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FRACTIONAL ELLIPTIC PROBLEMS WITH TWO CRITICAL SOBOLEV-HARDY EXPONENTS

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ABSTRACT. By using the mountain pass lemma and a concentration compactness principle, we obtain the existence of positive solutions to the fractional elliptic problem with two critical Hardy-Sobolev exponents at the origin.

1. INTRODUCTION

In this article, we study the following doubly critical problem involving the fractional Laplacian

$$(-\Delta)^s u - \gamma \frac{u}{|x|^{2s}} = \frac{|u|^{2_s^*(\alpha)-2}u}{|x|^{\alpha}} + \frac{|u|^{2_s^*(\beta)-2}u}{|x|^{\beta}}, \quad u > 0, \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $s \in (0,1)$, $0 < \alpha, \beta < 2s < n$ with $\alpha \neq \beta$, $\gamma < \gamma_H$ with

$$\gamma_H = 4^s \frac{\Gamma^2(\frac{n+2s}{4})}{\Gamma^2(\frac{n-2s}{4})}$$

being the fractional best Hardy constant on $\mathbb{R}^n$, and $2_s^*(\alpha) = 2(n-\alpha)/(n-2s)$ is the fractional critical Hardy-Sobolev exponent. The operator $(-\Delta)^s$ is the fractional Laplacian defined as

$$(-\Delta)^s u(x) = c_{n,s}\, \mathrm{pv} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy, \quad s \in (0,1),$$

where pv stands for the Cauchy principle value and

$$c_{n,s} = 2^{2s-1}\pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n+2s}{2}\right)}{|\Gamma(-s)|}$$

is the normalization constant so that the identity

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}u)) \quad \forall \xi \in \mathbb{R}^n,\ s \in (0,1),\ u \in \mathcal{S}(\mathbb{R}^n),$$

holds, here $\mathcal{F}u$ denotes the Fourier transform of u, $\mathcal{F}u(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} u(x)\, dx$, and $\mathcal{S}(\mathbb{R}^n)$ the Schwartz class, see [14] and references therein for the basics on the fractional Laplacian.

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In previous twenty years, the nonlocal elliptic problems have been investigated by many researchers, for example, [18, 27, 29, 30, 31] for the subcritical case, [3, 8, 23, 19, 28, 32, 33] for the critical case, [9, 10, 11] for the existence of solutions to fractional Laplacian system. Moreover, a great attention has been devoted to study the existence of solutions for the nonlocal problems with Hardy potential or nonlinearity term, we refer to see [1, 2, 4, 13, 15, 16, 34, 35, 36] and the references therein. In particular, the existence of solutions to the problem

$$(-\Delta)^s u - \gamma \frac{u}{|x|^{2s}} = \frac{u^{2_s^*(\alpha)-1}}{|x|^\alpha}, \quad u > 0 \quad \text{in } \mathbb{R}^n, \tag{1.2}$$

corresponds to the minimization problem

$$\mu_{s,\gamma,\alpha}(\mathbb{R}^n) = \inf_{u \in H^s(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |(-\Delta)^{s/2}u|^2\,dx - \gamma \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^{2s}}\,dx}{\big(\int_{\mathbb{R}^n} \frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha}\,dx\big)^{\frac{2}{2_s^*(\alpha)}}}. \tag{1.3}$$

Fall et al. [16] proved the existence of extremals for $\mu_{s,0,\alpha}(\mathbb{R}^n)$ in the case $s = \frac{1}{2}$. Yang [35] proved that there exists a positive, radially symmetric and non-increasing extremal for $\mu_{s,0,\alpha}(\mathbb{R}^n)$ when $s \in (0,1)$. Asymptotic properties of the positive solutions was given by Lei [24] and Yang-Yu [37]. The existence of extremals for $\mu_{s,\gamma,\alpha}(\mathbb{R}^n)$ in (1.3), when $\alpha \in [0, 2s)$ and $\gamma \in (-\infty, \gamma_H)$, was recently studied by Ghoussoub and Shakerian in [21]. Moreover, the authors in [21] used the mountain pass lemma to establish the existence of a nontrivial weak solution to the problem

$$(-\Delta)^s u - \gamma \frac{u}{|x|^{2s}} = |u|^{2_s^*-2}u + \frac{|u|^{2_s^*(\alpha)-2}u}{|x|^\alpha}, \quad u > 0, \quad \text{in } \mathbb{R}^n.$$

Furthermore, the authors in [36] showed the existence of nontrivial solutions for fractional elliptic problem in $\mathbb{R}^n$ with the critical nonlocal Hartree term and critical fractional Hardy-Sobolev term.

It is worth pointing out that in the local case, i.e. $s = 1$, the existence and multiplicity of solutions for the Laplacian problems with Hardy terms have been extensively studied, we refer the reader to [5, 7, 12, 17, 22] and references therein.

The aim of this paper is to consider the existence of nontrivial weak solutions of (1.1), which has a single pole with different powers of singularity and fractional critical Hardy-Sobolev exponents. We get the existence of nontrivial weak solutions of our problem by the Mountain Pass Lemma with concentration-compactness principle. Our result can be stated as follows.

Theorem 1.1. *Let $0 < s < 1$, $0 < \alpha, \beta < 2s < n$ with $\alpha \neq \beta$, and $\gamma < \gamma_H$. Then problem* (1.1) *admits a nontrivial solution.*

This article is organized as follows: in Section 2, we give some preliminaries about fractional Laplacian harmonic extension and function space, and also the fractional Hardy-Sobolev inequality. We prove the compactness of the energy in Section 3. Section 4 is concerned with the proof of our main result.

2. Preliminary results

In this section, we first introduce suitable function spaces for the variational principles that will be needed in the sequel. Caffarelli and Silvestre in [6] showed that the fractional Laplacian operator can be realized in a local way by using one more variable and the so-called $s-$harmonic extension, that is, for a function

$u \in H^s(\mathbb{R}^n)$, we say that $U = E_s(u)$ is its s-harmonic extension to the upper half-space, $\mathbb{R}^{n+1}_+$, i.e. it is a solution to the problem

$$\operatorname{div}(y^{1-2s}\nabla U) = 0 \quad \text{in } \mathbb{R}^{n+1}_+, U = u \quad \text{on } \mathbb{R}^n \times \{y = 0\}.$$

Define the space $X^s(\mathbb{R}^{n+1}_+)$ as the closure of $C_0^\infty(\overline{\mathbb{R}^{n+1}_+})$ with the norm

$$\|U\|_{X^s(\mathbb{R}^{n+1}_+)} := \Big(k_s \int_{\mathbb{R}^{n+1}_+} y^{1-2s}|\nabla U(x,y)|^2\, dx\, dy\Big)^{1/2},$$

where $k_s = \frac{\Gamma(s)}{2^{1-2s}\Gamma(1-s)}$ is a normalization constant chosen in such a way that the extension operator $U : H^s(\mathbb{R}^n) \to X^s(\mathbb{R}^{n+1}_+)$ is an isometry, that is, for any $u \in H^s(\mathbb{R}^n)$, we have

$$\|U\|_{X^s(\mathbb{R}^{n+1}_+)} = \|u\|_{H^s(\mathbb{R}^n)} = \|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^n)}. \tag{2.1}$$

Conversely, for a function $U \in X^s(\mathbb{R}^{n+1}_+)$, we denote its trace on $\mathbb{R}^n \times \{y = 0\}$ as $u = \operatorname{Tr}(U) := U(\cdot, 0)$. This trace operator is also well defined and satisfies

$$\|u\|_{H^s(\mathbb{R}^n)} = \|U(\cdot,0)\|_{H^s(\mathbb{R}^n)} \le \|U\|_{X^s(\mathbb{R}^{n+1}_+)}. \tag{2.2}$$

Caffarelli and Silvestre [6] showed that the extension function $U := E_s(u)$ is related to the fractional Laplacian of the original function u in the following way:

$$(-\Delta)^s u(x) = \frac{\partial U}{\partial \nu^s} := -k_s \lim_{y\to 0^+} y^{1-2s}\frac{\partial U}{\partial y}(x,y).$$

Thus, problem (1.1) can be written as the local problem

$$\begin{gathered} -\operatorname{div}(y^{1-2s}\nabla U) = 0 \quad \text{in } \mathbb{R}^{n+1}_+ \\ \frac{\partial U}{\partial \nu^s} = \gamma\frac{u}{|x|^{2s}} + \frac{|u|^{2^*_s(\alpha)-2}u}{|x|^\alpha} + \frac{|u|^{2^*_s(\beta)-2}u}{|x|^\beta} \quad \text{on } \mathbb{R}^n, \end{gathered} \tag{2.3}$$

where and in the follows $u = U(\cdot, 0)$. A function $U \in X^s(\mathbb{R}^{n+1}_+)$ is said to be a weak solution to (2.3), if for all $\Psi \in X^s(\mathbb{R}^{n+1}_+)$,

$$\begin{aligned} k_s \int_{\mathbb{R}^{n+1}_+} y^{1-2s}\langle \nabla U, \nabla \Psi\rangle\, dx\, dy &= \int_{\mathbb{R}^n} \gamma \frac{u}{|x|^{2s}}\psi\, dx + \int_{\mathbb{R}^n} \frac{|u|^{2^*_s(\alpha)-2}u}{|x|^\alpha}\psi\, dx \\ &\quad + \int_{\mathbb{R}^n} \frac{|u|^{2^*_s(\beta)-2}u}{|x|^\beta}\psi\, dx, \end{aligned}$$

where $\psi = \Psi(\cdot, 0)$. The energy functional corresponding to (2.3) is

$$\begin{aligned} J(U) = \frac{1}{2}\|U\|^2_{X^s(\mathbb{R}^{n+1}_+)} - \frac{\gamma}{2}\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^{2s}}\, dx - \frac{1}{2^*_s(\alpha)}\int_{\mathbb{R}^n} \frac{|u|^{2^*_s(\alpha)}}{|x|^s}\, dx \\ - \frac{1}{2^*_s(\beta)}\int_{\mathbb{R}^n} \frac{|u|^{2^*_s(\beta)}}{|x|^s}\, dx. \end{aligned}$$

We note that for any weak solution $U \in X^s(\mathbb{R}^{n+1}_+)$ to (2.3), the function $u = U(\cdot, 0)$ is in $H^s(\mathbb{R}^n)$ and is a weak solution to problem (1.1). Hence the associated trace of any critical point U of J in $X^s(\mathbb{R}^{n+1}_+)$ is a weak solution for (1.1). Let us recall the following results.

Lemma 2.1. *Assume that* $0 < s < 1$.

(i) (The fractional Hardy inequality [20]*) For all* $u \in H^s(\mathbb{R}^n)$*, we have*

$$\gamma_H \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^{2s}}\,dx \le \int_{\mathbb{R}^n} |(-\Delta)^{s/2}u|^2\,dx, \tag{2.4}$$

where $\gamma_H = 4^s \frac{\Gamma^2(\frac{n+2s}{4})}{\Gamma^2(\frac{n-2s}{4})}$ *is the best constant in the above inequality on* $\mathbb{R}^n$.

(ii) (The fractional Hardy-Sobolev inequality [21]*) Assume* $0 \le \alpha \le 2s < n$. *Then, there exist positive constants* c *and* C*, such that for all* $u \in H^s(\mathbb{R}^n)$,

$$\Big(\int_{\mathbb{R}^n} \frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2_s^*(\alpha)}} \le c\int_{\mathbb{R}^n} |(-\Delta)^{s/2}u|^2\,dx. \tag{2.5}$$

Moreover, if $\gamma < \gamma_H$*, then*

$$C\Big(\int_{\mathbb{R}^n} \frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2_s^*(\alpha)}} \le \int_{\mathbb{R}^n} |(-\Delta)^{s/2}u|^2\,dx - \gamma\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^{2s}}\,dx, \tag{2.6}$$

for all $u \in H^s(\mathbb{R}^n)$.

Remark 2.2. One can use (2.1) to rewrite inequalities (2.4), (2.5) and (2.6) as the following trace class inequalities:

$$\gamma_H \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^{2s}}\,dx \le \|U\|^2_{X^s(\mathbb{R}^{n+1}_+)}, \tag{2.7}$$

$$\Big(\int_{\mathbb{R}^n} \frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2_s^*(\alpha)}} \le c\|U\|^2_{X^s(\mathbb{R}^{n+1}_+)}, \tag{2.8}$$

$$C\Big(\int_{\mathbb{R}^n} \frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2_s^*(\alpha)}} \le \|U\|^2_{X^s(\mathbb{R}^{n+1}_+)} - \gamma\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^{2s}}\,dx. \tag{2.9}$$

In what follows, we will denote by $X^s(\mathbb{R}^{n+1}_+)$ the closure of $C_0^\infty(\overline{\mathbb{R}^{n+1}_+})$ for the following norm

$$\|U\| := \Big(k_s\int_{\mathbb{R}^{n+1}_+} y^{1-2s}|\nabla U|^2\,dx\,dy - \gamma\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^{2s}}\,dx\Big)^{1/2} \quad \text{for all } \gamma < \gamma_H. \tag{2.10}$$

Note that inequality (2.7) asserts that $X^s(\mathbb{R}^{n+1}_+)$ is embedded in the weighted space $L^2(\mathbb{R}^n, |x|^{-2s})$ and this embedding is continuous. Set $\gamma_+ = \max\{\gamma, 0\}$ and $\gamma_- = -\max\{\gamma, 0\}$. The following inequalities hold for any $u \in X^s(\mathbb{R}^{n+1}_+)$,

$$(1 - \frac{\gamma_+}{\gamma_H})\|U\|^2_{X^s(\mathbb{R}^{n+1}_+)} \le \|U\|^2 \le (1 + \frac{\gamma_-}{\gamma_H})\|U\|^2_{X^s(\mathbb{R}^{n+1}_+)}. \tag{2.11}$$

Thus, $\|\cdot\|$ is equivalent to the norm $\|\cdot\ \|_{X^s(\mathbb{R}^{n+1}_+)}$.

The best constant $\mu_{s,\gamma,\alpha}(\mathbb{R}^n)$ in inequality (2.6) can be written as

$$S(n,s,\gamma,\alpha) = \inf_{U \in X^s(\mathbb{R}^{n+1}_+)\setminus\{0\}} I_{\gamma,\alpha}(U), \quad \text{with}$$

$$I_{\gamma,\alpha}(U) = \frac{k_s\int_{\mathbb{R}^{n+1}_+} y^{1-2s}|\nabla U|^2\,dx\,dy - \gamma\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^{2s}}\,dx}{(\int_{\mathbb{R}^n} \frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha}\,dx)^{\frac{2}{2_s^*(\alpha)}}}.$$

If $S(n,s,\gamma,\alpha)$ is attained at some function $U \in X^s(\mathbb{R}^{n+1}_+)$, then $u = U(.,0)$ will be a function in $H^s(\mathbb{R}^n)$, where $\mu_{s,\gamma,\alpha}(\mathbb{R}^n)$ is attained. Recently, Ghoussoub and Shakerian [21] proved the extremal function of $S(n,s,\gamma,\alpha)$ is attained as following.

Lemma 2.3 ([21]). *Suppose* $0 < s < 1$, $0 \le \alpha < 2s < n$, *and* $\gamma < \gamma_H$. *Then*

1. *If* $\{\alpha > 0\}$ *or* $\alpha = 0$ *and* $\gamma \ge 0$, *then* $S(n,s,\gamma,\alpha)$ *is attained in* $X^s(\mathbb{R}^{n+1}_+)$ *by* $W_{\gamma,\alpha}$.
2. *If* $\alpha = 0$ *and* $\gamma < 0$, *then there are no extremals for* $S(n,s,\gamma,\alpha)$ *in* $X^s(\mathbb{R}^{n+1}_+)$.

3. Compactness lemmas

In this section, we study the compactness properties of the functional

$$J(U) = \frac{1}{2}\|U\|^2 - \frac{1}{2^*_s(\alpha)}\int_{\mathbb{R}^n}\frac{|u|^{2^*_s(\alpha)}}{|x|^\alpha}\,dx - \frac{1}{2^*_s(\beta)}\int_{\mathbb{R}^n}\frac{|u|^{2^*_s(\beta)}}{|x|^\beta}\,dx \tag{3.1}$$

for $U \in X^s(\mathbb{R}^{n+1}_+)$, where again $u := U(\cdot,0)$. From Lemma 2.1, we have that $J \in C^1(X^s(\mathbb{R}^{n+1}_+))$.

Definition 3.1. Let $c \in \mathbb{R}$, E be a Banach space and $J \in C^1(E,\mathbb{R})$.

(i) $\{u_k\}$ is a $(PS)_c$ sequence in E for J if $J(u_k) = c + o(1)$ and $J'(u_k) = o(1)$ strongly in E^* as $k \to \infty$.

(ii) We say that J satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence $\{u_k\}$ for J in E has a convergent subsequence.

Proposition 3.2. *Suppose* $0 < \alpha,\beta < 2s$ *and* $\gamma < \gamma_H$, *then the functional* J *defined in* (3.1) *satisfies the Palais-Smale condition* $(PS)_c$ *for* $c < c_*$, *where*

$$c_* := \min\Big\{\frac{2s-\alpha}{2(n-\alpha)}S(n,s,\gamma,\alpha)^{\frac{n-\alpha}{2s-\alpha}}, \frac{2s-\beta}{2(n-\beta)}S(n,s,\gamma,\beta)^{\frac{n-\beta}{2s-\beta}}\Big\}. \tag{3.2}$$

Proof. Let $\{U_k\}_{k\in\mathbb{N}}$ be the Palais-Smale sequence of the functional J, i.e.

$$J(U_k) \to c, \quad J'(U_k) \to 0 \quad \text{in } (X^s(\mathbb{R}^{n+1}_+))' \text{ as } k \to \infty.$$

Then

$$\begin{aligned} J(U_k) &= \frac{1}{2}\|U_k\|^2 - \frac{1}{2^*_s(\alpha)}\int_{\mathbb{R}^n}\frac{|u_k|^{2^*_s(\alpha)}}{|x|^\alpha}\,dx - \frac{1}{2^*_s(\beta)}\int_{\mathbb{R}^n}\frac{|u_k|^{2^*_s(\beta)}}{|x|^\beta}\,dx \\ &= c + o_k(1), \end{aligned} \tag{3.3}$$

and

$$\langle J'(U_k), U_k\rangle = \|U_k\|^2 - \int_{\mathbb{R}^n}\frac{|u_k|^{2^*_s(\alpha)}}{|x|^\alpha}\,dx - \int_{\mathbb{R}^n}\frac{|u_k|^{2^*_s(\beta)}}{|x|^\beta}\,dx = o_k(1)\|w_k\|, \tag{3.4}$$

where again $u_k = U_k(\cdot,0)$ and $o_k(1) \to 0$ as $k \to \infty$. From (3.3) and (3.4), we have

$$\begin{aligned} c + o_k(1)\|U_k\| &= J(U_k) - \frac{1}{2}\langle J'(U_k), U_k\rangle \\ &= \Big(\frac{1}{2} - \frac{1}{2^*_s(\alpha)}\Big)\int_{\mathbb{R}^n}\frac{|u_k|^{2^*_s(\alpha)}}{|x|^\alpha}\,dx + \Big(\frac{1}{2} - \frac{1}{2^*_s(\beta)}\Big)\int_{\mathbb{R}^n}\frac{|u_k|^{2^*_s(\beta)}}{|x|^\beta}\,dx. \end{aligned}$$

Since $2^*_s(\alpha) > 2$, $2^*_s(\beta) > 2$, we have

$$\int_{\mathbb{R}^n}\frac{|u_k|^{2^*_s(\alpha)}}{|x|^\alpha}\,dx \le C + o_k(1)\|U_k\|, \qquad \int_{\mathbb{R}^n}\frac{|u_k|^{2^*_s(\beta)}}{|x|^\beta}\,dx \le C + o_k(1)\|U_k\|. \tag{3.5}$$

By (3.4) and (3.5), we obtain

$$\|U_k\|^2 + o_k(1)\|U_k\| = \int_{\mathbb{R}^n}\frac{|u_k|^{2^*_s(\alpha)}}{|x|^\alpha}\,dx + \int_{\mathbb{R}^n}\frac{|u_k|^{2^*_s(\beta)}}{|x|^\beta}\,dx \le C + o_k(1)\|U_k\|, \tag{3.6}$$

which implies that $\{U_k\}_{k\in\mathbb{N}}$ is bounded in $X^s(\mathbb{R}^{n+1}_+)$. It follows that there exists a subsequence, still denote by U_k, such that $U_k \rightharpoonup U$ in $X^s(\mathbb{R}^{n+1}_+)$. For any $\Psi \in C_0^\infty(\mathbb{R}^{n+1}_+)$, we have

$$\begin{aligned} &o_k(1)\\ &= \langle J'(U_k), \Psi\rangle\\ &= k_s \int_{\mathbb{R}^{n+1}_+} y^{1-2s}\langle \nabla U_k, \nabla \Psi\rangle\,dx\,dy - \gamma\int_{\mathbb{R}^n} \frac{u_k(x)}{|x|^{2s}}\psi(x)\,dx \\ &\quad - \int_{\mathbb{R}^n} \frac{|u_k(x)|^{2_s^*(\alpha)-2}u_k(x)}{|x|^\alpha}\psi(x)\,dx - \int_{\mathbb{R}^n} \frac{|u_k(x)|^{2_s^*(\beta)-2}u_k(x)}{|x|^\beta}\psi(x)\,dx. \end{aligned} \tag{3.7}$$

Since $U_k \rightharpoonup U$ in $X^s(\mathbb{R}^{n+1}_+)$ as $k\to\infty$, we have that

$$\begin{aligned} &\int_{\mathbb{R}^{n+1}_+} y^{1-2s}\langle \nabla U_k, \nabla \Psi\rangle\,dx\,dy - \gamma\int_{\mathbb{R}^n} \frac{u_k(x)}{|x|^{2s}}\psi(x)\,dx\\ &\to \int_{\mathbb{R}^{n+1}_+} y^{1-2s}\langle \nabla U, \nabla \Psi\rangle\,dx\,dy - \gamma\int_{\mathbb{R}^n} \frac{u(x)}{|x|^{2s}}\psi(x)\,dx, \end{aligned}$$

for all $\Psi \in C_0^\infty(\mathbb{R}^{n+1}_+)$, where $u = U(\cdot, 0)$.

Moreover, the boundedness of U_k in $X^s(\mathbb{R}^{n+1}_+)$ implies that $|u_k|^{2_s^*(\alpha)-2}u_k$ and $|u_k|^{2_s^*(\beta)-2}u_k$ are bounded in $L^{\frac{2_s^*(\alpha)}{2_s^*(\alpha)-1}}(\mathbb{R}^n, |x|^{-\alpha})$ and $L^{\frac{2_s^*(\beta)}{2_s^*(\beta)-1}}(\mathbb{R}^n, |x|^{-\beta})$ respectively. Therefore,

$$\begin{aligned} |u_k|^{2_s^*(\alpha)-2}u_k &\rightharpoonup |u|^{2_s^*(\alpha)-2}u \quad \text{in } L^{\frac{2_s^*(\alpha)}{2_s^*(\alpha)-1}}(\mathbb{R}^n, |x|^{-\alpha}),\\ |u_k|^{2_s^*(\beta)-2}u_k &\rightharpoonup |u|^{2_s^*(\beta)-2}u \quad \text{in } L^{\frac{2_s^*(\beta)}{2_s^*(\beta)-1}}(\mathbb{R}^n, |x|^{-\beta}). \end{aligned}$$

Thus, taking limits as $k\to\infty$ in (3.7), we obtain

$$\begin{aligned} 0 &= \langle J'(U), \Psi\rangle\\ &= k_s \int_{\mathbb{R}^{n+1}_+} y^{1-2s}\langle \nabla U, \nabla \Psi\rangle\,dx\,dy - \gamma\int_{\mathbb{R}^n} \frac{u(x)}{|x|^{2s}}\psi(x)\,dx \\ &\quad - \int_{\mathbb{R}^n} \frac{|u(x)|^{2_s^*(\alpha)-2}u(x)}{|x|^\alpha}\psi(x)\,dx - \int_{\mathbb{R}^n} \frac{|u(x)|^{2_s^*(\beta)-2}u(x)}{|x|^\beta}\psi(x)\,dx. \end{aligned} \tag{3.8}$$

Hence U is a weak solution of (2.3).

The set $\mathbb{R}^n \cup \{\infty\}$ is compact for the standard topology which means that the measures can be identified as the dual space $C(\mathbb{R}^n \cup \{\infty\})$. For example, δ_∞ is well defined and $\delta_\infty = \varphi(\infty)$. By the concentration compactness principle [25, 26], there exist a subsequence, still denoted by U_k and real numbers μ_0, μ_∞, ν_0, ν_∞, η_0, η_∞ and ζ_0, ζ_∞ such that

$$\|U_k\|^2_{X^s(\mathbb{R}^{n+1}_+)} \rightharpoonup d\mu \ge \|U\|^2_{X^s(\mathbb{R}^{n+1}_+)} + \mu_0\delta_0 + \mu_\infty\delta_\infty, \tag{3.9}$$

$$|u_k|^2|x|^{-2s} \rightharpoonup d\nu = |u|^2|x|^{-2s} + \nu_0\delta_0 + \nu_\infty\delta_\infty, \tag{3.10}$$

$$|u_k|^{2_s^*(\alpha)}|x|^{-\alpha} \rightharpoonup d\eta = |u|^{2_s^*(\alpha)}|x|^{-\alpha} + \eta_0\delta_0 + \eta_\infty\delta_\infty, \tag{3.11}$$

$$|u_k|^{2_s^*(\beta)}|x|^{-\beta} \rightharpoonup d\zeta = |u|^{2_s^*(\beta)}|x|^{-\beta} + \zeta_0\delta_0 + \zeta_\infty\delta_\infty, \tag{3.12}$$

where δ_0 and δ_∞ are the Dirac mass at the origin and infinity respectively.

For $\varrho > 0$, define $B_\varrho^+ := \{(x,y) \in \mathbb{R}_+^{n+1} : |(x,y)| < \varrho\}$, $B_\varrho := \{x \in \mathbb{R}^n : |x| < \varrho\}$ and let $\Phi \in C_0^\infty(\mathbb{R}_+^{n+1})$ be a cut-off function such that $\Phi \equiv 1$ in $B_{\frac{1}{2}}^+$ and $0 \le \Phi \le 1$ in $\mathbb{R}_+^{n+1}$. We use ΦU_k as test function, we have

$$
\begin{aligned}
&\langle J'(U_k), \Phi U_k\rangle \\
&= k_s \int_{\mathbb{R}_+^{n+1}} y^{1-2s}\langle \nabla U_k, \nabla(\Phi U_k)\rangle\,dx\,dy - \gamma \int_{\mathbb{R}^n} \frac{u_k(x)^2\phi(x)}{|x|^{2s}}\,dx \\
&\quad - \int_{\mathbb{R}^n} \frac{|u_k(x)|^{2_s^*(\alpha)}\phi(x)}{|x|^\alpha}\,dx - \int_{\mathbb{R}^n} \frac{|u_k(x)|^{2_s^*(\beta)}\phi(x)}{|x|^\beta}\,dx \\
&= k_s \int_{\mathbb{R}_+^{n+1}} y^{1-2s}|\nabla U_k|^2\Phi(x)\,dx\,dy - \gamma \int_{\mathbb{R}^n} \frac{u_k(x)^2\phi(x)}{|x|^{2s}}\,dx \\
&\quad + k_s \int_{\mathbb{R}_+^{n+1}} y^{1-2s} U_k \langle \nabla U_k, \nabla \Phi\rangle\,dx\,dy \\
&\quad - \int_{\mathbb{R}^n} \frac{|u_k(x)|^{2_s^*(\alpha)}\phi(x)}{|x|^\alpha}\,dx - \int_{\mathbb{R}^n} \frac{|u_k(x)|^{2_s^*(\beta)}\phi(x)}{|x|^\beta}\,dx,
\end{aligned}
\tag{3.13}
$$

where $\phi = \Phi(\cdot, 0)$. First, we have

$$
\lim_{\varrho\to 0}\lim_{k\to\infty}\Big(k_s \int_{\mathbb{R}_+^{n+1}} y^{1-2s} U_k\langle \nabla U_k, \nabla\Phi\rangle\,dx\,dy\Big) = 0.
$$

Moreover, from (3.9)-(3.12), we obtain

$$
\lim_{\varrho\to 0}\lim_{k\to\infty}\Big(k_s \int_{\mathbb{R}_+^{n+1}} y^{1-2s}|\nabla U_k|^2\Phi\,dx\,dy\Big) \ge \mu_0,
$$

$$
\lim_{\varrho\to 0}\lim_{k\to\infty}\int_{\mathbb{R}^n} \frac{u_k(x)^2\phi(x)}{|x|^{2s}}\,dx = \nu_0,
$$

$$
\lim_{\varrho\to 0}\lim_{k\to\infty}\int_{\mathbb{R}^n} \frac{|u_k|^{2_s^*(\alpha)}\phi(x)}{|x|^\alpha}\,dx = \eta_0, \quad \lim_{\varrho\to 0}\lim_{k\to\infty}\int_{\mathbb{R}^n} \frac{|u_k|^{2_s^*(\beta)}\phi(x)}{|x|^\beta}\,dx = \zeta_0.
$$

Thus we obtain

$$
\lim_{\varrho\to 0}\lim_{k\to\infty}\langle J'(U_k), \Phi U_k\rangle \ge \mu_0 - \gamma\nu_0 - \eta_0 - \zeta_0. \tag{3.14}
$$

By the fractional Hardy-Sobolev inequalities, we have

$$
\eta_0^{\frac{2}{2_s^*(\alpha)}} S(n,s,\gamma,\alpha) \le \mu_0 - \gamma\nu_0, \quad \zeta_0^{\frac{2}{2_s^*(\beta)}} S(n,s,\gamma,\beta) \le \mu_0 - \gamma\nu_0. \tag{3.15}
$$

By (3.14) and (3.15), we find

$$
\eta_0^{\frac{2}{2_s^*(\alpha)}} S(n,s,\gamma,\alpha) \le \eta_0 + \zeta_0, \quad \zeta_0^{\frac{2}{2_s^*(\beta)}} S(n,s,\gamma,\beta) \le \eta_0 + \zeta_0. \tag{3.16}
$$

So

$$
\begin{gathered}
\eta_0^{\frac{2}{2_s^*(\alpha)}}\Big(1 - S(n,s,\gamma,\alpha)^{-1}\eta_0^{\frac{2_s^*(\alpha)-2}{2_s^*(\alpha)}}\Big) \le S(n,s,\gamma,\alpha)^{-1}\zeta_0, \\
\zeta_0^{\frac{2}{2_s^*(\beta)}}\Big(1 - S(n,s,\gamma,\beta)^{-1}\zeta_0^{\frac{2_s^*(\beta)-2}{2_s^*(\beta)}}\Big) \le S(n,s,\gamma,\beta)^{-1}\eta_0.
\end{gathered}
$$

Since $\{U_k\}_{k\in\mathbb{N}}$ is bounded in $X^s(\mathbb{R}^{n+1}_+)$, we have $\eta_0 \le c_1$ and $\zeta_0 \le c_2$ for positive constants c_1, c_2, thus

$$\eta_0^{\frac{2}{2_s^*(\alpha)}}\Big(1 - S(n,s,\gamma,\alpha)^{-1} c_1^{\frac{2_s^*(\alpha)-2}{2_s^*(\alpha)}}\Big) \le S(n,s,\gamma,\alpha)^{-1}\zeta_0,$$
$$\zeta_0^{\frac{2}{2_s^*(\beta)}}\Big(1 - S(n,s,\gamma,\beta)^{-1} c_2^{\frac{2_s^*(\beta)-2}{2_s^*(\beta)}}\Big) \le S(n,s,\gamma,\beta)^{-1}\eta_0.$$

Therefore, there exist constants $A = A(\alpha, 2_s^*(\alpha), c_1)$ and $B = B(\beta, 2_s^*(\beta), c_2)$ such that

$$\eta_0^{\frac{2}{2_s^*(\alpha)}} \le A\zeta_0, \quad \text{and} \quad \zeta_0^{\frac{2}{2_s^*(\beta)}} \le B\eta_0.$$

In particular, we have that either $\eta_0 = 0$ and $\zeta_0 = 0$, or

$$\eta_0 \ge S(n,s,\gamma,\alpha)^{\frac{n-\alpha}{2s-\alpha}}, \quad \zeta_0 \ge S(n,s,\gamma,\beta)^{\frac{n-\beta}{2s-\beta}}.$$

On the other hand, we know that

$$\begin{aligned} c &= J(U_k) - \frac{1}{2}\langle J'(U_k), U_k\rangle + o_k(1) \\ &\ge \frac{2s-\alpha}{2(n-\alpha)}\Big(\int_{\mathbb{R}^n} \frac{|u_k(x)|^{2_s^*(\alpha)}}{|x|^\alpha}\,dx + \eta_0\Big) \\ &\quad + \frac{2s-\beta}{2(n-\beta)}\Big(\int_{\mathbb{R}^n} \frac{|u_k(x)|^{2_s^*(\beta)}}{|x|^\beta}\,dx + \zeta_0\Big) \\ &\ge \frac{2s-\alpha}{2(n-\alpha)}\eta_0 + \frac{2s-\beta}{2(n-\beta)}\zeta_0. \end{aligned} \tag{3.17}$$

By the assumption that $c < c_*$, we obtain that $\eta_0 = 0$, $\zeta_0 = 0$.

For the concentration at infinity, we define $B_R^+ := \{(x,y) \in \mathbb{R}^{n+1}_+ : |(x,y)| < R\}$, $B_R := \{x \in \mathbb{R}^n : |x| < R\}$ and let $\Psi \in C_0^\infty(\mathbb{R}^{n+1}_+)$ be a cut-off function such that $\Psi = 0$ in B_R^+ and $\Psi \equiv 1$ in $\mathbb{R}^{n+1}_+ \backslash B_{2R}^+$ and $0 \le \Psi \le 1$ in $\mathbb{R}^{n+1}_+$. Consider

$$\begin{gathered} \mu_\infty = \lim_{R\to\infty}\lim_{k\to\infty}\sup\Big(k_s\int_{\mathbb{R}^{n+1}_+\backslash B_{2R}^+} y^{1-2s}|\nabla U_k|^2\Psi\,dx\,dy\Big), \\ \nu_\infty = \lim_{R\to\infty}\lim_{k\to\infty}\sup\int_{\mathbb{R}^n\backslash B_{2R}} \frac{u_k(x)^2\psi(x)}{|x|^{2s}}\,dx, \\ \eta_\infty = \lim_{R\to\infty}\lim_{k\to\infty}\sup\int_{\mathbb{R}^n\backslash B_{2R}} \frac{|u_k(x)|^{2_s^*(\alpha)}\psi(x)}{|x|^\alpha}\,dx, \\ \zeta_\infty = \lim_{R\to\infty}\lim_{k\to\infty}\sup\int_{\mathbb{R}^n\backslash B_{2R}} \frac{|u_k(x)|^{2_s^*(\beta)}\psi(x)}{|x|^\beta}\,dx. \end{gathered}$$

By the same arguments as the concentration at the origin, we can get the following facts: either $\eta_\infty = 0$ and $\zeta_\infty = 0$, or

$$\eta_\infty \ge S(n,s,\gamma,\alpha)^{\frac{n-\alpha}{2s-\alpha}}, \quad \zeta_\infty \ge S(n,s,\gamma,\beta)^{\frac{n-\beta}{2s-\beta}}.$$

As for (3.17), we obtain

$$c \ge \frac{2s-\alpha}{2(n-\alpha)}\eta_\infty + \frac{2s-\beta}{2(n-\beta)}\zeta_\infty. \tag{3.18}$$

By the assumption that $c < c_*$, we obtain that $\eta_\infty = 0$, $\zeta_\infty = 0$. Therefore, up to a subsequence $\{U_k\}_k$ converges strongly to U in $X^s(\mathbb{R}^{n+1}_+)$. $\square$

Let $W_{\gamma,\alpha}$ be the extremal function of $S(n,s,\gamma,\alpha)$ in $X^s(\mathbb{R}^{n+1}_+)$, whose existence was obtained by Ghoussoub and Shakerian in [21] for $\alpha>0$ or $\alpha=0$ and $0\le\gamma<\gamma_H$.

Lemma 3.3. *Let $0<s<1$, $0<\alpha,\beta<2s<n$, and $\gamma<\gamma_H$. Then*

$$\sup_{t\ge0} J(tW_{\gamma,\vartheta})<c_* \quad \text{for } \vartheta=\alpha,\beta,$$

where c_* is defined in Proposition 3.2.

Proof. For $\vartheta=\alpha$, we have

$$J(tW_{\gamma,\alpha})=\frac{t^2}{2}\|W_{\gamma,\alpha}\|^2-\frac{t^{2_s^*(\alpha)}}{2_s^*(\alpha)}\int_{\mathbb{R}^n}\frac{|w_{\gamma,\alpha}|^{2_s^*(\alpha)}}{|x|^\alpha}\,dx-\frac{t^{2_s^*(\beta)}}{2_s^*(\beta)}\int_{\mathbb{R}^n}\frac{|w_{\gamma,\alpha}|^{2_s^*(\beta)}}{|x|^\beta}\,dx.$$

where $w_{\gamma,\alpha}:=Tr(W_{\gamma,\alpha})=W_{\gamma,\alpha}(\cdot,0)$. By construction, we have that

$$J(tW_{\gamma,\alpha})\le f_\alpha(t):=\frac{t^2}{2}\|W_{\gamma,\alpha}\|^2-\frac{t^{2_s^*(\alpha)}}{2_s^*(\alpha)}\int_{\mathbb{R}^n}\frac{|w_{\gamma,\alpha}|^{2_s^*(\alpha)}}{|x|^\alpha}\,dx$$

Straightforward computations yield that $f_\alpha(t)$ attains its maximum at the point

$$\tilde t=\Big(\frac{\|W_{\gamma,\alpha}\|^2}{\int_{\mathbb{R}^n}\frac{|w_{\gamma,\alpha}|^{2_s^*(\alpha)}}{|x|^\alpha}\,dx}\Big)^{\frac{1}{2_s^*(\alpha)-2}}.$$

It follows that

$$\sup_{t\ge0} f_\alpha(t)=\frac{2s-\alpha}{2(n-\alpha)}\Big(\frac{\|W_{\gamma,\alpha}\|^2}{(\int_{\mathbb{R}^n}|w_{\gamma,\alpha}|^{2_s^*(\alpha)}/|x|^\alpha\,dx)^{\frac{2}{2_s^*(\alpha)}}}\Big)^{\frac{n-\alpha}{2s-\alpha}}.$$

Since $W_{\gamma,\alpha}$ is an extremal for $S(n,s,\gamma,\alpha)$ on $X^s(\mathbb{R}^{n+1}_+)$, we obtain that

$$\sup_{t\ge0} J(tW_{\gamma,\alpha})\le\sup_{t\ge0} f_\alpha(t)=\frac{2s-\alpha}{2(n-\alpha)}S(n,s,\gamma,\alpha)^{\frac{n-\alpha}{2s-\alpha}}. \tag{3.19}$$

We now need to show that equality does not hold in (3.19). Indeed, otherwise we would have that $\sup_{t\ge0} J(tW_{\gamma,\alpha})=\sup_{t\ge0} f_\alpha(t)$. Consider t_1 (resp. $t_2>0$) where $\sup_{t\ge0} J(tW_{\gamma,\alpha})$ (resp. $\sup_{t\ge0} f_\alpha(t)$) is attained. We obtain

$$f_\alpha(t_1)-\frac{t_1^{2_s^*(\beta)}}{2_s^*(\beta)}\int_{\mathbb{R}^n}\frac{|w_{\gamma,\alpha}|^{2_s^*(\beta)}}{|x|^\beta}\,dx=f_\alpha(t_2),$$

which means that $f_\alpha(t_1)>f_\alpha(t_2)$ since $t_1>0$. This contradicts the fact that t_2 is a maximum point of $f_\alpha(t)$, hence the strict inequality holds in (3.19).

Similarly, for $\vartheta=\beta$, we obtain

$$\sup_{t\ge0} J(tW_{\gamma,\beta})<\sup_{t\ge0} f_\beta(t)=\frac{2s-\beta}{2(n-\beta)}S(n,s,\gamma,\beta)^{\frac{n-\beta}{2s-\beta}}.$$

This completes the proof. □

4. Proof of main result

Proof of Theorem 1.1. For any $U \in X^s(\mathbb{R}^{n+1}_+)$, the energy functional to problem (2.3) is

$$J(U) = \frac{1}{2}\|U\|^2 - \frac{1}{2^*_s(\alpha)} \int_{\mathbb{R}^n} \frac{|u|^{2^*_s(\alpha)}}{|x|^\alpha}\,dx - \frac{1}{2^*_s(\beta)} \int_{\mathbb{R}^n} \frac{|u|^{2^*_s(\beta)}}{|x|^\beta}\,dx,$$

where again $u := Tr(U) = U(\cdot, 0)$. By fractional Hardy-Sobolev inequality, we have

$$\begin{aligned} J(U) &\geq \frac{1}{2}\|U\|^2 - \frac{1}{2^*_s(\alpha)} S(n,s,\gamma,\alpha)^{-\frac{2^*_s(\alpha)}{2}} \|U\|^{2^*_s(\alpha)} \\ &\quad - \frac{1}{2^*_s(\beta)} S(n,s,\gamma,\beta)^{-\frac{2^*_s(\beta)}{2}} \|U\|^{2^*_s(\beta)} \\ &= \Big(\frac{1}{2} - \frac{1}{2^*_s(\alpha)} S(n,s,\gamma,\alpha)^{-\frac{2^*_s(\alpha)}{2}} \|U\|^{2^*_s(\alpha)-2} \\ &\quad - \frac{1}{2^*_s(\beta)} S(n,s,\gamma,\beta)^{-\frac{2^*_s(\beta)}{2}} \|U\|^{2^*_s(\beta)-2}\Big) \|U\|^2. \end{aligned}$$

Since $\alpha, \beta \in (0, 2s)$, we have that $2^*_s(\alpha) > 2, 2^*_s(\beta) > 2$. By (2.11), we then get that there exists $R > 0$ such that $J(U) \geq \rho$ for all $U \in X^s(\mathbb{R}^{n+1}_+)$ with $\|U\|_{X^\alpha(\mathbb{R}^{n+1}_+)} = R$. Moreover, for $\vartheta = \alpha$ or $\vartheta = \beta$,

$$J(tW_{\gamma,\vartheta}) = \frac{t^2}{2}\|W_{\gamma,\vartheta}\|^2 - \frac{t^{2^*_s(\alpha)}}{2^*_s(\alpha)} \int_{\mathbb{R}^n} \frac{|w_{\gamma,\vartheta}|^{2^*_s(\alpha)}}{|x|^\alpha}\,dx - \frac{t^{2^*_s(\beta)}}{2^*_s(\beta)} \int_{\mathbb{R}^n} \frac{|w_{\gamma,\vartheta}|^{2^*_s(\beta)}}{|x|^\alpha}\,dx,$$

hence $\lim_{t\to+\infty} J(tW_{\gamma,\vartheta}) = -\infty$, then there exists $t_0 > 0$ such that $\|t_0 W_{\gamma,\vartheta}\| > R$ and $J(t_0 W_{\gamma,\vartheta}) < 0$. Set

$$c_\vartheta := \inf_{g\in\Gamma_\vartheta} \max_{t\in[0,1]} J(g(t)),$$

where

$$\Gamma_\vartheta := \big\{g \in C^0([0,1], X^s(\mathbb{R}^{n+1}_+)) : g(0) = 0,\ g(1) = t_0 W_{\gamma,\vartheta}\big\}.$$

Thus by Mountain Pass Lemma, there exists a sequence $\{U_k\}$ in $X^s(\mathbb{R}^{n+1}_+)$ such that

$$J(U_k) \to c, \quad J'(U_k) \to 0 \quad \text{in } (X^s(\mathbb{R}^{n+1}_+))' \text{ as } k \to \infty.$$

By Lemma 3.3, we have

$$0 < c \leq \sup_{t\in[0,1]} J(tt_0 W_{\gamma,\vartheta}) \leq \sup_{t>0} J(tW_{\gamma,\vartheta}) < c_*.$$

By Proposition 3.2, we deduce that $\{U_k\}$ has a subsequence, still denote by $\{U_k\}$, such that $U_k \to U$ strongly in $X^s(\mathbb{R}^{n+1}_+)$. Thus U is a nontrivial solution of problem (2.3), and $u := Tr(U) = U(\cdot, 0)$ is a nontrivial solution of problem (1.1). □

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