, it follows from the monotonicity of the principal eigenvalue with respect to the potential on the boundary (cf. [7, Proposition 3.5]) and (3.21) that

$$\sigma_1^\Omega\Big[-\Delta, \mathfrak{B}\Big(V(x) + \tilde{\gamma}b(x)\frac{u_2^q - u_1^q}{u_2 - u_1}\Big)\Big] > \sigma_1^\Omega\big[-\Delta, \mathfrak{B}\big(V(x) + \tilde{\gamma}b(x)u_2^{q-1}\big)\big] = \lambda$$

and hence,

$$\sigma_1^\Omega\Big[-\Delta - \lambda, \mathfrak{B}\Big(V(x) + \tilde{\gamma}b(x)\frac{u_2^q - u_1^q}{u_2 - u_1}\Big)\Big] > 0 \tag{3.26}$$

Then, since $\sigma_1^\Omega[-\Delta - \lambda, \mathfrak{B}(V + \tilde{\gamma}b(x)\frac{u_2^q - u_1^q}{u_2 - u_1})]$ is the least eigenvalue of (3.25) (cf. [4, Theorem 12.1]), and owing to (3.26), we obtain 0 is not an eigenvalue of (3.25) and therefore $\Theta = 0$, which contradicts (3.20). This completes the proof of the uniqueness of positive solution of (3.1) when it exists. The fact that u_λ is strongly positive in Ω follows from Theorem 2.1.

Now it will be proved that for each λ satisfying (3.3), the unique positive solution u_λ of (3.1) is linearly asymptotically stable. Indeed, the linearization of (3.1) in u_λ is given by

$$\begin{gathered}(-\Delta - \lambda)v = 0 \quad \text{in } \Omega \\ v = 0 \quad \text{on } \Gamma_0 \\ \Big(\partial + V(x) + \tilde{\gamma}qb(x)u_\lambda^{q-1}\Big)v = 0 \quad \text{on } \Gamma_1\end{gathered} \tag{3.27}$$

Since by (2.1),

$$\lambda = \sigma_1^\Omega[-\Delta, \mathfrak{B}(V(x) + \tilde{\gamma}b(x)u_\lambda^{q-1})],$$

and thanks to the facts that $b > 0$, $q > 1$ and $\tilde{\gamma} > 0$, it follows from the monotonicity of the principal eigenvalue with respect to the potential on the boundary that

$$\sigma_1^\Omega\big[-\Delta - \lambda, \mathfrak{B}(V(x) + q\tilde{\gamma}b(x)u_\lambda^{q-1})\big] > \sigma_1^\Omega\big[-\Delta - \lambda, \mathfrak{B}(V(x) + \tilde{\gamma}b(x)u_\lambda^{q-1})\big] = 0,$$

which proves that u_λ is linearly asymptotically stable. The proof of the fact that u_λ is globally asymptotically stable as steady state of the parabolic problem associated to (3.1) is given in Theorem 3.8-ii). This completes the proof of (ii).

(iii) Owing to (i), (ii) and (2.1), for each $\lambda \in (\sigma_1, \sigma_0)$ there exists a unique positive solution u_λ of (3.1) and the following holds

$$\lambda = \sigma_1^\Omega[-\Delta, \mathfrak{B}(V(x) + \tilde{\gamma} b(x) u_\lambda^{q-1})] \tag{3.28}$$

Also, differentiating with respect to λ in (3.1), we obtain

$$\begin{gathered} (-\Delta - \lambda)\dot{u}_\lambda = u_\lambda > 0 \quad \text{in } \Omega \\ \dot{u}_\lambda = 0 \quad \text{on } \Gamma_0 \\ (\partial + V(x) + \tilde{\gamma} b(x) q u_\lambda^{q-1})\dot{u}_\lambda = 0 \quad \text{on } \Gamma_1 \end{gathered} \tag{3.29}$$

Then, since $b > 0$, $\tilde{\gamma} > 0$ and $q > 1$, and owing to (3.28) and to the monotonicity of the principal eigenvalue with respect to the potential on the boundary, the following hold

$$\sigma_1^\Omega[-\Delta - \lambda, \mathfrak{B}(V(x) + \tilde{\gamma} b(x) q u_\lambda^{q-1})] > \sigma_1^\Omega[-\Delta - \lambda, \mathfrak{B}(V(x) + \tilde{\gamma} b(x) u_\lambda^{q-1})] = 0$$

Thus, the characterization of the strong maximum principle (cf. [5, Theorem 2.4]) establishes that $(-\Delta - \lambda, \mathfrak{B}(V(x) + \tilde{\gamma} b(x) q u_\lambda^{q-1}), \Omega)$ satisfies the strong maximum principle and therefore, (3.29) implies that $\dot{u}_\lambda$ is strongly positive in Ω. This completes the proof of (iii).

(iv). It is a straightforward consequence of *iii*), taking into account that owing to (3.3) and (3.5) the map $(\sigma_1, \sigma_0) \to \mathbb{R}^+$ given by

$$\lambda \mapsto \|u_\lambda\|_{L^\infty(\Omega)}$$

is increasing with λ and $u_{\sigma_1} = 0$. Then, if $[\alpha, \beta] \subset [\sigma_1, \sigma_0)$, we obtain

$$\|u_\lambda\|_{L^\infty(\Omega)} \le \|u_\beta\|_{L^\infty(\Omega)} \quad \text{for all } \lambda \in [\alpha, \beta]$$

This completes the proof of (iv).

Now we are going to prove (v). The fact that $\lambda = \sigma_1$ is the unique bifurcation value to positive solutions of (3.1) from the trivial branch $(\lambda, u) = (\lambda, 0)$ was proved in Theorem 2.2-i). The existence of the continuum $\mathfrak{C}^+$ of positive solutions of (3.1) emanating supercritically from the point $(\lambda, u) = (\sigma_1, 0)$ follows from Theorem 2.2 and Remark 2.3, taking into account that $\gamma := -\tilde{\gamma} < 0$ and $b > 0$ on Γ_1. Denoting by $\mathcal{P}_\lambda(\mathfrak{C}^+)$ the $\lambda-$projection of $\mathfrak{C}^+$ on the $\lambda-$axis, it follows from (3.3) that

$$\mathcal{P}_\lambda(\mathfrak{C}^+) \subset [\sigma_1, \sigma_0) \tag{3.30}$$

Owing to the fact that $(\sigma_1, 0)$ is the unique bifurcation point of (3.1) to positive solutions from the trivial branch $(\lambda, u) = (\lambda, 0)$, it follows from the global bifurcation theory (cf. Remark 2.4) that the continuum $\mathfrak{C}^+$ is unbounded in $\mathbb{R} \times L^\infty(\Omega)$ and since (3.30) holds, we obtain $\mathfrak{C}^+$ is unbounded in $L^\infty(\Omega)$, and therefore, $\mathfrak{C}^+$ must bifurcate to positive solutions from infinity at some value $\lambda^* \in [\sigma_1, \sigma_0]$. Now, the existence of uniform $L^\infty(\Omega)-$bounds for the positive solutions of (3.1) in compact intervals of λ contained in $[\sigma_1, \sigma_0)$, implies that $\mathfrak{C}^+$ must bifurcate from infinity in $L^\infty(\Omega)$ when $\lambda \uparrow \sigma_0$, and that $\lambda = \sigma_0$ is the unique bifurcation value to positive solutions of (3.1) from infinity. Then, since $\mathfrak{C}^+$ bifurcates to positive solutions from the trivial branch at $\lambda = \sigma_1$ and from infinity at $\lambda = \sigma_0$, since (3.3) holds and owing to the fact that $\mathfrak{C}^+$ is connected, we obtain

$$\mathcal{P}_\lambda(\mathfrak{C}^+) = [\sigma_1, \sigma_0) \tag{3.31}$$

The fact that all the positive solutions of (3.1) are contained in $\mathfrak{C}^+$ follows from (3.3), (3.31), from the fact that $\lambda = \sigma_1$ is the unique bifurcation value to positive solutions of (3.1) from the trivial branch and from the uniqueness of positive solution of (3.1) when it exists. Finally, the fact that $\mathfrak{C}^+$ is increasing in $L^\infty(\Omega)$ with λ, follows from (iii). This completes the proof of (v). $\square$

3.2. **Pointwise growth of positive solutions of** (3.1) **when** $\lambda \uparrow \sigma_0$**.** In this section we are going to analyze the pointing behavior of the positive solutions of (3.1) when $\lambda \uparrow \sigma_0$. The original ideas given in the previous works [19] and [20] will play a crucial role to obtain the results of this section. Owing to Theorem 3.1 it is known that $\lambda = \sigma_0$ is the unique bifurcation value from infinity to positive solutions of (3.1) and hence,

$$\lim_{\lambda\uparrow\sigma_0} \|u_\lambda\|_{L^\infty(\Omega)} = \infty$$

In this section we will prove that the growth to infinity of the positive solutions of (3.1) when $\lambda \uparrow \sigma_0$ is not concentrated in some particular region of $\Omega \cup \Gamma_1$, but it occurs uniformly in any compact subset of $\Omega \cup \Gamma_1$. The following result establishes the uniform growth to infinity in compact subsets of Ω of the positive solutions of (3.1) when $\lambda \uparrow \sigma_0$.

Theorem 3.4. *Assume* (3.2)*. Then*

$$\lim_{\lambda\uparrow\sigma_0} u_\lambda = \infty \quad \textit{and} \quad \lim_{\lambda\uparrow\sigma_0} \dot u_\lambda = \infty \tag{3.32}$$

uniformly in compact subsets of Ω*, where* $\dot u_\lambda := \frac{du_\lambda}{d\lambda}$

Proof. Let φ_0 be the principal eigenfunction of $-\Delta$ in the domain Ω under Dirichlet boundary conditions, normalized so that $\|\varphi_0\|_{L^\infty(\Omega)} = 1$, let us fix $\lambda_1 \in (\sigma_1, \sigma_0)$ and let u_{λ_1} be the unique positive solution of (3.1) for such a value $\lambda = \lambda_1$. Since φ_0 and u_{λ_1} are strongly positive in Ω, there exists $\alpha > 0$ such that

$$u_{\lambda_1} \gg \alpha\varphi_0 \quad \text{in } \Omega\,,$$

and since the branch of positive solutions $\mathfrak{C}^+ = \{u_\lambda : \lambda \in (\sigma_1, \sigma_0)\}$ is increasing with λ (cf. Theorem 3.1-iii)), we obtain

$$u_\lambda > u_{\lambda_1} > \alpha\varphi_0 \quad \text{in } \Omega\,,\ \lambda \in (\lambda_1, \sigma_0)\,. \tag{3.33}$$

Also, we obtain

$$\sigma_1^\Omega[-\Delta - \lambda, \mathcal{D}] = \sigma_0 - \lambda > 0\,, \quad \forall \lambda \in (\lambda_1, \sigma_0)\,. \tag{3.34}$$

Now, differentiating (3.1) with respect to λ gives

$$\begin{gathered} (-\Delta - \lambda)\dot u_\lambda = u_\lambda \quad \text{in } \Omega \\ \dot u_\lambda = 0 \quad \text{on } \Gamma_0 \\ (\partial + V(x) + q\tilde\gamma b(x) u_\lambda^{q-1})\dot u_\lambda = 0 \quad \text{on } \Gamma_1 \end{gathered} \tag{3.35}$$

and taking into account (3.4) and (3.33), we obtain for $\lambda \in (\lambda_1, \sigma_0)$,

$$\begin{gathered} (-\Delta - \lambda)\dot u_\lambda = u_\lambda > \alpha\varphi_0 \quad \text{in } \Omega \\ \dot u_\lambda = 0 \quad \text{on } \Gamma_0 \\ \dot u_\lambda > 0 \quad \text{on } \Gamma_1 \end{gathered} \tag{3.36}$$

Now, since by definition,

$$\varphi_0 = \begin{cases} \frac{1}{\sigma_0-\lambda}(-\Delta-\lambda)\varphi_0 & \text{in } \Omega, \\ 0 & \text{on } \partial\Omega = \Gamma_0 \cup \Gamma_1 \,, \end{cases}$$

Equation (3.36) becomes

$$\begin{gathered} (-\Delta-\lambda)\big(\dot{u}_\lambda - \frac{\alpha\varphi_0}{\sigma_0-\lambda}\big) > 0 \quad \text{in } \Omega \\ \dot{u}_\lambda - \frac{\alpha\varphi_0}{\sigma_0-\lambda} = 0 \quad \text{on } \Gamma_0 \\ \dot{u}_\lambda - \frac{\alpha\varphi_0}{\sigma_0-\lambda} > 0 \quad \text{on } \Gamma_1 \end{gathered}$$

and hence,

$$\begin{gathered} (-\Delta-\lambda)\big(\dot{u}_\lambda - \frac{\alpha\varphi_0}{\sigma_0-\lambda}\big) > 0 \quad \text{in } \Omega \\ \dot{u}_\lambda - \frac{\alpha\varphi_0}{\sigma_0-\lambda} > 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.37}$$

Owing to the characterization of the strong maximum principle (cf. [5, Theorem 2.4]) it follows from (3.34) that the problem $(-\Delta-\lambda,\Omega,\mathcal{D})$ satisfies the strong maximum principle for each $\lambda \in (\lambda_1,\sigma_0)$ and hence, (3.37) implies that

$$\dot{u}_\lambda > \frac{\alpha\varphi_0}{\sigma_0-\lambda}\,, \quad x \in \Omega\,, \quad \lambda \in (\lambda_1,\sigma_0) \tag{3.38}$$

Now, let $K \subset \Omega$ be a compact subset in Ω and let us denote $m_K := \min_{x\in K}\varphi_0 > 0$. Owing to (3.38) we obtain for $\lambda \in (\lambda_1,\sigma_0)$,

$$\dot{u}_\lambda > \frac{\alpha\varphi_0}{\sigma_0-\lambda} \geq \frac{\alpha m_K}{\sigma_0-\lambda}\,, \quad x \in K\,, \tag{3.39}$$

and taking limits in (3.39) when $\lambda \uparrow \sigma_0$ it is obtained that $\lim_{\lambda\uparrow\sigma_0}\dot{u}_\lambda = \infty$ uniformly in K. Finally, integrating (3.39) in $[\lambda_1,\lambda]$ gives

$$u_\lambda \geq u_{\lambda_1} + \alpha m_K \ln\big(\frac{\sigma_0-\lambda_1}{\sigma_0-\lambda}\big)\,, \quad x \in K \tag{3.40}$$

and therefore, $\lim_{\lambda\uparrow\sigma_0} u_\lambda = \infty$ uniformly in K. This completes the proof. □

To prove that the positive solutions of (3.1) also grow to infinity uniformly on Γ_1, we need the following lemmas. The following lemma gives a comparison result, and it may be proved following similar arguments to the used in the proof of [11, Proposition 3.2].

Lemma 3.5. *Let u_λ and θ_λ be a positive solution and a positive strict subsolution of* (3.1)*, respectively, for the value λ of the parameter. Then*

$$\theta_\lambda \ll u_\lambda \quad \text{in } \Omega \tag{3.41}$$

The following lemma will be proved adapting to our current framework some of the original ideas given in [19], assuming for it that the component Γ_1 of $\partial\Omega$ is of class $\mathcal{C}^3$ in $\mathbb{R}^N$.

Lemma 3.6. *Assume that the component Γ_1 of $\partial\Omega$ is of class $\mathcal{C}^3$ in $\mathbb{R}^N$, and let be the domain*

$$\Omega_\delta := \Omega \cup \{x \in \mathbb{R}^N \setminus \Omega : \operatorname{dist}(x,\Gamma_1) < \delta\}\,,$$

for $\delta > 0$ small enough, and φ_0 and φ_δ the principal eigenfunctions associated to $-\Delta$ operator in the domains Ω and Ω_δ, respectively, under Dirichlet boundary conditions, normalized so that

$$\|\varphi_0\|_{L^\infty(\Omega)} = 1\,, \quad \|\varphi_\delta\|_{L^\infty(\Omega_\delta)} = 1$$

Then, there exists $\bar{z}, \underline{z} \in \Gamma_1$ such that

$$\|\varphi_\delta\|_{L^\infty(\Gamma_1)} = -\partial\varphi_0(\bar{z})\delta + o(\delta), \tag{3.42}$$

$$\min_{x\in\Gamma_1} \varphi_\delta = -\partial\varphi_0(\underline{z})\delta + o(\delta) \tag{3.43}$$

Proof. Since Γ_1 is a compact surface of class $\mathcal{C}^3$ in $\mathbb{R}^N$, let $n := n(x) \in \mathcal{C}^2(\Gamma_1;\mathbb{R}^N)$ be the $\mathcal{C}^2$ outward unit normal field to Γ_1 and for $\delta > 0$ small enough, let

$$A_\delta := \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Gamma_1) < \delta\}$$

be a tubular $\delta-$neighborhood of Γ_1, and

$$\Gamma_\delta := \big\{x \in \mathbb{R}^N \setminus \Omega : \operatorname{dist}(x, \Gamma_1) = \delta\big\}$$

Then, for every $x \in A_\delta$, there exists a unique $y \in \Gamma_1$ and $\tau \in (-\delta, \delta)$ such that

$$x = y - \tau n(y) \tag{3.44}$$

By restricting $\delta > 0$ if it is necessary, the implicit function theorem gives the existence of two unique mappings $\tau \in \mathcal{C}^2(A_\delta;\mathbb{R})$ and $\pi \in \mathcal{C}^2(\mathcal{A}_\delta;\Gamma_1)$ such that

$$x = \pi(x) - \tau(x)n(\pi(x))\,, \quad x \in A_\delta \tag{3.45}$$

Let $\hat{\tau} \in \mathcal{C}(\bar{\Omega};\mathbb{R})$ be the extension to $\bar{\Omega}$ by $\hat{\tau}(x) = \delta$ if $\operatorname{dist}(x, \Gamma_1) \geq \delta$, $\hat{n} \in \mathcal{C}^2(\bar{\Omega};\mathbb{R}^N)$ any regular extension of the vector field $n(\pi(x))$ to $\bar{\Omega}$ and let us consider any function $\xi \in \mathcal{C}^3([0,\infty);[0,\infty))$ satisfying $\xi(0) = 1$, $\xi(\tau)\xi'(\tau) < 0$ for $\tau \in [0, \frac{\delta}{2})$ and $\xi(\tau) = 0$ for $\tau \geq \frac{\delta}{2}$. Now, let us consider the mapping

$$H(x) := \xi(\hat{\tau}(x))\hat{n}(x)\,, \quad x \in \bar{\Omega}$$

This map is of class $\mathcal{C}^2$ and it satisfies

$$H(x) = \begin{cases} 0 & \text{if } \operatorname{dist}(x, \Gamma_1) \geq \delta/2 \\ \xi(\tau(x))n(\pi(x)) & \text{if } 0 < \operatorname{dist}(x, \Gamma_1) < \delta/2 \\ n(x) & \text{if } x \in \Gamma_1 \end{cases}$$

Let us consider the mapping $T_\delta : \bar{\Omega} \mapsto \mathbb{R}^N$ defined by

$$T_\delta := I + \delta H\,,$$

where I stands for the identity map in $\mathbb{R}^N$. Owing to [19, Theorem 3.1], $T_\delta \in \mathcal{C}^2(\bar{\Omega};\mathbb{R}^N)$ and $T_\delta : \bar{\Omega} \mapsto \bar{\Omega}_\delta$ is a bijection and a real holomorphic family in $\delta \cong 0$(cf. [19, Section 2]). Now, set

$$y = T_\delta(x) \in \bar{\Omega}_\delta\,, \quad \psi_\delta(x) = \varphi_\delta(y) = \varphi_\delta(T_\delta(x))\,, \quad x \in \bar{\Omega}$$

By construction and definition we obtain $T_\delta(\Gamma_1) = \Gamma_\delta$ and

$$\psi_\delta|_{\Gamma_1} = (\varphi_\delta \circ T_\delta)|_{\Gamma_1} = \varphi_\delta|_{T_\delta(\Gamma_1)} = \varphi_\delta|_{\Gamma_\delta} = 0 \tag{3.46}$$

To prove (3.42), for each $\delta > 0$ small enough, let $\bar{y}_\delta \in \Gamma_1 \subset \Omega_\delta$ be such that

$$\|\varphi_\delta\|_{L^\infty(\Gamma_1)} = \varphi_\delta(\bar{y}_\delta)\,, \tag{3.47}$$

and $x_\delta \in \Omega$ such that $T_\delta(x_\delta) = \bar{y}_\delta$. Let $z_\delta \in \Gamma_1$ be such that $\pi(x_\delta) = z_\delta$. We have that $x_\delta = z_\delta - \delta n(z_\delta)$. Since $z_\delta \in \Gamma_1$, it follows from (3.46) that $\psi_\delta(z_\delta) = 0$ and since

$$\varphi_\delta(\bar{y}_\delta) = \varphi_\delta(T_\delta(x_\delta)) = \psi_\delta(x_\delta) = \psi_\delta(z_\delta - \delta n(z_\delta))\,,$$

arguing as in [19, Theorem 4.3] we find that

$$\begin{aligned} \frac{\varphi_\delta(\bar{y}_\delta)}{\delta} = \frac{\psi_\delta(x_\delta)}{\delta} &= \frac{\psi_\delta(z_\delta - \delta n(z_\delta)) - \psi_\delta(z_\delta)}{\delta} \\ &= -\int_0^1 \langle \nabla\psi_\delta(z_\delta - t\delta n(z_\delta)), n(z_\delta)\rangle\, dt \end{aligned} \tag{3.48}$$

Since $\{z_\delta : \delta > 0\} \subset \Gamma_1$ and Γ_1 is compact, taking limits when $\delta \to 0$, module some subsequence, we obtain there exists $\bar{z} \in \Gamma_1$ such that $\lim_{\delta\downarrow 0} z_\delta = \bar{z} \in \Gamma_1$. Then, taking limits in (3.48) when $\delta \to 0$ we find that

$$\lim_{\delta\to 0} \frac{\varphi_\delta(\bar{y}_\delta)}{\delta} = -\lim_{\delta\to 0}\int_0^1 \langle \nabla\psi_\delta(z_\delta - t\delta n(z_\delta)), n(z_\delta)\rangle\, dt = -\partial\varphi_0(\bar{z})\,, \tag{3.49}$$

and therefore, (3.47) and (3.49) imply

$$\|\varphi_\delta\|_{L^\infty(\Gamma_1)} = \varphi_\delta(\bar{y}_\delta) = -\partial\varphi_0(\bar{z})\delta + o(\delta)$$

This completes the proof of (3.42).

The proof of (3.43) follows exactly the same steps than in the proof of (3.42), changing in (3.47) the existence of $\bar{y}_\delta \in \Gamma_1 \subset \Omega_\delta$ satisfying (3.47), by the existence of $\tilde{y}_\delta \in \Gamma_1 \subset \Omega_\delta$ such that

$$\min_{x\in\Gamma_1} \varphi_\delta = \varphi_\delta(\tilde{y}_\delta) > 0\,,$$

where later, module some subsequence, we will get that

$$\lim_{\delta\downarrow 0} \tilde{z}_\delta = \underline{z} \in \Gamma_1\,,$$

being $\tilde{x}_\delta \in \Omega$, $T_\delta(\tilde{x}_\delta) = \tilde{y}_\delta$ and $\pi(\tilde{x}_\delta) = \tilde{z}_\delta$. This completes the proof of (3.43) and of the result. □

The following result establishes the uniform growth to infinity on Γ_1 of the positive solutions of (3.1) when $\lambda \uparrow \sigma_0$. Part of its proof is based in some of the original ideas given in [19].

Theorem 3.7. *Assume* (3.2) *and that the component* Γ_1 *of* $\partial\Omega$ *is of class* $\mathcal{C}^3$ *in* $\mathbb{R}^N$. *Then*

$$\lim_{\lambda\uparrow\sigma_0} u_\lambda(x) = \infty \quad \textit{uniformly on } \Gamma_1 \tag{3.50}$$

Proof. Set, for $\delta > 0$ small enough, the domain

$$\Omega_\delta := \Omega \cup \{x \in \mathbb{R}^N \setminus \Omega : \operatorname{dist}(x, \Gamma_1) < \delta\}\,,$$

and let (σ_0, φ_0) and $(\sigma^\delta, \varphi_\delta)$ be the principal eigen-pairs associated to $-\Delta$ operator in the domains Ω and Ω_δ, respectively, under Dirichlet boundary conditions, with the eigenfunctions normalized so that

$$\|\varphi_0\|_{L^\infty(\Omega)} = 1\,, \quad \|\varphi_\delta\|_{L^\infty(\Omega_\delta)} = 1$$

By definition and since φ_0 and φ_δ are strongly positive in Ω and Ω_δ, respectively, we obtain

$$\varphi_0(x) = 0\,, \quad \partial\varphi_0(x) < 0 \quad \forall x \in \partial\Omega = \Gamma_0 \cup \Gamma_1\,,$$

and since $\Gamma_1 \subset \Omega_\delta$ for all $\delta > 0$,

$$\varphi_\delta(x) > 0 \quad \forall x \in \Gamma_1 \tag{3.51}$$

Let us denote

$$\alpha_0(x) := -\partial\varphi_0(x) > 0 \quad \forall x \in \partial\Omega\,, \tag{3.52}$$

$$\underline{\alpha}_0 := \min_{x\in\Gamma_1} \alpha_0(x) > 0\,, \quad \bar{\alpha}_0 := \|\alpha_0(x)\|_{L^\infty(\Gamma_1)}\,, \tag{3.53}$$

$$\bar{b} := \|b(x)\|_{L^\infty(\Gamma_1)} > 0\,, \quad \bar{V} := \|V(x)\|_{L^\infty(\Gamma_1)} \geq 0 \tag{3.54}$$

Let us fix $\varepsilon > 0$ such that

$$0 < \varepsilon < \frac{\underline{\alpha}_0}{1+\bar{V}} \tag{3.55}$$

and $k_0 > 0$ large enough such that

$$0 < \frac{1}{k_0} < \underline{\alpha}_0 - \varepsilon(1+\bar{V}) \tag{3.56}$$

Thanks to the monotonicity of the principal eigenvalue with respect to the domain (cf. [18], [7, Proposition 3.2]), it is known that

$$\sigma^\delta < \sigma_0 \quad \text{for all } \delta > 0, \tag{3.57}$$

and by the continuous dependence of the principal eigenvalue with respect to perturbations of the domain around its Dirichlet boundary (cf. [18, Theorem 4.2], [7, Theorems 7.1 and 7.4]), we obtain

$$\lim_{\delta\downarrow 0} \sigma^\delta = \sigma_0\,. \tag{3.58}$$

Then, by (1.5), (3.57) and (3.58), there exists $\delta_0 > 0$ such that

$$\sigma_1 < \sigma^\delta < \sigma_0 \quad \forall \delta \in (0, \delta_0] \tag{3.59}$$

Also, by construction, owing to the regularity of the principal eigenfunctions φ_δ and φ_0 and to the results in [19], there exists $\delta_1 \in (0, \delta_0]$ such that

$$\partial\varphi_\delta(x) \leq \partial\varphi_0(x) + \varepsilon\,, \quad \forall \delta \in (0, \delta_1]\,,\ \forall x \in \Gamma_1, \tag{3.60}$$

$$\varphi_\delta(x) \leq \varphi_0(x) + \varepsilon = \varepsilon\,, \quad \forall \delta \in (0, \delta_1]\,,\ \forall x \in \Gamma_1 \tag{3.61}$$

Let $\bar{y}_\delta \in \Gamma_1$ be such that

$$\varphi_\delta(\bar{y}_\delta) = \|\varphi_\delta\|_{L^\infty(\Gamma_1)} \tag{3.62}$$

(cf. (3.47)). Owing to (3.42) we obtain the existence of $\bar{z} \in \Gamma_1$ such that

$$\varphi_\delta(\bar{y}_\delta) = \|\varphi_\delta\|_{L^\infty(\Gamma_1)} = \alpha_0(\bar{z})\delta + o(\delta)\,. \tag{3.63}$$

Now, let us consider the function defined in Ω_δ by

$$v_\delta := C(\delta)\varphi_\delta\,, \tag{3.64}$$

where

$$C = C(\delta) := \frac{1}{(k_0\bar{b}\tilde{\gamma})^{\frac{1}{q-1}}(\bar{\alpha}_0\delta)^{\frac{q}{q-1}}} > 0\,, \tag{3.65}$$

being $\bar{\alpha}_0$, $\bar{b}$ and k_0 defined by (3.53), (3.54) and (3.56), respectively. We are going to prove that there exists $\delta_2 \in (0, \delta_1]$, such that for each $\delta \in (0, \delta_2]$ and $\lambda \in (\sigma^\delta, \sigma_0)$, the function

$$u_\delta := v_\delta|_{\bar{\Omega}} = C(\delta)\varphi_\delta|_{\bar{\Omega}}$$

is a positive strict subsolution of (3.1). Indeed, by construction the following holds in Ω for each $\delta \in (0, \delta_1]$ and $\lambda \in (\sigma^\delta, \sigma_0)$

$$(-\Delta - \lambda)u_\delta = C(\delta)(\sigma^\delta - \lambda)\varphi_\delta < 0 \tag{3.66}$$

In regard to the boundary conditions on Γ_1, by (3.52), (3.53), (3.54), (3.60), (3.61), (3.62), (3.63) and (3.65), the following hold on Γ_1:

$$\begin{aligned}
&(\partial + V(x))\, u_\delta(x) + \tilde{\gamma} b(x) u_\delta^q(x) \\
&= C\left(\partial\varphi_\delta(x) + V(x)\varphi_\delta(x) + \tilde{\gamma} b(x) C^{q-1} \varphi_\delta^q(x)\right) \\
&\le C\left(\partial\varphi_0(x) + \varepsilon(1+\bar{V}) + \tilde{\gamma}\bar{b} C^{q-1}\varphi_\delta^q(y_\delta)\right) \\
&= C\left(-\alpha_0(x) + \varepsilon(1+\bar{V}) + \tilde{\gamma}\bar{b} C^{q-1}(\alpha_0(z_0)\delta + o(\delta))^q\right) \\
&\le C(-\underline{\alpha}_0 + \varepsilon(1+\bar{V}) + \tilde{\gamma}\bar{b} C^{q-1}(\bar{\alpha}_0\delta + o(\delta))^q) \\
&= C(-\underline{\alpha}_0 + \varepsilon(1+\bar{V}) + \tilde{\gamma}\bar{b} C^{q-1}\left(\bar{\alpha}_0^q\delta^q + o(\delta^q)\right)) \\
&= C\Big(-\underline{\alpha}_0 + \varepsilon(1+\bar{V}) + \frac{1}{k_0} + \frac{1}{k_0\bar{\alpha}_0^q}\frac{o(\delta^q)}{\delta^q}\Big).
\end{aligned}$$

Then, by (3.56), there exists $\delta_2 \in (0, \delta_1]$ such that for each $\delta \in (0, \delta_2]$ and $x \in \Gamma_1$,

$$(\partial + V(x))\, u_\delta(x) + \tilde{\gamma} b(x) u_\delta^q(x) \le C\Big(-\underline{\alpha}_0 + \varepsilon(1+\bar{V}) + \frac{1}{k_0} + \frac{1}{k_0\bar{\alpha}_0^q}\frac{o(\delta^q)}{\delta^q}\Big) < 0$$

and hence,

$$\partial u_\delta + V(x)u_\delta + \tilde{\gamma} b(x) u_\delta^q < 0 \quad \text{on } \Gamma_1\,,\ \delta \in (0, \delta_2] \tag{3.67}$$

Also, by construction

$$u_\delta\big|_{\Gamma_0} = C(\delta)\varphi_\delta|_{\Gamma_0} = 0 \tag{3.68}$$

Then, (3.66), (3.67) and (3.68) give that for $\delta \in (0, \delta_2]$ and $\lambda \in (\sigma^\delta, \sigma_0)$

$$\begin{gathered}
(-\Delta - \lambda)u_\delta < 0 \quad \text{in } \Omega \\
u_\delta = 0 \quad \text{on } \Gamma_0 \\
\partial u_\delta + V(x)u_\delta + \tilde{\gamma} b(x) u_\delta^q < 0 \quad \text{on } \Gamma_1
\end{gathered} \tag{3.69}$$

and therefore, u_δ is a positive strict subsolution of (3.1) for $\delta > 0$ small enough and $\lambda \in (\sigma^\delta, \sigma_0)$.

Now we prove (3.50). From (3.51), for each $\delta \in (0, \delta_2]$ there exists $\tilde{y}_\delta \in \Gamma_1$ such that

$$\min_{x\in\Gamma_1} \varphi_\delta(x) = \varphi_\delta(\tilde{y}_\delta) > 0\,, \tag{3.70}$$

and from (3.43), there exists $\underline{z} \in \Gamma_1$ such that

$$\varphi_\delta(\tilde{y}_\delta) = \alpha_0(\underline{z})\delta + o(\delta) \tag{3.71}$$

Thus, from (3.70), (3.71) and the definition of the constants $C = C(\delta) > 0$ and $\underline{\alpha}_0$ (cf. (3.65), (3.53)), for each $x \in \Gamma_1$ and $\delta \in (0, \delta_2]$ we obtain

$$u_\delta(x) = C\varphi_\delta(x) \ge C\varphi_\delta(\tilde{y}_\delta) = \frac{\alpha_0(\underline{z})\delta + o(\delta)}{\left(k_0\bar{b}\tilde{\gamma}\right)^{\frac{1}{q-1}}\left(\bar{\alpha}_0\delta\right)^{\frac{q}{q-1}}} \ge \frac{\underline{\alpha}_0\delta + o(\delta)}{\left(k_0\bar{b}\tilde{\gamma}\right)^{\frac{1}{q-1}}\left(\bar{\alpha}_0\delta\right)^{\frac{q}{q-1}}}\,,$$

and therefore, since $q > 1$, it holds

$$\begin{aligned}\liminf_{\delta\downarrow 0} u_\delta(x) &\geq \lim_{\delta\downarrow 0} \frac{\underline{\alpha}_0\delta + o(\delta)}{k_0^{\frac{1}{q-1}}\tilde{\gamma}^{\frac{1}{q-1}}\bar{b}^{\frac{1}{q-1}}\bar{\alpha}_0^{\frac{q}{q-1}}\delta^{\frac{q}{q-1}}} \\ &= \lim_{\delta\downarrow 0} \frac{\underline{\alpha}_0}{k_0^{\frac{1}{q-1}}\tilde{\gamma}^{\frac{1}{q-1}}\bar{b}^{\frac{1}{q-1}}\bar{\alpha}_0^{\frac{q}{q-1}}\delta^{\frac{1}{q-1}}} = \infty\end{aligned} \tag{3.72}$$

uniformly on Γ_1. Now, due to the fact that $u_\delta(x)$ is a positive strict subsolution of (3.1) for any $\delta \in (0, \delta_2]$ and $\lambda \in (\sigma^\delta, \sigma_0)$, it follows from Lemma 3.5 and from the existence and uniqueness of positive solution u_λ of (3.1) for each $\lambda \in (\sigma^\delta, \sigma_0)$ (cf. (3.59) and Theorem 3.1) that

$$u_\delta < u_\lambda \text{ in } \Omega, \quad 0 < \delta \leq \delta_2\,, \quad \sigma^\delta < \lambda < \sigma_0\,, \tag{3.73}$$

and therefore, (3.72), (3.73) and (3.58) imply (3.50). This completes the proof. $\square$

3.3. **Dynamics of the positive solutions of the parabolic problem for $\gamma < 0$.** In this section we will analyze, in the particular case when $\gamma < 0$, depending on the values of the parameter λ, the longtime behavior of the positive solutions of the parabolic problem associated to (3.1), given by

$$\begin{gathered} w_t - \Delta w = \lambda w \quad \text{in } \Omega \times (0,\infty) \\ w = 0 \quad \text{on } \Gamma_0 \times (0,\infty) \\ \partial w + V(x)w = \gamma b(x) w^q \quad \text{on } \Gamma_1 \times (0,\infty) \\ w(x,0) = u_0 > 0 \quad \text{in } \Omega\,. \end{gathered} \tag{3.74}$$

In this section $\Theta_\lambda(x,t;u_0)$ stands for the solution of (3.74) for the value λ of the parameter, u_λ will stand for the unique positive solution of (3.1) when it exists, that is, when $\lambda \in (\sigma_1, \sigma_0)$ and $T(t)$ will stand for the L_p-evolution operator associated with $\Delta + \lambda$ under the linear homogeneous mixed boundary conditions given by the boundary operator $\mathfrak{B}(V(x))$. The solution $\Theta_\lambda(x,t;u_0)$ is globally defined in time, since it satisfies

$$\begin{gathered} (\Theta_\lambda(\cdot,t;u_0))_t - (\Delta+\lambda)\Theta_\lambda(\cdot,t;u_0) = 0 \quad \text{in } \Omega \times (0,\infty) \\ \Theta_\lambda(\cdot,t;u_0) = 0 \quad \text{on } \Gamma_0 \times (0,\infty) \\ (\partial + V(x))\Theta_\lambda(\cdot,t;u_0) < 0 \quad \text{on } \Gamma_1 \times (0,\infty) \\ \Theta_\lambda(\cdot,0;u_0) = u_0 > 0 \quad \text{in } \Omega\,; \end{gathered}$$

that is, it is a positive strict subsolution of the linear heat equation in the domain Ω under the linear homogeneous mixed boundary conditions

$$\mathfrak{B}(V(x))\Theta_\lambda = 0\,,$$

and therefore, by the parabolic maximum principle we obtain

$$0 \ll \Theta_\lambda(x,t;u_0) \ll T(t)u_0\,.$$

Hereafter we will say that a non-negative steady-state $\tilde{u}_\lambda$ of (3.1) for the value λ of the parameter is globally asymptotically stable, if

$$\lim_{t\uparrow\infty} \|\Theta_\lambda(\cdot,t;u_0) - \tilde{u}_\lambda(\cdot)\|_{L^\infty(\Omega)} = 0\,,$$

for any initial data $u_0 > 0$.

Arguing as in [20, Theorem 2.2] and by Lemma 3.2, Lemma 3.3 and Theorem 3.1, we obtain the following result, which gives the dynamics of the solution $\Theta_\lambda(\cdot,t;u_0)$

of the parabolic problem (3.74), with $u_0 > 0$, depending on the value λ of the parameter.

Theorem 3.8. *Under the assumptions of Theorem 3.1 the following hold:*

(i) *If $\lambda \leq \sigma_1$, then $u = 0$ is globally asymptotically stable for* (3.74).

(ii) *If $\sigma_1 < \lambda < \sigma_0$, then u_λ is globally asymptotically stable for* (3.74).

(iii) *If $\lambda \geq \sigma_0$, then for any $u_0 > 0$,*

$$\lim_{t\uparrow\infty} \Theta_\lambda(\cdot, t; u_0) = \infty \tag{3.75}$$

uniformly in compact subsets $K \subset \Omega$. If in addition Γ_1 is of class $\mathcal{C}^3$, then (3.75) *holds uniformly in compact subsets $K \subset \Omega \cup \Gamma_1$.*

Proof. (i) Let $u_0 > 0$ be, fix $t_1 > 0$ and let us consider $\tilde{u}_0 = u(\cdot, t_1; u_0)$. Owing to the parabolic maximum principle we know that

$$\tilde{u}_0 \gg 0 \quad \text{in } \Omega \tag{3.76}$$

Since $\lambda \leq \sigma_1 < \sigma_0$, it follows from Lemma 3.3 the existence of a positive strict supersolution $\bar{u}_\lambda$ of (3.1), strongly positive in Ω, such that

$$\tilde{u}_0 < \bar{u}_\lambda \,,$$

and owing to the parabolic maximum principle, we obtain

$$0 \ll \Theta_\lambda(\cdot, t; u_0) = \Theta_\lambda(\cdot, t - t_1; \tilde{u}_0) \leq \Theta_\lambda(\cdot, t - t_1; \bar{u}_\lambda) \tag{3.77}$$

Now, thanks to the results in [28], we know that $\Theta_\lambda(\cdot, t - t_1; \bar{u}_\lambda)$ is a decreasing function in $t > t_1$ which converges to a non-negative solution of (3.1), and due to the fact that $u = 0$ is the unique nonnegative solution of (3.1) for $\lambda \leq \sigma_1$ (cf. Theorem 3.1), taking limits in (3.77) when $t \uparrow \infty$ we obtain

$$\lim_{t\uparrow\infty} \Theta_\lambda(\cdot, t; u_0) = \lim_{t\uparrow\infty} \Theta_\lambda(\cdot, t - t_1; \bar{u}_\lambda) = 0 \,,$$

which completes the proof of (i).

(ii) Let $u_0 > 0$ be, fix $t_1 > 0$ and let us consider $\tilde{u}_0 = u(\cdot, t_1; u_0) \gg 0$ (cf. (3.76)). Since $\lambda \in (\sigma_1, \sigma_0)$, owing to Lemma 3.2 and Lemma 3.3, we obtain the existence of a positive strict subsolution $\underline{u}_\lambda$ of (3.1) and a positive strict supersolution $\bar{u}_\lambda$ of (3.1), both of them strongly positive in Ω, such that

$$0 < \underline{u}_\lambda < \tilde{u}_0 < \bar{u}_\lambda \quad \text{in } \Omega$$

Then, by the parabolic maximum principle, we obtain

$$0 \ll \Theta_\lambda(\cdot, t - t_1; \underline{u}_\lambda) \leq \Theta_\lambda(\cdot, t; u_0) = \Theta_\lambda(\cdot, t - t_1; \tilde{u}_0) \leq \Theta_\lambda(\cdot, t - t_1; \bar{u}_\lambda) \tag{3.78}$$

On the other hand, thanks to the results in [28], we know that $\Theta_\lambda(\cdot, t - t_1; \underline{u}_\lambda)$ and $\Theta_\lambda(x, t - t_1; \bar{u}_\lambda)$ are an increasing and a decreasing function in $t > t_1$, respectively, converging to a nonnegative solution of (3.1). Then, since for $\lambda \in (\sigma_1, \sigma_0)$ there exists a unique positive solution u_λ of (3.1) (cf. Theorem 3.1), we obtain

$$\lim_{t\uparrow\infty} \Theta_\lambda(\cdot, t - t_1, \bar{u}_\lambda) = u_\lambda \,, \quad \lim_{t\uparrow\infty} \Theta_\lambda(\cdot, t - t_1, \underline{u}_\lambda) = u_\lambda \,,$$

and therefore, taking limits in (3.78) when $t \uparrow \infty$, we have

$$\lim_{t\uparrow\infty} \Theta_\lambda(\cdot, t; u_0) = u_\lambda \,,$$

which completes the proof of (ii).

(iii) Let $u_0 > 0$ be and let us consider the global solution $\Theta_\lambda := \Theta_\lambda(\cdot, t; u_0) > 0$ of the parabolic problem (3.74) for the value λ of the parameter with initial data $u_0 > 0$. Since $\lambda \geq \sigma_0$, for all $\varepsilon > 0$ small enough, $\lambda \geq \sigma_0 > \sigma_0 - \varepsilon > \sigma_1$. Then

$$(\Theta_\lambda)_t - \Delta\Theta_\lambda = \lambda\Theta_\lambda > (\sigma_0 - \varepsilon)\Theta_\lambda\,, \quad \text{in } \Omega \times (0, \infty)\,,$$

and hence, Θ_λ is a positive strict supersolution of the problem

$$\begin{gathered} w_t - \Delta w = (\sigma_0 - \varepsilon)w \quad \text{in } \Omega \times (0, \infty) \\ w = 0 \quad \text{on } \Gamma_0 \times (0, \infty) \\ \partial w + V(x)w = \gamma b(x)w^q \quad \text{on } \Gamma_1 \times (0, \infty) \\ w(\cdot, 0) = u_0 > 0 \quad \text{in } \Omega \end{gathered} \tag{3.79}$$

Then, by the parabolic maximum principle,

$$\Theta_\lambda(\cdot, t; u_0) > \Theta_{\sigma_0-\varepsilon}(\cdot, t; u_0)\,, \tag{3.80}$$

where $\Theta_{\sigma_0-\varepsilon}(\cdot, t, u_0)$ stands for the solution of (3.79). Now, since $\sigma_0 - \varepsilon \in (\sigma_1, \sigma_0)$, owing to $ii)$ we obtain $u_{\sigma_0-\varepsilon}$ is globally asymptotically stable for (3.74). Then

$$\lim_{t\uparrow\infty} \|\Theta_{\sigma_0-\varepsilon}(\cdot, t; u_0) - u_{\sigma_0-\varepsilon}(\cdot)\|_{L^\infty(\Omega)} = 0\,, \tag{3.81}$$

and hence, (3.80) and (3.81) imply

$$\liminf_{t\uparrow\infty} \Theta_\lambda(\cdot, t; u_0) \geq u_{\sigma_0-\varepsilon}(\cdot)\,. \tag{3.82}$$

Now, taking limits in (3.82) when $\varepsilon \downarrow 0$ we obtain

$$\liminf_{t\uparrow\infty} \Theta_\lambda(\cdot, t; u_0) \geq \lim_{\varepsilon\downarrow 0} u_{\sigma_0-\varepsilon}(\cdot)\,, \tag{3.83}$$

and since by Theorem 3.4

$$\lim_{\varepsilon\downarrow 0} u_{\sigma_0-\varepsilon}(\cdot) = \infty \tag{3.84}$$

uniformly in compact subsets $K \subset \Omega$, we obtain that

$$\liminf_{t\uparrow\infty} \Theta_\lambda(\cdot, t; u_0) = \infty \tag{3.85}$$

uniformly in compact subsets $K \subset \Omega$. If in addition Γ_1 is of class $\mathcal{C}^3$ in $\mathbb{R}^N$, then owing to Theorem 3.7 and (3.83), (3.84) holds uniformly on Γ_1 and therefore (3.85) holds uniformly in compact subsets of $\Omega \cup \Gamma_1$. This completes the proof. $\square$

4. The case $\gamma > 0$

In this section we obtain, in the particular case when $\gamma > 0$, some results about existence and non existence of positive solutions of (1.1), about the stability of them and about the structure of the global bifurcation diagram of positive solutions of (1.1). Also, we will obtain some results about the dynamics of the positive solutions of the parabolic problem associated to (1.1), given by

$$\begin{gathered} v_t - \Delta v = \lambda v \quad \text{in } \Omega \times (0, T)\,, \\ v = 0 \quad \text{on } \Gamma_0 \times (0, T)\,, \\ \partial v + V(x)v = \gamma b(x)v^q \quad \text{on } \Gamma_1 \times (0, T)\,, \quad q > 1 \\ v(\cdot, 0) = u_0 > 0 \quad \text{in } \Omega\,, \end{gathered} \tag{4.1}$$

where $T > 0$ is its time of existence.

4.1. **Dynamics of positive solutions of the parabolic problem for $\gamma > 0$.** In this section we analyze the longtime behavior or blow up in finite time of the positive solutions of (4.1). In this section $\Theta_\lambda(x,t;u_0)$ will stand for the solution of (4.1) for the value λ of the parameter. The main result of this section is the following.

Theorem 4.1. *Assume $\gamma > 0$. Then, the following hold:*

(i) *For any λ sufficiently negative, $\Theta_\lambda(x,t;u_0)$ is globally defined in time and*
$$\|\Theta_\lambda(\cdot,t;u_0)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty \tag{4.2}$$

(ii) *If (3.2) holds, $\lambda \geq \sigma_1$ and u_0 is large enough, then $\Theta_\lambda(x,t;u_0)$ blows up in finite time uniformly in compact subsets $Q \subset \Omega \cup \Gamma_1$.*

Proof. (i) Let us take some $p > 2q-1$, let us fix some $\alpha > 0$ satisfying
$$0 < \alpha < \big(\gamma \|b\|_{L^\infty(\Gamma_1)}\big)^{-\frac{1}{q-1}}, \tag{4.3}$$
let $\tilde{V}$ be any extension of V from Γ_1 to $\partial\Omega = \Gamma_0 \cup \Gamma_1$ with $\tilde{V} \in \mathcal{C}^1(\partial\Omega)$, and let us consider the parabolic problem
$$\begin{gathered} w_t - \Delta w = -w^p \quad \text{in } \Omega \times (0,T) \\ \partial w + \tilde{V}(x) w = w^q \quad \text{on } \partial\Omega \times (0,T) \\ w(x,0) = \frac{1}{\alpha} u_0 > 0 \quad \text{in } \Omega\,. \end{gathered} \tag{4.4}$$
By [11, Lemma 4.5], obtained adapting to our framework the original ideas given in [6, Lemma 5.1] and [6, Theorem 2.3] for $m=1$ therein (also cf. [12],[25]), the solution $w(x,t)$ of (4.4) is globally defined in time ($T=\infty$) and globally bounded in $(x,t) \in \bar{\Omega} \times [0,\infty)$. Now, let us consider the function
$$\bar{v}(x,t) := \alpha e^{-t} w(x,t)$$
Since $w(x,t)$ is globally bounded in $\bar{\Omega} \times [0,\infty)$, we obtain
$$\lim_{t \uparrow \infty} \|\bar{v}(x,t)\|_{L^\infty(\Omega)} = 0 \tag{4.5}$$
Now we prove that for $\alpha > 0$ satisfying (4.3) and λ sufficiently negative, the function $\bar{v}$ is a positive strict supersolution of (4.1). Indeed, since $w > 0$ is globally bounded in $\bar{\Omega} \times [0,\infty)$, by construction the following hold in $\Omega \times [0,\infty)$ for any $\alpha > 0$ and λ negative enough
$$\begin{aligned} \bar{v}_t - \Delta \bar{v} - \lambda \bar{v} &= \alpha e^{-t} w\big(-(1+\lambda) - w^{p-1}\big) \\ &\geq \alpha e^{-t} w\big(-(1+\lambda) - \|w\|^{p-1}_{L^\infty(\Omega\times[0,\infty))}\big) > 0 \end{aligned} \tag{4.6}$$
On the other hand, as for the boundary conditions, owing to (4.3) and since $q > 1$, $\gamma > 0$ and $b > 0$, the following hold on Γ_1
$$\begin{aligned} \partial \bar{v} + V(x)\bar{v} - \gamma b(x) \bar{v}^q &= \alpha e^{-t} w^q \big(1 - \gamma b(x) \alpha^{q-1} e^{-(q-1)t}\big) \\ &\geq \alpha e^{-t} w^q \left(1 - \gamma \|b\|_{L^\infty(\Gamma_1)} \alpha^{q-1}\right) > 0 \end{aligned} \tag{4.7}$$
Also, since $w > 0$ in $\bar{\Omega}$, we obtain
$$\bar{v} \geq 0 \quad \text{on } \Gamma_0 \tag{4.8}$$
Finally,
$$\bar{v}(x,0) = \alpha w(x,0) = u_0(x) \tag{4.9}$$

Thus (4.6), (4.7), (4.8) and (4.9) show that for any λ negative enough, $\bar{v}$ is a positive strict supersolution of (4.1) and hence, owing to the parabolic maximum principle we obtain

$$0 < \Theta_\lambda(x,t;u_0) \le \bar{v}(x,t)\,, \quad (x,t) \in \Omega \times [0,\infty)\,, \tag{4.10}$$

where $\Theta_\lambda(x,t;u_0)$ stands for the solution of (4.1). Now, (4.10) and (4.5) give (4.2) and complete the proof of (i).

(ii) Let us denote $\tilde{d}(x) := \operatorname{dist}(x,\Gamma_0)$, and for each $\delta > 0$ set

$$\Gamma_0^\delta := \{x \in \bar{\Omega} : \tilde{d}(x) < \delta\}$$

Since $\partial\Omega = \Gamma_0 \cup \Gamma_1 \in \mathcal{C}^2$, there exists $\delta > 0$ small enough such that $\tilde{d} \in \mathcal{C}^2(\Gamma_0^\delta)$ and for each $x \in \Gamma_0^\delta$, there exists a unique $y(x) \in \Gamma_0$ such that $\tilde{d}(x) = |y(x) - x|$ and if $n = n(x)$ denotes the unit (outward) normal to Γ_0 at $y(x)$, then

$$|\nabla\tilde{d}|^2 = \Big(\frac{\partial\tilde{d}}{\partial n}\Big)^2 = 1\,, \quad \|\Delta\tilde{d}\|_{L^\infty(\Gamma_0^\delta)} \le C\,,$$

for some $C > 0$ depending only on the curvature of $\partial\Omega$ (cf. [23, Lemma 3.1], [14, Lemmas 14.16 and 14.17]). Let us take any smooth extension $d \in \mathcal{C}^2(\bar{\Omega})$ of $\tilde{d}$, from Γ_0^δ to $\bar{\Omega}$, satisfying

$$|\nabla d| \ge \mu > 0 \quad \text{for all } x \in \Omega\,, \tag{4.11}$$

for some $\mu > 0$, and

$$d(x) > 0 \quad \text{for all } x \in \Omega \cup \Gamma_1 \quad \text{and} \quad \|\Delta d\|_{L^\infty(\Omega)} \le \tilde{C}\,, \tag{4.12}$$

for some $\tilde{C} > 0$.

Now, let φ be the solution of the initial-value problem

$$\begin{gathered} \varphi'(t) = \varphi^q(t)\,, \quad t > 0 \\ \varphi(0) = \alpha > 0\,, \end{gathered} \tag{4.13}$$

for some fixed α satisfying

$$\alpha > \Big(\frac{1 + |\sigma_1|\|d\|_{L^\infty(\Omega)} + \|\Delta d\|_{L^\infty(\Omega)}}{q\mu^2}\Big)^{\frac{1}{q-1}}\,. \tag{4.14}$$

The solution is

$$\varphi(t) := \frac{\alpha}{(1-(q-1)\alpha^{q-1}t)^{\frac{1}{q-1}}}\,.$$

This function blows up in finite time, at

$$T^* = T^*(\alpha,q) := \frac{1}{(q-1)\alpha^{q-1}}\,, \tag{4.15}$$

and since $\varphi(t) > 0$ for all $t \in [0,T^*)$ and $\varphi' = \varphi^q > 0$, we obtain φ is increasing and the following hold

$$\varphi(t) \ge \varphi(0) = \alpha > 0\,, \quad \varphi' = \varphi^q \ge \alpha^q > 0\,, \quad \varphi'' = q\varphi^{q-1}\varphi' \ge q\alpha^{q-1}\varphi' > 0 \tag{4.16}$$

Hereafter we will denote

$$s = s(x,t) := t + d(x)\,.$$

By construction we obtain if $x \in \Omega \cup \Gamma_1$, then $s > t$, and if $x \in \Gamma_0$, then $s = t$. Also, set

$$d_1 := \min_{x\in\Gamma_1} d(x) > 0\,.$$

Since φ is increasing, for each $x \in \Gamma_1$ the following holds

$$\varphi(d(x)) \ge \varphi(d_1) > \varphi(0)\,. \tag{4.17}$$

Now, let us consider for each $x \in \Gamma_1$ the function

$$g(x,t) := \frac{\varphi(t)}{\varphi(t+d(x))} = \frac{\varphi(t)}{\varphi(s)}\,.$$

Since $s > t$ on Γ_1 and φ is increasing and positive, we obtain

$$g_t(x,t) = \frac{\partial g}{\partial t}(x,t) = \frac{\varphi(t)}{\varphi(s)}\left(\varphi^{q-1}(t) - \varphi^{q-1}(s)\right) < 0\,, \quad x \in \Gamma_1$$

Hence, for each fixed $x \in \Gamma_1$, $g(x,t)$ is decreasing in t and owing to (4.17) we obtain

$$g(x,t) < g(x,0) = \frac{\varphi(0)}{\varphi(d(x))} \le \frac{\varphi(0)}{\varphi(d_1)} := \varepsilon < 1 \tag{4.18}$$

Also, for each $x \in \Gamma_1$ we obtain

$$\frac{(\varphi(s)-\varphi(t))^q}{\varphi'(s)} = \frac{(\varphi(s)-\varphi(t))^q}{\varphi^q(s)} = \Big(1 - \frac{\varphi(t)}{\varphi(s)}\Big)^q = (1-g(x,t))^q \tag{4.19}$$

Now, let us consider the function

$$\tilde{u}(x,t) = k\left(\varphi(t+d(x)) - \varphi(t)\right)\,, \tag{4.20}$$

where

$$k > \Big(\frac{\left(\|\partial d\|_{L^\infty(\Gamma_1)} + \|V\|_{L^\infty(\Gamma_1)}\|d\|_{L^\infty(\Omega)}\right)}{\gamma \underline{b}(1-\varepsilon)^q}\Big)^{\frac{1}{q-1}} > 0\,, \tag{4.21}$$

being $\varepsilon \in (0,1)$ and $\mu > 0$ defined by (4.18) and (4.11), respectively. It must be pointed out that for each $x \in \Omega \cup \Gamma_1$, the function $\tilde{u}(x,t)$ blows up in finite time $t^*(x) = T^* - d(x)$, where T^* is the time defined by (4.15).

To prove the result we are going to prove that if $u_0 > 0$ is large enough, then $\tilde{u}(x,t)$ is a positive subsolution of (4.1) which blows up in finite time. The construction of the subsolution $\tilde{u}(x,t)$ made to prove the result, is strongly motivated, up the necessary and involved changes and modifications to adapt it to our mixed boundary conditions with spatial heterogeneities, for the previous constructions made in [33] and [34] for other kind of problems. Indeed, by construction

$$\tilde{u}_t = k(\varphi'(s) - \varphi'(t))\,, \quad \Delta\tilde{u} = k(\varphi''(s)|\nabla d|^2 + \varphi'(s)\Delta d), \tag{4.22}$$

$$\partial\tilde{u} = k\varphi'(s)\partial d\,, \quad x \in \Gamma_1\,. \tag{4.23}$$

Owing to (4.16), if $x \in \Omega \cup \Gamma_1$ there exists $\xi_1(x,t) \in (t,s)$ such that

$$\varphi'(t)d(x) \le \varphi(s) - \varphi(t) = \varphi'(\xi_1(x,t))d(x) \le \varphi'(s)d(x) \tag{4.24}$$

Also, for each $\lambda \ge \sigma_1$ and $x \in \Omega$ we obtain

$$(\lambda - \sigma_1)(\varphi(s) - \varphi(t)) \ge 0 \tag{4.25}$$

Then, owing to (4.20), (4.22), (4.16), (4.11), (4.25), (4.24) and (4.14), the following hold in Ω,

$$\begin{aligned}
&\frac{1}{k}(\tilde{u}_t - \Delta\tilde{u} - \lambda\tilde{u})\\
&= \varphi'(s) - \varphi'(t) - \varphi''(s)|\nabla d|^2 - \varphi'(s)\Delta d - \lambda(\varphi(s) - \varphi(t))\\
&\le \varphi'(s) - \varphi'(t) - q\alpha^{q-1}\varphi'(s)|\nabla d|^2 - \varphi'(s)\Delta d - \lambda(\varphi(s) - \varphi(t))\\
&\le \varphi'(s)\left[1 - q\alpha^{q-1}\mu^2 - \Delta d\right] - \varphi'(t) - (\lambda - \sigma_1)(\varphi(s) - \varphi(t)) - \sigma_1(\varphi(s) - \varphi(t))\\
&\le \varphi'(s)\left[1 - q\alpha^{q-1}\mu^2 - \Delta d\right] + |\sigma_1|(\varphi(s) - \varphi(t))\\
&\le \varphi'(s)\left[1 - q\alpha^{q-1}\mu^2 - \Delta d\right] + |\sigma_1|\varphi'(s)\|d\|_{L^\infty(\Omega)}
\end{aligned}$$

$$\leq \varphi'(s)\left[1-q\alpha^{q-1}\mu^2+\|\Delta d\|_{L^\infty(\Omega)}+|\sigma_1|\|d\|_{L^\infty(\Omega)}\right]<0\,,$$

and therefore,

$$\tilde{u}_t-\Delta\tilde{u}-\lambda\tilde{u}<0 \quad \text{if} \quad (x,t)\in\Omega\times(0,t^*(x)) \tag{4.26}$$

As for the boundary conditions, owing to (4.20), (4.23), (4.24), (4.19), (4.18), (3.2) and (4.21), the following hold on Γ_1,

$$\begin{aligned}
&\partial\tilde{u} \quad +V(x)\tilde{u}-\gamma b(x)\tilde{u}^q\\
&=k(\varphi'(s)\partial d+V(x)(\varphi(s)-\varphi(t))-\gamma b(x)k^q(\varphi(s)-\varphi(t))^q\\
&=k\varphi'(s)\big[\partial d+V(x)\frac{\varphi(s)-\varphi(t)}{\varphi'(s)}-\gamma b(x)k^{q-1}\frac{(\varphi(s)-\varphi(t))^q}{\varphi'(s)}\big]\\
&\leq k\varphi'(s)\left[\|\partial d\|_{L^\infty(\Gamma_1)}+\|V\|_{L^\infty(\Gamma_1)}\|d\|_{L^\infty(\Omega)}-\gamma b(x)k^{q-1}\left(1-g(x,t)\right)^q\right]\\
&\leq k\varphi'(s)\left[\|\partial d\|_{L^\infty(\Gamma_1)}+\|V\|_{L^\infty(\Gamma_1)}\|d\|_{L^\infty(\Omega)}-\gamma\underline{b}k^{q-1}(1-\varepsilon)^q\right]<0\,;
\end{aligned}$$

therefore

$$\partial\tilde{u}+V(x)\tilde{u}-\gamma b(x)\tilde{u}^q<0 \quad \text{if} \quad (x,t)\in\Gamma_1\times(0,t^*(x)) \tag{4.27}$$

Also, since $d(x)=0$ for all $x\in\Gamma_0$, we obtain

$$\tilde{u}(x,t)=0 \quad \text{for all } x\in\Gamma_0,\ t\in[0,T^*) \tag{4.28}$$

Finally, if the initial data u_0 is large enough to satisfy

$$\tilde{u}(x,0)=k(\varphi(d(x))-\varphi(0))=k(\varphi(d(x))-\alpha)\leq u_0(x)\,, \tag{4.29}$$

then, by (4.26), (4.27), (4.28) and (4.29), the following holds

$$\begin{gathered}
\tilde{u}_t-\Delta\tilde{u}-\lambda\tilde{u}<0 \quad \text{if } (x,t)\in\Omega\times(0,t^*(x))\,,\\
\tilde{u}=0 \quad \text{if } (x,t)\in\Gamma_0\times(0,t^*(x))\,,\\
\partial\tilde{u}+V(x)\tilde{u}<\gamma b(x)\tilde{u}^q \quad \text{if } (x,t)\in\Gamma_1\times(0,t^*(x))\,,\\
\tilde{u}(x,0)\leq u_0(x) \quad \text{if } x\in\Omega\,;
\end{gathered}$$

that is, $\tilde{u}(x,t)$ is a positive strict subsolution of (4.1) for each initial data $u_0>0$ satisfying (4.29). Then, owing to the parabolic maximum principle we obtain the solution $\Theta_\lambda(x,t;u_0)$ of (4.1) holds

$$\Theta_\lambda(x,t;u_0)\geq\tilde{u}(x,t)\,, \quad (x,t)\in\Omega\times[0,t^*(x)) \tag{4.30}$$

Now, let us consider a compact set $Q\subset\Omega\cup\Gamma_1$. Since $d(x)>0$ for all $x\in Q$, we obtain there exists $d_Q>0$ such that

$$d(x)\geq d_Q>0 \quad \text{for all } x\in Q. \tag{4.31}$$

Thus, from (4.30), (4.24) and (4.31), we obtain that if $(x,t)\in Q\times[0,T^*)$ then

$$\Theta_\lambda(x,t;u_0)\geq\tilde{u}(x,t)=k(\varphi(s)-\varphi(t))\geq k\varphi'(t)d(x)=k\varphi^q(t)d(x)\geq kd_Q\varphi^q(t)\,, \tag{4.32}$$

and therefore, since φ blows up in finite time $t=T^*$, owing to (4.32), so do $\tilde{u}(x,t)$ and $\Theta_\lambda(x,t;u_0)$ uniformly in Q. This completes the proof of (ii), and the result. □

Remark 4.2. If the domain Ω is in some sense a nice domain, then

$$d(x)=\tilde{d}(x)=\operatorname{dist}(x,\Gamma_0) \quad \text{for all } x\in\bar{\Omega}\,,$$

that is, in the proof of Theorem 4.1-ii) is not necessary to take an smooth extension $d \in \mathcal{C}^2(\bar{\Omega})$ of $\tilde{d}$, from Γ_0^δ to $\bar{\Omega}$, different from $\tilde{d}(x)$, because $\tilde{d}$ already satisfies (4.11) and (4.12). For instance, if Ω is the annulus of $\mathbb{R}^N, N > 1$ given by

$$\Omega := \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$$

with

$$\Gamma_0 := \{x \in \mathbb{R}^N : |x| = R_1\}\,, \quad \Gamma_1 := \{x \in \mathbb{R}^N : |x| = R_2\}\,,$$

for some $0 < R_1 < R_2$, then for all $x \in \bar{\Omega}$ we obtain

$$\tilde{d}(x) = \operatorname{dist}(x, \Gamma_0) = |x| - R_1\,, \quad |\nabla d(x)| = 1\,, \quad |\Delta d(x)| = \frac{N-1}{|x|} \le \frac{N-1}{R_1}\,,$$

and (4.11) and (4.12) hold for

$$d(x) = \tilde{d}(x)\,, \quad \mu = 1\,, \quad \tilde{C} = \frac{N-1}{R_1} > 0.$$

Remark 4.3. In the particular case when $\gamma > 0$, $\lambda \ge \sigma_1$ and (3.2) holds, taking into account in the proof of Theorem 4.1(ii) the definition and properties of the functions $\varphi(t)$ and $d(x)$ (cf. (4.13), (4.11), (4.12)) and the constant k (cf. (4.21)), we obtain (4.29) gives a structure for the initial data $u_0 > 0$, in terms of $d(x)$, to ensure that the positive solution $\Theta_\lambda(x, t; u_0)$ of (4.1) blows up in finite time

Corollary 4.4. *Under the general assumptions, let us consider the superlinear parabolic problem*

$$\begin{gathered} w_t - \Delta w = \lambda w + a(x) w^p \quad \text{in } \Omega \times (0,T)\,,\ p > 1 \\ w = 0 \quad \text{on } \Gamma_0 \times (0,T) \\ \partial w + V(x) w = \gamma b(x) w^q \quad \text{on } \Gamma_1 \times (0,T)\,, \quad q > 1 \\ w(\cdot, 0) = u_0 > 0 \quad \text{in } \Omega\,, \end{gathered} \tag{4.33}$$

where the potential $a \in \mathcal{C}(\bar{\Omega})$, $a > 0$ in Ω, $\gamma > 0$, $\lambda \ge \sigma_1$ and (3.2) *holds. Then, if u_0 is large enough, the solution $w_\lambda(x, t; u_0)$ of* (4.33) *blows up in finite time uniformly in compact subsets $Q \subset \Omega \cup \Gamma_1$.*

Proof. The result follows from the parabolic maximum principle and Theorem 4.1(ii), taking into account that the solution $\Theta_\lambda(x, t; u_0)$ of (4.1) is a positive strict subsolution of (4.33). $\square$

4.2. **Structure of the set of positive solutions of** (1.1)**.** In this section we analyze, in the particular case when $\gamma > 0$, the structure of the set of positive solutions of (1.1) and the linear stability of them. The main result of this section is the following.

Theorem 4.5. *Assume $\gamma > 0$. Then, the following hold:*

(i) *If u_λ is a positive solution of* (1.1) *for the value λ of the parameter, then*
$$\lambda < \sigma_1 \tag{4.34}$$
(ii) *There exists $\lambda^* < 0$, such that* (1.1) *does not admit a positive solution for $\lambda < \lambda^*$.*

(iii) *The value* $\lambda = \sigma_1$ *is the unique bifurcation value to positive solutions from the trivial branch* $(\lambda, u) = (\lambda, 0)$. *From it emanates subcritically a continuum* $\mathfrak{C}^+$ *of positive solutions of* (1.1)*, which also bifurcates to positive solutions from infinity at least in some value* $\lambda_\infty \in [\lambda^*, \sigma_1]$. *In particular,* $\mathfrak{C}^+$ *is unbounded in* $L^\infty(\Omega)$ *and*

$$\mathcal{P}_\lambda(\mathfrak{C}^+ \setminus \{(\sigma_1, 0)\}) \subset [\lambda^*, \sigma_1) . \tag{4.35}$$

(iv) *All the positive solutions of* (1.1) *are unstable.*

Proof. (i) Let $u_\lambda > 0$ be a positive solution of (1.1) for the value λ of the parameter. Then, since $b > 0$, $\gamma > 0$ and owing to (2.1) and to the monotonicity of the principal eigenvalue with respect to the potential on the boundary conditions, we obtain

$$\lambda = \sigma_1^\Omega[-\Delta, \mathfrak{B}(V(x) - \gamma b(x) u_\lambda^{q-1})] < \sigma_1^\Omega[-\Delta, \mathfrak{B}(V(x))] = \sigma_1 ,$$

which proves (4.34).

(ii) Arguing in a similar way to [13, Lemma 4.4] and [11, Theorem 4.6], the result follows from Theorem 4.1-i).

(iii) The existence of the continuum $\mathfrak{C}^+$ of positive solutions of (1.1) emanating subcritically from the trivial branch at the unique bifurcation value to positive solutions $\lambda = \sigma_1$, follows from Theorem 2.2 and Remark 2.3, taking into account that $b > 0$ and $\gamma > 0$. On the other hand, (4.35) follows from *i*) and *ii*). Now, since $\mathfrak{C}^+$ is unbounded in $\mathbb{R} \times L^\infty(\Omega)$ (cf. Remark 2.4) and since by (4.35) $\mathcal{P}_\lambda(\mathfrak{C}^+)$ is bounded in $\mathbb{R}$, we obtain $\mathfrak{C}^+$ is unbounded in $L^\infty(\Omega)$ and therefore, it must bifurcate from infinity at least in some value $\lambda_\infty \in [\lambda^*, \sigma_1]$.

(iv) To prove the result we will prove that the principal eigenvalue of the linearization of (1.1) at any positive solution u_λ of it is negative. Indeed, let $u_\lambda > 0$ be a positive solution of (1.1) for some value λ of the parameter with $\lambda < \sigma_1$. The linearization of (1.1) at u_λ is given by

$$\begin{gathered} (-\Delta - \lambda) w = 0 \quad \text{in } \Omega \\ \mathfrak{B}\big(V(x) - \gamma q b(x) u_\lambda^{q-1}\big) w = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{4.36}$$

Then, taking into account that $\gamma > 0$, $b > 0$, $u_\lambda > 0$ and $q > 1$, it follows from (2.1) and from the monotonicity of the principal eigenvalue with respect to the potential on the boundary condition that

$$\sigma_1^\Omega\big[-\Delta - \lambda, \mathfrak{B}\big(V(x) - \gamma q b(x) u_\lambda^{q-1}\big)\big] < \sigma_1^\Omega\big[-\Delta - \lambda, \mathfrak{B}\big(V(x) - \gamma b(x) u_\lambda^{q-1}\big)\big] = 0$$

which completes the proof of (iv), and of the theorem. $\square$

Remark 4.6. In the particular case when $\gamma > 0$, if we denote

$$\begin{gathered} \Lambda := \{\lambda \in \mathbb{R} : (1.1) \text{ possesses positive solution } \} \subset [\lambda^*, \sigma_1), \\ \tilde\lambda := \inf \Lambda \geq \lambda^* , \quad \hat\lambda := \inf\{ \lambda : (\lambda, u) \in \mathfrak{C}^+ \} , \end{gathered}$$

it might occur that $\tilde\lambda < \hat\lambda$, by the existence of an isola $\mathfrak{G}$ of positive solutions of (1.1) such that

$$\lambda^* \leq \tilde\lambda \leq \inf\{ \lambda : (\lambda, u) \in \mathfrak{G}\} < \hat\lambda$$

In fact, Λ might be unconnected, since it might occur that

$$\tilde\lambda \leq \inf\{\lambda : (\lambda, u) \in \mathfrak{G}\} < \bar\lambda := \sup\{ \lambda : (\lambda, u) \in \mathfrak{G}\} < \hat\lambda$$

and that (1.1) does not possess positive solutions for $\lambda \in (\bar\lambda, \hat\lambda)$. In this kind of problems, if u_1 is a positive solution of (1.1) for $\lambda = \lambda_1$, then small positive multiples of u_1 are not positive subsolutions of (1.1) for $\lambda > \lambda_1$, and large positive multiples of u_1 are not positive supersolutions of (1.1) for $\lambda < \lambda_1$.

Acknowledgments. This research was supported by the Ministry of Economy and Competivity under grant MTM2015-65899-P. The author wants to thank the anonymous referees for their useful comments and remarks which have contributed to improve the earlier version of this paper.

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