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SOLUTION TO A MULTI-DIMENSIONAL ISENTROPIC QUANTUM DRIFT-DIFFUSION MODEL FOR BIPOLAR SEMICONDUCTORS

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Abstract. We study the existence of weak solution and semiclassical limit for mixed Dirichlet-Neumann boundary value problem of 1,2,3-dimensional isentropic transient quantum drift-diffusion models for bipolar semiconductors. A time-discrete approximate scheme for the model constructed employing the quantum quasi-Fermi potential is composed of non-degenerate elliptic systems, and the system in each time step has a solution in which the components of carrier's densities are strictly positive. Some stability estimates guarantee convergence of the approximate solutions and performance of the semiclassical limit.

1. Introduction

In this article we consider the bipolar isentropic quantum drift-diffusion model

$$\begin{aligned} \frac{\partial n}{\partial t} &= \operatorname{div}\Big(-\varepsilon^2 n\nabla\Big(\frac{\Delta\sqrt{n}}{\sqrt{n}}\Big)+\theta\nabla n^r - n\nabla V\Big),\\ \frac{\partial p}{\partial t} &= \operatorname{div}\Big(-\varepsilon^2 p\nabla\Big(\frac{\Delta\sqrt{p}}{\sqrt{p}}\Big)+\theta\nabla p^r + p\nabla V\Big),\\ \lambda^2\Delta V &= n-p-f \quad \text{in } \Omega\times(0,T), \end{aligned} \tag{1.1}$$

where Ω is a bounded domain of R^d $(d=1,2,3)$ occupied by semiconductor, n,p are the electron and hole's densities, V is the electrostatic potential and $f(x)$ is doping profile which describes the fixed background charges. The parameters $\varepsilon>0$, $\lambda>0$, $\theta>0$ are the scaled Planck constant, permittivity and temperature, respectively. $r>1$ is a constant. The model is the relaxation time limit version of the quantum hydrodynamic model which is derived from the mixed state Schrödinger-Poisson system or, equivalently, the Wigner-Poisson system.

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The system is supplemented with the following initial and mixed Dirichlet-Neumann boundary conditions which are physically motivated and commonly employed in the quantum semiconductor modeling

$$
\begin{gathered}
(n,p,V)=(n_D,p_D,V_D), \quad \varepsilon^2\Delta\sqrt{n}=\varepsilon^2\Delta\sqrt{p}=0 \quad \text{on } \Gamma_D,\\
\nabla n^r\cdot\nu=\nabla p^r\cdot\nu=\nabla V\cdot\nu=\varepsilon^2\nabla\Big(\frac{\Delta\sqrt{n}}{\sqrt{n}}\Big)\cdot\nu=\varepsilon^2\nabla\Big(\frac{\Delta\sqrt{p}}{\sqrt{p}}\Big)\cdot\nu=0\\
\text{on } \Gamma_N,\\
n(x,0)=n_0(x), \quad p(x,0)=p_0(x) \quad \text{in } \Omega.
\end{gathered} \tag{1.2}
$$

The boundary $\partial\Omega\in C^{0,1}$ is piecewise regular and splits into two disjoint parts Γ_D(Ohmic contacts), Γ_N(insulating parts). ν denotes the unit outward normal vector on $\partial\Omega$. Putting $\varepsilon=0$ formally in (1.1), (1.2), we obtain mixed boundary value problem of classical drift-diffusion model. The geometrical figure and boundary conditions affect greatly the flow of carriers because the device becomes more and more smaller nowadays. Hence, the investigation of the multi-dimensional mixed boundary value problem is very important for engineers, especially.

From the view point of mathematics the model is a fourth-order parabolic system for carrier's densities n,p coupled with the Poisson equation for electrostatic potential V and the main difficulty in the investigation is that maximum principle is not available in general for fourth-order parabolic equation to ensure the positivity of the carrier's densities. Another difficulty is that one could not expect, in general, smooth solution due to the mixed boundary condition in (1.2) and singular behavior of the solution may occur near $\overline{\Gamma}_D\cap\overline{\Gamma}_N$ no matter how smooth the known data are.

For the 1-dimensional problem, many results were obtained. In [5, 8, 9, 19, 20] the existence of weak solution and semiclassical or quasineutral limit were studied for various boundary value problems. In [24] the existence of a unique strong solution near the stationary solution and classical limit were studied for the Dirichlet boundary value problem when the scaled Planck constant and disturbance of boundary data are small. In [23] it was proved the unique existence of strong solution of initial value problem which has the character of self-similarity in large time when the doping profile is zero.

Also, for the multi-dimensional problem and related models, some results were obtained only in the case of single boundary value problems. The existence of strong solution for initial value problem near the stationary solution was proved by the relaxation time limit argument of quantum hydrodynamic model in [16] and in [6, 7] Neumann or periodic boundary value problem was studied. Concerning the zero-electric field and zero-temperature approximation of the model we refer [13, 14, 17].

The investigation of stationary problem with both single and mixed boundary value problems is very active and we refer [1, 3, 10, 11, 15, 28]. However, there is no any result for the multi-dimensional transient quantum drift-diffusion model (1.1) with mixed boundary conditions (1.2) which we are interested in.

In this article we prove the existence of weak solution and semiclassical limit for (1.1), (1.2). For this, we construct a time-discrete approximate system for the original transient system using the quantum quasi-Fermi potential. The approximate system in each time step is non-degenerate elliptic and the approximate solution of the system exists. Furthermore, the approximate carrier's densities in each time step are strictly positive and some stability estimates needed for convergence of the approximate solutions and semiclassical limit hold.

Throughout this paper we assume the following: Γ_D is nonempty open subset of $\partial\Omega$, $\Gamma_N = \partial\Omega\backslash\overline{\Gamma}_D$. $f \in L_\infty(\Omega)$ and $r > 1$. V_D is a trace of some function $\widetilde{V}_D \in W^1_\infty(\Omega)$. $n_0, p_0 \in L_\infty(\Omega) \cap H^1(\Omega)$, $\exists m_0 > 0; n_0, p_0 \geq m_0, n_0|_{\Gamma_D} = n_D, p_0|_{\Gamma_D} = p_D$. $n_D, p_D > 0$ are constants. With out loss of generality, we assume $n_D, p_D < 1$. In fact, if $n_D, p_D < k$, $k > 1$, then the new functions $\zeta = n/k$, $\xi = p/k$ satisfy (1.1), (1.2) with λ^2/k, θk^{r-1}, f/k instead of λ^2,θ, f, and with n_D/k, p_D/k, n_0/k, p_0/k instead of n_D, p_D, n_0, p_0.

The quantum quasi-Fermi potentials F, G for (1.1) are defined as

$$F = -\varepsilon^2 \frac{\Delta\sqrt{n}}{\sqrt{n}} + \theta h(n) - V, \quad G = -\varepsilon^2 \frac{\Delta\sqrt{p}}{\sqrt{p}} + \theta h(p) + V. \tag{1.3}$$

where $h(x) = \frac{r}{r-1}(x^{r-1} - 1)$. For the time-discretization of (1.1), we divide the time interval $(0, T]$ into N subintervals $(t_{i-1}, t_i]$, $i = 1, 2, \dots, N$ with mesh size $\tau = t_i - t_{i-1} = \frac{T}{N}$ and $t_0 = 0$. Given ρ_{i-1}, η_{i-1}, $i = 1, 2, \dots, N$, we solve the following elliptic system recursively

$$\begin{gathered}
\frac{1}{\tau}((\rho_i + a(\tau))^2 - (\rho_{i-1} + a(\tau))^2) = \operatorname{div}((\rho_i + a(\tau))^2 \nabla F_i), \\
\varepsilon^2 \Delta\rho_i = \rho_i(\theta h(\rho_i^2) + \tau \ln \rho_i^2 - F_i - V_i), \\
\frac{1}{\tau}((\eta_i + a(\tau))^2 - (\eta_{i-1} + a(\tau))^2) = \operatorname{div}((\eta_i + a(\tau))^2 \nabla G_i), \\
\varepsilon^2 \Delta\eta_i = \eta_i(\theta h(\eta_i^2) + \tau \ln \eta_i^2 - G_i + V_i), \\
\lambda^2 \Delta V_i = (\rho_i + a(\tau))^2 - (\eta_i + a(\tau))^2 - f \quad \text{in } \Omega, \\
(\rho_i, \eta_i, F_i, G_i, V_i) = (\rho_D, \eta_D, F_D, G_D, V_D) \quad \text{on } \Gamma_D, \\
\nabla\rho_i \cdot \nu = \nabla\eta_i \cdot \nu = \nabla F_i \cdot \nu = \nabla G_i \cdot \nu = \nabla V_i \cdot \nu = 0 \quad \text{on } \Gamma_N.
\end{gathered} \tag{1.4}$$

where $(\rho_0, \eta_0) = (\sqrt{n_0}, \sqrt{p_0})$, $(\rho_D, \eta_D) = (\sqrt{n_D}, \sqrt{p_D})$, $F_D = \theta h(n_D) + \tau \ln n_D - V_D$, $G_D = \theta h(p_D) + \tau \ln p_D + V_D$ and continuous function $a(\tau)$ satisfies $a(\tau) > 0$, there exists $c > 0$ such that $\frac{a(\tau)}{\tau^2} \leq c$,for all $\tau > 0$ and $a(0) = 0$. We define the approximate solutions for (1.1), (1.2) as follows

$$\big(\rho^{(N)}, \eta^{(N)}, F^{(N)}, G^{(N)}, V^{(N)}\big)(x, t) = (\rho_i, \eta_i, F_i, G_i, V_i), \quad t \in (t_{i-1}, t_i]. \tag{1.5}$$

where $(\rho_i, \eta_i, F_i, G_i, V_i)$ is the solution to (1.4). Unlike the previous works (see [5, 19, 20]) where the embedding $H^1(\Omega) \hookrightarrow L_\infty(\Omega)$ in 1-dimensional case and exponential transformation are used essentially, we introduce a new "relaxation parameter" $a(\tau)$ in the semi-discretization to ensure non-degeneracy of the first and third equations. The appearance of $\tau \ln \rho_i^2, \tau \ln \eta_i^2$ in the second and forth equations gives the positive lower bounds of ρ_i, η_i in each steps. So, we can prove the existence of the weak solution to (1.4) using the Stampacchia's truncation method and Leray-Schauder fixed point theorem. (Theorem 1.1)

Theorem 1.1. *Let $N = N_0, N_0 + 1, \dots$ be the integers such that*

$$\frac{T}{2\theta N} \|\nabla V_D\|^2_{L_\infty(\Omega)} \leq \min\{-h(n_D), -h(p_D)\}. \tag{1.6}$$

Then there exist weak solutions $(\rho_i, \eta_i, F_i, G_i, V_i) \in (L_\infty(\Omega) \cap H^1(\Omega))^5$, $i = 1, 2, \dots N$ *to the recurrent elliptic system* (1.4) *satisfying*

$$m_{i,N} \le \rho_i, \eta_i \le M_{i,N}$$

for some $m_{i,N}, M_{i,N} > 0$.

The entropy inequality in the previous works (see [18, 19]) which show boundedness of the first-order derivatives of the approximate solutions for the case of 1-dimensional model also holds for our case (Lemma 3.2). This is enough for the upper bound of the approximate carrier's densities independent of the mesh size in 1-dimensional case because of the embedding $H^1(\Omega) \hookrightarrow L_\infty(\Omega)$. Furthermore, the upper bound gives boundedness of their second-order derivatives needed for convergence of the scheme. Hence, for 1-dimensional problems the embedding $H^1(\Omega) \hookrightarrow L_\infty(\Omega)$ plays a crucial role in the convergence of the approximate solution as well as in the existence of the approximate solution. However, such embedding does not hold in multi-dimensional case. So, we employ the functions

$$\frac{\rho_i - \rho_D}{\rho_i + a(\tau)}, \frac{\eta_i - \eta_D}{\eta_i + a(\tau)} \in H_0^1(\Omega \cup \Gamma_N) := \{u \in H^1(\Omega); u = 0 \text{ on } \Gamma_D\}$$

as test functions of the first and third equation in (1.4) and, with careful calculation, get the boundedness of $\{(\varepsilon\frac{\Delta\rho^{(N)}}{\sqrt{\rho^{(N)}}}, \varepsilon\frac{\Delta\eta^{(N)}}{\sqrt{\eta^{(N)}}})\}$ and $\{(\varepsilon\nabla(\rho^{(N)})^{2r}, \varepsilon\nabla(\eta^{(N)})^{2r})\}$. Using these facts and employing the Stampacchia's truncation method, we obtain boundedness of $\{(\varepsilon\Delta\rho^{(N)}, \varepsilon\Delta\eta^{(N)})\}$. Through the obtained estimates and the compactness result for piecewise constant functions in time(see [12]) we prove for any fixed $\varepsilon \in (0,1)$ the compactness of $\{(\rho^{(N)}, \eta^{(N)})\}$ in $L_p(0,T; H^1(\Omega))$ for all $p \in (1, \infty)$. Furthermore, when $r \ge 9/5$, some estimates of the solutions independent of $\varepsilon \in (0,1)$ as well as N are obtained. See the details of these stability estimates in section 3.

Theorem 1.2. *For each fixed* $\varepsilon \in (0,1)$ *there exist* (ρ, η, V) *and a subsequence of approximate solutions obtained in Theorem 1.1 (again denoted by* $(\rho^{(N)}, \eta^{(N)}, V^{(N)})$*) such that*

$$\begin{gathered}
\rho^{(N)} \to \rho, \quad \eta^{(N)} \to \eta \quad *\text{-weakly in } X \cap L_{4/3}(0,T; H^{3/2-\delta}(\Omega)), \quad \delta > 0, \\
\rho^{(N)} \to \rho, \quad \eta^{(N)} \to \eta \quad \text{in } L_p(0,T; H^1(\Omega)), \quad \forall p \in (1, \infty), \\
\nabla(\rho^{(N)})^{2r} \to \nabla\rho^{2r}, \quad \nabla(\eta^{(N)})^{2r} \to \nabla\eta^{2r} \quad \text{weakly in } L_1(0,T; L_{6/5}(\Omega)), \\
V^{(N)} \to V \quad *\text{-weakly in } L_\infty(0,T; H^1(\Omega))
\end{gathered} \tag{1.7}$$

as $N \to \infty$ where $X := \{u \in L_\infty(0,T;H^1(\Omega))\ \Delta u \in L_{4/3}(0,T;L_2(\Omega)),\ \nabla u \cdot \nu = 0$ on $\Gamma_N\}$. Furthermore, (ρ^2, η^2, V) is a solution of (1.1), (1.2) in the sense of

$$
\begin{aligned}
&\rho, \eta \geq 0, \quad \frac{\partial \rho^2}{\partial t}, \frac{\partial \eta^2}{\partial t} \in L_2(0,T;W^{-1}_{r+1}(\Omega \cup \Gamma_N)),\\
&\rho - \rho_D, \eta - \eta_D, V - V_D \in L_\infty(0,T;H^1_0(\Omega \cup \Gamma_N)),\\
&\int_0^T \langle \frac{\partial \rho^2}{\partial t}, \phi\rangle\,dt = -2\varepsilon^2 \int_Q \Delta\rho\nabla\rho \cdot \nabla\phi\,dx\,dt - \varepsilon^2 \int_Q \Delta\rho\rho\Delta\phi\,dx\,dt\\
&\qquad - \theta \int_Q \nabla\rho^{2r} \cdot \nabla\phi\,dx\,dt + \int_Q \rho^2 \nabla V \cdot \nabla\phi\,dx\,dt,\\
&\int_0^T \langle \frac{\partial \eta^2}{\partial t}, \phi\rangle\,dt = -2\varepsilon^2 \int_Q \Delta\eta\nabla\eta \cdot \nabla\phi\,dx\,dt - \varepsilon^2 \int_Q \Delta\eta\eta\Delta\phi\,dx\,dt\\
&\qquad - \theta \int_Q \nabla\eta^{2r} \cdot \nabla\phi\,dx\,dt - \int_Q \eta^2 \nabla V \cdot \nabla\phi\,dx\,dt,\\
&-\lambda^2 \int_Q \nabla V \cdot \nabla\phi\,dx\,dt = \int_Q (\rho^2 - \eta^2 - f)\phi\,dx\,dt,\ \forall \phi \in C_0^\infty(Q)
\end{aligned} \tag{1.8}
$$

where $W^{-1}_{r+1}(\Omega \cup \Gamma_N)$ is dual space of $\{u \in W^1_{(r+1)/r}(\Omega); u = 0$ on $\Gamma_D\}$.

Remark 1.3. We note that in the case of unipolar model one can also obtain such kind of existence result by the same method. However, we discuss here only the bipolar model.

Theorem 1.4. *Let $(n^{(\varepsilon)}, p^{(\varepsilon)}, V^{(\varepsilon)}), \varepsilon \in (0,1)$ be the solution of (1.1), (1.2) with $r \geq 9/5$ obtained in the Theorem 1.2. Then there exist some n, p, V and sequence $\varepsilon \to 0$ such that*

$$
\begin{aligned}
&n^{(\varepsilon)} \to n, \quad p^{(\varepsilon)} \to p \quad *\text{-weakly in } L_\infty(0,T;L_r(\Omega)),\\
&\nabla (n^{(\varepsilon)})^r \to \nabla n^r, \quad \nabla (p^{(\varepsilon)})^r \to \nabla p^r \quad \text{weakly in } L_1(Q),\\
&V^{(\varepsilon)} \to V \quad *\text{-weakly in } L_\infty(0,T;H^1(\Omega)).
\end{aligned} \tag{1.9}
$$

Furthermore, (n, p, V) is a weak solution to the mixed boundary value problem of classical drift-diffusion model in the sense of

$$
\begin{aligned}
&n, p \geq 0, \quad \frac{\partial n}{\partial t}, \frac{\partial p}{\partial t} \in L_2(0,T;W^{-1}_{r+1}(\Omega)),\\
&\int_0^T \langle \frac{\partial n}{\partial t}, \phi\rangle\,dt = -\theta \int_Q \nabla n^r \cdot \nabla\phi\,dx\,dt + \int_Q n\nabla V \cdot \nabla\phi\,dx\,dt,\\
&\int_0^T \langle \frac{\partial p}{\partial t}, \phi\rangle\,dt = -\theta \int_Q \nabla p^r \cdot \nabla\phi\,dx\,dt - \int_Q p\nabla V \cdot \nabla\phi\,dx\,dt,\\
&-\lambda^2 \int_Q \nabla V \cdot \nabla\phi\,dx\,dt = \int_Q (n - p - f)\phi\,dx\,dt, \quad \forall \phi \in C_0^\infty(Q).
\end{aligned} \tag{1.10}
$$

The article is organized as follows. In section 2 we prove the Theorem 1.1. Some stability estimates of the approximate solutions are presented in section 3. Section 4 is devoted to the proof of the convergence of the approximate solutions and semiclassical limit.

2. Existence of approximate solutions

First of all, we introduce Stampacchia's lemma which will be used later.

Lemma 2.1 ([15, lemma 5.2.4]). *Let $\varphi : (a,b) \to R^1$ be a nonnegative, nonincreasing function where $a < b \le +\infty$. Suppose that there exist constants $K > 0, r > 0, \alpha > 1$ such that*

$$\varphi(\xi) \le K(\xi-\zeta)^{-r}\varphi(\zeta)^{\alpha}, \quad a < \zeta < \xi < b.$$

If the number $\xi^* = K^{\frac{1}{r}} 2^{\frac{\alpha}{\alpha-1}} \varphi(a)^{\frac{\alpha-1}{r}}$ is such that $a + \xi^* < b$, then $\varphi(a+\xi^*) = 0$.

Lemma 2.2. *The weak solution $u \in H^1(\Omega)$ to*

$$\begin{gathered} \operatorname{div}(a(x)\nabla u) = f_1^2 - f_2^2 - f_0, \ \text{in } \Omega, \\ u = u_D \ \text{on } \Gamma_D, \quad \frac{\partial u}{\partial \nu} = 0 \ \text{on } \Gamma_N. \end{gathered} \tag{2.1}$$

with $f_i \in L_4(\Omega)$, $i = 0,1,2$, $u_D \in W^1_\infty(\Omega)$ and $a(\cdot) \in L_\infty(\Omega)$ satisfying $a(x) \ge a_0$ for some constant $a_0 > 0$ satisfies for some constant $c(\Omega) > 0$ depending only on Ω

$$\begin{gathered} u(x) \le u^* := \|u_D\|_{L_\infty(\Gamma_D)} + c(\Omega)a_0^{-1}(\|f_2\|^2_{L_4(\Omega)} + \|f_0\|_{L_2(\Omega)}), \\ u(x) \ge -u_* := -\Big(\|u_D\|_{L_\infty(\Gamma_D)} + c(\Omega)a_0^{-1}(\|f_1\|^2_{L_4(\Omega)} + \|f_0\|_{L_2(\Omega)})\Big), \end{gathered} \tag{2.2}$$

a.e. in Ω.

The above lemma can be proved as in [15, Lemma 5.2.5], using Lemma 2.1.

For $\rho_{i-1}, \eta_{i-1} \in H^1(\Omega) \cap L_\infty(\Omega)$ satisfying

$$\rho_{i-1}|_{\Gamma_D} = \rho_D, \quad \eta_{i-1}|_{\Gamma_D} = \eta_D, \quad \exists m_{i-1} > 0; \rho_{i-1}, \eta_{i-1} \ge m_{i-1},$$

we consider the auxiliary boundary-value problem

$$\begin{gathered} \operatorname{div}((S_M(\rho) + a(\tau))^2 \nabla F) = \frac{\delta}{\tau}((\rho + a(\tau))^2 - (\xi + a(\tau))^2), \\ \varepsilon^2 \Delta\rho = \rho_+(\theta\delta h(S_M^2(\rho)) + \tau\delta \ln \rho^2 - F - \delta V), \\ \operatorname{div}((S_M(\eta) + a(\tau))^2 \nabla G) = \frac{\delta}{\tau}((\eta + a(\tau))^2 - (\zeta + a(\tau))^2), \\ \varepsilon^2 \Delta\eta = \eta_+(\theta\delta h(S_M^2(\eta)) + \tau\delta \ln \eta^2 - G + \delta V), \\ \lambda^2 \Delta V = (\rho + a(\tau))^2 - (\eta + a(\tau))^2 - f \quad \text{in } \Omega, \end{gathered} \tag{2.3}$$

with

$$\begin{gathered} (\rho, \eta, F, G, V) = (\rho_{D\delta}, \eta_{D\delta}, F_{D\delta}, G_{D\delta}, V_D) \quad \text{on } \Gamma_D, \\ \nabla\rho \cdot \nu = \nabla\eta \cdot \nu = \nabla F \cdot \nu = \nabla G \cdot \nu = \nabla V \cdot \nu = 0 \quad \text{on } \Gamma_N \end{gathered} \tag{2.4}$$

where $\delta \in (0,1]$, $M \ge 1$ are constants, $S_M(\varphi) = \min\{M, \max\{\varphi, 0\}\}$, $\varphi_+ = \max\{\varphi, 0\}$, and

$$\begin{gathered} \xi = \delta\rho_{i-1}, \quad \zeta = \delta\eta_{i-1}, \quad \rho_{D\delta} = \delta\rho_D, \quad \eta_{D\delta} = \delta\eta_D, \\ F_{D\delta} = \delta(\theta h(\delta^2\rho_D^2) + \tau \ln(\delta\rho_D)^2 - V_D), \\ G_{D\delta} = \delta(\theta h(\delta^2\eta_D^2) + \tau \ln(\delta\eta_D)^2 + V_D). \end{gathered}$$

Lemma 2.3. *The weak solution $(\rho, \eta, F, G, V) \in (H^1(\Omega))^5$ to (2.3), (2.4) satisfies*

$$\rho(x), \eta(x) \ge m, \quad a.\ e.\ in\ \Omega, \tag{2.5}$$

$$\|(\rho,\eta)\|_{(L_\infty(\Omega))^2} \le c(\|(\rho,\eta)\|_{(L_4(\Omega))^2}) \tag{2.6}$$

for a constant $m > 0$, and $c(\|(\rho,\eta)\|_{(L_4(\Omega))^2})$ which is also bounded if $\|(\rho,\eta)\|_{(L_4(\Omega))^2}$ is bounded.

We note that the L_∞-bound depends on M in the definition of the function $S_M(\cdot)$.

Proof. Taking $\rho_- = \min\{\rho, 0\}$, $\eta_- = \min\{\eta, 0\} \in H_0^1(\Omega \cup \Gamma_N)$ as test functions of the second and fourth equation of (2.3), respectively, we can easily verify the nonnegativity of ρ, η. Also, by Lemma 2.2 we obtain the lower and upper bounds for F, G, V

$$\begin{gathered}
V(x) \le V^* := \|V_D\|_{L_\infty(\Gamma_D)} + c(\Omega,\lambda)\Big(\|\eta + a(\tau)\|^2_{L_4(\Omega)} + \|f\|_{L_\infty(\Omega)}\Big), \\
V(x) \ge -V_* := -\Big(\|V_D\|_{L_\infty(\Gamma_D)} + c(\Omega,\lambda)(\|\rho + a(\tau)\|^2_{L_4(\Omega)} + \|f\|_{L_\infty(\Omega)})\Big), \\
F(x) \le F^* := \|F_{D\delta}\|_{L_\infty(\Gamma_D)} + c(\Omega,\tau,\delta)\|\xi + a(\tau)\|^2_{L_4(\Omega)}, \\
F(x) \ge -F_* := -\Big(\|F_{D\delta}\|_{L_\infty(\Gamma_D)} + c(\Omega,\tau,\delta)\|\rho + a(\tau)\|^2_{L_4(\Omega)}\Big), \\
G(x) \le G^* := \|G_{D\delta}\|_{L_\infty(\Gamma_D)} + c(\Omega,\tau,\delta)\|\zeta + a(\tau)\|^2_{L_4(\Omega)}, \\
G(x) \ge -G_* := -\Big(\|G_{D\delta}\|_{L_\infty(\Gamma_D)} + c(\Omega,\tau,\delta)\|\eta + a(\tau)\|^2_{L_4(\Omega)}\Big)
\end{gathered} \tag{2.7}$$

for some constants $c(\Omega,\lambda), c(\Omega,\tau,\delta) > 0$. Hence, we have

$$\rho(x) \le K_\rho := \max\{1, \exp(\frac{1}{2\tau\delta}(F^* + V^*))\} = c(\|\xi\|_{L_4(\Omega)}, \|\eta\|_{L_4(\Omega)}), \tag{2.8}$$

a. e. in Ω. The second equation in (2.3) yields

$$\begin{aligned}
\varepsilon^2 \int_\Omega |\nabla(\rho - K_\rho)_+|^2 dx &= -\int_\Omega \rho_+(\theta\delta h(S_M^2(\rho)) + \tau\delta\ln\rho^2 - F - \delta V)(\rho - K_\rho)_+ dx \\
&\le \int_\Omega \rho_+(F^* + V^* - \tau\delta\ln K_\rho^2)(\rho - K_\rho)_+ dx \le 0.
\end{aligned}$$

In the same way we obtain the upper bound for η as

$$\eta(x) \le K_\eta := \max\{1, \exp(\frac{1}{2\tau\delta}(G^* + V_*))\} = c(\|\zeta\|_{L_4(\Omega)}, \|\rho\|_{L_4(\Omega)}), \tag{2.9}$$

a. e. in Ω.

From (2.8), (2.9) we obtain (2.6). To obtain the lower bounds

$$\begin{gathered}
\rho(x) \ge m_\rho := \min\{\delta\rho_D, \exp(-\frac{1}{2\tau\delta}(F_* + V_*))\}, \quad \text{a.e. in } \Omega, \\
\eta(x) \ge m_\eta := \min\{\delta\eta_D, \exp(-\frac{1}{2\tau\delta}(G_* + V^*))\}, \quad \text{a.e. in } \Omega,
\end{gathered} \tag{2.10}$$

we take $(\rho - m_\rho)_-$, $(\eta - m_\eta)_- \in H_0^1(\Omega \cup \Gamma_N)$ as test functions of the second and fourth equation of (2.3) respectively and use (2.7). $\square$

Lemma 2.4. *Assume that $\tau = N/T$ satisfies* (1.6)*. Then the weak solution $(\rho, \eta, F, G, V) \in (H^1(\Omega))^5$ to* (2.3)*,* (2.4) *satisfies*

$$\|(\rho,\eta)\|_{(H^1(\Omega))^2} \le c \tag{2.11}$$

for some constant $c > 0$ independent of the solution, $\delta \in (0, 1]$ and the choice of $M \ge 1$.

Proof. By Lemma 2.3,

$$F=-\varepsilon^2\frac{\Delta\rho}{\rho}+\theta\delta h(S_M^2(\rho))+\tau\delta\ln\rho^2-\delta V,\quad G=-\varepsilon^2\frac{\Delta\eta}{\eta}+\theta\delta h(S_M^2(\eta))+\tau\delta\ln\eta^2+\delta V.$$

Now, we take $(F-F_{D\delta})\in H_0^1(\Omega\cap\Gamma_N)$ as test function of the first equation in (2.3) to obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (S_M(\rho)+a(\tau))^2|\nabla F_{D\delta}|^2dx\\
&\geq \frac{1}{2}\int_\Omega (S_M(\rho)+a(\tau))^2|\nabla F|^2dx\\
&\quad+\frac{\delta}{\tau}\int_\Omega(\rho^2-\xi^2+2a(\tau)(\rho-\xi))\Big(-\varepsilon^2\frac{\Delta\rho}{\rho}\Big)\,dx\\
&\quad+\delta^2\int_\Omega(\rho^2-\xi^2+2a(\tau)(\rho-\xi))\ln\rho^2dx\\
&\quad+\frac{\delta^2}{\tau}\int_\Omega((\rho+a(\tau))^2-(\xi+a(\tau))^2)(-V+V_D)\,dx\\
&\quad+\frac{\theta\delta^2}{\tau}\int_\Omega(\rho^2-\xi^2+2a(\tau)(\rho-\xi))h(S_M^2(\rho))\,dx\\
&\quad+\frac{\delta}{\tau}\int_\Omega((\rho+a(\tau))^2-(\xi+a(\tau))^2)(-F_{D\delta}-\delta V_D)\,dx\\
&=\sum_{j=1}^6R_j.
\end{aligned}\tag{2.12}$$

We estimate term by term. Integration by parts and Young's inequality yield

$$\begin{aligned}
R_2&=\frac{\delta\varepsilon^2}{\tau}\int_\Omega\nabla\Big(\frac{\rho^2-\xi^2}{\rho}\Big)\cdot\nabla\rho dx+2\frac{\delta\varepsilon^2a(\tau)}{\tau}\int_\Omega\nabla\Big(\frac{\rho-\xi}{\rho}\Big)\cdot\nabla\rho dx\\
&=\frac{\delta\varepsilon^2}{\tau}\Big(\int_\Omega|\nabla\rho|^2dx-\int_\Omega|\nabla\xi|^2dx+\int_\Omega|\nabla\xi-\frac{\xi}{\rho}\nabla\rho|^2dx\Big)\\
&\quad+2\frac{\delta\varepsilon^2a(\tau)}{\tau}\int_\Omega\Big(\frac{\xi}{\rho^2}|\nabla\rho|^2-\frac{\nabla\rho\cdot\nabla\xi}{\rho}\Big)\,dx\\
&\geq\frac{\delta\varepsilon^2}{\tau}\int_\Omega|\nabla\rho|^2dx-\frac{\delta\varepsilon^2}{\tau}\int_\Omega|\nabla\xi|^2dx-\frac{\delta\varepsilon^2a(\tau)}{\tau}\int_\Omega\frac{|\nabla\xi|^2}{\xi}dx.
\end{aligned}\tag{2.13}$$

The estimate for R_3 is

$$\begin{aligned}
R_3&=\delta^2\int_\Omega(\rho^2-\xi^2)\ln\rho^2dx+4\delta^2a(\tau)\int_\Omega(\rho-\xi)\ln\rho dx\\
&\geq\delta^2\int_\Omega(H(\rho^2)-H(\xi^2))\,dx+4\delta^2a(\tau)\int_\Omega(H(\rho)-H(\xi))\,dx,
\end{aligned}\tag{2.14}$$

where $H(\alpha):=\alpha(\ln\alpha-1)+1$ and $\alpha>0$, which is well-known (see [19, Lemma 2.2]). We rewrite R_5 as

$$\begin{aligned}
R_5&=\frac{r\theta\delta^2}{(r-1)\tau}\int_{\Omega(\rho>1,\rho>\xi)}((\rho^2-\xi^2)+2a(\tau)(\rho-\xi))(S_M^{2(r-1)}(\rho)-1)\,dx\\
&\quad+\frac{r\theta\delta^2}{(r-1)\tau}\int_{\Omega(\rho>1,\rho\leq\xi)}((\rho^2-\xi^2)+2a(\tau)(\rho-\xi))(S_M^{2(r-1)}(\rho)-1)\,dx
\end{aligned}$$

$$
\begin{aligned}
&+\frac{r\theta\delta^2}{(r-1)\tau}\int_{\Omega(\rho\le 1)}((\rho^2-\xi^2)+2a(\tau)(\rho-\xi))(S_M^{2(r-1)}(\rho)-1)\,dx\\
&=R_{5,1}+R_{5,2}+R_{5,3}.
\end{aligned}
$$

Since $R_{5,1}\ge 0$,

$$
\begin{aligned}
R_{5,2}&\ge -\frac{r\theta\delta^2}{(r-1)\tau}\int_{\Omega(\rho>1,\rho\le\xi)}(\xi^2+2a(\tau)\xi)\Big(S_M^{2(r-1)}(\rho)-1\Big)\,dx\\
&\ge -\frac{r\theta\delta^2}{(r-1)\tau}\int_{\Omega(\rho>1,\rho\le\xi)}(\xi^{2r}+2a(\tau)\xi^{2r-1})\,dx,
\end{aligned}
$$

and

$$
R_{5,3}\ge -\frac{r\theta\delta^2}{(r-1)\tau}\int_{\Omega(\rho\le 1)}(\rho^2+2a(\tau)\rho)\,dx\ge -\frac{r\theta\delta^2(1+2a(\tau))}{(r-1)\tau}\operatorname{meas}(\Omega),
$$

we obtain

$$
R_5\ge -\frac{r\theta\delta^2}{(r-1)\tau}\Big(\int_\Omega(\xi^{2r}+2a(\tau)\xi^{2r-1})\,dx+(1+2a(\tau))\operatorname{meas}(\Omega)\Big). \tag{2.15}
$$

Using the above inequalities we have

$$
\begin{aligned}
&\varepsilon^2\int_\Omega|\nabla\rho|^2dx+\tau\delta\int_\Omega H(\rho^2)\,dx+4\tau\delta a(\tau)\int_\Omega H(\rho)\,dx\\
&+\frac{\tau}{2\delta}\int_\Omega(S_M(\rho)+a(\tau))^2|\nabla F|^2dx-\delta\int_\Omega((\rho+a(\tau))^2-(\xi+a(\tau))^2)(V-V_D)\,dx\\
&\le \varepsilon^2\int_\Omega|\nabla\xi|^2dx+\tau\delta\int_\Omega H(\xi^2)\,dx+4\tau\delta a(\tau)\int_\Omega H(\xi)\,dx\\
&\quad+\varepsilon^2a(\tau)\int_\Omega\frac{|\nabla\xi|^2}{\xi}dx+\frac{\tau}{2\delta}\int_\Omega(\rho+a(\tau))^2|\nabla F_{D\delta}|^2dx\\
&\quad+\frac{r\theta\delta}{(r-1)}(\int_\Omega(\xi^{2r}+2a(\tau)\xi^{2r-1})\,dx+(1+2a(\tau))\operatorname{meas}(\Omega))\\
&\quad+\int_\Omega((\rho+a(\tau))^2-(\xi+a(\tau))^2)(F_{D\delta}+\delta V_D)\,dx.
\end{aligned}
$$

We obtain a similar inequality for η by taking $(G-G_{D\delta})\in H_0^1(\Omega\cap\Gamma_N)$ as test function of the third equation in (2.3). Adding the resulting two inequalities and taking into account that

$$
\begin{aligned}
&\frac{\tau}{2\delta}\Big(\int_\Omega(\rho+a(\tau))^2|\nabla F_{D\delta}|^2dx+\int_\Omega(\eta+a(\tau))^2|\nabla G_{D\delta}|^2dx\Big)\\
&\le\frac{\delta\tau}{2}\|\nabla V_D\|^2_{L_\infty(\Omega)}\int_\Omega((\rho+a(\tau))^2+(\eta+a(\tau))^2)\,dx,
\end{aligned}
$$

$$
\begin{aligned}
&\int_\Omega(\rho+a(\tau))^2(\delta V_D+F_{D\delta})\,dx+\int_\Omega(\eta+a(\tau))^2(G_{D\delta}-\delta V_D)\,dx\\
&\le\theta\delta h(\rho_D^2)\int_\Omega(\rho+a(\tau))^2dx+\theta\delta h(\eta_D^2)\int_\Omega(\eta+a(\tau))^2dx,
\end{aligned}
$$

$$
\begin{aligned}
&- \delta \int_\Omega (((\rho + a(\tau))^2 - (\eta + a(\tau))^2) - ((\xi + a(\tau))^2 - (\zeta + a(\tau))^2))(V - V_D)\, dx \\
&= \delta\lambda^2 \int_\Omega \nabla(V - V_D) \cdot \nabla((V - V_D) - (V' - V_D))\, dx \\
&\geq \frac{\delta\lambda^2}{2}\Big(\int_\Omega |\nabla(V - V_D)|^2 dx - \int_\Omega |\nabla(V' - V_D)|^2 dx\Big)
\end{aligned}
$$

we have

$$
\begin{aligned}
&\varepsilon^2 \int_\Omega (|\nabla\rho|^2 + |\nabla\eta|^2)\, dx - \theta\delta\Big(h(\rho_D^2)\int_\Omega (\rho + a(\tau))^2 dx + h(\eta_D^2)\int_\Omega (\eta + a(\tau))^2 dx\Big) \\
&\leq \varepsilon^2 \int_\Omega (|\nabla\xi|^2 + |\nabla\zeta|^2)\, dx + \tau \int_\Omega (H(\xi^2) + H(\zeta^2))\, dx \\
&\quad + 4\tau a(\tau) \int_\Omega (H(\xi) + H(\zeta))\, dx + \frac{\lambda^2}{2}\int_\Omega |\nabla(V' - V_D)|^2 dx \\
&\quad + \frac{\delta\tau}{2}\|\nabla V_D\|^2_{L_\infty(\Omega)} \int_\Omega ((\rho + a(\tau))^2 + (\eta + a(\tau))^2)\, dx \\
&\quad + \varepsilon^2 a(\tau) \int_\Omega \Big(\frac{|\nabla\xi|^2}{\xi} + \frac{|\nabla\zeta|^2}{\zeta}\Big)\, dx + c\int_\Omega ((\xi + a(\tau))^2 + (\zeta + a(\tau))^2)\, dx \\
&\quad + \frac{r\theta}{(r-1)}\Big(\int_\Omega (\xi^{2r} + \zeta^{2r} + 2a(\tau)(\xi^{2r-1} + \zeta^{2r-1}))\, dx + 2(1 + 2a(\tau))\operatorname{meas}(\Omega)\Big)
\end{aligned}
$$

for some $c > 0$ independent of the solution, $\delta \in (0,1]$ and the choice of $M \geq 1$. Here V' is the solution to

$$
\begin{gathered}
\lambda^2 \triangle V = (\xi + a(\tau))^2 - (\zeta + a(\tau))^2 - f \quad \text{in } \Omega, \\
V = V_D \text{ on } \Gamma_D, \quad \frac{\partial V}{\partial \nu} = 0 \text{ on } \Gamma_N,
\end{gathered}
$$

This completes the proof with the help of the choice of τ and $\xi = \delta\rho_{i-1}, \zeta = \delta\eta_{i-1}$. $\square$

Proof of Theorem 1.1. We define a fixed point mapping $R : (L_4(\Omega))^2 \times [0,1] \to (L_4(\Omega))^2$ as follows. Let $\delta \in [0,1]$, $u, w \in (L_4(\Omega))^2$ be given. Then, by the theory of elliptic equation, there exists a unique weak solution $(\rho, \eta, F, G, V) \in (H^1(\Omega))^5$ to

$$
\begin{gathered}
\operatorname{div}((S_M(u) + a(\tau))^2 \nabla F) = \frac{\delta}{\tau}((u + a(\tau))^2 - (\xi + a(\tau))^2), \\
\varepsilon^2 \Delta\rho = u_+(\theta\delta h(S_M^2(u)) + \tau\delta \ln u^2 - F - \delta V), \\
\operatorname{div}((S_M(w) + a(\tau))^2 \nabla G) = \frac{\delta}{\tau}((w + a(\tau))^2 - (\zeta + a(\tau))^2), \\
\varepsilon^2 \Delta\eta = w_+(\theta\delta h(S_M^2(w)) + \tau\delta \ln w^2 - G + \delta V), \\
\lambda^2 \Delta V = (u + a(\tau))^2 - (w + a(\tau))^2 - f \quad \text{in } \Omega
\end{gathered}
\tag{2.16}
$$

with the boundary condition (2.4) and $\xi = \delta\rho_{i-1}$, $\zeta = \delta\eta_{i-1}$. Hence, the mapping given by $R((u,w),\delta) = (\rho,\eta)$ is well defined. Moreover, $R((u,w),0) = 0$ for any $u, w \in L_4(\Omega)$. We can easily verify that R is continuous and compact by standard argument. Lemma 2.4 shows that there is a constant $c > 0$ such that for all $\rho, \eta \in L_4(\Omega), \delta \in [0,1]$ satisfying $R((\rho,\eta),\delta) = (\rho,\eta)$ it holds $\|(\rho,\eta)\|_{(L_4(\Omega))^2} \leq c$. Hence, by the Leray-Schauder fixed point theorem there exists (ρ,η) satisfying $R((\rho,\eta),1) = (\rho,\eta)$ which is the solution to (2.3), (2.4) with $\delta = 1$ depending on the choice of M. However, taking into account of the estimate in Lemmas 2.3 and 2.4, and the embedding $H^1(\Omega) \hookrightarrow L_4(\Omega)$, the solution satisfies $\rho(x), \eta(x) \le M$ for some M large enough. This completes the proof. $\square$

3. Stability estimates

Let $N = N_0, N_0+1, \dots$, $(\rho_i, \eta_i, F_i, G_i, V_i) \in (L_\infty(\Omega) \cap H^1(\Omega))^5$, $i = 1, 2, \dots N$ be the recursively defined solutions to (1.4) and $(\rho^{(N)}, \eta^{(N)}, F^{(N)}, G^{(N)}, V^{(N)})$ be the approximate solutions defined by (1.5).

Lemma 3.1. *There exist constant $c > 0$ and integer $N^* \ge N_0$ such that for all $N = N^*, N^*+1, \dots$ and $\varepsilon \in (0,1)$ it holds*

$$\begin{aligned}
&\|(\rho^{(N)}, \eta^{(N)})\|^2_{(L_\infty(0,T;L_2(\Omega)))^2} + \|(\varepsilon \frac{\Delta\rho^{(N)}}{\sqrt{\rho^{(N)}}}, \varepsilon \frac{\Delta\eta^{(N)}}{\sqrt{\eta^{(N)}}})\|^2_{(L_2(Q))^2} \\
&+ \tau \|(\frac{\nabla\rho^{(N)}}{\sqrt{\rho^{(N)}}}, \frac{\nabla\eta^{(N)}}{\sqrt{\eta^{(N)}}})\|^2_{(L_2(Q))^2} + \sum_{i=1}^{N} \int_\Omega (\frac{(\rho_i - \rho_{i-1})^2}{\rho_i + a(\tau)} + \frac{(\eta_i - \eta_{i-1})^2}{\eta_i + a(\tau)})\, dx \\
&+ \|(\nabla(\rho^{(N)})^{r-1/2}, \nabla(\eta^{(N)})^{r-1/2})\|^2_{(L_2(Q))^2} \le c.
\end{aligned} \tag{3.1}$$

Proof. We take $\phi = \frac{\rho_i - \rho_D}{\rho_i + a(\tau)} \in H^1_0(\Omega \cup \Gamma_N)$ as test function in the first equation of (1.4) to obtain

$$\begin{aligned}
&- \int_\Omega (\rho_i + a(\tau))^2 \nabla F_i \cdot \nabla \frac{\rho_i - \rho_D}{\rho_i + a(\tau)} dx \\
&= \frac{1}{\tau} \int_\Omega ((\rho_i + a(\tau))^2 - (\rho_{i-1} + a(\tau))^2) \frac{\rho_i - \rho_D}{\rho_i + a(\tau)} dx.
\end{aligned} \tag{3.2}$$

Using the equality $\alpha(\alpha - \beta) = \frac{1}{2}(\alpha^2 - \beta^2 + (\alpha-\beta)^2)$, we rewrite the right hand side of (3.2) as

$$\begin{aligned}
R &= \frac{1}{\tau} \int_\Omega \Big(2(\rho_i + a(\tau))(\rho_i - \rho_{i-1}) - (\rho_i - \rho_{i-1})^2\Big) \frac{\rho_i - \rho_D}{\rho_i + a(\tau)} dx \\
&= \frac{1}{\tau} \Big(\int_\Omega (\rho_i - \rho_D)^2 dx + \int_\Omega \frac{\rho_D + a(\tau)}{\rho_i + a(\tau)} (\rho_i - \rho_{i-1})^2 dx - \int_\Omega (\rho_{i-1} - \rho_D)^2 dx \Big).
\end{aligned}$$

The left-hand side of (3.2) can be written as

$$\begin{aligned}
L &= - \int_\Omega (\rho_i + a(\tau))^2 \nabla \Big(-\varepsilon^2 \frac{\Delta\rho_i}{\rho_i} + \tau \ln \rho_i^2 + \theta h(\rho_i^2) - V_i \Big) \cdot \nabla \frac{\rho_i - \rho_D}{\rho_i + a(\tau)} dx \\
&= -(\rho_D + a(\tau)) \int_\Omega \frac{\varepsilon^2 (\Delta\rho_i)^2 + 2\tau |\nabla\rho_i|^2}{\rho_i} dx \\
&\quad - (\rho_D + a(\tau)) \Big(2r\theta \int_\Omega \rho_i^{2r-3} |\nabla\rho_i|^2 dx - \int_\Omega \nabla V_i \cdot \nabla(\rho_i - \rho_D)\, dx \Big).
\end{aligned}$$

Hence, from (3.2) we have

$$\begin{aligned}
&\frac{1}{\rho_D + a(\tau)} \int_\Omega (\rho_i - \rho_D)^2 dx + \int_\Omega \frac{(\rho_i - \rho_{i-1})^2}{\rho_i + a(\tau)} dx \\
&+ \tau \int_\Omega \frac{\varepsilon^2 (\Delta\rho_i)^2 + 2\tau |\nabla\rho_i|^2}{\rho_i} dx + 2r\theta\tau \int_\Omega \rho_i^{2r-3} |\nabla\rho_i|^2 dx \\
&= \frac{1}{\rho_D + a(\tau)} \int_\Omega (\rho_{i-1} - \rho_D)^2 dx + \tau \int_\Omega \nabla V_i \cdot \nabla(\rho_i - \rho_D)\, dx.
\end{aligned} \tag{3.3}$$

Similar inequality can be obtained by taking $\eta = \frac{\eta_i-\eta_D}{\eta_i+a(\tau)}$ as test function in the third equation of (1.4). We observe that

$$
\begin{aligned}
&\tau \int_\Omega \nabla V_i \cdot \nabla((\rho_i-\rho_D)-(\eta_i-\eta_D))\,dx \\
&= \tau\lambda^{-2}\int_\Omega (f-(\rho_i+a(\tau))^2+(\eta_i+a(\tau))^2)\Big((\rho_i+a(\tau))-(\eta_i+a(\tau)) \\
&\quad -(\rho_D-\eta_D)\Big)\,dx \\
&\le c\tau\int_\Omega ((\rho_i-\rho_D)^2+(\eta_i-\eta_D)^2)\,dx + c\tau
\end{aligned}
$$

for some constant $c>0$. Here and in the following we denote the constants independent of $N=N_0,N_0+1,\dots,$ $i=1,2,\dots,N$, $\varepsilon\in(0,1)$ and solution as c. Add the resulting inequalities for ρ,η with respect $i=1,2,\dots,k\le N$ to have

$$
\begin{aligned}
&\Big(\frac{1}{\rho_D+a(\tau)}-c\tau\Big)\int_\Omega(\rho_k-\rho_D)^2dx+\Big(\frac{1}{\eta_D+a(\tau)}-c\tau\Big)\int_\Omega(\eta_k-\eta_D)^2dx \\
&+\sum_{i=1}^k\int_\Omega\Big(\frac{(\rho_i-\rho_{i-1})^2}{\rho_i+a(\tau)}+\frac{(\eta_i-\eta_{i-1})^2}{\eta_i+a(\tau)}\Big)\,dx \\
&+\tau\sum_{i=1}^k\int_\Omega\Big(\frac{\varepsilon^2(\Delta\rho_i)^2+2\tau|\nabla\rho_i|^2}{\rho_i}+\int_\Omega\frac{\varepsilon^2(\Delta\eta_i)^2+2\tau|\nabla\eta_i|^2}{\eta_i}\Big)\,dx \\
&+\frac{2r\theta\tau}{(r+\frac12)^2}\sum_{i=1}^k\Big(\int_\Omega|\nabla\rho_i^{r-1/2}|^2dx+\int_\Omega|\nabla\eta_i^{r-1/2}|^2dx\Big) \\
&\le\frac{1}{\rho_D+a(\tau)}\int_\Omega(\rho_0-\rho_D)^2dx+\frac{1}{\eta_D+a(\tau)}\int_\Omega(\eta_0-\eta_D)^2dx \\
&\quad+c\tau\sum_{i=1}^{k-1}\int_\Omega((\rho_i-\rho_D)^2+(\eta_i-\eta_D)^2)\,dx+cT.
\end{aligned}
$$

This proves (3.1) with the help of the discrete Gronwall lemma [2, Theorem 2.18]. □

Lemma 3.2. *There exist a constants $c>0$ independent $N=N^*,N^*+1,\dots$ and $\varepsilon\in(0,1)$ such that*

$$
\begin{aligned}
&\|(\varepsilon\rho^{(N)},\varepsilon\eta^{(N)},V^{(N)})\|^2_{(L_\infty(0,T;H^1(\Omega)))^3}+\|(\rho^{(N)},\eta^{(N)})\|^{2r}_{(L_\infty(0,T;L_{2r}(\Omega)))^2} \\
&+\|((\rho^{(N)}+a(\tau))\nabla F^{(N)},(\eta^{(N)}+a(\tau))\nabla G^{(N)})\|^2_{(L_2(Q))^2}\le c
\end{aligned}
\tag{3.4}
$$

Proof. Using

$$
\begin{aligned}
F_i-F_D&=-\varepsilon^2\frac{\Delta\rho_i}{\rho_i}+\theta h(\rho_i^2)+\tau\ln\rho_i^2-V-F_D,\\
G_i-G_D&=-\varepsilon^2\frac{\Delta\eta_i}{\eta_i}+\theta h(\eta_i^2)+\tau\ln\eta_i^2+V-G_D.
\end{aligned}
$$

as test functions in the first and third equations of (1.4) similarly to the proof of Lemma 2.4, we obtain that for any $i=1,2,\dots,N$,

$$
\varepsilon^2\int_\Omega|\nabla\rho_i|^2dx+\tau\int_\Omega H(\rho_i^2)\,dx+4\tau a(\tau)\int_\Omega H(\rho_i)\,dx
$$

$$
\begin{aligned}
&+\frac{\tau}{2}\int_\Omega(\rho_i+a(\tau))^2|\nabla F_i|^2dx-\int_\Omega((\rho_i+a(\tau))^2-(\rho_{i-1}+a(\tau))^2)(V_i-V_D)\,dx\\
&+\frac{\theta}{r-1}\int_\Omega\Big((\rho_i^{2r}-\rho_{i-1}^{2r})+\frac{2a(\tau)r}{2r-1}(\rho_i^{2r-1}-\rho_{i-1}^{2r-1})\Big)\,dx\\
&-\frac{\theta r}{r-1}\int_\Omega((\rho_i^2-\rho_{i-1}^2)+2a(\tau)(\rho_i-\rho_{i-1}))\,dx\\
&\le\varepsilon^2\int_\Omega|\nabla\rho_{i-1}|^2dx+\tau\int_\Omega H(\rho_{i-1}^2)\,dx+4\tau a(\tau)\int_\Omega H(\rho_{i-1})\,dx\\
&\quad+\varepsilon^2a(\tau)\int_\Omega\frac{|\nabla\rho_{i-1}|^2}{\rho_{i-1}}dx+\frac{\tau}{2}\int_\Omega(\rho_i+a(\tau))^2|\nabla F_D|^2dx\\
&\quad+\int_\Omega((\rho_i+a(\tau))^2-(\rho_{i-1}+a(\tau))^2)(F_D+V_D)\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
&\varepsilon^2\int_\Omega|\nabla\eta_i|^2dx+\tau\int_\Omega H(\eta_i^2)\,dx+4\tau a(\tau)\int_\Omega H(\eta_i)\,dx\\
&+\frac{\tau}{2}\int_\Omega(\eta_i+a(\tau))^2|\nabla G_i|^2dx+\int_\Omega((\eta_i+a(\tau))^2-(\eta_{i-1}+a(\tau))^2)(V_i-V_D)\,dx\\
&+\frac{\theta}{r-1}\int_\Omega((\eta_i^{2r}-\eta_{i-1}^{2r})+\frac{2a(\tau)r}{2r-1}(\eta_i^{2r-1}-\eta_{i-1}^{2r-1}))\,dx\\
&-\frac{\theta r}{r-1}\int_\Omega((\eta_i^2-\eta_{i-1}^2)+2a(\tau)(\eta_i-\eta_{i-1}))\,dx\\
&\le\varepsilon^2\int_\Omega|\nabla\eta_{i-1}|^2dx+\tau\int_\Omega H(\eta_{i-1}^2)\,dx+4\tau a(\tau)\int_\Omega H(\eta_{i-1})\,dx\\
&\quad+\varepsilon^2a(\tau)\int_\Omega\frac{|\nabla\eta_{i-1}|^2}{\eta_{i-1}}dx+\frac{\tau}{2}\int_\Omega(\eta_i+a(\tau))^2|\nabla G_D|^2dx\\
&\quad+\int_\Omega((\eta_i+a(\tau))^2-(\eta_{i-1}+a(\tau))^2)(G_D-V_D)\,dx.
\end{aligned}
$$

Here we use Young's inequality $\alpha\beta\le\frac{1}{p}\alpha^p+\frac{1}{q}\beta^q,\frac{1}{p}+\frac{1}{q}=1$ to estimate the terms of the type $((\alpha+a(\tau))^2-(\beta+a(\tau))^2)h(\alpha^2)$. Adding the above two inequalities similarly to the proof of Lemma 2.4 and again adding the resulting inequalities for $i=1,2,\dots,k\le N$, we obtain the conclusion with the help of Lemma 3.1. $\square$

Lemma 3.3. *There exist constants $c>0$ independent $N=N^*,N^*+1,\dots$ and $\varepsilon\in(0,1)$ such that*

$$
\begin{gathered}
\|(\varepsilon\nabla(\rho^{(N)})^{2r},\varepsilon\nabla(\eta^{(N)})^{2r})\|_{(L_1(0,T;L_{6/5}(\Omega)))^2}\le c\,,\\
\|(\nabla(\rho^{(N)})^{2r-1},\nabla(\eta^{(N)})^{2r-1})\|_{(L_1(0,T;L_{3/2}(\Omega)))^2}\le c.
\end{gathered}\tag{3.5}
$$

Especially, if $r\ge 9/5$, then

$$
\begin{gathered}
\|(|\varepsilon^{3/2}\Delta\rho^{(N)}\nabla\rho^{(N)}|,|\varepsilon^{3/2}\Delta\eta^{(N)}\nabla\eta^{(N)}|)\|_{(L_1(Q))^2}\le c,\\
\|(\varepsilon\Delta\rho^{(N)}\rho^{(N)},\varepsilon\Delta\eta^{(N)}\eta^{(N)})\|_{(L_1(Q))^2}\le c,\\
\|(\nabla(\rho^{(N)})^{2r},\nabla(\eta^{(N)})^{2r})\|_{(L_1(Q))^2}\le c.
\end{gathered}\tag{3.6}
$$

Proof. We estimate only ρ because the same estimates hold also for η. By the Sobolev's imbedding theorem and Lemma 3.1, it holds

$$\|\rho^{(N)}\|^{2r-1}_{L_{2r-1}(0,T;L_{6r-3}(\Omega))} = \|(\rho^{(N)})^{r-1/2}\|^2_{L_2(0,T;L_6(\Omega))} \le c.$$

We apply Hölder's inequality to obtain

$$\begin{aligned}
&\|\varepsilon\nabla(\rho^{(N)})^{2r}\|_{L_1(0,T;L_{6/5}(\Omega))} \le 2r\|(\rho^{(N)})^{2r-1}\|_{L_1(0,T;L_3(\Omega))}\|\varepsilon\nabla\rho^{(N)}\|_{L_\infty(0,T;L_2(\Omega))}\\
&= 2r\|\rho^{(N)}\|^{2r-1}_{L_{2r-1}(0,T;L_{6r-3}(\Omega))}\|\varepsilon\nabla\rho^{(N)}\|_{L_\infty(0,T;L_2(\Omega))} \le c,\\
&\qquad\|\nabla(\rho^{(N)})^{2r-1}\|_{L_1(0,T;L_{3/2}(\Omega))}\\
&\qquad\le 2\|\nabla(\rho^{(N)})^{r-1/2}\|_{L_2(Q)}\|(\rho^{(N)})^{r-1/2}\|_{L_2(0,T;L_6(\Omega))} \le c.
\end{aligned}$$

In the $r \ge 9/5$, using the interpolation $\alpha = \frac{4r}{10r-3}$,

$$[L_\infty(0,T;L_{2r}(\Omega)), L_{2r-1}(0,T;L_{6r-3}(\Omega))]_\alpha = L_{\frac{10r-3}{3}}(Q) \subseteq L_5(Q)$$

and Lemma 3.1, we have

$$\begin{aligned}
\||\sqrt{\varepsilon\rho^{(N)}}\nabla\rho^{(N)}|\|^2_{L_2(Q)} &= \frac{1}{2}\int_0^T\int_\Omega \varepsilon\nabla\rho^{(N)}\cdot\nabla((\rho^{(N)})^2-\rho_D^2)\,dx\,dt\\
&= -\frac{1}{2}\int_0^T\int_\Omega \varepsilon\frac{\Delta\rho^{(N)}}{\sqrt{\rho^{(N)}}}\sqrt{\rho^{(N)}}((\rho^{(N)})^2-\rho_D^2)\,dx\,dt \le c
\end{aligned}$$

which yields

$$\|\varepsilon^{3/2}\Delta\rho^{(N)}\nabla\rho^{(N)}\|_{L_1(Q)} = \|\varepsilon\frac{\Delta\rho^{(N)}}{\sqrt{\rho^{(N)}}}\sqrt{\varepsilon\rho^{(N)}}\nabla\rho^{(N)}\|_{L_1(Q)} \le c.$$

By Lemmas 3.1 and 3.2 it follows the second estimate of (3.6),

$$\|\varepsilon\Delta\rho^{(N)}\rho^{(N)}\|_{L_1(Q)} = \|\varepsilon\frac{\Delta\rho^{(N)}}{\sqrt{\rho^{(N)}}}(\rho^{(N)})^{3/2}\|_{L_1(Q)} \le c.$$

The third estimate of (3.6) is immediate consequence of Lemma 3.1 and

$$\begin{gathered}
\nabla(\rho^{(N)})^{2r} = \frac{4r}{2r-1}(\rho^{(N)})^{r+1/2}\nabla(\rho^{(N)})^{r-1/2},\\
\|(\rho^{(N)})^{r+1/2}\|_{L_{\frac{20r-6}{6r+3}}(Q)} = \|\rho^{(N)}\|^{\frac{2r+1}{2}}_{L_{\frac{10r-3}{3}}(Q)} \le c,\\
\frac{20r-6}{6r+3} \ge 2.
\end{gathered}$$

□

In the following we denote all constants dependent on $\varepsilon \in (0,1)$ as $c(\varepsilon)$.

Lemma 3.4. *There exists a constant $c(\varepsilon) > 0$ independent $N = N^*, N^*+1, \dots$ such that*

$$\begin{gathered}
\|(\rho^{(N)},\eta^{(N)})\|_{(L_2(0,T;L_\infty(\Omega))\cap L_{4/3}(0,T;H^{3/2-\delta}(\Omega)))^2} \le c(\varepsilon),\\
\|(\Delta\rho^{(N)},\Delta\eta^{(N)})\|_{(L_{4/3}(0,T;L_2(\Omega)))^2} \le c(\varepsilon).
\end{gathered}\tag{3.7}$$

where $\delta \in (0,3/2)$.

Proof. By Lemma 3.1 and 3.2, and Sobolev's embedding theorem, we have

$$\begin{aligned} \|\Delta\rho^{(N)}\|^2_{L_2(0,T;L_{12/7}(\Omega))} &\le \int_0^T \Big(\int_\Omega |\frac{\Delta\rho^{(N)}}{\sqrt{\rho^{(N)}}}|^2 dx\Big)\Big(\int_\Omega |\rho^{(N)}|^6 dx\Big)^{1/6} dt \\ &\le c(\varepsilon). \end{aligned} \tag{3.8}$$

Here we use the embedding $H^1(\Omega) \hookrightarrow L_6(\Omega)$ for the $d(\le 3)$-dimensional domain Ω and $\|\rho^{(N)}\|_{L_\infty(0,T;H^1(\Omega))} \le c(\varepsilon)$.

Using this fact, let us prove that the set $\{\rho^{(N)} : N = N^*, N^*+1, \dots\}$ is bounded in $L_2(0,T;L_\infty(\Omega))$. We set $Z_i := \rho_i(\theta h(\rho_i^2) + \tau \ln \rho_i^2 - F_i - V_i)$ and take $\phi = (\rho_i - K)_+ \in H_0^1(\Omega \cup \Gamma_N)$, $K \ge \rho_D$ as test function of the second equation of (1.4). Then we have

$$\varepsilon^2 \int_\Omega |\nabla(\rho_i - K)_+|^2 dx \le \|Z_i\|_{L_{12/7}(\Omega)} \|(\rho_i - K)_+\|_{L_6(\Omega)} |\Omega(\rho_i > K)|^{1/4}.$$

Using the Sobolev's embedding theorem, from the above inequality we obtain

$$\begin{aligned} (K' - K)|\Omega(\rho_i > K')|^{1/6} &\le \|(\rho_i - K)_+\|_{L_6(\Omega)} \\ &\le c\varepsilon^{-2}\|Z_i\|_{L_{12/7}(\Omega)} |\Omega(\rho_i > K)|^{1/4}, \quad K' > K \end{aligned}$$

which allows us to use Lemma 2.1. Hence it follows

$$\begin{gathered} \rho_i \le \rho_D + c\varepsilon^{-2}\|Z_i\|_{L_{12/7}(\Omega)} = \rho_D + c\|\Delta\rho_i\|_{L_{12/7}(\Omega)}, \\ \|\rho^{(N)}\|^2_{L_2(0,T;L_\infty(\Omega))} \le \tau \sum_{i=1}^N \Big(\rho_D + c\|\Delta\rho_i\|_{L_{12/7}(\Omega)}\Big)^2 \le c(\varepsilon). \end{gathered} \tag{3.9}$$

Also, by (3.9) and Lemma 3.1 we obtain

$$\begin{aligned} \|\Delta\rho^{(N)}\|^{4/3}_{L_{4/3}(0,T;L_2(\Omega))} &\le \int_0^T \Big(\int_\Omega |\frac{\Delta\rho^{(N)}}{\sqrt{\rho^{(N)}}}|^2 dx\Big)^{2/3} \|\rho^{(N)}(t)\|^{2/3}_{L_\infty(\Omega)} dt \\ &\le c(\varepsilon). \end{aligned} \tag{3.10}$$

Furthermore, by [25, Theorem 1] it holds

$$\|\rho_i\|_{H^{3/2-\delta}(\Omega)} \le c(\|\rho_i\|_{H^1(\Omega)} + \|Z_i\|_{L_2(\Omega)} + \|\rho_D\|_{L_2(\Omega)}).$$

which with (3.10) implies

$$\|\rho^{(N)}\|_{L_{4/3}(0,T;H^{3/2-\delta}(\Omega))} \le c(\varepsilon). \tag{3.11}$$

We can obtain similar estimates to (3.9)-(3.11) for η. □

4. Convergence

Lemma 4.1. *For any fixed $\varepsilon \in (0,1)$ the set $\{(\rho^{(N)}, \eta^{(N)}); N = N^*, N^*+1, \dots\}$ is precompact in $(L_p(0,T;H^1(\Omega)))^2$ for all $p \in (1,\infty)$.*

Proof. Dividing the first equation in (1.4) by $\rho_i + a(\tau)$, we have

$$\widehat{\rho^{(N)}} = \frac{1}{2(\rho^{(N)} + a(\tau))} \operatorname{div}\Big((\rho^{(N)} + a(\tau))^2 \nabla F^{(N)}\Big) + g^{(N)} \tag{4.1}$$

where

$$\widehat{\rho^{(N)}}(x,t) = \frac{\rho_i - \rho_{i-1}}{\tau}, \quad g^{(N)}(x,t) = \frac{(\rho_i - \rho_{i-1})^2}{2\tau(\rho_i + a(\tau))}, \quad t \in (t_{i-1}, t_i].$$

By Lemma 3.1,

$$\|g^{(N)}\|_{L_1(Q)} \le c. \tag{4.2}$$

We estimate the first term of the right-hand side in (4.1) as

$$\begin{aligned}
&\frac{1}{2(\rho^{(N)}+a(\tau))}\operatorname{div}\Big((\rho^{(N)}+a(\tau))^2\nabla F^{(N)}\Big)\\
&=\frac{1}{2}\operatorname{div}((\rho^{(N)}+a(\tau))\nabla F^{(N)})+\tau\frac{|\nabla\rho^{(N)}|^2}{\rho^{(N)}}-\frac{1}{2}\nabla\rho^{(N)}\cdot\nabla V^{(N)}\\
&\quad-\frac{\varepsilon^2}{2}\nabla\rho^{(N)}\cdot\nabla\Big(\frac{\Delta\rho^{(N)}}{\rho^{(N)}}\Big)+r\theta(\rho^{(N)})^{2r-3}|\nabla\rho^{(N)}|^2\\
&=\sum_{i=1}^{5}R_i^{(N)}.
\end{aligned}$$

By Lemmas 3.1 and 3.2, for some $c(\varepsilon)>0$ it holds

$$\|R_1^{(N)}\|_{L_2(0,T;H^{-1}(\Omega))},\ \|R_2^{(N)}\|_{L_1(Q)},\|R_3^{(N)}\|_{L_\infty(0,T;L_1(\Omega))},\ \|R_5^{(N)}\|_{L_1(Q)}\le c(\varepsilon).$$

Now, let us estimate R_4. Lemma3.1 and 3.2, and the equality

$$\rho^{(N)}\nabla F^{(N)}=-\varepsilon^2\rho^{(N)}\nabla(\frac{\Delta\rho^{(N)}}{\rho^{(N)}})+2r\theta(\rho^{(N)})^{2(r-1)}\nabla\rho^{(N)}-\rho^{(N)}\nabla V^{(N)}$$

yield the estimate: for some $c(\varepsilon)>0$,

$$\|\rho^{(N)}\nabla(\frac{\Delta\rho^{(N)}}{\rho^{(N)}})\|_{L_1(Q)}\le c(\varepsilon).$$

Hence Lemma 3.2 and 3.4, and the equality

$$\begin{aligned}
\nabla\rho^{(N)}\cdot\nabla(\frac{\Delta\rho^{(N)}}{\rho^{(N)}})&=\operatorname{div}(\nabla\rho^{(N)}\frac{\Delta\rho^{(N)}}{\rho^{(N)}})-\frac{(\Delta\rho^{(N)})^2}{\rho^{(N)}}\\
&=\operatorname{div}(\nabla(\Delta\rho^{(N)})-\rho^{(N)}\nabla(\frac{\Delta\rho^{(N)}}{\rho^{(N)}}))-\frac{(\Delta\rho^{(N)})^2}{\rho^{(N)}}
\end{aligned}$$

give the estimate: for some $c(\varepsilon)>0$,

$$\|R_4^{(N)}\|_{L_1(0,T;Z)}\le c(\varepsilon),$$

where $Z=H^{-2}(\Omega)$. Thus we obtain the estimate

$$\frac{1}{\tau}\|\rho^{(N)}(\cdot)-\rho^{(N)}(\cdot-\tau)\|_{L_1(0,T;Z)}=\|\widehat{\rho^{(N)}}\|_{L_1(0,T;Z)}\le c(\varepsilon). \tag{4.3}$$

This fact and Lemma 3.4 allows us to use the compactness theorem for piecewise constant functions [12, Theorem 1] to derive the compactness of $\{\rho^{(N)};N=N^*,N^*+1,\dots\}$ in $L_p(0,T;H^1(\Omega))$, $\forall p\in(1,\infty)$. We can also prove the compactness of $\{\eta^{(N)};N=N^*,N^*+1,\dots\}$ similarly. $\square$

Proof of Theorem 1.2. Using Lemmas 3.2 and 3.4, and (3.5) of Lemmas 3.3 and 4.1, we can easily verify the convergence estimate (1.7) for some ρ,η,V.

It remains only to prove that ρ, η, V satisfy (1.8). From Theorem 1.1 and Lemma 3.2 we have

$$\begin{aligned}
&\frac{1}{\tau}(n^{(N)} - \sigma_N n^{(N)}) = \operatorname{div}((\rho^{(N)} + a(\tau))^2 \nabla F^{(N)}),\\
&\|\frac{1}{\tau}(n^{(N)} - \sigma_N n^{(N)})\|_{L_2(0,T;W^{-1}_{r+1}(\Omega\cup\Gamma_N))}\\
&= \|(\rho^{(N)} + a(\tau))^2 \nabla F^{(N)}\|_{L_2(0,T;L_{(r+1)/r}(\Omega))} \le c
\end{aligned} \tag{4.4}$$

for some constant $c > 0$ independent of $N = N^*, N^* + 1, \dots$ and $\varepsilon \in (0, 1)$ where $n^{(N)}$ is defined as

$$n^{(N)}(x, t) = (\rho_i + a(\tau))^2, t \in (t_{i-1}, t_i]$$

and σ_N is the shift operator

$$\sigma_N n^{(N)}(x, t) = (\rho_{i-1} + a(\tau))^2, t \in (t_{i-1}, t_i].$$

Since it holds by the strong convergence of $\rho^{(N)}$ to ρ

$$n^{(N)} \to \rho^2 \quad \text{in } L_2(Q)$$

as $N \to \infty$, we can easily verify in the sense of distribution $[0, T] \to W^{-1}_{r+1}(\Omega\cup\Gamma_N)$,

$$\frac{1}{\tau}\Big(n^{(N)} - \sigma_N n^{(N)}\Big) \to \frac{\partial \rho^2}{\partial t}.$$

This and (4.4) gives us, up to a subsequence,

$$\frac{1}{\tau}\Big(n^{(N)} - \sigma_N n^{(N)}\Big) \rightharpoonup \frac{\partial \rho^2}{\partial t} \quad \text{weakly in } L_2(0, T; W^{-1}_{r+1}(\Omega \cup \Gamma_N)). \tag{4.5}$$

We take $\phi \in C_0^\infty(Q)$ as test function of the first equation in (4.4) to obtain

$$\begin{aligned}
&\int_0^T \langle \frac{1}{\tau}(n^{(N)} - \sigma_N n^{(N)}), \phi\rangle \,dt\\
&= -\varepsilon^2 \int_Q \frac{\Delta\rho^{(N)}}{\rho^{(N)}} \operatorname{div}((\rho^{(N)} + a(\tau))^2 \nabla\phi) \,dx\,dt\\
&\quad - \theta \int_Q (\rho^{(N)} + a(\tau))^2 \nabla h((\rho^{(N)})^2) \cdot \nabla\phi \,dx\,dt\\
&\quad - 2\tau \int_Q (\rho^{(N)} + a(\tau))^2 \frac{\nabla\rho^{(N)}}{\rho^{(N)}} \cdot \nabla\phi \,dx\,dt + \int_Q (\rho^{(N)} + a(\tau))^2 \nabla V^{(N)} \cdot \nabla\phi \,dx\,dt\\
&= -2\varepsilon^2 \int_Q \Delta\rho^{(N)} \nabla\rho^{(N)} \cdot \nabla\phi \,dx\,dt - \varepsilon^2 \int_Q \Delta\rho^{(N)} \rho^{(N)} \Delta\phi \,dx\,dt\\
&\quad - \theta \int_Q \nabla(\rho^{(N)})^{2r} \cdot \nabla\phi \,dx\,dt + \int_Q (\rho^{(N)})^2 \nabla V^{(N)} \cdot \nabla\phi \,dx\,dt\\
&\quad - 2\tau \int_Q \rho^{(N)} \nabla\rho^{(N)} \cdot \nabla\phi \,dx\,dt - a(\tau)^2 \int_Q \nabla F^{(N)} \cdot \nabla\phi \,dx\,dt\\
&\quad - 2a(\tau) \int_Q \Big(\varepsilon^2 \frac{\Delta\rho^{(N)}}{\rho^{(N)}} \nabla\rho^{(N)} \cdot \nabla\phi + \varepsilon^2 \Delta\rho^{(N)} \Delta\phi\Big) \,dx\,dt\\
&\quad - 2a(\tau) \int_Q \Big(\frac{2\theta r}{2r-1} \nabla(\rho^{(N)})^{2r-1} + 2\tau \nabla\rho^{(N)} - \rho^{(N)} \nabla V^{(N)}\Big) \cdot \nabla\phi \,dx\,dt\\
&= -2\varepsilon^2 \int_Q \Delta\rho^{(N)} \nabla\rho^{(N)} \cdot \nabla\phi \,dx\,dt - \varepsilon^2 \int_Q \Delta\rho^{(N)} \rho^{(N)} \Delta\phi \,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&-\theta\int_Q \nabla(\rho^{(N)})^{2r}\cdot\nabla\phi\,dx\,dt+\int_Q(\rho^{(N)})^2\nabla V^{(N)}\cdot\nabla\phi\,dx\,dt\\
&-2\tau R_1^N-a(\tau)^2R_2^N-2a(\tau)R_3^N.
\end{aligned}\tag{4.6}
$$

By Lemma 3.2,

$$
\begin{gathered}
2\tau R_1^N=2\tau\int_Q\rho^{(N)}\nabla\rho^{(N)}\cdot\nabla\phi\,dx\,dt,\\
a(\tau)^2R_2^N=a(\tau)^2\int_Q\nabla F^{(N)}\cdot\nabla\phi\,dx\,dt
\end{gathered}
$$

approaches zero as $N\to\infty(\tau\to0)$. Also, considering the property $a(\tau)\le c\tau^2$ of $a(\tau)$ (see page 3) and Lemmas 3.1 and 3.2, (3.5) of Lemmas 3.3 and 3.4, it follows that

$$
\begin{aligned}
2a(\tau)R_3^N&=2a(\tau)\int_Q(\varepsilon^2\frac{\Delta\rho^{(N)}}{\sqrt{\rho^{(N)}}}\frac{\nabla\rho^{(N)}}{\sqrt{\rho^{(N)}}}\cdot\nabla\phi+\varepsilon^2\Delta\rho^{(N)}\Delta\phi)\,dx\,dt\\
&\quad+2a(\tau)\int_Q(\frac{2\theta r}{2r-1}\nabla(\rho^{(N)})^{2r-1}+2\tau\nabla\rho^{(N)}-\rho^{(N)}\nabla V^{(N)})\cdot\nabla\phi\,dx\,dt\\
&\to0\quad\text{as }N\to\infty.
\end{aligned}
$$

Thus, using the obtained convergence estimates (1.7) and taking the limit $\tau\to0$ in the above equation (4.6), we arrive at the first equation of (1.8). Similarly, we obtain the second equation in (1.8) for η . The third equation of (1.8) can also be obtained by the limit of

$$
\lambda^2\Delta V^{(N)}=(\rho^{(N)}+a(\tau))^2-(\eta^{(N)}+a(\tau))^2-f.
$$

□

Proof of Theorem 1.4. Using (3.6) of lemmas 3.2 and 3.3, (4.4) and (4.5), we can easily verify for some $c>0$

$$
\begin{gathered}
\|(\frac{\partial n^{(\varepsilon)}}{\partial t},\frac{\partial p^{(\varepsilon)}}{\partial t})\|_{(L_2(0,T;W^{-1}_{r+1}(\Omega\cup\Gamma_N)))^2}\le c,\\
\|(\nabla(n^{(\varepsilon)})^r,\nabla(p^{(\varepsilon)})^r)\|_{(L_1(Q))^2}\le c,\\
\|V^{(\varepsilon)}\|_{L_\infty(0,T;H^1(\Omega))}\le c,\quad\forall\varepsilon\in(0,1).
\end{gathered}\tag{4.7}
$$

Hence the set $\{n^{(\varepsilon)},p^{(\varepsilon)};\varepsilon\in(0,1)\}$ is precompact in $L_p(0,T;L_r(\Omega))$ for all $p<\infty$ (cf. [15, Theorem 3.2.2], [26, Theorem 6]). This allows us to take the limit $\varepsilon\to0$ in (1.8) and arrive at (1.9) and (1.10) with the help of Lemmas 3.2 and 3.3. □

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