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SUFFICIENT CONDITIONS FOR THE EXISTENCE OF NON-OSCILLATORY SOLUTIONS TO FIRST-ORDER DIFFERENTIAL EQUATIONS WITH MULTIPLE ADVANCED ARGUMENTS

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Abstract. This article concerns the existence of non-oscillatory solutions to the equation

$$x'(t) = \sum_{k=1}^{m} a_k(t)x(h_k(t)),$$

where $a_k \geq 0$ and $h_k(t) \geq t$. We generalize existing results and then give an answer to the open question stated in [4]. Moreover we present a new condition based on the integral of $(\sum a_k)^2$.

1. Introduction

In this article we show sufficient conditions for the existence of non-oscillatory solutions to the equation

$$x'(t) = \sum_{k=1}^{m} a_k(t)x(h_k(t))\,, \tag{1.1}$$

where a_k, and h_k are in $C([t_0,\infty),\mathbb{R})$ with $a_k(t) \geq 0$, and $h_k(t) \geq t$.

Advanced equations have applications in real world problems where the current rate of change may depend on future events. Such phenomena happen in population dynamics and in economics, see [7]. For additional information on advanced equations see the books [1, 2, 8, 9, 10, 11].

By a solution we mean a nontrivial function $x \in C^1([t_0,\infty),\mathbb{R})$ that satisfies (1.1). A solution is called oscillatory if it has arbitrarily large zeros; otherwise is non-oscillatory. Note that if x is a positive solution of (1.1) for sufficient large t, then $-x$ is also a solution; so among non-oscillatory solutions, we only consider positive solutions.

The main two approaches for solving (1.1) are: using a fixed point argument, and solving an inequality that implies the solvability of (1.1). In this article we use the second approach. It is well known that the existence of positive solutions to

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(1.1) is equivalent to the existence of a solution to the inequality

$$\lambda(t) \geq \sum_{k=1}^{m} a_k(t) \exp\Big(\int_t^{h_k(t)} \lambda(s)\,ds\Big) \quad \forall t \geq t_0\,. \tag{1.2}$$

However there is no specific way for finding solutions to (1.2). Our strategy is to define solutions λ of special form, and then find the corresponding inequalities for $\sum a_k$ that guarantee the existence of solutions to (1.2). These inequalities can include values $\sum a_k(t)$, or integrals of $\sum a_k$. In some publications, the inequalities are called pointwise and integral conditions, respectively. Here, we generalize some of the existing conditions and then give an answer to the open question stated in [4] (see inequality (3.6)). Also we present a new condition based on the integral of $(\sum a_k)^2$.

The following theorem shows sufficient conditions for the existence of solutions to (1.1); see [1, Thm. 5.1], and for $m = 1$ with $h_1(t) = t + \tau$, see [6, Thm. 1]. The converse of the theorem is easy to prove by using $\lambda(t) = x'(t)/x(t)$.

Theorem 1.1. *Suppose that inequality* (1.2) *has a non-negative solution that is integrable on each interval* $[t_0, b]$. *Then* (1.1) *has a positive solution for* $t \geq t_0$.

Note that letting

$$h(t) = \max\{h_k(t) : k = 1, 2, \dots, m\}\,, \tag{1.3}$$

it follows that solutions to

$$\lambda(t) \geq \sum_{k=1}^{m} a_k(t) \exp\Big(\int_t^{h(t)} \lambda(s)\,ds\Big) \quad \forall t \geq t_0\,. \tag{1.4}$$

are also solutions to (1.2). Furthermore, if there is a constant $\tau > 0$ such that

$$h(t) - t \leq \tau, \quad \forall t \geq t_0\,, \tag{1.5}$$

then solutions to

$$\lambda(t) \geq \sum_{k=1}^{m} a_k(t) \exp\Big(\int_t^{t+\tau} \lambda(s)\,ds\Big) \quad \forall t \geq t_0 \tag{1.6}$$

are also solutions to (1.4), and hence to (1.2).

2. Conditions using point values of $\sum a_k$

Initially we look for a constant solution to (1.6). This leads to an extension of the well-known condition $1/(\tau e) \geq a_1(t)$, see [11].

Theorem 2.1. *Under assumption* (1.5)*, the condition*

$$\frac{1}{\tau e} \geq \sum_{k=1}^{m} a_k(t) \quad \forall t \geq t_0 \geq 1 \tag{2.1}$$

is sufficient for the existence of solutions to (1.1).

Proof. Let $\lambda(t) = \lambda_0$ be a constant. In this case, assuming (1.5), condition (1.6) is implied by

$$\lambda_0 \geq \Big(\sum_{k=1}^{m} a_k(t)\Big) e^{\lambda_0 \tau} \quad \forall t \geq t_0\,.$$

Note that the mapping $\lambda_0 \mapsto \lambda_0/e^{\lambda_0\tau}$ has its maximum at $\lambda_0 = 1/\tau$. With this value we obtain (2.1) which implies the existence of a solution to (1.6), and hence the existence of a positive solution to (1.1). $\square$

Next we perturb the constant solution $1/\tau$, with a positive function that tends to zero as $t \to 0$ and leads to a simple integral in (1.6).

Theorem 2.2. *Under assumption* (1.5)*, the condition*

$$\frac{1}{\tau e}\big(1+\frac{\tau}{2t}\big)/\big(1+\frac{\tau}{t}\big)^{1/2} \ge \sum_{k=1}^m a_k(t) \quad \forall t \ge t_0 \tag{2.2}$$

is sufficient for the existence of solutions to (1.1)*.*

Proof. We look for solutions, to (1.6), of the form $\lambda(t) = \frac{1}{\tau} + \frac{\alpha}{t}$, where $\alpha \ge 0$ and $t \ge 1$. In this case (1.6) is implied by

$$\frac{1}{\tau} + \frac{\alpha}{t} \ge \Big(\sum_{k=1}^m a_k(t)\Big) \exp\Big(\int_t^{t+\tau} (\frac{1}{\tau} + \frac{\alpha}{s})\, ds = \Big(\sum_{k=1}^m a_k(t)\Big) e\big(1+\frac{\tau}{t}\big)^\alpha$$

which is equivalent to

$$\frac{1}{\tau e}\big(1+\frac{\alpha\tau}{t}\big)/\big(1+\frac{\tau}{t}\big)^{\alpha} \ge \sum_{k=1}^m a_k(t) \quad \forall t \ge 1\,. \tag{2.3}$$

Note that for $t \ge 1$ and $0 \le \alpha \le 1$, using two terms of the Taylor series for the mapping $x \mapsto (1+x)^\alpha$ about $x = 0$, we have

$$\big(1+\frac{\alpha\tau}{t}\big)/\big(1+\frac{\tau}{t}\big)^{\alpha} \ge \big(1+\frac{\alpha\tau}{t}\big)/\big(1+\frac{\alpha\tau}{t}\big) = 1\,,$$

with equality when $\alpha = 0$ or $\alpha = 1$. Therefore (2.3) is less restrictive, on $\sum a_k$, than (2.1). Setting $\alpha = 1/2$, we have condition (2.2) which implies the existence of solutions to (1.6); thus the existence of solutions to (1.1). $\square$

Another perturbation of the constant solution $1/\tau$ is $\lambda(t) = \frac{1}{\tau} + \frac{\alpha}{t^2}$ with $\alpha \ge 0$ and $t \ge 1$. In this case assuming (1.5), condition (1.6) is implied by

$$\frac{1}{\tau} + \frac{\alpha}{t^2} \ge \Big(\sum_{k=1}^m a_k(t)\Big) \exp\Big(\int_t^{t+\tau} (\frac{1}{\tau} + \frac{\alpha}{s^2})\, ds\Big) = \Big(\sum_{k=1}^m a_k(t)\Big) e \exp\Big(\frac{\alpha\tau}{t(t+\tau)}\Big)\,.$$

Then the condition

$$\frac{1}{\tau e}\big(1+\frac{\alpha\tau}{t^2}\big)/\exp\big(\frac{\alpha\tau}{t(t+\tau)}\big) \ge \sum_{k=1}^m a_k(t) \quad \forall t \ge 1\,. \tag{2.4}$$

is sufficient for the existence of a positive solution to (1.1). Since

$$\exp\big(\frac{\alpha\tau}{t(t+\tau)}\big) \ge 1 + \frac{\alpha\tau}{t(t+\tau)} + \frac{1}{2}\Big(\frac{\alpha\tau}{t(t+\tau)}\Big)^2 \ge 1 + \frac{\alpha\tau}{t^2}\,,$$

the coefficient of $\frac{1}{\tau e}$ in (2.4) is less than 1. Therefore, (2.4) is more restrictive, on $\sum a_k$, than (2.1). Then there is no advantage in using (2.4) instead of (2.1).

Next we consider the perturbed function $\lambda(t) = \frac{1}{\tau} + \frac{\alpha}{t\ln(t)}$ with $\alpha \ge 0$ and $t \ge e$. Under assumption (1.5), condition (1.6) is implied by

$$\frac{1}{\tau} + \frac{\alpha}{t\ln(t)} \ge \Big(\sum_{k=1}^m a_k(t)\Big) \exp\Big(\int_t^{t+\tau} (\frac{1}{\tau} + \frac{\alpha}{s\ln(s)})\, ds\Big)$$

$$= \Big(\sum_{k=1}^m a_k(t)\Big) e\Big(\frac{\ln(t+\tau)}{\ln(t)}\Big)^{\alpha}.$$

Then the condition

$$\frac{1}{\tau e}\Big(1+\frac{\alpha\tau}{t\ln(t)}\Big)/\Big(\frac{\ln(t+\tau)}{\ln(t)}\Big)^{\alpha} \geq \sum_{k=1}^m a_k(t) \quad \forall t \geq e\,. \tag{2.5}$$

is sufficient for the existence of a positive solution (1.1). There is no optimal value for α in (2.5), and if its exists, it depends on τ.

To compare (2.5) with (2.2) we graph the upper bounds in both inequalities. The graph corresponding to (2.5) is eventually above the graph corresponding to (2.2) for each α. This indicates that (2.5) is less restrictive, on $\sum a_k$, than (2.2).

Next we consider the perturbed function $\lambda(t) = \frac{1}{\tau} + \frac{\alpha}{t\ln(t)\ln(\ln(t))}$ with $\alpha \geq 0$ and $t \geq e^e$. This leads to the condition

$$\frac{1}{\tau e}\Big(1+\frac{\alpha\tau}{t\ln(t)\ln(\ln(t))}\Big)/\Big(\frac{\ln\big(\ln(t+\tau)\big)}{\ln\big(\ln(t)\big)}\Big)^{\alpha} \geq \sum_{k=1}^m a_k(t) \quad \forall t \geq e^e\,. \tag{2.6}$$

which is sufficient for the existence of a positive solution (1.1). Graphing their upper bounds, we conclude that (2.6) is less restrictive than (2.2), but more restrictive than (2.5). Apparently further iterations of the logarithm do not lead to less restrictive conditions.

Combining two of the perturbations above, we have $\lambda(t) = \frac{1}{\tau} + \frac{1}{2t} + \frac{1}{2t\ln(t)}$. For this λ, we obtain the condition

$$\frac{1}{\tau e}\Big(1+\frac{\tau}{2t\ln(t)}\Big)/\Big(\frac{\ln(t+\tau)}{\ln(t)}\Big)^{1/2} \geq \sum_{k=1}^m a_k(t)\,. \tag{2.7}$$

Using the the same λ and Taylor polynomials, Diblik [3] obtained the condition

$$\frac{1}{\tau e}\Big(1+\frac{\tau^2}{8t^2}\Big) \geq a_1(t)\,. \tag{2.8}$$

Comparing the graphs of their bounds, we realize that (2.7) is less restrictive than (2.8). Some other functions λ have been considered in [6], such as

$$\lambda(t) = \frac{1}{\tau}\Big(1 + \frac{\tau}{2t} + \frac{\tau}{2t\ln(t)} + \dots + \frac{\tau}{2t\ln(t)\ln(\ln(t))\cdots\ln(\dots\ln(t))}\Big).$$

We conclude this section by stating that there are many good choices for λ, but there is no known optimal choice.

3. Conditions using weighted integrals of $\sum a_k$

Initially we look for solutions to (1.4), of the form $\lambda(t) = \sum_{k=1}^m a_k(t)e^{\delta}$, where δ is a constant. Under assumption (1.3), condition (1.4) is implied by

$$e^{\delta}\Big(\sum_{k=1}^m a_k(t)\Big) \geq \Big(\sum_{k=1}^m a_k(t)\Big)\exp\Big(e^{\delta}\int_t^{h(t)}\sum_{k=1}^m a_k(s)\,ds\Big)$$

which is equivalent to

$$\frac{\delta}{e^{\delta}} \geq \int_t^{h(t)}\sum_{k=1}^m a_k(s)\,ds\,.$$

Note that the mapping $\delta \mapsto \delta/e^\delta$ has its maximun when $\delta = 1$. Therefore we set

$$\frac{1}{e} \geq \int_t^{h(t)} \sum_{k=1}^m a_k(s)\,ds \quad \forall t \geq t_0 \tag{3.1}$$

as a sufficient condition for the existence of positive solutions for (1.1). This condition is the same as in [1, Corollary 5.1]. Also when $m = 1$ and $h(t) = t + \tau$, condition (3.1) becomes the classical condition $1/e \geq \int_t^{t+\tau} a_1(s)\,ds$. It is also well known that if $\liminf_{t\to\infty} \int_t^{t+\tau} a_1(s)\,ds > 1/e$, then every solution of (1.1) with $m = 1$ is oscillatory; see [8, p. 31]

Now we consider a perturbation of the solution $\lambda(t) = \sum_{k=1}^m a_k(t)e$, by using a non-negative function w that decays to zero as $t \to \infty$. Let

$$\lambda(t) = \sum_{k=1}^m a_k(t)e^{1+w(t)}\,.$$

Then under assumption (1.3), condition (1.4) is implied by

$$\Big(\sum_{k=1}^m a_k(t)\Big)e^{1+w(t)} \geq \Big(\sum_{k=1}^m a_k(t)\Big)\exp\Big(\int_t^{h(t)} e^{1+w(s)} \sum_{k=1}^m a_k(s)\,ds\Big)$$

which is equivalent to the condition

$$\frac{1}{e}\big(1 + w(t)\big) \geq \int_t^{h(t)} e^{w(s)} \sum_{k=1}^m a_k(s)\,ds \quad \forall t \geq t_0\,, \tag{3.2}$$

which correspond to [4, Lemma 1]. This is a sufficient condition, but it is an implicit condition. With the goal of separating the variables w and $\sum a_k(s)$ in the above inequality, we assume that w is non-decreasing. Then we obtain the condition

$$\frac{1}{e}\big(1 + w(t)\big)/e^{w(t)} \geq \int_t^{h(t)} \sum_{k=1}^m a_k(s)\,ds \quad \forall t \geq t_0\,. \tag{3.3}$$

The variables are separated, but (3.3) is more restrictive than (3.2), because of the inequality $1 + w \leq e^w$.

Next we follow the approach in [4] of partitioning the interval $[t, t + \tau]$ into n subintervals of equal length, τ/n.

Theorem 3.1. *Assume* (1.4), $n \geq 2$, *and* w *is non-increasing. Also assume that there exists a non-increasing function* $\beta_{n,\tau}$ *such that*

$$\int_t^{t+\tau/n} \sum_{k=1}^m a_k(s)\,ds \leq \beta_{n,\tau}(t), \tag{3.4}$$

and

$$\frac{1}{e}\big(1 + w(t)\big) \geq \beta_{n,\tau}(t + \frac{\tau}{n}) \sum_{i=1}^n \exp\big(w(t + \frac{(i-1)\tau}{n})\big), \tag{3.5}$$

for all $t \geq t_0$. *Then* (3.2) *is satisfied so that there exists a positive solution to* (1.1).

Proof. Note that the right-hand side of (3.2) satisfies

$$\int_t^{t+\tau} e^{w(s)} \sum_{k=1}^m a_k(s)\,ds = \sum_{i=1}^n \int_{t+(i-1)\tau/n}^{t+i\tau/n} e^{w(s)} \sum_{k=1}^m a_k(s)\,ds$$

$$
\begin{aligned}
&\le \sum_{i=1}^{n} \exp\big(w(t+(i-1)\tau/n)\big) \int_{t+(i-1)\tau/n}^{t+i\tau/n} \sum_{k=1}^{m} a_k(s)\,ds \\
&\le \beta_{n,\tau}(t+\frac{\tau}{n}) \sum_{i=1}^{n} \exp\big(w(t+\frac{(i-1)\tau}{n})\big)\,.
\end{aligned}
$$

This inequality and (3.5) imply (3.2); thus there exists a positive solution to (1.1). □

A concrete example of Theorem 3.1 was obtained in [4], where

$$
\beta_{n,\tau}(t) = \frac{1}{ne} + \frac{\mu(n-1)^2\tau^2}{8n^3t^2e} \quad \text{and} \quad w(t) = \frac{(n-1)\tau}{2nt}\,,
$$

with $0 < \mu < 1$ and $n \ge 2$.

Trying to address the case $n = 1$, Diblik [4] formulated the open question: Prove of disprove that if for some $\mu \in (0,1)$ and $t_0 > 0$,

$$
\sup_{t \ge t_0} t^2 \Big(\int_t^{t+\tau} a_1(s)\,ds - \frac{1}{e} \Big) \le \frac{\mu\tau^2}{8e}, \tag{3.6}
$$

then there exists a positive solution to (1.1) with $m = 1$ and $h(t) = t+\tau$. A positive answer to this question is found as a particular case of the next Theorem. First we rewrite (3.6) as (3.7) below.

Theorem 3.2. *Assume that* (1.5) *holds and*

$$
\int_t^{t+\tau} \sum_{k=1}^{m} a_k(s)\,ds \le \frac{1}{e}\Big(1 + \frac{\mu\tau^2}{8t^2}\Big) \tag{3.7}
$$

for some $\mu \in (0,1)$ and all $t \ge t_0$. Then there exist a positive solution to (1.1).

Proof. Our strategy is to show that (2.2) is satisfied, so that Theorem 2.2 can be applied. Since $\sum a_k$ is a continuous function, by the mean value theorem for integrals, there exists $\xi \in [t, t+\tau]$ such that

$$
\int_t^{t+\tau} \sum_{k=1}^{m} a_k(s)\,ds = \tau \sum_{k=1}^{m} a_k(\xi)\,.
$$

Note that ξ depends on t, τ, a_k. Then by (3.7), and since $t \le \xi \le t+\tau$, we have

$$
\sum_{k=1}^{m} a_k(\xi) \le \frac{1}{e\tau}\Big(1 + \frac{\mu\tau^2}{8\xi^2}\Big) \le \frac{1}{e\tau}\Big(1 + \frac{\mu\tau^2}{8(t-\tau)^2}\Big)\,.
$$

To show that $\sum a_k$ satisfies (2.2), we need to show that

$$
\frac{1}{\tau e}\big(1 + \frac{\tau}{2t}\big)/\big(1 + \frac{\tau}{t}\big)^{1/2} \ge \frac{1}{e\tau}\Big(1 + \frac{\mu\tau^2}{8(t-\tau)^2}\Big)\,. \tag{3.8}
$$

Squaring both sides, this inequality is equivalent to

$$
\begin{aligned}
&1 + \frac{\tau}{t} + \frac{\tau^2}{4t^2} \\
&\ge 1 + \frac{\tau}{t} + \frac{\mu\tau^2}{4(t-\tau)^2} + \frac{\mu\tau^3}{4t(t-\tau)^2} + \frac{\mu^2\tau^4}{64(t-\tau)^4} + \frac{\mu^2\tau^5}{64t(t-\tau)^4}\,.
\end{aligned}
$$

Note that

$$\frac{\tau^2}{4t^2} - \frac{\mu\tau^2}{4(t-\tau)^2} = \frac{\tau^2}{4}\Big(\frac{1-\mu}{(t-\tau)^2} - \frac{2\tau}{t(t-\tau)^2} + \frac{\tau^2}{t^2(t-\tau)^2}\Big)\,.$$

Since $\frac{1-\mu}{(t-\tau)^2} > 0$ and its decay is slower that the other two terms as $t \to \infty$, the above inequality is true for t large enough. Therefore $\sum a_k$ satisfies (2.2), and by Theorem 2.2 there exists a positive solution to (1.1). The proof is complete. $\square$

We conclude this section by remarking that (3.7) is less restrictive than (3.1). Also for some examples the pair (3.4)-(3.5) is less restrictive than (3.7), but for other examples is more restrictive.

4. Conditions using integrals of $(\sum a_k)^2$

In this section we define solutions of (1.4) depending on $\sum a_k$ and obtain integral conditions for $(\sum a_k)^2$.

Theorem 4.1. *Under assumption* (1.3)*, the existence of a non-negative and continuous function g such that*

$$g^2(t)/\Big(\int_t^{h(t)} e^{2g(s)}\,ds\Big) \geq \int_t^{t+\tau} \Big(\sum_{k=1}^m a_k(s)\Big)^2\,ds \quad \forall t \geq t_0 \tag{4.1}$$

implies the existence of a positive solution to (1.1)*.*

Proof. Letting $\lambda(t) = \sum_{k=1}^m a_k(t)e^{g(t)}$, condition (1.4) is implied by

$$\Big(\sum_{k=1}^m a_k(t)\Big)e^{g(t)} \geq \Big(\sum_{k=1}^m a_k(t)\Big)\exp\Big(\int_t^{h(t)} \sum_{k=1}^m a_k(s)e^{g(s)}\,ds\Big)$$

which is equivalent to

$$g(t) \geq \int_t^{h(t)} e^{g(s)} \sum_{k=1}^m a_k(s)\,ds\,.$$

From (4.1) and the Cauchy-Schwarz inequality, we have

$$g^2(t) \geq \Big(\int_t^{h(t)} e^{2g(s)}\,ds\Big)\Big(\int_t^{h(t)} \big(\sum_{k=1}^m a_k(s)\big)^2\,ds\Big) \geq \Big(\int_t^{h(t)} e^{g(s)} \sum_{k=1}^m a_k(s)\,ds\Big)^2\,.$$

Therefore $\lambda(t)$ is a solution of (1.4), which implies the existence of a positive solution to (1.1). $\square$

Now we consider $g(t) = R$, where $R > 0$. Then (4.1) is implied by

$$\frac{R^2}{(h(t)-t)e^{2R}} \geq \int_t^{h(t)} \big(\sum_{k=1}^m a_k(s)\big)^2\,ds\,.$$

Assuming (1.5) and recalling that $R/e^R \leq 1/e$ for all R, we have the condition

$$\frac{1}{\tau e^2} \geq \int_t^{t+\tau} \big(\sum_{k=1}^m a_k(s)\big)^2\,ds \quad \forall t \geq t_0\,. \tag{4.2}$$

This is a sufficient condition for the existence of a positive solution to (1.1), because $\lambda(t) = \sum_{k=1}^m a_k(t)e$ is a solution of (1.4).

Now we consider the perturbed function $g(t) = 1 + \frac{1}{2}\ln(1+\tau/t)$. Then

$$\int_t^{t+\tau} e^{2g(s)}\,ds = \int_t^{t+\tau} e^2(1+\tau/s)\,ds = \tau e^2\Big(1+\ln\big(1+\tau/t\big)\Big).$$

Following the above process we set

$$\frac{1}{\tau e^2}\big(1+\frac{1}{2}\ln(1+\tau/t)\big)^2/\big(1+\ln(1+\tau/t)\big) \geq \int_t^{t+\tau}\big(\sum_{k=1}^m a_k(s)\big)^2\,ds \quad \forall t\geq t_0 \tag{4.3}$$

as a sufficient condition for the existence of a positive solution to (1.1). Note that expanding the square above, we have

$$\begin{aligned}\big(1+\frac{1}{2}\ln(1+\tau/t)\big)^2 &= 1+\ln(1+\tau/t)+\big(\frac{1}{2}\ln(1+\tau/t)\big)^2\\ &\geq 1+\ln(1+\tau/t).\end{aligned}$$

Therefore the coefficient of $1/(\tau e^2)$ in (4.3) is greater than 1; thus (4.3) is less restrictive than (4.2).

Now we consider the perturbed function $g(t) = 1+\frac{1}{2}\ln\big(1+\tau/(t\ln(t)\big)$. Then

$$\int_t^{t+\tau} e^{2g(s)}\,ds = \int_t^{t+\tau} e^2(1+\tau/(s\ln(s))\,ds = \tau e^2\Big(1+\ln\big(\frac{\ln(t+\tau)}{\ln(t)}\big)\Big).$$

Following the above process we set

$$\frac{1}{\tau e^2}\Big(1+\frac{1}{2}\ln\big(1+\tau/(t\ln(t))\big)\Big)^2/\Big(1+\ln\big(\frac{\ln(1+\tau)}{\ln(t)}\big)\Big) \geq \int_t^{t+\tau}\big(\sum_{k=1}^m a_k(s)\big)^2\,ds \tag{4.4}$$

as a sufficient condition for the existence of a positive solution to (1.1). Graphing their upper bounds we notice that (4.3) is less restrictive than (4.4). As an example of a function that satisfies (4.4) but not (4.3), we have $a_1(t) = (1+\tau/(t^2)$, when $m = 1$ in (1.1).

We conclude this section with a theorem that uses $\int(\sum a_k)^2$ to establish the oscillation of all solutions.

Theorem 4.2. *Assume that $m = 1$, $0 \leq a_1(t) \leq a_{\max} < \infty$ and $h(t) = t+\tau$ for all $t \geq t_0$. If*

$$\frac{1}{\tau e^2} < \liminf_{t\to\infty}\int_t^{t+\tau} a_1^2(s)\,ds, \tag{4.5}$$

then all solutions to (1.1) are oscillatory.

Proof. The strategy is to show that $1/e < \liminf_{t\to\infty}\int_t^{t+\tau} a_1(s)\,ds$ which implies the oscillation of all solutions. From (4.5), there exist t_1 and $\epsilon > 0$ such that

$$\frac{1}{\tau e^2}+\epsilon \leq \int_t^{t+\tau} a_1^2(s)\,ds \quad \forall t \geq t_1.$$

Then

$$\begin{aligned}\epsilon &\leq \int_t^{t+\tau}\big(a_1^2(s)-\frac{1}{\tau^2e^2}\big)\,ds = \int_t^{t+\tau}\big(a_1(s)+\frac{1}{\tau e}\big)\big(a_1(s)-\frac{1}{\tau e}\big)\,ds\\ &\leq \big(a_{\max}+\frac{1}{\tau e}\big)\int_t^{t+\tau}\big(a_1(s)-\frac{1}{\tau e}\big)\,ds\end{aligned}$$

Therefore,

$$\frac{\epsilon}{(a_{\max} + \frac{1}{\tau e})} \leq \int_t^{t+\tau} a_1(s)\,ds - \int_t^{t+\tau} \frac{1}{\tau e}\,ds \quad \forall t \geq t_1\,,$$

which implies

$$\frac{1}{e} + \frac{\epsilon}{(a_{\max} + \frac{1}{\tau e})} \leq \liminf_{t\to\infty} \int_t^{t+\tau} a_1(s)\,ds\,.$$

Thus, all solutions of (1.1) are oscillatory. □

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Addendum posted on October 21, 2019

In response to a message from a reader, we want to make two corrections:

(1) Ignore the comment after equation (2.4), i.e. delete "Since ... instead of (2.1)". This is done because the inequality in that comment does not hold.

(2) To clarify the statement before (4.2), replace "we have the condition" with "when $R = 1$, we have the condition"

End of addendum.

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