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STABILITY OF WEAK SOLUTIONS OF A NON-NEWTONIAN POLYTROPIC FILTRATION EQUATION

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Abstract. We study a non-Newtonian polytropic filtration equation with a convection term. We introduce new type of weak solutions and show the existence of weak solutions. We show that when $\int_\Omega [a(x)]^{-1(p-1)}dx < \infty$, the stability of weak solutions is based on the usual initial-boundary value conditions. When $1 < p < 2$, under the given conditions on the diffusion coefficient and the convection term, the stability of weak solutions can be proved without any boundary value condition. In particular, the stability results are presented based on the given optimal boundary value condition.

1. Introduction

Consider the non-Newtonian polytropic filtration equation

$$u_t = \operatorname{div}\big(a(x)|\nabla u^m|^{p-2}\nabla u^m\big) + \vec{b}(x)\cdot\nabla u^q,\ (x,t)\in\Omega\times(0,T), \tag{1.1}$$

where $p > 1$, $m > 0$, $q > 0$, $a(x) \geq 0$, $a(x) \in C^1(\overline{\Omega})$, $\vec{b(x)} = (b_i(x))$, $i = 1, 2, \dots, N$, $b_i(x) \in C^1(\overline{\Omega})$, and $\Omega \subset \mathbb{R}^N$ is a bounded domain with the smooth boundary. Equation (1.1) arises from a variety of diffusion phenomena such as soil physics, fluid dynamics, combustion theory, and reaction chemistry [1, 2, 3, 4].

When $a(x) \equiv 1$, equation (1.1) with the usual initial-boundary value conditions

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega\times(0,T), \tag{1.3}$$

has been extensively studied, see [5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and the references therein. In this study, we restrict our attention to the case of $a(x) \geq 0$ and the stability of weak solutions of (1.1). Note that when $p > 1$, $|\nabla u^m|^{p-2}$ may be singular or degenerate on $\overline{\Omega}$.

Definition 1.1. A function $u(x,t)$ is said to be the weak solution of type 1 of (1.1), if $u(x,t)$ satisfies

$$u \in L^\infty(Q_T), \quad \frac{\partial u^m}{\partial t} \in L^2(Q_T), \quad a(x)|\nabla u^m|^p \in L^1(Q_T), \tag{1.4}$$

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and for any functions $\varphi_1 \in L^1(0,T;C_0^1(\Omega))$, $\varphi_2 \in L^\infty(Q_T)$, and $\varphi_2(x,\cdot) \in W^{1,p}_{loc}(\Omega)$ for any given $t \in [0,T)$, it holds

$$\begin{aligned} &\iint_{Q_T} \big[\frac{\partial u}{\partial t}(\varphi_1\varphi_2) + a(x)|\nabla u^m|^{p-2}\nabla u^m \cdot \nabla(\varphi_1\varphi_2)\big]\,dx\,dt \\ &+ \iint_{Q_T} [u^q b_i(x)(\varphi_1\varphi_2)_{x_i} + u^q b_{ix_i}\varphi_1\varphi_2]\,dx\,dt = 0. \end{aligned} \tag{1.5}$$

The initial value condition (1.2) is satisfied in the sense of

$$\lim_{t\to 0}\int_\Omega |u(x,t) - u_0(x)|dx = 0. \tag{1.6}$$

With the assumption that

$$a(x) > 0,\ x \in \Omega \quad \text{and} \quad a(x) = 0,\ x \in \partial\Omega, \tag{1.7}$$

we can obtain the existence of weak solutions of (1.1) with the initial value condition (1.2) as follows.

Theorem 1.2. *Suppose that $p \ge 2$, $m > 0$, $q \ge 1 + \frac{m-1}{2}$, and $u_0 \ge 0$ satisfies*

$$u_0 \in L^\infty(\Omega), \quad a(x)|\nabla u_0^m|^p \in L^1(\Omega). \tag{1.8}$$

If $a(x)$ satisfies (1.7) *and*

$$\int_\Omega [a(x)]^{-\frac{2}{p-2}}|\vec{b}(x)|^{\frac{2p}{p-2}}dx < \infty, \tag{1.9}$$

then there exists a nonnegative weak solution of type 1 to equation (1.1)*.*

Clearly, if we denote $\phi(v) = v^{\frac{1}{m}}$ and $v = A(u) = u^m$, then (1.4) is equivalent to

$$v \in L^\infty(Q_T), \quad \frac{\partial \phi(v)}{\partial t} \in L^2(Q_T),\ a(x)|\nabla v|^p \in L^1(Q_T), \tag{1.10}$$

and (1.5) is equivalent to

$$\begin{aligned} &\iint_{Q_T} \big[\frac{\partial \phi(v)}{\partial t}(\varphi_1\varphi_2) + a(x)|\nabla v|^{p-2}\nabla v \cdot \nabla(\varphi_1\varphi_2)\big]\,dx\,dt \\ &+ \iint_{Q_T} [v^{\frac{q}{m}} b_i(x)(\varphi_1\varphi_2)_{x_i} + v^{\frac{q}{m}} b_{ix_i}(x)\varphi_1\varphi_2]\,dx\,dt = 0. \end{aligned} \tag{1.11}$$

The following theorems relate to the stability of weak solutions.

Theorem 1.3. *Let $u(x,t)$ and $v(x,t)$ be two nonnegative weak solutions of type 1 of* (1.1) *when $p > 1$, $q \ge \max\{m,1\}$ and $m > 0$. Suppose that condition* (1.3) *holds and*

$$\int_\Omega [a(x)]^{-\frac{1}{p-1}}dx < \infty. \tag{1.12}$$

Then we have

$$\int_\Omega |\ \phi(u) - \phi(v)\ |\ (x,t)dx \le \int_\Omega |\ \phi(u_0) - \phi(v_0)\ |\ (x)dx,\ \ a.e.\ t \in [0,T). \tag{1.13}$$

Condition (1.12) ensures that the homogeneous boundary value condition is true in the sense of trace. However, such a homogeneous boundary value condition may be overdetermined.

Theorem 1.4. *Suppose that $u(x,t)$ and $v(x,t)$ are two nonnegative weak solutions of type 1 of* (1.1) *when $1 < p \leq 2$ and $q \geq m > 0$. If condition* (1.12) *holds, then the stability of weak solutions holds in the sense of* (1.13).

Definition 1.5. A function $u(x,t)$ is said to be a weak solution of type 2 of (1.1), if (1.10) is true, and for any function $g(s) \in C^1(\mathbb{R})$ with $g(0) = 0$, and $\varphi_1 \in C_0^1(\Omega)$ and $\varphi_2 \in L^\infty(0,T;W_{loc}^{1,p}(\Omega))$, it holds

$$\begin{aligned} &\iint_{Q_T} \big[\phi_t(u)g(\varphi_1\varphi_2) + a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla g(\varphi_1\varphi_2)\big]\,dx\,dt \\ &+ \iint_{Q_T} \big[u^{\frac{q}{m}} b_{ix_i}(x)g(\varphi_1\varphi_2) + u^{\frac{q}{m}} b_i(x)g'(\varphi_1\varphi_2)(\varphi_1\varphi_2)_{x_i}\big]\,dx\,dt = 0. \end{aligned} \tag{1.14}$$

The initial value condition is satisfied in the sense of (1.6).

The existence of weak solution of type 2 can be stated in a similar way as Theorem 1.2, so we omit it, and focus on the stability.

Theorem 1.6. *Let $u(x,t)$ and $v(x,t)$ be two nonnegative weak solutions of type 2 of* (1.1). *If $b_i(x) \equiv a(x)$, $p > 1$, $q \geq \max\{m,1\}$, $m > 0$, and*

$$\int_\Omega [a(x)]^{-(p-1)}dx < \infty, \tag{1.15}$$

then stability (1.13) *holds.*

Note that when $m = 1$, the usual initial-boundary value problem was investigated in [13]. We now present the stability of weak solutions based on an optimal partial boundary value condition.

Theorem 1.7. *Let $u(x,t)$ and $v(x,t)$ be two weak solutions of type 2 of the initial-boundary value problem of* (1.1) *with the same partial boundary value condition*

$$u(x,t) = v(x,t) = 0, \quad (x,t) \in \Sigma_p \times (0,T), \tag{1.16}$$

where

$$\Sigma_p = \{x \in \partial\Omega : b_i(x)n_i < 0\}, \tag{1.17}$$

and $\vec{n} = \{n_i\}$ is the inner normal vector of Ω. Suppose that $2 > p > 1$, $q \geq m$, $a(x)$ satisfies (1.7), *and*

$$c_1 d^p(x) \leq a(x) \leq c_2 d(x), \quad x \in \Omega \setminus \Omega_\lambda, \tag{1.18}$$

$$|b_i(x)| \leq c d(x), i = 1,2,\dots,N, \quad x \in \Omega \setminus \Omega_\lambda, \tag{1.19}$$

where c_1 and c_2 are two constants, $d(x)$ is the distance function from the boundary $\partial\Omega$, $\Omega_\lambda = \{x \in \Omega : d(x) > \lambda\}$, and λ is a small positive number. Then the stability (1.13) *holds.*

The article is organized as follows. In Section 2, we prove the existence of weak solutions of type 1 to equation (1.1). In Section 3, we prove Theorems 1.3 and 1.4. In Section 4, we prove Theorem 1.6. The last section is devoted to the stability of weak solutions only dependent on the partial boundary value condition.

2. Proof of existence

Consider the regularized equation

$$u_t = \operatorname{div}\left((a(x)+\varepsilon)(|\nabla u^m|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u^m\right) + \overrightarrow{b}(x)\cdot\nabla u^q,\ \ (x,t)\in\ Q_T, \tag{2.1}$$

with the initial-boundary value conditions

$$u(x,0) = u_{0\varepsilon}(x)+\varepsilon, \quad x\in\Omega, \tag{2.2}$$

$$u(x,t)=\varepsilon, \quad (x,t)\in\partial\Omega\times(0,T), \tag{2.3}$$

where $\varepsilon>0$ is small such that $0\le u_{0\varepsilon}\in C_0^\infty(\Omega)$, $\|u_{0\varepsilon}\|_{L^\infty(\Omega)}$ and

$$\|(a(x)+\varepsilon)|\nabla u_{0\varepsilon}^m|^p\|_{L^1(\Omega)}$$

are uniformly bounded, and $u_{0\varepsilon}$ converges to u_0 in $W_0^{1,p}(\Omega)$. It is well-known that the problem (2.1)-(2.3) has a unique nonnegative classical solution [11, 12].

Proof of Theorem 1.2. Multiplying (2.1) by u_ε^m and integrating it over $Q_t=\Omega\times(0,t)$ for any $t\in[0,T)$ yields

$$\begin{aligned}
&\frac{1}{m+1}\Big[\int_\Omega u_\varepsilon^{m+1}(x,t)\,dx - \int_\Omega (u_{0\varepsilon}(x)+\varepsilon)^{m+1}\,dx\Big]\\
&= \int_0^t\int_{\partial\Omega}(a(x)+\varepsilon)(|\nabla u^m|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u^m\cdot\vec{n}u_\varepsilon^m d\Sigma\,dt\\
&\quad - \iint_{Q_t}(a(x)+\varepsilon)(|\nabla u^m|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u_\varepsilon^m|^2\,dx\,dt\\
&\quad + \iint_{Q_t}\overrightarrow{b}(x)\cdot\nabla u_\varepsilon^q u_\varepsilon^m\,dx\,dt.
\end{aligned} \tag{2.4}$$

It is easy to see that

$$\iint_{Q_t} a(x)|\nabla u_\varepsilon^m|^p\,dx\,dt \le \iint_{Q_t}(a(x)+\varepsilon)(|\nabla u^m|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u_\varepsilon^m|^2\,dx\,dt\le c. \tag{2.5}$$

Then

$$\int_0^t\int_{\Omega_\delta}|\nabla u_\varepsilon^m|^p\,dx\,dt\le c(\delta,T) \tag{2.6}$$

for any $\Omega_\delta=\{x\in\Omega, d(x,\partial\Omega)>\delta\}\subseteq\Omega$, where δ is a small constant.

Multiplying (2.1) by $\frac{\partial u_\varepsilon^m}{\partial t}$ and integrating it over Q_t leads to

$$\begin{aligned}
\iint_{Q_t} u_{\varepsilon t}\frac{\partial u_\varepsilon^m}{\partial t}\,dx\,dt &= \iint_{Q_t}\operatorname{div}(a(x)+\varepsilon)(|\nabla u^m|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon^m)\cdot\frac{\partial u_\varepsilon^m}{\partial t}\,dx\,dt\\
&\quad + \iint_{Q_t}\overrightarrow{b}(x)\cdot\nabla u_\varepsilon^q\frac{\partial u_\varepsilon^m}{\partial t}\,dx\,dt.
\end{aligned} \tag{2.7}$$

Note that

$$(|\nabla u_\varepsilon^m|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon^m\cdot\nabla\frac{\partial u_\varepsilon^m}{\partial t} = \frac{1}{2}\frac{d}{dt}\int_0^{|\nabla u_\varepsilon^m(x,t)|^2+\varepsilon} s^{\frac{p-2}{2}}\,ds.$$

Then

$$
\begin{aligned}
&\iint_{Q_t} \operatorname{div}((a(x)+\varepsilon)(|\nabla u_\varepsilon^m|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon^m \frac{\partial u_\varepsilon^m}{\partial t}\,dx\,dt \\
&= -\iint_{Q_t} (a(x)+\varepsilon)(|\nabla u_\varepsilon^m|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon^m \nabla\frac{\partial u_\varepsilon^m}{\partial t}\,dx\,dt \\
&= -\frac{1}{2}\iint_{Q_t} (a(x)+\varepsilon)\frac{d}{dt}\int_0^{|\nabla u_\varepsilon^m(x,t)|^2+\varepsilon} s^{\frac{p-2}{2}}ds\,dx\,dt \\
&= -\frac{1}{2}\int_\Omega (a(x)+\varepsilon)\int_0^{|\nabla u_\varepsilon^m(x,t)|^2+\varepsilon} s^{\frac{p-2}{2}}\,ds\,dx \\
&\quad + \frac{1}{2}\int_\Omega (a(x)+\varepsilon)\int_0^{|\nabla u_\varepsilon^m(x,0)|^2+\varepsilon} s^{\frac{p-2}{2}}\,ds\,dx.
\end{aligned} \tag{2.8}
$$

By Young's inequality, we obtain

$$
\begin{aligned}
&\iint_{Q_t} \frac{\partial u_\varepsilon^m}{\partial t}\overrightarrow{b}(x)\cdot\nabla u_\varepsilon^q\,dx\,dt \\
&\le \frac{1}{2}\iint_{Q_t} |\overrightarrow{b}(x)\cdot\nabla u_\varepsilon^q\sqrt{mu_\varepsilon^{m-1}}|^2 + \frac{1}{2}\iint_{Q_t} |\sqrt{mu_\varepsilon^{m-1}}\frac{\partial u_\varepsilon}{\partial t}|^2\,dx\,dt \\
&\le c\iint_{Q_t} \Big[(a^{-\frac{2}{p}}|\overrightarrow{b}(x)u_\varepsilon^{q-1-\frac{m-1}{2}}|^2)^{\frac{p}{p-2}} + a|\nabla u_\varepsilon^m|^p\Big]\,dx\,dt \\
&\quad + \frac{1}{2}\iint_{Q_t} |u_{\varepsilon t}\frac{\partial u_\varepsilon^m}{\partial t}|\,dx\,dt.
\end{aligned} \tag{2.9}
$$

According to (2.7)-(2.9), in view of $q \ge 1+\frac{m-1}{2}$ and (1.9) we deduce that

$$
\iint_{Q_t} |u_{\varepsilon t}\frac{\partial u_\varepsilon^m}{\partial t}|\,dx\,dt \le c
$$

and

$$
\iint_{Q_t} |\frac{\partial u_\varepsilon^m}{\partial t}|^2\,dx\,dt = \iint_{Q_t} |mu_\varepsilon^{m-1}u_{\varepsilon t}\frac{\partial u_\varepsilon^m}{\partial t}|\,dx\,dt \le c. \tag{2.10}
$$

By (2.5)-(2.6) and (2.10), according to the Sobolev embedding theorem there exists a function $v \in L^\infty(Q_T)$ such that

$$
u_\varepsilon^m \to v, \quad \text{a.e. } \in Q_T. \tag{2.11}
$$

Let the nonnegative function u satisfy $u^m = v$. Then $u_\varepsilon^m \to u^m$ a.e. in Q_T, and so

$$
u_\varepsilon \to u, \quad u_\varepsilon^q \to u^q, \quad \text{a.e. in } Q_T. \tag{2.12}
$$

Since for any $\varphi \in C_0^1(Q_T)$, it follows that

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\iint_{Q_T} \frac{\partial u_\varepsilon^m}{\partial t}\varphi\,dx\,dt &= -\lim_{\varepsilon\to 0}\iint_{Q_T} u_\varepsilon^m\varphi_t\,dx\,dt \\
&= \iint_{Q_T} u^m\varphi_t\,dx\,dt = \iint_{Q_T} \frac{\partial u^m}{\partial t}\varphi\,dx\,dt.
\end{aligned}
$$

By a process of limit, the above calculation is also true for any $\varphi \in L^2(Q_T)$, then we have

$$
\frac{\partial u_\varepsilon^m}{\partial t} \rightharpoonup \frac{\partial u^m}{\partial t}, \quad \text{in } L^2(Q_T). \tag{2.13}
$$

From the above discussions, we also obtain

$$\iint_{Q_T} \varepsilon^{\frac{p-2}{2}} a(x)|\nabla u_\varepsilon^m|^2 \,dx\,dt \le c. \tag{2.14}$$

Since

$$\begin{aligned}
&\iint_{Q_T} |a(x)(|\nabla u_\varepsilon^m|^2+\varepsilon)^{\frac{p-2}{2}} \frac{\partial u_\varepsilon^m}{\partial x_i}|^{\frac{p}{p-1}} \,dx\,dt \\
&\le c\iint_{Q_T} |a^{\frac{p-1}{p}}(x)(|\nabla u_\varepsilon^m|^2+\varepsilon)^{\frac{p-2}{2}} \frac{\partial u_\varepsilon^m}{\partial x_i}|^{\frac{p}{p-1}} \,dx\,dt \\
&\le c\iint_{Q_T} a(x)(|\nabla u_\varepsilon^m|^2+\varepsilon)^{\frac{p(p-2)}{2(p-1)}} |\nabla u_\varepsilon^m|^{\frac{p}{p-1}} \,dx\,dt \\
&\le c\iint_{Q_T} [a(x)|\nabla u_\varepsilon^m|^p + \varepsilon^{\frac{p(p-2)}{2(p-1)}} a(x)|\nabla u_\varepsilon^m|^{\frac{p}{p-1}}] \,dx\,dt \\
&\le c\iint_{Q_T} a(x)|\nabla u_\varepsilon^m|^p \,dx\,dt + c\varepsilon^{\frac{p(p-2)}{2(p-1)}-\frac{p-2}{2}} \le c,
\end{aligned}$$

this implies that there exists an $n-$dimensional vector $\overrightarrow{\zeta} = (\zeta_1, \dots, \zeta_n)$, $\zeta_i \in L^{\frac{p}{p-1}}(Q_T)$ such that

$$a(x)(|\nabla u_\varepsilon^m|^2+\varepsilon)^{\frac{p-2}{2}} \frac{\partial u_\varepsilon^m}{\partial x_i} \rightharpoonup \zeta_i, \quad \text{in } L^{\frac{p}{p-1}}(Q_T). \tag{2.15}$$

To prove that u is the solution of (1.1), we note that for any function $\varphi \in C_0^1(Q_T)$, we have

$$\begin{aligned}
&\iint_{Q_T} \big[u_{\varepsilon t}\varphi + (a(x)+\varepsilon)(|\nabla u_\varepsilon^m|^2+\varepsilon)^{\frac{p(x)-2}{2}} \nabla u_\varepsilon^m \cdot \nabla\varphi\big] \,dx\,dt \\
&+ \iint_{Q_T} u_\varepsilon^q [b_{ix_i}(x)\varphi + b_i(x)\varphi_{x_i}] \,dx\,dt = 0.
\end{aligned} \tag{2.16}$$

Since $a(x) > 0$ when $x \in \Omega$, we have $c > \sup_{supp\varphi} \frac{|\nabla\varphi|}{a(x)} > 0$ because $\varphi \in C_0^1(Q_T)$, and

$$\begin{aligned}
&\varepsilon|\iint_{Q_T} (|\nabla u_\varepsilon^m|^2+\varepsilon)^{p-2} \nabla u_\varepsilon \cdot \nabla\varphi \,dx\,dt| \\
&\le \varepsilon \sup_{\operatorname{supp}\varphi} \frac{|\nabla\varphi|}{a(x)} \iint_{Q_T} a(x)(|\nabla u_\varepsilon^m|^p + c) \,dx\,dt \to 0,
\end{aligned} \tag{2.17}$$

as $\varepsilon \to 0$. Based on this inequality, we find that

$$\begin{aligned}
&(|\nabla u_\varepsilon^m|^2+\varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon \\
&= |\nabla u_\varepsilon^m|^{p-2}\nabla u_\varepsilon^m + \varepsilon\frac{p-2}{2}\int_0^1 (|\nabla u_\varepsilon^m|^2+\varepsilon s)^{p-4} ds \nabla u_\varepsilon^m.
\end{aligned} \tag{2.18}$$

Just like for the general evolutionary p-Laplician equation [12], by (2.16)-(2.18) we derive that

$$\iint_{Q_T} [u_t\varphi + \vec{\varsigma}\cdot\nabla\varphi + u^q(b_{ix_i}(x)\varphi + b_i(x)\varphi_{x_i})] \,dx\,dt = 0, \tag{2.19}$$

$$\iint_{Q_T} a(x)|\nabla u^m|^{p-2}\nabla u \cdot \nabla\varphi \,dx\,dt = \iint_{Q_T} \overrightarrow{\zeta} \cdot \nabla\varphi \,dx\,dt \tag{2.20}$$

for any function $\varphi \in C_0^1(Q_T)$. Then

$$\iint_{Q_T} [u_t\varphi + a(x)|\nabla u^m|^{p-2}\nabla u^m \cdot \nabla\varphi + u^q(b_{ix_i}(x)\varphi + b_i(x)\varphi_{x_i})]\,dx\,dt = 0. \tag{2.21}$$

If we denote $\Omega_\varphi = \operatorname{supp}\varphi$, then

$$\int_0^T \int_{\Omega_\varphi} [u_t\varphi + a(x)|\nabla u^m|^{p-2}\nabla u^m \cdot \nabla\varphi + u^q(b_{ix_i}(x)\varphi + b_i(x)\varphi_{x_i})]\,dx\,dt = 0. \tag{2.22}$$

For any functions $\varphi_1 \in L^1(0,T;C_0^1(\Omega))$ and $\varphi_2 \in L^\infty(Q_T)$, and for any given $t \in [0,T)$ $\varphi_2(x,\cdot) \in W_{loc}^{1,p}(\Omega)$, we know that $\varphi_1\varphi_2 \in W_0^{1,p}(\Omega_{\varphi_1})$. By the fact of that $C_0^\infty(\Omega_{\varphi_1})$ is dense in $W_0^{1,p}(\Omega_{\varphi_1})$, by a process of limit, we have

$$\begin{aligned}
&\int_0^T \int_{\Omega_{\varphi_1}} [u_t(\varphi_1\varphi_2) + a(x)|\nabla u^m|^{p-2}\nabla u^m \cdot \nabla(\varphi_1\varphi_2)]\,dx\,dt \\
&+ \int_0^T \int_{\Omega_{\varphi_1}} u^q[b_{ix_i}(x)(\varphi_1\varphi_2) + b_i(x)(\varphi_1\varphi_2)_{x_i}]\,dx\,dt = 0,
\end{aligned} \tag{2.23}$$

which implies

$$\begin{aligned}
&\int_0^T \int_{\Omega} [u_t(\varphi_1\varphi_2) + a(x)|\nabla u^m|^{p-2}\nabla u^m \cdot \nabla(\varphi_1\varphi_2)]\,dx\,dt \\
&+ \int_0^T \int_{\Omega} u^q[b_{ix_i}(x)(\varphi_1\varphi_2) + b_i(x)(\varphi_1\varphi_2)_{x_i}]\,dx\,dt = 0.
\end{aligned} \tag{2.24}$$

Finally, we can obtain (1.6) by applying a similar method as for the usual evolutionary $p-$Laplacian equation [12]. Thus, u satisfies equation (1.1) in the sense of Definition 1.1. □

Proposition 2.1. *Let $u(x,t)$ be a weak solution of type 1 of* (1.1)*. Then*

$$\frac{\partial\phi(v)}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(\Omega)). \tag{2.25}$$

Proof. For any $\varphi(x,t) \in L^p((0,T;W_0^p(\Omega))$, by (2.5) we have

$$\begin{aligned}
\langle \frac{\partial\phi(v)}{\partial t}, \varphi \rangle &= \iint_{Q_T} \frac{\partial\phi(v)}{\partial t}\varphi\,dx\,dt \\
&= -\iint_{Q_T} \big[a(x)|\nabla v|^{p-2}\nabla v \cdot \nabla\varphi + v^{\frac{q}{m}} b_i(x)\varphi_{x_i} + v^{\frac{q}{m}} b_{ix_i}\varphi\big]\,dx\,dt \\
&\leq c\iint_{Q_T} [a(x)|\nabla v|^p + |\nabla\varphi|^p + 1]\,dx\,dt \leq c.
\end{aligned}$$

□

3. Proofs of Theorems 1.3 and 1.4

Definition 3.1. A function $u(x,t)$ is said to be a weak solution of (1.1) with the initial-boundary conditions (1.2)-(1.3), if u is a weak solution of (1.1) in the sense of Definition 1.1, and the boundary value condition (1.3) is satisfied in the sense of trace.

Lemma 3.2 ([13]). *Let u be a weak solution of type 1 of* (1.1). *If* $\int_\Omega [a(x)]^{-\frac{1}{p-1}}dx \leq c$*, then*

$$\iint_{Q_T} |\nabla u^m|\, dx\, dt \leq c. \tag{3.1}$$

By Theorem 1.2 and Lemma 3.2, we have the following theorem.

Theorem 3.3. *Let u be be a weak solution of type 1 of* (1.1) *with the initial value u_0. If $a(x)$ satisfies* (1.12)*, then there exists a weak solution of* (1.1) *with the usual initial-boundary value conditions* (1.2)-(1.3).

Proof of Theorem 1.3. Suppose that u and v are two nonnegative solutions of type 1 of (1.1) with the same homogeneous boundary value (1.3). In view of the definition of weak solution of type 1, we let $\varphi_1 = \varphi \in L^1(0,T;C_0^1(\Omega))$ and $\varphi_2 \equiv 1$. Then

$$\begin{aligned}&\int_\Omega \varphi\frac{\partial(\phi(u)-\phi(v))}{\partial t}dx + \int_\Omega a(x)(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\cdot\nabla\varphi dx\\ &+\int_\Omega (b_{ix_i}\varphi + b_i\varphi_{x_i})(u^{\frac{q}{m}} - v^{\frac{q}{m}})dx = 0.\end{aligned} \tag{3.2}$$

For a small $\eta > 0$, let

$$S_\eta(s) = \int_0^s h_\eta(\tau)d\tau, \quad h_\eta(s) = \frac{2}{\eta}\Big(1 - \frac{|\, s\,|}{\eta}\Big)_+.$$

Obviously, $h_\eta(s) \in C(\mathbb{R})$, and

$$\lim_{\eta\to 0} S_\eta(s) = \operatorname{sign} s, \quad \lim_{\eta\to 0} sS_\eta'(s) = 0. \tag{3.3}$$

Choosing $\varphi = S_\eta(u-v)$ as the test function, we have

$$\begin{aligned}&\int_\Omega S_\eta(u-v)\frac{\partial(\phi(u)-\phi(v))}{\partial t}dx\\ &+\int_\Omega a(x)(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\cdot\nabla(u-v)S_\eta'(u-v)dx\\ &= -\int_\Omega b_i(x)(u^{\frac{q}{m}} - v^{\frac{q}{m}})(u-v)_{x_i}S_\eta'(u-v)dx\\ &\quad -\int_\Omega (u^{\frac{q}{m}} - v^{\frac{q}{m}})b_{ix_i}(x)S_\eta(u-v)dx.\end{aligned} \tag{3.4}$$

Since $\frac{\partial\phi(v)}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, by [8, Lemma 2.2], we obtain

$$\begin{aligned}&\int_0^t \Big\langle S_\eta(\phi(u)-\phi(v)), \frac{\partial(\phi(u)-\phi(v))}{\partial t}\Big\rangle dt\\ &= \int_\Omega I_\eta(\phi(u)-\phi(v))(x,t)\, dx - \int_\Omega I_\eta(\phi(u)-\phi(v))(x,0)\, dx.\end{aligned} \tag{3.5}$$

While, from [12] we know that

$$S_\eta(u-v) \to \operatorname{sign}(u-v), \quad \text{in } W_0^{1,p}(\Omega),$$
$$S_\eta(\phi(u)-\phi(v)) \to \operatorname{sign}(\phi(u)-\phi(v)) = \operatorname{sign}(u-v), \ \text{ in } W_0^{1,p}(\Omega).$$

So have

$$
\begin{aligned}
&\lim_{\eta\to 0}|\big\langle S_\eta(u-v)-S_\eta(\phi(u)-\phi(v)),\frac{\partial(\phi(u)-\phi(v))}{\partial t}\big\rangle| \\
&\le \lim_{\eta\to 0}\|S_\eta(u-v)-S_\eta(\phi(u)-\phi(v))\|_{W_0^{1,p}(\Omega)}\|\frac{\partial(\phi(u)-\phi(v))}{\partial t}\|_{W^{-1,p'}(\Omega)} \\
&=0.
\end{aligned} \tag{3.6}
$$

By (3.5)-(3.6), we have

$$
\begin{aligned}
&\lim_{\eta\to 0}\int_0^t\big\langle S_\eta(u-v),\frac{\partial(\phi(u)-\phi(v))}{\partial t}\big\rangle dt \\
&=\int_\Omega|\phi(u)-\phi(v))(x,t)|dx-\int_\Omega|\phi(u)-\phi(v))(x,0)|dx.
\end{aligned} \tag{3.7}
$$

To evaluate the second part on the right hand side of (3.4), we have

$$
\int_\Omega a(x)|(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla(u-v)S_\eta'(u-v)\mathrm{d}x\ge 0. \tag{3.8}
$$

Since $q\ge m$, and $a(x)$ satisfies

$$
\int_\Omega[a(x)]^{\frac{-1}{p-1}}dx<\infty,
$$

it follows by Lebesgue's dominated convergence theorem and (3.5) that

$$
\begin{aligned}
&\lim_{\eta\to 0}\big|\int_\Omega(u^{\frac{q}{m}}-v^{\frac{q}{m}})b_i(x)(u-v)_{x_i}S_\eta'(u-v)dx\big| \\
&\le \lim_{\eta\to 0}\Big(\int_\Omega|b_i(x)(u^{\frac{q}{m}}-v^{\frac{q}{m}})S_\eta'(u-v)a(x)^{-1/p}|^{\frac{p}{p-1}}dx\Big)^{\frac{p-1}{p}} \\
&\quad\times\Big(\int_\Omega a(x)(|\nabla u|^p+|\nabla v|^p)dx\Big)^{1/p} \\
&\le c\lim_{\eta\to 0}\Big(\int_\Omega|(u-v)S_\eta'(u-v)a(x)^{-1/p}|^{\frac{p}{p-1}}dx\Big)^{\frac{p-1}{p}}=0.
\end{aligned} \tag{3.9}
$$

Meanwhile, since $q\ge 1$ and $|b_{ix_i}|\le c$, it follows that

$$
\begin{aligned}
&\lim_{\eta\to 0}\big|\int_\Omega(u^{\frac{q}{m}}-v^{\frac{q}{m}})b_{ix_i}(x)S_\eta(u-v)dx\big| \\
&\le c\int_\Omega|(u^{\frac{q}{m}}-v^{\frac{q}{m}})\mathrm{sign}(u-v)|dx \\
&=\int_\Omega|(\phi^q(u)-\phi^q(u))\mathrm{sign}(\phi(u)-\phi(v))|dx \\
&\le c\int_\Omega|\phi(u)-\phi(v)|dx.
\end{aligned} \tag{3.10}
$$

Let $\eta\to 0$ in (3.2). Then we arrive at the desired result (1.13). □

Proof of Theorem 1.4. By Definition 1.1, for any functions $\varphi_1 \in L^1(0,T;C_0^1(\Omega))$, $\varphi_2 \in L^\infty(Q_T)$, and for any given $t\in[0,T)$ $\varphi_2(x,\cdot)\in W^{1,p}_{loc}(\Omega)$, we have

$$\begin{aligned}
&\iint_{Q_T}\Big[\frac{\partial(\phi(u)-\phi(v))}{\partial t}(\varphi_1\varphi_2)+a(x)(|\nabla u|^{p-2}\nabla u\\
&-|\nabla v|^{p-2}\nabla v)\cdot\nabla(\varphi_1\varphi_2)\Big]\,dx\,dt \qquad (3.11)\\
&+\iint_{Q_T}\big[b_i(x)(u^{\frac{q}{m}}-v^{\frac{q}{m}})(\varphi_1\varphi_2)_{x_i}+b_{ix_i}(x)(u^{\frac{q}{m}}-v^{\frac{q}{m}})(\varphi_1\varphi_2)\big]\,dx\,dt=0.
\end{aligned}$$

For a small positive constant $\lambda>0$, we let $\Omega_\lambda=\{x\in\Omega: a(x)>\lambda\}$ in this section and define

$$\varphi_\lambda(x)=\begin{cases}1, & \text{if } x\in\Omega_\lambda,\\ \frac{1}{\lambda}a(x), & \text{if } x\in\Omega\setminus\Omega_\lambda.\end{cases} \qquad (3.12)$$

Choosing $\varphi_1=\varphi_\lambda(x)\chi_{[\tau,s]}$ and $\varphi_2=S_\eta(u-v)$, and integrating it over Ω, we have

$$\begin{aligned}
&\int_\tau^s\int_\Omega\varphi_\lambda(x)S_\eta(u-v)\frac{\partial(\phi(u)-\phi(v))}{\partial t}\,dx\,dt\\
&+\int_\tau^s\int_\Omega\varphi_\lambda(x)a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla(u-v)S_\eta'(u-v)\,dx\,dt\\
&+\int_\tau^s\int_\Omega a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla\varphi_\lambda(x)S_\eta(u-v)\,dx\,dt\\
&+\int_\tau^s\int_\Omega b_i(x)(u^{\frac{q}{m}}-v^{\frac{q}{m}})[\varphi_\lambda(x)S_\eta'(u-v)(u-v)_{x_i}+S_\eta(u-v)\varphi_{\lambda x_i}(x)]\,dx\,dt\\
&+\int_\tau^s\int_\Omega b_{ix_i}(u^{\frac{q}{m}}-v^{\frac{q}{m}})\varphi_\lambda(x)S_\eta(u-v)=0.
\end{aligned} \qquad (3.13)$$

Moreover,

$$\int_\Omega\varphi_\lambda(x)a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla(u-v)S_\eta'(u-v)dx\ge 0, \qquad (3.14)$$

and

$$\begin{aligned}
&\Big|\int_\Omega a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla\varphi_\lambda(x)S_\eta(u-v)dx\Big|\\
&\le\int_{\Omega\setminus\Omega_\lambda}a(x)\big|(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla\varphi_\lambda(x)S_\eta(u-v)\big|dx\\
&\le\int_{\Omega\setminus\Omega_\lambda}a(x)|(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)||\nabla\varphi_\lambda(x)|dx\\
&\le\frac{c}{\lambda}\Big[\int_{\Omega\setminus\Omega_\lambda}a(x)|\nabla u|^{p-1}|\nabla a|dx+\int_\tau^s\int_{\Omega\setminus\Omega_\lambda}a(x)|\nabla v|^{p-1}|\nabla a|dx\Big].
\end{aligned} \qquad (3.15)$$

Since $1<p\le 2$, $|\nabla a|\le c$, and

$$\int_{\Omega\setminus\Omega_\lambda}|\nabla a|^p\,dx\le c\lambda\le c\lambda^{p-1},$$

it follows that

$$\frac{c}{\lambda}\Big(\int_{\Omega\setminus\Omega_\lambda}a(x)|\nabla a|^pdx\Big)^{1/p}\le\frac{c}{\lambda}\Big(\lambda\int_{\Omega\setminus\Omega_\lambda}|\nabla a|^pdx\Big)^{1/p}\le c. \qquad (3.16)$$

By (3.15)-(3.16), and Hölder's inequality it follows that

$$\begin{aligned}
&\Big|\int_\Omega a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla\varphi_\lambda(x)S_\eta(u-v)dx\Big|\\
&\le \frac{c}{\lambda}\Big[\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla u|^{p-1}|\nabla a|dx+\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla v|^{p-1}|\nabla a|dx\Big]\\
&\le \frac{c}{\lambda}\Big(\int_{\Omega\setminus\Omega_\lambda} a|\nabla a|^p dx\Big)^{1/p}\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla u|^p dx\Big)^{\frac{p-1}{p}}\\
&\quad+\frac{c}{\lambda}\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla a|^p dx\Big)^{1/p}\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla v|^p dx\Big)^{\frac{p-1}{p}}\\
&\le c\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla u|^p dx\Big)^{\frac{p-1}{p}}+c\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla v|^p dx\Big)^{\frac{p-1}{p}}.
\end{aligned}\tag{3.17}$$

Thus, we obtain

$$\lim_{\lambda\to0}\Big|\int_\Omega a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla\varphi_\lambda(x)S_\eta(u-v)dx\Big|=0.\tag{3.18}$$

In view of (1.12) and (3.5), and Lebesgue's dominated convergence theorem it follows that

$$\lim_{\eta\to0}\int_\Omega \varphi_\lambda b_i(x)(u^{\frac{q}{m}}-v^{\frac{q}{m}})S_\eta'(u-v)(u-v)_{x_i}dx=0.\tag{3.19}$$

Since $q\ge1$, by using (1.12) and (3.16) we obtain

$$\begin{aligned}
&\lim_{\lambda\to0}\Big|\int_\Omega \varphi_{\lambda x_i}b_i(x)(u^{\frac{q}{m}}-v^{\frac{q}{m}})S_\eta(u-v)dx\Big|\\
&\le \lim_{\lambda\to0}\frac{c}{\lambda}\int_{\Omega\setminus\Omega_\lambda}|\nabla a|dx\\
&\le \lim_{\lambda\to0}\frac{c}{\lambda}\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla a|^p dx\Big)^{1/p}\Big(\int_{\Omega\setminus\Omega_\lambda} a^{-\frac{1}{p-1}}(x)dx\Big)^{\frac{p-1}{p}}=0.
\end{aligned}\tag{3.20}$$

Similar to (3.10), we have

$$\lim_{\lambda\to0}\Big|\int_\Omega \varphi_\lambda b_{ix_i}(x)(u^{\frac{q}{m}}-v^{\frac{q}{m}})S_\eta(u-v)dx\Big|\le\|\phi(u)-\phi(v)\|_{L^1(\Omega)}.\tag{3.21}$$

Processing as we discussed in the proof of Theorem 1.3, we have

$$\begin{aligned}
&\lim_{\eta\to0}\lim_{\lambda\to0}\int_\tau^s\int_\Omega \varphi_\lambda(x)S_\eta(u-v)\frac{\partial(\phi(u)-\phi(v))}{\partial t}\,dx\,dt\\
&=\lim_{\eta\to0}\int_\tau^s\int_\Omega S_\eta(u-v)\frac{\partial(\phi(u)-\phi(v))}{\partial t}\,dx\,dt\\
&=\int_\Omega|(\phi(u)-\phi(v))(x,s)|dx-\int_\Omega|(\phi(u)-\phi(v))(x,\tau)|dx.
\end{aligned}\tag{3.22}$$

Letting $\lambda\to0$ and $\eta\to0$ in (3.13), in view of the arbitrariness of τ and (3.18)–(3.22), we obtain

$$\int_\Omega|\phi(u)(x,s)-\phi(v)(x,s)|dx\le\int_\Omega|\phi(u(x,0))-\phi(v(x,0))|dx.$$

□

4. Proof Theorem 1.6

To prove Theorem 1.6, we start with a more general case.

Theorem 4.1. *Let u and v be two weak solutions of type 2 of* (1.1) *with the initial values $u_0(x)$ and $v_0(x)$ respectively. When $p > 1$, $q \geq 1$ and $0 < m \leq 1$, we suppose that*

$$\int_\Omega \frac{|b_i(x)\nabla a|}{a}|u^{\frac{q}{m}}|dx \leq c, \quad \int_\Omega \frac{|b_i(x)\nabla a|}{a}|v^{\frac{q}{m}}|dx \leq c, \tag{4.1}$$

$$\int_\Omega |a^{-1}b_i^p|^{\frac{1}{p-1}}dx < \infty, \tag{4.2}$$

and condition (1.15) *holds. Then stability* (1.13) *holds.*

Apparently, if $b_i(x) \equiv a(x)$, conditions (4.1)-(4.2) hold naturally. Thus, Theorem 1.6 is a particular consequence of Theorem 4.1.

Proof of Theorem 4.1. Let u and v be two solutions of type 2 of (1.1) with the initial values $u_0(x)$ and $v_0(x)$ respectively. We choose $\chi_{[\tau,s]}S_\eta(a^\beta(u-v))$ as the test function. Then

$$\begin{aligned}
&\int_\tau^s \int_\Omega S_\eta(a^\beta(u-v))\frac{\partial(\phi(u)-\phi(v))}{\partial t}\,dx\,dt\\
&+\int_\tau^s \int_\Omega a^{\beta+1}(x)(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\cdot\nabla(u-v)S_\eta'(a^\beta(u-v))\,dx\,dt\\
&+\int_\tau^s \int_\Omega a(x)(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\cdot\nabla a^\beta(u-v)S_\eta'(a^\beta(u-v))\,dx\,dt\\
&+\int_\tau^s \int_\Omega (u^{\frac{q}{m}} - v^{\frac{q}{m}})b_{ix_i}(x)S_\eta(a^\beta(u-v))\,dx\,dt\\
&+\int_\tau^s \int_\Omega (u^{\frac{q}{m}} - v^{\frac{q}{m}})b_i(x)S_\eta'(a^\beta(u-v))\big[\beta a^{\beta-1}(u-v)a_{x_i}\\
&+a^\beta(u-v)_{x_i}\big]\,dx\,dt = 0.
\end{aligned} \tag{4.3}$$

Similar to the proof of Theorem 1.3, we have

$$\begin{aligned}
&\lim_{\eta\to 0}\int_\tau^s \int_\Omega S_\eta(a^\beta(u-v))\frac{\partial(\phi(u)-\phi(v))}{\partial t}dx\\
&= \int_\Omega \big|\phi(u(x,\tau)) - \phi(v(x,\tau))\big|\,dx,
\end{aligned} \tag{4.4}$$

$$\int_\Omega a^{\beta+1}(x)(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\cdot\nabla(u-v)S_\eta'(a^\beta(u-v))\,dx \geq 0. \tag{4.5}$$

In view of $|\nabla a(x)| \leq c$, in Ω we have

$$\begin{aligned}
&\Big|\int_\Omega a(x)(u-v)S_\eta'(a^\beta(u-v))(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\cdot\nabla a^\beta dx\Big|\\
&\leq c\Big|\int_\Omega a^\beta(u-v)S_\eta'(a^\beta(u-v))(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)dx\Big|,
\end{aligned} \tag{4.6}$$

and

$$
\begin{aligned}
&\Big|\int_\Omega a^\beta(u-v)S_\eta'(a^\beta(u-v))(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)dx\Big|\\
&=\Big|\int_{\{\Omega:a^\beta|u-v|<\eta\}}a^{-\frac{p-1}{p}}a^\beta(u-v)S_\eta'(a^\beta(u-v))a^{\frac{p-1}{p}}\\
&\quad\times(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)dx\Big|\\
&\le\Big(\int_{\{\Omega:a^\beta|u-v|<\eta\}}|a^{-\frac{p-1}{p}}a^\beta(u-v)S_\eta'(a^\beta(u-v))|^pdx\Big)^{1/p}\\
&\quad\times\Big(\int_{\{\Omega:a^\beta|u-v|<\eta\}}a(x)(|\nabla u|^p+|\nabla v|^p)dx\Big)^{\frac{p-1}{p}}.
\end{aligned}\tag{4.7}
$$

If $\{x\in\Omega:u-v=0\}$ has the zero measure, by (1.15) we have

$$
\int_{\{\Omega:a^\beta|u-v|<\eta\}}|a^{-\frac{p-1}{p}}a^\beta(u-v)S_\eta'(a^\beta(u-v))|^pdx<\infty,
$$

and

$$
\begin{aligned}
&\lim_{\eta\to0}\Big(\int_{\{\Omega:a^\beta|u-v|<\beta\}}a(x)(|\nabla u|^p+|\nabla v|^p)dx\Big)^{\frac{p-1}{p}}\\
&=\Big(\int_{\{\Omega:|u-v|=0\}}a(x)(|\nabla u|^p+|\nabla v|^p)dx\Big)^{\frac{p-1}{p}}=0.
\end{aligned}\tag{4.8}
$$

If $\{x\in\Omega:u-v=0\}$ has a positive measure, then

$$
\begin{aligned}
&\lim_{\eta\to0}\Big(\int_{\{\Omega:a^\beta|u-v|<\eta\}}|a^{-\frac{p-1}{p}}a^\beta(u-v)S_\eta'(a^\beta(u-v))|^pdx\Big)^{1/p}\\
&=\Big(\int_{\{\Omega:|u-v|=0\}}\lim_{\eta\to0}|a^{-\frac{p-1}{p}}a^\beta(u-v)S_\eta'(a^\beta(u-v))|^pdx\Big)^{1/p}=0.
\end{aligned}\tag{4.9}
$$

In view of (3.3) and condition (1.15), by Lebesgue's dominated convergence theorem, for both cases we have

$$
\lim_{\eta\to0}\Big|\int_\Omega a^\beta(u-v)S_\eta'(a^\beta(u-v))(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)dx\Big|=0,\tag{4.10}
$$

and from (3.5) we obtain

$$
\begin{aligned}
&\Big|\int_\Omega(u^{\frac{q}{m}}-v^{\frac{q}{m}})b_i(x)S_\eta'(a^\beta(u-v))(\beta a^{\beta-1}(u-v)a_{x_i}dx\Big|\\
&\le c\int_\Omega(|u|^{\frac{q}{m}}+|v|^{\frac{q}{m}})\frac{|b_i(x)\nabla a|}{a}a^\beta(u-v)S_\eta'(a^\beta(u-v))dx\to0,
\end{aligned}\tag{4.11}
$$

as $\eta\to0$.

Since $q \geq m$, from (3.5) it holds

$$\begin{aligned} &\big|\int_\Omega (u^{\frac{q}{m}} - v^{\frac{q}{m}}) b_i(x) S'_\eta(a^\beta(u-v)) a^\beta (u-v)_{x_i} dx\big| \\ &= \big|\int_\Omega a^{\beta-\frac{1}{p}} (u^{\frac{q}{m}} - v^{\frac{q}{m}}) b_i(x) S'_\eta(a^\beta(u-v)) a^{-1/p} (u-v)_{x_i} dx\big| \\ &\leq c\Big(\int_\Omega |b_i(x) a^{-1/p} a^\beta (u-v) S'_\eta(a^\beta(u-v))|^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}} \\ &\quad \times \Big(\int_\Omega a(x)(|\nabla u|^p + |\nabla v|^p)\Big)^{1/p} \to 0, \end{aligned} \tag{4.12}$$

as $\eta \to 0$. In view of $q \geq 1$, we deduce that

$$\begin{aligned} &\lim_{\eta\to 0} \big|\int_\Omega (u^{\frac{q}{m}} - v^{\frac{q}{m}}) b_{ix_i}(x) S_\eta(a^\beta(u-v)) dx\big| \\ &= \big|\int_\Omega (\phi^q(u) - \phi^q(v)) b_{ix_i}(x) sign(a^\beta(u-v)) dx\big| \\ &= \big|\int_\Omega (\phi^q(u) - \phi^q(v)) b_{ix_i}(x) sign(\phi(u) - \phi(v)) dx\big| \\ &\leq c\int_\Omega |\phi^q(u) - \phi^q(v)| dx \\ &\leq c\int_\Omega |\phi(u) - \phi(v)| dx. \end{aligned} \tag{4.13}$$

Let $\eta \to 0$ in (4.3). Because of the arbitrariness of τ we obtain

$$\int_\Omega |\phi(u) - \phi(v)|(x,s) dx \leq c\int_\Omega |\phi(u_0) - \phi(v_0)|(x) dx, \quad \forall s \in [0,T).$$

□

5. Partial boundary value condition

Proof of Theorem 1.7. Let $u(x,t)$ and $v(x,t)$ be two weak solutions of type 2 of the initial-boundary value problem of (1.1). Let $\Omega_\lambda = \{x \in \Omega : d(x) > \lambda\}$ in what follows and

$$\varphi_\lambda(x) = \begin{cases} 1, & \text{if } x \in \Omega_\lambda, \\ \frac{1}{\lambda} d(x), & \text{if } x \in \Omega \setminus \Omega_\lambda. \end{cases}$$

Choosing $S_\eta(\varphi_\lambda(u-v))$ as the test function, we deduce that

$$\begin{aligned}
&\int_\Omega S_\eta(\varphi_\lambda(u-v))\frac{\partial(\phi(u)-\phi(v))}{\partial t}dx\\
&+\int_\Omega a(x)\varphi_\lambda(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla(u-v)S'_\eta(\varphi_\lambda(u-v))dx\\
&+\int_\Omega a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla\varphi_\lambda(u-v)S'_\eta(\varphi_\lambda(u-v))dx\\
&+\int_\Omega b_{ix_i}(x)(u^{\frac{q}{m}}-v^{\frac{q}{m}})S_\eta(\varphi_\lambda(u-v))dx\\
&+\int_\Omega \varphi_\lambda b_i(x)(u^{\frac{q}{m}}-v^{\frac{q}{m}})\cdot(u-v)_{x_i}S'_\eta(\varphi_\lambda(u-v))dx\\
&+\int_\Omega b_i(x)(u^{\frac{q}{m}}-v^{\frac{q}{m}})\cdot\varphi_{\lambda x_i}(u-v)S'_\eta(\varphi_\lambda(u-v))dx=0.
\end{aligned}\tag{5.1}$$

As in the proof Theorem 1.3, one can obtain

$$\begin{aligned}
&\lim_{\eta\to 0}\int_0^t\int_\Omega S_\eta(\varphi_\lambda(u-v))\frac{\partial(\phi(u)-\phi(v))}{\partial t}\,dx\,dt\\
&=\int_\Omega|\phi(u)-\phi(v)|(x,t)dx-\int_\Omega|\phi(u)-\phi(v)|(x,0)dx,
\end{aligned}\tag{5.2}$$

$$\int_\Omega a(x)\varphi_\lambda(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla(u-v)S'_\eta(\varphi_\lambda(u-v))dx\geq 0,\tag{5.3}$$

and

$$\begin{aligned}
&\Big|\int_\Omega a(u-v)S'_\eta(\varphi_\lambda(u-v))(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\nabla\varphi_\lambda dx\Big|\\
&=\Big|\int_{\{\Omega:\varphi_\lambda|u-v|<\eta\}}a^{-\frac{p-1}{p}}a(u-v)S'_\eta(\varphi_\lambda(u-v))a^{\frac{p-1}{p}}(|\nabla u|^{p-2}\nabla u\\
&\quad-|\nabla v|^{p-2}\nabla v)\nabla\varphi_\lambda dx\Big|\\
&\leq\Big(\int_{\{\Omega:\varphi_\lambda|u-v|<\eta\}}|a^{1/p}(u-v)S'_\eta(\varphi_\lambda(u-v))\nabla\varphi_\lambda|^p dx\Big)^{1/p}\\
&\quad\times\Big(\int_{\{\Omega:\varphi_\lambda|u-v|<\eta\}}a(x)(|\nabla u|^p+|\nabla v|^p)dx\Big)^{\frac{p-1}{p}}.
\end{aligned}\tag{5.4}$$

If $\{x\in\Omega:u-v=0\}$ has the zero measure, in view of condition (1.18), $|\nabla d|=1$, $1<p<2$, $c_2d(x)\geq a(x)\geq c_1d^p(x)$, and

$$\int_\Omega a(x)|\frac{\nabla\varphi_\lambda}{\varphi_\lambda}|^p dx=\int_{\Omega\setminus\Omega_\lambda}\frac{a(x)}{d^p}dx\leq c_2\int_{\Omega\setminus\Omega_\lambda}d^{1-p}(x)dx<\infty,$$

then we have

$$\begin{aligned}
&\Big|\int_{\{\Omega:\varphi_\lambda|u-v|<\eta\}}|a^{1/p}\nabla\varphi_\lambda(u-v)S'_\eta(\varphi_\lambda(u-v))|^p dx\Big|\\
&=\int_{\{\Omega:\varphi_\lambda|u-v|<\eta\}}\Big|a^{1/p}\frac{\nabla\varphi_\lambda}{\varphi_\lambda}\varphi_\lambda(u-v)S'_\eta(\varphi_\lambda(u-v))\Big|^p dx\leq c,
\end{aligned}\tag{5.5}$$

and

$$\begin{aligned} &\lim_{\eta\to 0}\Big(\int_{\{\Omega:\varphi_\lambda|u-v|<\eta\}} a(x)(|\nabla u|^p+|\nabla v|^p)dx\Big)^{\frac{p-1}{p}} \\ &=\Big(\int_{\{\Omega:|u-v|=0\}} a(x)(|\nabla u|^p+|\nabla v|^p)dx\Big)^{\frac{p-1}{p}}=0. \end{aligned} \tag{5.6}$$

If $\{x\in\Omega: u-v=0\}$ has positive measure, by Lebesgue's dominated convergence theorem it follows that

$$\begin{aligned} &\lim_{\eta\to 0}\Big(\int_{\{\Omega:\varphi_\lambda|u-v|<\eta\}} \big|[a(x)]^{1/p}\frac{\nabla\varphi_\lambda}{\varphi_\lambda}\varphi_\lambda(u-v)S'_\eta(\varphi_\lambda(u-v))\big|^p dx\Big)^{1/p} \\ &=\Big(\int_{\{\Omega:|u-v|=0\}} a(x)\big|\frac{\nabla\varphi_\lambda}{\varphi_\lambda}\big|^p\lim_{\eta\to 0}|\varphi_\lambda(u-v)S'_\eta((u-v)\varphi_\lambda)|^p dx\Big)^{1/p}=0. \end{aligned} \tag{5.7}$$

So for both cases we have

$$\lim_{\eta\to 0}\big|\int_\Omega a(x)\varphi_\lambda(u-v)S'_\eta(\varphi_\lambda(u-v))(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\nabla\varphi_\lambda dx\big|=0. \tag{5.8}$$

Denote

$$\Omega_1=\big\{x\in\Omega: -\sum_{i=1}^N b_i(x)d_{x_i}(x)>0\big\}.$$

Then

$$\begin{aligned} &-\int_\Omega b_i(x)(u^{\frac{q}{m}}-v^{\frac{q}{m}})\cdot\varphi_{\lambda x_i}(u-v)S'_\eta(\varphi_\lambda(u-v))dx \\ &=-\int_{\Omega\setminus\Omega_\lambda}\frac{b_i(x)d_{x_i}}{d(x)}(u^{\frac{q}{m}}-v^{\frac{q}{m}})\varphi_\lambda(u-v)S'_\eta(\varphi_\lambda(u-v))dx \\ &\le -\int_{(\Omega\setminus\Omega_\lambda)\bigcap\Omega_1}\frac{b_i(x)d_{x_i}}{d(x)}(u^{\frac{q}{m}}-v^{\frac{q}{m}})\varphi_\lambda(u-v)S'_\eta(\varphi_\lambda(u-v))dx \\ &\le -\int_{(\Omega\setminus\Omega_\lambda)\bigcap\Omega_1}\frac{b_i(x)d_{x_i}}{d(x)}|(u^{\frac{q}{m}}-v^{\frac{q}{m}})\varphi_\lambda(u-v)S'_\eta(\varphi_\lambda(u-v))|dx \\ &\le -c\int_{(\Omega\setminus\Omega_\lambda)\bigcap\Omega_1}\frac{b_i(x)d_{x_i}}{d(x)}|u-v|dx. \end{aligned}$$

In view of conditions (1.16)-(1.19) and $|b_i(x)|\le cd(x)$, since $\lim_{\lambda\to 0}\Omega_1=\Sigma_p$, we have

$$\begin{aligned} &-\lim_{\lambda\to 0}\int_\Omega b_i(x)(u^{\frac{q}{m}}-v^{\frac{q}{m}})\varphi_{\lambda x_i}(u-v)|S'_\eta(\varphi_\lambda(u-v))dx \\ &\le -c\lim_{\lambda\to 0}\int_{(\Omega\setminus\Omega_\lambda)\bigcap\Omega_1}\frac{b_i(x)d_{x_i}}{d(x)}|u-v|dx \\ &\le c\lim_{\lambda\to 0}\int_{(\Omega\setminus\Omega_\lambda)\bigcap\Omega_1}|u-v|dx \\ &=\int_{\Sigma_p}|u-v|d\Sigma=0. \end{aligned} \tag{5.9}$$

Moreover, since $|b_i(x)|\le cd(x)$ and $c_2d(x)\ge a(x)\ge c_1d^p(x)$, we have

$$b_i(x)a^{-1/p}(x)\le c.$$

Using Lebesgue's dominated convergence theorem leads to

$$\begin{aligned}
&\lim_{\eta\to 0}\Big|\int_\Omega b_i(x)\varphi_\lambda(x)(u^{\frac{q}{m}}-v^{\frac{q}{m}}){S_\eta}'(\varphi_\lambda(u-v))(u-v)_{x_i}dx\Big| \\
&\leq c\lim_{\eta\to 0}\Big(\int_\Omega a(x)(|\nabla u|^p+|\nabla v|^p)dx\Big)^{1/p} \\
&\quad\times\Big(\int_\Omega |b_i(x)a^{-1/p}(x)\varphi_\lambda x)(u^{\frac{q}{m}}-v^{\frac{q}{m}}){S_\eta}'(\varphi_\lambda(u-v))|^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}} \\
&=0.
\end{aligned}\tag{5.10}$$

When $q\geq m$ and $\phi(s)=s^{\frac{1}{m}}$, there holds

$$\begin{aligned}
&\lim_{\eta\to 0}\Big|\int_\Omega (b_{ix_i}(x))(u^{\frac{q}{m}}-v^{\frac{q}{m}})S_\eta(\varphi_\lambda(u-v))dx\Big| \\
&\leq c\int_\Omega |\phi(u)(x,t)-\phi(v)(x,t)|dx.
\end{aligned}\tag{5.11}$$

Lettin $\eta\to 0$ and $\lambda\to 0$ in (5.1) we have

$$\int_\Omega |\phi(u)(x,t)-\phi(v)(x,t)|dx-\int_\Omega |\phi(u)(x,0)-\phi(v)(x,0)|dx\leq \int_0^t\int_\Omega |\phi(u)(x,t)-\phi(v)(x,t)|dx.$$

By using Gronwall's inequality, we obtain

$$\int_\Omega |\phi(u)(x,t)-\phi(v)(x,t)|dx\leq c(T)\int_\Omega |\phi(u_0)(x)-\phi(v_0)(x)|dx,\quad \forall t\in[0,T).$$

□

To conclude this article, we give an explanation why the partial boundary value condition (1.17) imposed on (1.1) sounds optimal. Let us consider a simple case as an example that $p=2$ and $m=1=q$. Then (1.1) becomes

$$\frac{\partial u}{\partial t}-\operatorname{div}(a(x)\nabla u)-\sum_{i=1}^N b_i(x)D_iu=0,\tag{5.12}$$

which is a linear degenerate parabolic equation. According to the Fichera-Oleinik theory [15, 16], the optimal partial boundary value condition matching up with equation (5.12) is

$$u(x,t)=0,\quad (x,t)\in\Sigma\times[0,T),\tag{5.13}$$

with

$$\Sigma=\{x\in\partial\Omega: b_i(x)n_i(x)<0\},\tag{5.14}$$

where $\vec{n}=\{n_i\}$ is the inner normal vector of Ω. Here, (5.14) agrees well with condition (1.17).

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