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BLOW-UP FOR A SEMILINEAR HEAT EQUATION WITH MOVING NONLINEAR REACTION

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Abstract. We study the behavior of solutions of the semilinear problem

$$u_t = u_{xx} + (1 + (T-t)^{-\alpha}\chi_{\{|x|<(T-t)^{1/2}\}})u^p, \quad x \in \mathbb{R},\ t \in (0,T),$$
$$u(x,0) = u_0(x) \geq 0, \quad x \in \mathbb{R},$$

with $\alpha > 0$ and $p > 0$. We describe, in terms of the parameters when the solution is bounded and when it blows up. For blowing up solutions we find the blow-up rate and the blow-up set.

1. Introduction

In this article we study the Cauchy problem, for the equation

$$u_t = u_{xx} + h(x,t)u^p, \tag{1.1}$$

with $\alpha > 0$, $p > 0$ and

$$h(x,t) = 1 + (T-t)^{-\alpha}\chi_{\{|x|<(T-t)^{1/2}\}}.$$

Moreover, we assume that u_0 is a nonnegative, nontrivial regular function.

Existence of a solution can be easily achieved, and it is given by the representation formula

$$\begin{aligned} u(x,t) = &\int_{\mathbb{R}} \Gamma(x-y,t)u_0(y)\,dy + \int_0^t \int_{\mathbb{R}} \Gamma(x-y,t-s)u^p(y,s)\,dy\,ds \\ &+ \int_0^t \int_{\{|y|<(T-s)^{1/2}\}} \Gamma(x-y,t-s)u^p(y,s)(T-s)^{-\alpha}\,dy\,ds. \end{aligned}$$

where $\Gamma(x,t) = (4\pi t)^{-1/2}e^{-x^2/4t}$ is the heat Kernel.

Uniqueness is standard for $p \geq 1$, but for $0 < p < 1$ the reaction term $f(x,t,u) = h(x,t)u^p$ is non-Lipschitz on u and the uniqueness fails, see [1]. Nevertheless, in this case we can we can construct a maximal solution just by taking the limit

$$\overline{u} = \lim_{\varepsilon \to 0} u^{(\varepsilon)},$$

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where $u^{(\varepsilon)}$ is the unique solution to our problem with initial condition $u^{(\varepsilon)}(x,0) = u_0(x)+\varepsilon$, and with the reaction $f(x,t,u)$ replaced by

$$f_{(\varepsilon)}(x,t,u) = h(x,t)\begin{cases} u^p & \text{if } s \ge \varepsilon, \\ \varepsilon^{p-1}u & \text{if } s < \varepsilon, \end{cases}$$

see [10]. We note that the maximal solution satisfies the above representation formula. Moreover, a comparison principle among maximal solutions can be easily obtained.

Equation (1.1) can be considered in some way as a perturbation of the system

$$\begin{gathered} u_t = u_{xx} + u^{p_1}v^{q_1} \\ v_t = v_{xx} + u^{p_2}v^{q_2} \end{gathered}$$

Indeed, if we assume that the v component blows up in a finite time T, typically the behaviour near the blow-up time is

$$v \sim \begin{cases} (T-t)^{-q} & |x| < (T-t)^{1/2} \\ 1 & |x| > (T-t)^{1/2} \end{cases}$$

see [4], [13] ($f \sim g$ means that there exist finite positive constants c_1, c_2 such that $c_1 g \le f \le c_2 g$). Now including this behaviour on the equation of u, we obtain an equation like (1.1).

Systems of this kind are common in population dynamics. In this context u and v represent two different species with a symbiotic behaviour. The cooperation between them is represented by the coupled reaction terms.

Probably the first study in blow-up for the semilinear heat equation is given by Fujita, [6], where the case $h = 1$ is studied. He proves that for $1 < p < 1+2/n$ every non-trivial positive solution blows up in finite time, while for $p > 1+2/n$ there are both blow up and global solutions. The values $p_g = 1$ and $p_c = 1+2/n$ are called the critical global exponent and the critical blow-up exponent, respectively. In the border case $p = p_c$ every positive solution blow up, see [9, 15].

For the case of localized reaction ($h(x,t) = h(x)$ with compact support) it is known that $p_g = 1$ and $p_c = 2$ if $n = 1$ while $p_g = p_c = 1$ for $n \ge 2$, see [11]. For further result on the blow-up phenomena in one dimensional case with localized reaction we refer to [5].

If the coefficient reaction is given by $h(x,t) = t^s|x|^\sigma$, Qi shows in [12] that $p_g = 1$ and $p_c = 1 + (2+2s+\sigma)/n$. Since, in our problem, the reaction coefficient blows up as $t \to T$, the critical exponents are independent of p, see Theorem 1.1 below.

Blow-up phenomena has attracted an increasing interest among researchers in the last years, not only for its wide variety of applications, but also motivated by the mathematical analysis behind those kind of equations, see for instance the books [7, 14] and the surveys [3, 8].

Our first objective is to identify when either the solution is bounded or it blows up.

Theorem 1.1.
- (i) *If $p \le 1$ and $\alpha < 1$ all the solutions to problem* (1.1) *are bounded;*
- (ii) *If $p \le 1$and $\alpha \ge 1$ all the solutions to problem* (1.1) *blow up a time $t = T$;*
- (iii) *If $p > 1$ and $\alpha \ge 1$ all the solutions to problem* (1.1) *blow up a time $T^* < T$;*
- (iv) *If $p > 1$ and $\alpha < 1$ solutions may blow up at time $T^* \le T$ or not depending on the size of initial data.*

Once we have characterized the exponents giving rise to either blow-up or bounded solutions, we want to study the way the blowing up solutions behave as approaching the blow-up time. This means that we must investigate the speed at which they blow up, the blow-up rate and where the solutions blow up, the blow-up set.

We note that if the blow-up time $T^* < T$, the reaction coefficient $h(x,t)$ is bounded. Then, in the study of the blow up does not play any role and the solution behaves like the solution of

$$w_t = w_{xx} + w^p,$$

see [2]. Thus, we only study the range $p \le 1 \le \alpha$ where the maximal solution blows up at time T independently of initial data.

Theorem 1.2. *Let u be the maximal solution of* (1.1)*. As t approaches T we have that*

(i) *For $p < 1$ and $\alpha > 1$,*

$$\|u(\cdot,t)\|_\infty \sim \ (T-t)^{-\frac{\alpha-1}{1-p}}.$$

(ii) *For $p < 1$ and $\alpha = 1$,*

$$\|u(\cdot,t)\|_\infty \sim \Big(\log\big(\frac{T}{T-t}\big)\Big)^{\frac{1}{(1-p)}}.$$

(iii) *For $p = 1$ and $\alpha > 1$,*

$$\lim_{t\to T}(T-t)^{1-\alpha}\log\|u(\cdot,t)\|_\infty = \frac{1}{1-\alpha}.$$

(iv) *For $p = 1 = \alpha$, there exists $\gamma_* \in (0,1)$ such that for all $\varepsilon > 0$*

$$C_\varepsilon(T-t)^{-\gamma_*+\varepsilon} \le \|u(\cdot,t)\|_\infty \le C_2(T-t)^{-\gamma_*}.$$

Finally, we study the blow-up set. Which it is defined by

$$B(u) = \{x \in \mathbb{R} : \exists x_n \to x, t_n \nearrow T \text{ with } u(x_n,t_n) \to \infty\}.$$

Theorem 1.3. *Let u a blow-up solution of* (1.1)*. Then, $B(u) = \{0\}$ for $p < 1 \le \alpha$, while for $p = 1$ we have*

$$B(u) = \begin{cases} (-\infty,\infty) & \text{for } \alpha > 2 \\ [-2,2] & \text{for } \alpha = 2 \\ \{0\} & \text{for } \alpha \in [1,2) \end{cases}$$

This paper is organized as follows. In the next Section, we study in terms of the parameters p and α, when the solution is bounded and it blows up, namely we prove Theorem 1.1. In Section 3 we find the blow rate for the range $p \le 1 \le \alpha$, Theorem 1.2. Finally, in Section 4 Theorem 1.3 are proved.

2. Blow-up versus boundedness

Lemma 2.1. *If $\alpha < 1$ and $p \le 1$ then the solution u is bounded.*

Proof. It follows by comparison with the flat supersolution

$$\begin{gathered} w'(t) = (1 + (T-t)^{-\alpha})w^p, \quad t > 0, \\ w(0) = \|u_0\|_\infty. \end{gathered} \tag{2.1}$$

Which is given by

$$v(t) = \begin{cases} \Big(v_0^{1-p} + (1-p)\big(t - \frac{1}{1-\alpha}\big((T-t)^{1-\alpha} - T^{1-\alpha}\big)\big)\Big)^{1/(1-p)}, & p<1, \\ v_0 \exp\Big(t - \frac{1}{1-\alpha}\big((T-t)^{1-\alpha} - T^{1-\alpha}\big)\Big), & p = 1. \end{cases} \tag{2.2}$$

□

Lemma 2.2. *If $\alpha < 1$ and $p > 1$, then there are both: bounded and blow-up solutions.*

Proof. The fact that for small initial data the solution is bounded follows by comparison with the flat supersolution (2.1).

On the other hand, the blow-up, is given by comparison with the problem

$$\begin{gathered} v_t = v_{xx} + v^p \quad (x,t) \in (-L,L) \times (0,T) \\ v(\pm L, t) = 0 \quad t \in (0,T) \\ v(x,0) = v_0(x) \quad x \in (-L,L) \end{gathered}$$

Applying Kapplan's method it is well known that we can take v_0 large enough to ensure that v blows up before time T. □

Lemma 2.3. *Let $\alpha \geq 1$ then the solution blows up at a finite time $T^* \leq T$. Moreover, if $T^* = T$, then $u(0,t) \to \infty$ as $t \to T$.*

Proof. First we observe that by comparison with the heat equation we get that there exists $m > 0$ such that

$$u \geq m \quad (x,t) \in (-T^{1/2}, T^{1/2}) \times (0,T]. \tag{2.3}$$

Using the representation formula, we get

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}} \Gamma(x-y,t) u_0(y)\,dy + \int_0^t \int_{\mathbb{R}} \Gamma(x-y,t-s) u^p(y,s)\,dy\,ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \Gamma(x-y,t-s) u^p(y,s)(T-s)^{-\alpha} \chi_{\{|y|<(T-s)^{1/2}\}}\,dy\,ds \\ &\geq \int_0^t \int_{\mathbb{R}} \Gamma(x-y,t-s) m^p (T-s)^{-\alpha} \chi_{\{|y|<(T-s)^{1/2}\}}\,dy\,ds. \end{aligned}$$

Now, we observe that for $x = 0$ and $t = T$ the above integral becomes

$$\begin{aligned} &C \int_0^T (T-t)^{-\alpha} \int_{|y|<(T-s)^{1/2}} \frac{1}{(T-s)^{1/2}} e^{-y^2/(4(T-s))}\,dy\,ds \\ &= C \int_0^T (T-t)^{-\alpha} \int_{-1/2}^{1/2} e^{-r^2}\,dr\,ds. \end{aligned}$$

Which is divergent for $\alpha \geq 1$. □

Lemma 2.4. *Let $\alpha \geq 1$. If $p \leq 1$ the maximal solution u blows up at time $T^* = T$, while for $p > 1$ the blow up time satisfies that $T^* < T$.*

Proof. The fact that for $p \leq 1$ the solution blows up at time T, is given by comparison with the flat supersolution given in (2.1).

Now, for $p > 1$ we assume that u blows up at time T and we define

$$v(\xi,\tau) = u(x,t) \quad \xi = x(T-t)^{-1/2},\ \tau = -\log\big(\frac{T-t}{T}\big),$$

which satisfies the equation

$$v_\tau = v_{\xi\xi} - \frac{1}{2}\xi v_\xi + \Big(Te^{-\tau} + T^{1-\alpha}e^{(\alpha-1)\tau}\chi_{\{|\xi|\le 1\}}\Big)v^p.$$

Notice that

- As v is symmetric and decreasing for $\xi > 0$ the term $-\xi v_\xi$ is non-negative.
- From (2.3) we have that $v > m$ in $[-1,1]\times[0,\infty]$.
- $\alpha \ge 1$ implies that $e^{\alpha-1}\tau \ge 1$.

Therefore, for all $1 < q < p$ the function v is a supersolution of the problem

$$\begin{gathered} w_\tau = w_{\xi\xi} + T^{1-\alpha}\chi_{\{|\xi|\le 1\}}m^{p-q}w^q \quad \mathbb{R}\times(\tau_0,\infty) \\ w(\xi,\tau_0) = v(\xi,\tau_0) \end{gathered}$$

For this problem it is well know that for all non-negative initial data the solution blows up at finite time if $1 < q \le 2$, see [5, 11]. Then, the function u blows up before time T. □

3. Blow-up rates

Since for $\alpha \ge 1$ and $p \le 1$ the solution of Problem (1.1) blows up at time T, the comparison with the flat supersolution defined in (2.1) gives us the upper blow-up rate. However, as we see below, this is not the correct blow-up rate in the linear case $p = \alpha = 1$. More precisely,

Lemma 3.1. *Let u be a solution of (1.1). Then, there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_\infty \le C\begin{cases} (T-t)^{-\frac{\alpha-1}{1-p}} & p < 1 < \alpha \\ e^{\frac{(T-t)^{1-\alpha}}{\alpha-1}} & p = 1 < \alpha \\ \big(\log\big(\frac{1}{T-t}\big)\big)^{\frac{1}{1-p}} & p < 1 = \alpha \end{cases}$$

To study the lower blow-up rate we consider different cases.

Lemma 3.2. *Let $p < 1$ and $\alpha > 1$. Then*

$$\|u(\cdot,t)\|_\infty \ge (T-t)^{-\frac{\alpha-1}{1-p}}.$$

Proof. Let ϕ_1 be the first eigenfunction of the Laplacian in $(-1,1)$ normalized according to $\|\phi_1\|_\infty = 1$, and consider the function

$$\underline{u}(x,t) = A(T-t)^{-\frac{\alpha-1}{1-p}}\phi_1(x(T-t)^{-1/2}).$$

To see that $\underline{u}$ is subsolution we need:

• Comparison of the initial data. Notice that u is a supersolution of the heat equation, then $u(x,t) \ge m > 0$ in $(-T^{1/2},T^{1/2})\times(0,T)$. Thus taking A small enough, $\underline{u}(x,0) \le u_0(x)$.

• An inequality for the equation: substituting $\underline{u}$ in the equation we need that

$$A\frac{\alpha-1}{1-p}\phi(\xi) + \frac{A}{2}\xi\phi_1'(\xi) \le -\lambda_1 A\phi_1(\xi) + ((T-t)^\alpha + 1)A^p\phi_1^p(\xi)$$

where $\xi = x(T-t)^{-1/2}$. Observe that $\frac{A}{2}\xi\phi_1'(\xi) < 0$ and $(T-t)^\alpha A^p\phi_1^p(\xi) > 0$, therefore the above inequality holds provided that

$$A^{1-p}\phi_1^{1-p}(\xi)\Big(\frac{\alpha-1}{1-p} + \lambda_1\Big) \le 1.$$

Thus, taking A small enough we are done.

This implies that $u(x,t) \geq \underline{u}(x,t)$ and lower blow-up rate follows. $\square$

Lemma 3.3. *Let $p = 1$ and $\alpha > 1$. Then, for $A > 0$ small enough*

$$\|u(\cdot,t)\|_\infty \geq A(T-t)^{\frac{\pi^2}{4}} e^{\frac{(T-t)^{1-\alpha}}{\alpha-1}}.$$

Proof. As in the previous Lemma, we use a comparison argument. In this case we consider the subsolution

$$\underline{u} = A(T-t)^{\lambda_1} e^{\frac{(T-t)^{1-\alpha}}{\alpha-1}} \phi_1(x(T-t)^{-1/2}), \quad \gamma > 0,$$

where $\lambda_1 = \pi^2/4$ and ϕ_1 are the first eigenvalue and the first eigenfunction of the Laplacian in $(-1,1)$. $\square$

We remark that in this case the extra term, $(T-t)^{\pi^2/4}$ appears. We conjecture that this extra term is technical and it can be avoided. Using the comparison with the function w given in (2.1) it is easy to obtain the following blow-up rate.

Corollary 3.4. *Let $p = 1$ and $\alpha > 1$. Then*

$$\lim_{t\to T} (T-t)^{1-\alpha} \log(u(0,t)) = \frac{1}{\alpha-1}.$$

Lemma 3.5. *Let $\alpha = 1$ and $p < 1$. Then*

$$\|u(\cdot,t)\|_\infty \geq \Big(\log\big(\frac{T}{T-t}\big)\Big)^{\frac{1}{(1-p)}}$$

Proof. Using the representation formula

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}} \Gamma(x-y,t)u_0(y)\,dy + \int_0^t \int_{\mathbb{R}} \Gamma(x-y,t-s)u^p(y,s)\,dy\,ds \\ &\quad + \int_0^t \int_{\{|y|<(T-s)^{1/2}\}} \Gamma(x-y,t-s)u^p(y,s)(T-s)^{-1}\,dy\,ds. \end{aligned}$$

Observe that the first two integrals are positive, then

$$\begin{aligned} u(x,t) &\geq \int_0^t \int_{\{|y|<(T-s)^{1/2}\}} \Gamma(x-y,t-s)u^p(y,s)(T-s)^{-1}\,dy\,ds \\ &= \frac{1}{\sqrt{\pi}} \int_0^t \int_{\frac{x-(T-s)^{1/2}}{2(t-s)^{1/2}}}^{\frac{x+(T-s)^{1/2}}{2(t-s)^{1/2}}} u^p(x-2(t-s)^{1/2}z,s)\frac{1}{T-s}e^{-z^2}\,dz\,ds. \end{aligned}$$

As $0 < s < t < T$, we note that for $0 \leq x \leq (T-t)^{1/2}$,

$$\frac{x-(T-s)^{1/2}}{2(t-s)^{1/2}} \leq 0 \quad \text{and} \quad \frac{x+(T-s)^{1/2}}{2(t-s)^{1/2}} \geq \frac{(T-s)^{1/2}}{2(t-s)^{1/2}} \geq \frac{1}{2},$$

Therefore,

$$u(x,t) \geq \frac{1}{\sqrt{\pi}} \int_0^t \int_0^{1/2} u^p(x-2(t-s)^{1/2}z,s)\frac{1}{T-s}e^{-z^2}\,dz\,ds. \tag{3.1}$$

On the other hand, by comparison with the heat equation, there exists $C_0 > 0$ such that

$$u(x,t) \geq C_0, \quad x \in (-T^{1/2},T^{1/2}),\ t \in (0,T). \tag{3.2}$$

Since $-T^{1/2} \le x - t^{1/2} \le x - 2(t-s)^{1/2}z \le x \le T^{1/2}$, we can use this lower estimate in (3.1) to improve it as follows

$$u(x,t) \ge \frac{C_0^p}{\sqrt{\pi}} \int_0^t \int_0^{1/2} \frac{1}{T-s} e^{-z^2}\,dz\,ds = C_1 \log\big(\frac{T}{T-t}\big),$$

where

$$C_1 = AC_0^p \quad \text{and} \quad A = \frac{1}{T\sqrt{\pi}} \int_0^{1/2} e^{-z^2}\,dz.$$

Which is a better lower estimate. Using this new lower estimate in (3.1)

$$\begin{aligned} u(x,t) &\ge \frac{C_1^p}{\sqrt{\pi}} \int_0^t \int_0^{1/2} \Big(\log\big(\frac{T}{T-s}\big)\Big)^p \frac{1}{T-s} e^{-z^2}\,dz\,ds \\ &= C_1^p \frac{A}{2} \Big(\log\big(\frac{T}{T-t}\big)\Big)^{p+1}. \end{aligned}$$

Iterating this procedure we get that

$$u(x,t) \ge C_k \Big(\log\big(\frac{T}{T-t}\big)\Big)^{\gamma_k},$$

with $\gamma_0 = 0$, C_0 given in (3.2) and

$$\gamma_{k+1} = p\gamma_k + 1, \quad C_{k+1} = C_k^p \frac{A}{\gamma_{k+1}}.$$

It is easy to see that as $k \to \infty$,

$$\gamma_k = \sum_{j=0}^{k-1} p^j \to \frac{1}{1-p} \quad \text{and} \quad C_k \to \Big((1-p)A\Big)^{\frac{1}{1-p}}.$$

Therefore, for $0 \le x \le (T-t)^{1/2}$,

$$u(x,t) \ge \Big((1-p)A\Big)^{\frac{1}{1-p}} \Big(\log\big(\frac{T}{T-t}\big)\Big)^{\frac{1}{1-p}}$$

and the proof is complete. $\square$

For the linear case $p = \alpha = 1$ we perform the self-similar change of variables

$$u(x,t) = Ae^t (T-t)^{-\gamma} v(\xi,\tau) \quad \xi = |x|(T-t)^{-1/2},\ \tau = \log\big(\frac{1}{T-t}\big).$$

It is easy to see that the rescaled function satisfies

$$v_\tau = v_{\xi\xi} - \frac{1}{2}\xi v_\xi + (\chi_1 - \gamma)v. \tag{3.3}$$

Lemma 3.6. *There exits a unique (γ_*, F_*) such that $\gamma_* \in (0,1)$ and F_* is an even, positive non-increasing (for $\xi > 0$) stationary solution of (3.3).*

Proof. We look for solutions of the problem

$$\begin{gathered} F_\gamma'' - \frac{1}{2}\xi F_\gamma' + (\chi_1 - \gamma)F_\gamma = 0, \quad \xi > 0, \\ F_\gamma(0) = 1, \quad F_\gamma'(0) = 0. \end{gathered} \tag{3.4}$$

To study this problem, we use a shooting method. We define

$$\begin{gathered} \Lambda_+ = \big\{\gamma \in (0,1) : F_\gamma > 0 \text{ and there exists } x_\gamma \text{ such that } F_\gamma(x_\gamma) > 1\big\} \\ \Lambda_- = \big\{\gamma \in (0,1) : \text{there exists } x_\gamma \text{ such that } F_\gamma(x_\gamma) < 0\big\} \end{gathered}$$

$$\Lambda_* = \big\{\gamma \in (0,1) : F_\gamma \geq 0 \text{ and } F'_\gamma \leq 0\big\}.$$

Notice that these sets are disjoint. Moreover, by continuous dependence of the solution with respect to the parameter γ both sets, Λ_+ and Λ_- are open sets. Then, if we prove that both sets are non-empty, we get that Λ_* is a non-empty closed set and the result follows.

To prove that we consider the boundary cases $\gamma = 1$ and $\gamma = 0$.

(i) For $\gamma = 1$ it is trivial to see that $F_1(\xi) = 1$ for $0 \leq \xi \leq 1$. Moreover $F_1''(1^+) = 1$, then the profile $F_1(\xi) > 1$ for $\xi > 1$. Now, applying the continuous dependence of the solution with respect to the parameter γ, we get that $\gamma \in \Lambda_+$ for $\gamma \sim 1$.

(ii) For $\gamma = 0$ we rewrite the equation (3.4) as

$$(e^{-\xi^2/4}F_0')' = -\chi_1 F_0 e^{-\xi^2/4},$$

to get that for $0 \leq \xi \leq 1$, the profile satisfies that $F_0' < 0$ in the positivity set of F_0. Therefore if there exists $\xi_0 \in (0,1]$ such that $F_0(\xi_0) = 0$, the profile is negative in $(\xi_0, \xi_0 + \varepsilon)$. On the other hand, if $F_0(\xi) > 0$ in $0 \leq \xi \leq 1$, we have that $F_0'(1) < 0$ and

$$F_0'(\xi) = F_0'(1)e^{\frac{\xi^2-1}{4}} < 0, \quad \xi > 1.$$

Therefore, the profile crosses the axis at some point ξ_0. The continuous dependence of the solution with respect to the parameter γ implies that $\gamma \in \Lambda_-$ for $\gamma \sim 0$.

The uniqueness follows from the fact that $G_\gamma = e^{-\xi^2/4}F_\gamma$ is a solution of

$$G_\gamma'' + \frac{1}{2}\xi G_\gamma' + (\chi_1 + \frac{1}{2} - \gamma)G_\gamma = 0.$$

Let us suppose that there exists $\gamma_1 \neq \gamma_2$ in Λ_*. Then

$$\begin{aligned}
\gamma_1 \int_{\mathbb{R}} F_{\gamma_1} G_{\gamma_2}\, d\xi &= \int_{\mathbb{R}} (F''_{\gamma_1} - \frac{1}{2}\xi F'_{\gamma_1} + \chi_1 F_{\gamma_1}) G_{\gamma_2}\, d\xi \\
&= \int_{\mathbb{R}} F_1 (G''_{\gamma_2} + \frac{1}{2}\xi G'_{\gamma_2} + (\chi_1 + \frac{1}{2}) G_{\gamma_2})\, d\xi \\
&= \gamma_2 \int_{\mathbb{R}} F_{\gamma_1} G_{\gamma_2}\, d\xi,
\end{aligned}$$

which is a contradiction. $\square$

Remark 3.7. The parameter γ_* can be seen as the first eigenvalue of the operator $L(w) = w'' - \frac{1}{2}\xi w' + \chi_1 w$. Then

$$-\gamma_* = \inf_{w \in X} \frac{\int_{\mathbb{R}} |w'|^2 e^{-\xi^2/4}\, d\xi - \int_{\mathbb{R}} \chi_1 w^2 e^{-\xi^2/4}\, d\xi}{\int_{\mathbb{R}} w^2 e^{-\xi^2/4}\, d\xi},$$

where X is the weighted $H^1_\rho(\mathbb{R})$ space with weight $\rho(\xi) = e^{-\xi^2/4}$.

Lemma 3.8. *Let v be a solution of* (3.3) *with $\gamma = \gamma_*$. Then, v is bounded.*

Proof. Let us define

$$H(v) = \int_{\mathbb{R}} |v_\xi|^2 e^{-\xi^2/4}\, d\xi - \int_{\mathbb{R}} (\chi_1 - \gamma_*) v^2 e^{-\xi^2/4}\, d\xi.$$

Notice that by the definition of γ_* in Remark 3.7, $H(v)\geq 0$. On the other hand, multiplying the equation (3.3) by $e^{\xi^2/4}v$ we get

$$\frac{\partial}{\partial\tau}2\int_{\mathbb{R}} v^2 e^{-\xi^2/4}\,d\xi = -H(v)\leq 0.$$

This monotonicity implies that v is bounded almost everywhere. Therefore, there exists $\xi_0\in\mathbb{R}\setminus[-1,1]$ such that $v(\xi_0,\tau)\leq C$. Now we observe that for $\xi>\xi_0$, the function v is a subsolution of

$$\begin{gathered} w_\tau = w_{\xi\xi}-\frac{1}{2}\xi w_\xi-\gamma_* w \quad \xi>\xi_0,\ \tau>0\\ w(\xi_0,\tau)=C\\ w(\xi,0)=v_0(\xi). \end{gathered}$$

But for this problem $\overline{w}(\xi)=C$ is a supersolution. Then, v is uniformly bounded for $\xi\geq\xi_0$. The same argument provides that v is uniformly bounded for $\xi\leq-\xi_0$. Finally, for $\xi\in[-\xi_0,\xi_0]$, v is a subsolution of the problem

$$\begin{gathered} w_\tau = w_{\xi\xi}-\frac{1}{2}\xi w_\xi+(\chi_1-\gamma_*)w \quad \xi\in(-\xi_0,\xi_0),\ \tau>0\\ w(\pm\xi_0,\tau)=C\\ w(\xi,0)=v_0(\xi). \end{gathered}$$

Observe that the function $\overline{w}(\xi)=AF_*(\xi)$ is a supersolution for A large enough. Thus, v is uniformly bounded. $\square$

From these results we obtain the following blow-up rates.

Lemma 3.9. *Let $p=\alpha=1$ and (γ_*,F_*) given in Lemma 3.6. Then, there exists a positive constant C_1 such that*

$$\|u(\cdot,t)\|_\infty\leq C_1(T-t)^{-\gamma_*}.$$

Moreover:

(1) *If for some $0\leq t_0<T$, $u(x,t_0)>F_*(x)$ for x large, then there exists $C_2>0$ such that*

$$\|u(\cdot,t)\|_\infty\geq C_2(T-t)^{-\gamma_*}.$$

(2) *In the general case, we have that for all $\varepsilon>0$ there exists $C_\varepsilon>0$ such that*

$$\|u(\cdot,t)\|_\infty\geq C_\varepsilon(T-t)^{-\gamma_*+\varepsilon}.$$

Proof. The upper blow-up rate follows by the fact that v is bounded.

If for some time t_0 and x large enough $u(x,t_0)>F_*(x)$, we only note that for A small enough the function $\underline{u}=A(T-t)^{-\gamma_*}F_*(x(T-t)^{-1/2})$ is a subsolution of the problem (1.1). Then, the lower blow-up rate follows.

For the general case we can use the profiles given in Lemma 3.6 with $\gamma\in\Lambda_-$ to obtain the subsolution $\underline{u}=A(T-t)^{-\gamma}\max\{F(x(T-t)^{1/2}),\,0\}$. $\square$

4. Blow-up set

Using the blow-up rate we can construct subsolutions and supersolutions which determine the blow-up set. We consider the problem

$$\begin{gathered} w_t = w_{xx} + \lambda w \quad x > x_0\,,\ 0 < t < T \\ w(x_0,t) = K(T-t)^{\nu} e^{\frac{(T-t)^{-n}}{\alpha-1}} \quad 0 < t < T \\ w(x,0) = w_0(x) \quad x > x_0 \end{gathered} \tag{4.1}$$

It is well know that, see for instance [7],

$$B(w) = \begin{cases} [x_0,\infty) & \text{for } n > 1 \\ \big[x_0, x_0 + 2\big(\frac{1}{\alpha-1}\big)^{1/2}\big] & \text{for } n = 1 \\ \{x_0\} & \text{for } n \in (0,1) \\ \{x_0\} & \text{for } n = 0 \text{ and } \nu < 0 \end{cases}$$

Notice that for fixed $n > 0$ the blow-up set is independent of ν and K.

Lemma 4.1. *Let $p < 1 \le \alpha$. Then the blow-up set of a solution of* (1.1) *is the origin, that is,* $B(u) = \{0\}$.

Proof. We first note that u is a supersolution of the equation

$$v_t = v_{xx} + v^p.$$

Since $p < 1$ we get that $v \ge ((1-p)t)^{1/(1-p)}$, see [1]. Then

$$u(x,t) \ge v(x,t) \ge ((1-p)t)^{1/(1-p)}. \tag{4.2}$$

Now, we assume that $x_1 > 0$ is a blow-up point, that is $x_1 \in B(u)$, and define $0 < x_0 < x_1$ and t_0 such that $x_0 = (T-t_0)^{1/2}$. Using the estimate (4.2),

$$u^p(x,t) \le \lambda u, \quad \lambda = \Big(\frac{p-1}{t_0}\Big)^{-1},$$

for $t \ge t_0$. On the other hand, from the upper blow-up rate is

$$u(x_0,t) \le C(T-t)^{-(\alpha-1)/(1-p)} \quad \text{for } \alpha < 1,$$

while

$$u(x_0,t) \le C\Big(\log\big(\frac{T}{T-t}\big)\Big)^{\frac{1}{1-p}} \le C(T-t)^{-\frac{1}{1-p}}, \quad \text{for } \alpha = 1.$$

Summing up we obtain that for $\alpha \le 1$ the solution u is a subsolution of the problem

$$\begin{gathered} v_t = v_{xx} + \lambda v, \quad (x_0,\infty)\times(t_0,T) \\ v(x_0,t) = K(T-t)^{-\gamma} \\ v(x,t_0) = u(x,t_0) \end{gathered}$$

Note that this is problem (4.1) with $\lambda = ((p-1)t_0)^{-1}$, $\nu = \gamma$ and $n = 0$. Then, $B(v) = \{x_0\}$. Hence, by comparison $u(x_1,t)$ is bounded. A contradiction. $\square$

Lemma 4.2. *Let $p = 1$ and u a solution of* (1.1). *Then, the blow-up set is*

$$B(u) = \begin{cases} (-\infty,\infty) & \textit{for } \alpha > 2 \\ [-2,2] & \textit{for } \alpha = 2 \\ \{0\} & \textit{for } \alpha \in [1,2) \end{cases}$$

Proof. Note that from Lemmas 3.9 and 3.1, u is a subsolution of (4.1) with $\lambda = 1$, $\nu = \gamma_*$, $n = \alpha - 1 > 0$ and k large enough. Arguing as in the previous Lemma we obtain $B(u) = \{0\}$ for $\alpha \in [1, 2)$. While for $\alpha = 2$ we get that if $x_0 \in B(u)$ then $u(x, t)$ is bounded for $x > x_0 + 2$.

On the other hand, Lemma 3.3 provides a lower bound of $u(0, t)$, then u is a supersolution of (4.1) with $\lambda = 1$, $\nu = \pi^2/4$, $n = \alpha - 1$ and k small enough. Then, $\mathbb{R} = B(w) \subset B(u)$ for $\alpha > 2$, while for $\alpha = 2$ we have that $[-2, 2] = B(w) \subset B(u)$.

Finally, for the critical case $\alpha = 2$, we observe that for all $\varepsilon > 0$, the point $x_0 = \varepsilon \in B(u)$, then $u(x, t)$ is bounded for $x \in (\varepsilon + 2, \infty)$. Passing to the limit as $\epsilon \to 0$ we obtain that $B(u) = [-2, 2]$. □

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