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VARIATIONAL METHODS FOR KIRCHHOFF TYPE PROBLEMS WITH TEMPERED FRACTIONAL DERIVATIVE

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Abstract. In this article, using variational methods, we study the existence of solutions for the Kirchhoff-type problem involving tempered fractional derivatives

$$M\Big(\int_{\mathbb{R}}|\mathbb{D}_{+}^{\alpha,\lambda}u(t)|^2dt\Big)\mathbb{D}_{-}^{\alpha,\lambda}(\mathbb{D}_{+}^{\alpha,\lambda}u(t))=f(t,u(t)),\quad t\in\mathbb{R},$$
$$u\in W_{\lambda}^{\alpha,2}(\mathbb{R}),$$

where $\mathbb{D}_{\pm}^{\alpha,\lambda}u(t)$ are the left and right tempered fractional derivatives of order $\alpha\in(1/2,1]$, $\lambda>0$, $W_{\lambda}^{\alpha,2}(\mathbb{R})$ represent the fractional Sobolev space, $f\in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $M\in C(\mathbb{R}^+,\mathbb{R}^+)$.

1. Introduction

Fractional calculus is a natural extension of ordinary calculus, where integrals and derivatives are defined for arbitrary real orders. Since 17th century, when fractional calculus was born, several kinds of fractional derivatives have been proposed. Examples include Riemann-Liouville, Hadamard, Grunwald-Letnikov, Caputo, tempered, etc. [4, 5, 9, 11, 12, 13, 14, 17, 19], each of them having its own advantages and disadvantages. The choice of an appropriate fractional derivative, depending on the system under consideration, has led to a variety of researches for fractional differential equations involving different fractional derivatives. For details and examples, we refer the reader to a series of papers [2, 3, 26, 27, 30, 31, 32, 33, 34, 35] and the references cited therein. One of the simplest description of a fractional derivative relies on Fourier transform. If $f(x)$ is a function with Fourier transform $\widehat{f}(w)$, then the Riemann-Liouville fractional derivative $D^\alpha f(x)$ is the function with Fourier transform $(iw)^\alpha\widehat{f}(w)$, which is an extension of familiar integer-order formula [9, 17]. The foregoing arguments have motivated the researchers to investigate the tempered fractional derivative $\mathbb{D}^{\alpha,\lambda}f(t)$, defined in terms of a function having Fourier transform $(\lambda+iw)^\alpha\widehat{u}(w)$ with the tempered fractional integral $\mathbb{I}^{\alpha,\lambda}f(t)$ as its inverse, having Fourier transform $(\lambda+iw)^{-\alpha}\widehat{f}(w)$ [4, 5, 12, 13, 19]. In this paper, we apply variational methods to establish the existence of infinitely many solutions

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to the following Kirchhoff-type problem involving tempered fractional derivatives:

$$
\begin{gathered}
M\Big(\int_{\mathbb{R}} |\mathbb{D}_{+}^{\alpha,\lambda} u(t)|^2 dt\Big) \mathbb{D}_{-}^{\alpha,\lambda}(\mathbb{D}_{+}^{\alpha,\lambda} u(t)) = f(t,u(t)), \quad t \in \mathbb{R}, \\
u \in W_{\lambda}^{\alpha,2}(\mathbb{R}),
\end{gathered} \tag{1.1}
$$

where $\mathbb{D}_{\pm}^{\alpha,\lambda} u(t)$ denote the left and right tempered fractional derivatives of order $\alpha \in (1/2, 1]$, $\lambda > 0$, $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $M \in C(\mathbb{R}^+,\mathbb{R}^+)$.

In recent years, there has been a growing interest in the study of fractional differential equations by means of variational methods and critical point theory. One of the pioneering works in this direction was due to Jiao and Zhou [7], who investigated the following fractional boundary value problem by using Mountain Pass Theorem,

$$
\begin{gathered}
\frac{d}{dt}\Big(\frac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(t))\Big) + \nabla F(t,u(t)) = 0, \quad \text{a.e. } t \in [0,T], \\
u(0) = u(T) = 0,
\end{gathered}
$$

where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \le \beta < 1$ respectively. For more examples, we refer the reader to a series of papers [8, 22, 23, 24, 28, 29, 15, 16, 36] and the references cited therein.

In the sequel, we need the following assumptions:

- (A1) $M \in C(\mathbb{R}^+,\mathbb{R}^+)$ and there exists $\Upsilon > 1$ such that $M(t)t \le \Upsilon \widehat{M}(t)$ for all $t \in [0,\infty)$, and for all $\delta > 0$ there exists $\varrho = \varrho(\delta) > 0$ such that $M(t) \ge \varrho$ for all $t \ge \delta$, where $\widehat{M}(t) = \int_0^t M(s)ds$;
- (A2) $M \in C(\mathbb{R}^+,\mathbb{R}^+)$ and there exist three constants $0 < m_1 \le m_2 < \infty$ and $1 < \beta < \infty$ such that
$$
m_1 t^{\beta} \le \widehat{M}(t) \le m_2 t^{\beta}, \quad \forall t \in \mathbb{R}^+; \tag{1.2}
$$
- (A3) $f(t,u) = o(|u|)$ as $|u| \to 0$ uniformly for $t \in \mathbb{R}$,
- (A4) $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ such that there exist $b \in C(\mathbb{R},\mathbb{R}^+)$ with $\lim_{t\to+\infty} b(t) = 0$ and $2 < q < +\infty$ such that
$$
f(t,s) \le b(t)|s|^{q-1}, \quad \forall (t,s) \in \mathbb{R}\times\mathbb{R};
$$
- (A5) There exist $\mu > 2\Upsilon$ with $\Upsilon > 1$ such that
$$
0 < \mu F(t,\zeta) \le \zeta f(t,\zeta), \quad \forall \zeta > 0,
$$
where $F(t,u) = \int_0^u f(t,s)ds$;
- (A6) $\lim_{\zeta\to 0} \frac{F(t,\zeta)}{|\zeta|^{2\Upsilon}} = 0$ uniformly for a.e. $t \in \mathbb{R}$;
- (A7) There exist two constants $b_1 > 0$, $1 < \gamma_0 < 2$ such that
$$
F(t,s) \ge b_1|s|^{\gamma_0}, \quad \forall (t,s) \in \mathbb{R}\times\mathbb{R};
$$
- (A8) f is odd in x, i.e. $f(t,-x) = -f(t,x)$, $\forall (t,x) \in \mathbb{R}\times\mathbb{R}$.

The rest of the paper is organized as follows. In Section 2, we describe some basic concepts related to our main results (Theorems 3.7–3.12). In Section 3, the existence of infinitely many solutions to the problem (1.1) is established.

2. Preliminaries

Let us recall some basic definitions and lemmas that we need in the forthcoming analysis.

Definition 2.1. For any $\lambda > 0$, we define the positive tempered fractional integral of a function $f \in L^p(\mathbb{R})$ with $1 \le p < \infty$ as

$$\mathbb{I}_{+}^{\alpha,\lambda} f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} f(\xi)(x-\xi)^{\alpha-1} e^{-\lambda(x-\xi)} d\xi, \tag{2.1}$$

and the negative tempered fractional integral by

$$\mathbb{I}_{-}^{\alpha,\lambda} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{+\infty} f(\xi)(\xi-x)^{\alpha-1} e^{-\lambda(\xi-x)} d\xi. \tag{2.2}$$

If $\lambda = 0$, these formulae reduce to the well-known Riemann-Liouville fractional integrals [9, 14].

Definition 2.2. The positive and negative tempered fractional derivatives of order $0 < \alpha < 1$ for a function $f : \mathbb{R} \to \mathbb{R}$ are defined by

$$\mathbb{D}_{+}^{\alpha,\lambda} f(x) = \lambda^{\alpha} f(x) + \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{f(x)-f(\xi)}{(x-\xi)^{\alpha+1}} e^{-\lambda(x-\xi)} d\xi, \tag{2.3}$$

$$\mathbb{D}_{-}^{\alpha,\lambda} f(x) = \lambda^{\alpha} f(x) + \frac{\alpha}{\Gamma(1-\alpha)} \int_{x}^{+\infty} \frac{f(x)-f(\xi)}{(\xi-x)^{\alpha+1}} e^{-\lambda(\xi-x)} d\xi, \tag{2.4}$$

for any $\lambda > 0$.

Define the fractional space

$$W_{\lambda}^{\alpha,2}(\mathbb{R}) = \Big\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (\lambda^2 + \omega^2)^{\alpha} |\widehat{f}(\omega)|^2 d\omega < \infty \Big\}, \tag{2.5}$$

which is a Banach space with the norm

$$\|f\|_{\alpha,\lambda} = \Big(\int_{\mathbb{R}} (\lambda^2 + \omega^2)^{\alpha} |\widehat{f}(\omega)|^2 d\omega \Big)^{1/2}. \tag{2.6}$$

For any $f \in W_{\lambda}^{\alpha,2}(\mathbb{R})$, let $\mathbb{D}_{\pm}^{\alpha,\lambda} f(x)$ denote the functions with Fourier transform $(\lambda \pm i\omega)^{\alpha} \widehat{f}(\omega)$ ([20]), where the Fourier transform of $u(x)$ is defined as follows

$$\mathcal{F}(u)(\xi) = \int_{-\infty}^{\infty} e^{-ix\cdot\xi} u(x) dx.$$

Now we state the following known results.

Lemma 2.3 (See [20])**.** (i) *For any $\alpha, \lambda > 0$ and $f \in L^2(\mathbb{R})$, we have*

$$\mathbb{D}_{\pm}^{\alpha,\lambda} \mathbb{I}_{\pm}^{\alpha,\lambda} f(x) = f(x), \tag{2.7}$$

and for any $f \in W_{\lambda}^{\alpha,2}(\mathbb{R})$, we have

$$\mathbb{I}_{\pm}^{\alpha,\lambda} \mathbb{D}_{\pm}^{\alpha,\lambda} f(x) = f(x). \tag{2.8}$$

(ii) *For any $\alpha, \lambda > 0$ and $f, g \in W_{\lambda}^{\alpha,2}(\mathbb{R})$, we have*

$$\langle f, \mathbb{D}_{+}^{\alpha,\lambda} g \rangle_{L^2(\mathbb{R})} = \langle \mathbb{D}_{-}^{\alpha,\lambda} f, g \rangle_{L^2(\mathbb{R})}. \tag{2.9}$$

Lemma 2.4 (See [12]). (i) *For any $\alpha, \lambda > 0$ and $p \geq 1$, $\mathbb{I}_{\pm}^{\alpha,\lambda} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ are bounded linear operators with*

$$\|\mathbb{I}_{\pm}^{\alpha,\lambda} f\|_{L^p(\mathbb{R})} \leq \lambda^{-\alpha} \|f\|_{L^p(\mathbb{R})}. \tag{2.10}$$

(ii) *For any $\alpha, \beta, \lambda > 0$ and $f \in L^p(\mathbb{R})$, we have*

$$\mathbb{I}_{\pm}^{\alpha,\lambda} \mathbb{I}_{\pm}^{\beta,\lambda} f(x) = \mathbb{I}_{\pm}^{\alpha+\beta,\lambda} f(x). \tag{2.11}$$

(iii) *For any $\alpha, \lambda > 0$ and $f, g \in L^2(\mathbb{R})$, we have*

$$\langle f, \mathbb{I}_{+}^{\alpha,\lambda} g \rangle_{L^2(\mathbb{R})} = \langle \mathbb{I}_{-}^{\alpha,\lambda} f, g \rangle_{L^2(\mathbb{R})}. \tag{2.12}$$

Next, for $0 < \alpha < 1$, we define fractional Sobolev space $H^\alpha(\mathbb{R})$ as follows

$$H^\alpha(\mathbb{R}) = \overline{C_0^\infty(\mathbb{R})}^{\|\cdot\|_\alpha},$$

endowed with the norm

$$\|u\|_\alpha = \Big(\int_{\mathbb{R}} |u(t)|^2 dt + \int_{\mathbb{R}} |\omega|^{2\alpha} |\widehat{u}(\omega)|^2 d\omega\Big)^{1/2}. \tag{2.13}$$

For $0 < \alpha < 1$, we have

$$2^{\frac{\alpha-1}{2}} \|u\|_\alpha \leq \|u\|_{\alpha,1} \leq \|u\|_\alpha, \tag{2.14}$$

$$\|u\|_{\alpha,1} \leq \|u\|_{\alpha,\lambda} \leq \lambda^\alpha \|u\|_{\alpha,1}, \tag{2.15}$$

$$\|u\|_{\alpha,\lambda} \leq \|u\|_{\alpha,1} \leq \lambda^{-\alpha} \|u\|_{\alpha,\lambda}, \tag{2.16}$$

where $\|u\|_{\alpha,1}$ is the norm on $W_1^{\alpha,2}(\mathbb{R})$ and so $W_1^{\alpha,2}(\mathbb{R}) = H^\alpha(\mathbb{R})$ with equivalent norms.

Lemma 2.5 (See [1]). *Let $\alpha > 1/2$. Then any $u \in W_\lambda^{\alpha,2}(\mathbb{R})$ is uniformly continuous, bounded and there exists a constant $C = C_\alpha$ such that*

$$\sup_{t \in \mathbb{R}} |u(t)| \leq C \|u\|_{\alpha,\lambda}. \tag{2.17}$$

Remark 2.6. From Lemma 2.5 and (2.13)-(2.15), we have the following implication: if $u \in W_\lambda^{\alpha,2}$ with $\frac{1}{2} < \alpha < 1$, then $u \in L^q(\mathbb{R})$ for all $q \in [2, \infty)$ as

$$\int_{\mathbb{R}} |u(t)|^q dt \leq \|u\|_\infty^{q-2} \|u\|_{L^2(\mathbb{R})}^2 \leq 2^{1-\alpha} C^{q-2} \|u\|_{\alpha,\lambda}^q.$$

Remark 2.7 ([21]). The imbedding of $W_\lambda^{\alpha,2}$ in $L^q(-T,T)$ is compact for $q \in (2, \infty)$ and any $T > 0$.

3. Main results

Definition 3.1. For every $u, v \in W_\lambda^{\alpha,2}(\mathbb{R})$, a weak solution of problem (1.1) is

$$M\Big(\int_{\mathbb{R}} |\mathbb{D}_{+}^{\alpha,\lambda} u(t)|^2 dt\Big) \int_{\mathbb{R}} (\lambda^2 + \omega^2)^\alpha \widehat{u}(\omega) \overline{\widehat{v}(\omega)} d\omega = \int_{\mathbb{R}} f(t, u(t)) v(t) dt,$$

that is,

$$M\Big(\int_{\mathbb{R}} |\mathbb{D}_{+}^{\alpha,\lambda} u(t)|^2 dt\Big) \int_{\mathbb{R}} \mathbb{D}_{+}^{\alpha,\lambda} u(t) \mathbb{D}_{+}^{\alpha,\lambda} v(t) dt = \int_{\mathbb{R}} f(t, u(t)) v(t) dt.$$

Definition 3.2. We say that the functional Φ satisfies the Palais-Smale condition if any sequence $\{u_n\}_{n \in \mathbb{N}} \subset X$ has a convergent subsequence provided $\{\Phi(u_n)\}_{n \in \mathbb{N}}$ is bounded and $\Phi'(u_n) \to 0$ as $n \to +\infty$.

Theorem 3.3 ([6, Theorem 2.2]). *Let X be a real infinite dimensional Banach space and $K \in C^1(X)$ be a functional satisfying the Palais-Smale condition and that*

(i) *$K(0) = 0$ and there exist two constants $\widetilde{\alpha} > 0$ and $\rho > 0$ such that $\varphi|_{\partial B_\rho} \geq \widetilde{\alpha}$, where $B_\rho = \{u \in X : \|u\| < \rho\}$;*
(ii) *K is even;*
(iii) *for all finite dimensional subspace $\widetilde{X} \subset X$, there is $R = R(\widetilde{X})$ such that $\varphi(u) \leq 0$ on $X \setminus B_{R(\widetilde{X})}$.*

Then the functional K possess an unbounded sequence of critical values characterized by a minimax argument.

Consider a functional $\Phi : W_\lambda^{\alpha,2}(\mathbb{R}) \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{2}\widehat{M}\Big(\int_{\mathbb{R}} |\mathbb{D}_+^{\alpha,\lambda} u(t)|^2 dt\Big) - \int_{\mathbb{R}} F(t, u(t))dt, \quad \forall u \in W_\lambda^{\alpha,2}(\mathbb{R}), \tag{3.1}$$

where $\widehat{M}(t) = \int_0^t M(s)ds$. Obviously, by the conditions (A1) (or (A2)) and (A4), $\Phi \in C^1(W_\lambda^{\alpha,2}(\mathbb{R}), \mathbb{R})$, and

$$\begin{aligned}\Phi'(u)v = M\Big(\int_{\mathbb{R}} |\mathbb{D}_+^{\alpha,\lambda} u(t)|^2 dt\Big) \int_{\mathbb{R}} \Big(\mathbb{D}_+^{\alpha,\lambda} u(t) \mathbb{D}_+^{\alpha,\lambda} v(t)\Big) dt \\ - \int_{\mathbb{R}} f(t, u(t))v(t)dt,\end{aligned} \tag{3.2}$$

for every $u, v \in W_\lambda^{\alpha,2}(\mathbb{R})$.

Lemma 3.4. *Assume that* (A1), (A4)-(A6) *hold. Then there exist $\rho, \beta > 0$ such that*

$$\Phi(u) \geq \beta, \quad \forall u \in W_\lambda^{\alpha,2}(\mathbb{R}), \|u\|_{\alpha,\lambda} = \rho.$$

Proof. In view of (A6), for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|F(t, u)| \leq \varepsilon |u|^{2\Upsilon}, \quad \forall\, (t, u) \in \mathbb{R} \times [0, \delta). \tag{3.3}$$

Also, by (A4), for $u > \delta$, there is a $T > 0$ such that

$$|F(t, u)| \leq \varepsilon |u|^q, \tag{3.4}$$

for $|t| > T$. Set $b_T := \max_{t \in [-T,T]} b(t)$, then we have

$$|F(t, u)| \leq b_\varepsilon |u|^q, \tag{3.5}$$

for $u > \delta$, where $b_\varepsilon = max\{b_T, \varepsilon\}$.

As $F(t, \cdot)$ is even, it follows by (3.3) and (3.5) that for all $\varepsilon > 0$, there is a $b_\varepsilon > 0$ such that

$$|F(t, u)| \leq \varepsilon |u|^{2\Upsilon} + b_\varepsilon |u|^q, \quad \text{for all } (t, u) \in \mathbb{R} \times \mathbb{R}. \tag{3.6}$$

Moreover, from (2.6) and (A1), we have $M(t) > 0$ for all $t > 0$ and

$$\widehat{M}(t) \geq \widehat{M}(1) t^\Upsilon, \quad \text{for all } t \in [0, 1]. \tag{3.7}$$

From (3.6) and (3.7), for all $u \in W_\lambda^{\alpha,2}(\mathbb{R})$ with $\|u\|_{\alpha,\lambda} \leq 1$, we get

$$\begin{aligned}\Phi(u) &\geq \frac{1}{2}\widehat{M}\big(\int_{\mathbb{R}} |\mathbb{D}_+^{\alpha,\lambda} u(t)|^2 dt\big) - \varepsilon \int_{\mathbb{R}} |u(t)|^{2\Upsilon} dt - a_\varepsilon \int_{\mathbb{R}} |u(t)|^q dt \\ &\geq \frac{1}{2}\widehat{M}(1)\|u\|_{\alpha,\lambda}^{2\Upsilon} - \varepsilon 2^{1-\alpha} C^{2\Upsilon-2} \|u\|_{\alpha,\lambda}^{2\Upsilon} - a_\varepsilon 2^{1-\alpha} C^{q-2} \|u\|_{\alpha,\lambda}^q.\end{aligned} \tag{3.8}$$

Choosing
$$\varepsilon := \frac{\widehat{M}(1)}{42^{1-\alpha}C^{2\Upsilon-2}},$$
one obtains
$$\begin{aligned}\Phi(u) &\geq \frac{\widehat{M}(1)}{4}\|u\|_{\alpha,\lambda}^{2\Upsilon} - a_\varepsilon 2^{1-\alpha}C^{q-2}\|u\|_{\alpha,\lambda}^{q}\\ &\geq \|u\|_{\alpha,\lambda}^{2\Upsilon}\Big(\frac{\widehat{M}(1)}{4} - a_\varepsilon 2^{1-\alpha}C^{q-2}\|u\|_{\alpha,\lambda}^{q-2\Upsilon}\Big).\end{aligned}$$
Hence, for all $u \in W_\lambda^{\alpha,2}(\mathbb{R})$ with $\|u\|_{\alpha,\lambda} = \rho$ and $0 < \rho < 1$ small enough, we have
$$\frac{\widehat{M}(1)}{4} - a_\varepsilon 2^{1-\alpha}C^{q-2}\rho^{q-2\Upsilon} > 0.$$
Therefore, by taking
$$\beta := \rho^{2\Upsilon}\Big(\frac{\widehat{M}(1)}{4} - a_\varepsilon 2^{1-\alpha}C^{q-2}\rho^{q-2\Upsilon}\Big),$$
we get $\Phi(u) \geq \beta$ for all $u \in W_\lambda^{\alpha,2}(\mathbb{R})$, $\|u\|_{\alpha,\lambda} = \rho$. Thus the conclusion is achieved. $\square$

Lemma 3.5. *Assume that* (A1), (A3)–(A6) *hold. Then, for any finite dimensional subspace E of $W_\lambda^{\alpha,2}$, there exists $R_1 = R_1(E) > 0$ such that*
$$\Phi(u) \leq 0, \quad \forall u \in W_\lambda^{\alpha,2}(\mathbb{R}) \setminus B_{R_1(E)},$$
where $B_{R_1(E)} = \{u \in W_\lambda^{\alpha,2} : \|u\|_{\alpha,\lambda} < R_1\}$.

Proof. In a straightforward manner, one can obtain $F(t,\zeta) \geq K|\zeta|^\mu$, where
$$K := \frac{1}{r^\mu}\inf_{\zeta\in\mathbb{R},|\zeta|=r} F(t,\zeta) > 0.$$
Then, by (A3)–(A5), there exists $M > 0$ such that
$$F(t,\zeta) \geq K|\zeta|^\mu - M|\zeta|^2, \quad \text{for all } (t,\zeta) \in \mathbb{R}\times\mathbb{R}. \tag{3.9}$$
Also, by assumption (A1), we have
$$\widehat{M}(t) \leq \widehat{M}(1)t^\Upsilon. \tag{3.10}$$
Let $E \subset W_\lambda^{\alpha,2}(\mathbb{R})$ be a fixed finite dimensional. Now, for any $u \in E$ with $\|u\|_{\alpha,\lambda} = 1$, by Remark 2.6, (3.9) and (3.10), we have
$$\begin{aligned}\Phi(su) &= \frac{1}{2}\widehat{M}(s^2) - \int_{\mathbb{R}} F(t,su(t))dt\\ &\leq \frac{1}{2}s^{2\Upsilon}\widehat{M}(1) - Ks^\mu\int_{\mathbb{R}}|u(t)|^\mu dt + M\int_{\mathbb{R}}|u(t)|^2dt\\ &\leq \frac{1}{2}s^{2\Upsilon}\widehat{M}(1) - Ks^\mu M_E^\mu\|u\|_{\alpha,\lambda}^\mu + M2^{1-\alpha}\|u\|_{\alpha,\lambda}^2\\ &= \frac{1}{2}s^{2\Upsilon}\widehat{M}(1) - Ks^\mu M_E^\mu + M2^{1-\alpha} \to -\infty, \quad \text{as } s\to\infty,\end{aligned}$$
where $M_E > 0$ such that $\|u\|_{L^p} \geq M_E\|u\|_{\alpha,\lambda}$ for all $u \in E$. As $R_1 \to \infty$, we have
$$\sup_{u\in E,\|u\|_{\alpha,\lambda}=R_1}\Phi(u) = \sup_{u\in E,\|u\|_{\alpha,\lambda}=1}\Phi(R_1u) \to -\infty.$$

Therefore, there exists $R_0 > 0$ large enough such that $\Phi(u) \le 0$ for all $u \in E$ with $\|u\|_{\alpha,\lambda} = R_1$ and $R_1 \ge R_0$. This completes the proof. □

Lemma 3.6. *Assume that* (A1), (A4), (A5) *hold. Then* Φ *satisfies Palais-Smale condition.*

Proof. Assume that $\{u_n\}_{n\in\mathbf{N}} \subset W_\lambda^{\alpha,2}(\mathbb{R})$ is a sequence such that $\{\Phi(u_n)\}_{n\in\mathbb{N}}$ is bounded and $\Phi'(u_n) \to 0$ as $n \to \infty$. Then there exists a constant $D > 0$ such that

$$|\Phi(u_n)| \le D \quad \text{and} \quad \|\Phi'(u_n)\|_{(W_\lambda^{\alpha,2}(\mathbb{R}))^*} \le D, \tag{3.11}$$

for any $n \in \mathbb{N}$, where $(W_\lambda^{\alpha,2}(\mathbb{R}))^*$ is the dual space of $W_\lambda^{\alpha,2}(\mathbb{R})$.

Firstly, we show that $\{u_n\}_{n\in\mathbf{N}}$ is bounded. Without loss of generality, we assume that $\inf_n \|u_n\|_{\alpha,\lambda} = \eta > 0$, denote by $\varrho = \varrho(\eta)$ the number corresponding to $\delta = \eta^2$ in (A1) such that

$$M(\|u_n\|^2_{\alpha,\lambda}) \ge \varrho \quad \text{for all } n. \tag{3.12}$$

In view of (A5) and (3.12), one gets

$$\begin{aligned}
D + D\|u_n\|_{\alpha,\lambda} &\ge \Phi(u_n) - \frac{1}{\mu}\Phi'(u_n)u_n \\
&= \frac{1}{2}\widehat{M}(\|u_n\|^2_{\alpha,\lambda}) - \frac{1}{\mu}M(\|u_n\|^2_{\alpha,\lambda})\|u_n\|^2_{\alpha,\lambda} \\
&\quad - \frac{1}{\mu}\int_{\mathbb{R}} (\mu F(t,u_n(t)) - f(t,u_n(t))u_n(t))dt \\
&\ge \big(\frac{1}{2\Upsilon} - \frac{1}{\mu}\big)M(\|u_n\|^2_{\alpha,\lambda})\|u_n\|^2_{\alpha,\lambda} \\
&\ge \varrho\big(\frac{1}{2\Upsilon} - \frac{1}{\mu}\big)\|u_n\|^2_{\alpha,\lambda}.
\end{aligned}$$

Since $\mu > 2\Upsilon$, the boundedness of $\{u_n\}_{n\in\mathbf{N}}$ follows directly. So, there exist a subsequence $\{u_n\}_{n\in\mathbf{N}}$, and $u \in W_\lambda^{\alpha,2}$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_\lambda^{\alpha,2}(\mathbb{R}), \tag{3.13}$$

which yields

$$\begin{aligned}
\Phi'(u_n)(u_n - u) &= M(\|u_n\|^2_{\alpha,\lambda})\int_{\mathbb{R}} \Big(\mathbb{D}_+^{\alpha,\lambda}u_n \mathbb{D}_+^{\alpha,\lambda}(u_n - u)\Big)dt \\
&\quad - \int_{\mathbb{R}} f(t,u_n)(u_n - u)dt \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.14}$$

Now we show that $\lim_{n\to\infty}\int_{\mathbb{R}} f(t,u_n)(u_n - u)dt = 0$. To this end, by (3.13), there is some constant $d > 0$ such that

$$\begin{gathered}
\|u_n\|_{\alpha,\lambda} < d \quad \text{and} \quad \|u\|_{\alpha,\lambda} < d, \quad \text{for } n \in \mathbb{N}, \\
u_n \to u \quad \text{strongly in } L^q(\mathbb{R}) \text{ and a.e. in } \mathbb{R}.
\end{gathered}$$

Moreover, for any $\varepsilon > 0$, (A4) implies that there exists $T > 0$ such that

$$f(t,u_n) \le \varepsilon|u_n|^{q-1}, \quad \text{for } |t| > T. \tag{3.15}$$

Then, for n large enough, from (2.17), Remark 2.6 and Young inequality, we obtain

$$\Big|\int_{\mathbb{R}} f(t,u_n)(u_n - u)dt\Big|$$

$$
\begin{aligned}
&\le \int_{\mathbb{R}} |f(t,u_n)||u_n-u|dt \\
&\le \int_{-T}^{T} |f(t,u_n)||u_n-u|dt + \int_{|t|>T} |f(t,u_n)||u_n-u|dt \\
&\le \varepsilon\|u_n\|_\infty + \varepsilon \int_{|t|>T} |u_n|^{q-1}|u_n-u|dt \\
&\le \varepsilon C\|u_n\|_{\alpha,\lambda} + \varepsilon \int_{|t|>T} \Big(\frac{q-1}{q}|u_n|^q + \frac{1}{\mu}|u_n-u|^q\Big)dt \\
&\le \varepsilon C\|u_n\|_{\alpha,\lambda} + \frac{q-1}{q}\varepsilon 2^{1-\alpha}C^{q-2}\|u_n\|^q_{\alpha,\lambda} + \varepsilon\frac{1}{\mu}2^{1-\alpha}C^{q-2}\|u_n-u\|^q_{\alpha,\lambda} \\
&\le \varepsilon Cd + \frac{q-1}{q}\varepsilon 2^{1-\alpha}C^{q-2}d^q + \varepsilon\frac{1}{\mu}2^{1-\alpha}C^{q-2}\|u_n-u\|^q_{\alpha,\lambda}.
\end{aligned}
$$

Then

$$
\lim_{n\to\infty}\int_{\mathbb{R}} f(t,u_n)(u_n-u)dt = 0.
$$

Therefore, by (3.14), we have

$$
M(\|u_n\|^2_{\alpha,\lambda})\int_{\mathbb{R}}\Big(\mathbb{D}^{\alpha,\lambda}_{+}u_n\mathbb{D}^{\alpha,\lambda}_{+}(u_n-u)\Big)dt \to 0, \quad \text{as } n\to\infty.
$$

Thus, by (3.12) and the boundedness of $M(\|u_n\|^2_{\alpha,\lambda})$, one can get

$$
\int_{\mathbb{R}}\Big(\mathbb{D}^{\alpha,\lambda}_{+}u_n\mathbb{D}^{\alpha,\lambda}_{+}(u_n-u)\Big)dt \to 0, \quad \text{as } n\to\infty. \tag{3.16}
$$

In a similar manner, we can get

$$
\int_{\mathbb{R}}\Big(\mathbb{D}^{\alpha,\lambda}_{+}u\mathbb{D}^{\alpha,\lambda}_{+}(u_n-u)\Big)dt \to 0, \quad \text{as } n\to\infty. \tag{3.17}
$$

Combining (3.16) and (3.17), we obtain

$$
\int_{\mathbb{R}}\Big(\mathbb{D}^{\alpha,\lambda}_{+}(u_n-u)\mathbb{D}^{\alpha,\lambda}_{+}(u_n-u)\Big)dt \to 0, \quad \text{as } n\to\infty.
$$

Hence, $\|u_n-u\|_{\alpha,\lambda}\to 0$ as $n\to\infty$ and then Φ satisfies Palais-Smale condition. □

Theorem 3.7. *Assume that* (A1), (A3)–(A6), (A8) *hold. Then problem* (1.1) *has infinitely many nontrivial solutions.*

Proof. Assumption (A8) implies that $F(t,\cdot)$ is even in $\mathbb{R}$ and so is Φ. Since $\Phi(0)=0$, it follows from Lemmas 3.4-3.6 and Theorem 3.3 that there exists an unbounded sequence of weak solutions of problem (1.1). □

To prove our second result, we will use the genus properties. So we recall the following definitions and results (see [18]). Let X be a Banach space, $g\in C^1(X,\mathbb{R})$ and $c\in\mathbb{R}$. Set

$$
\begin{gathered}
\Sigma = \{A\subset X\setminus\{0\} : A \text{ is closed in } X \text{ and symmetric with respect to } 0)\},\\
K_c = \{x\in X : g(x)=c,\ g'(x)=0\},\\
g^c = \{x\in X : g(x)\le c\}.
\end{gathered}
$$

Definition 3.8 ([10])**.** For $A \in \Sigma$, we say genus of A is j (denoted by $\gamma(A) = j$) if there is an odd map $\psi \in C(A, \mathbb{R}^j \setminus \{0\})$, and j is the smallest integer with this property.

Theorem 3.9. *Let g be an even C^1 functional on X which satisfies the Palais-Smale condition. For $j \in \mathbb{N}$, let*

$$\Sigma_j = \{A \in \Sigma : \ \gamma(A) \geq j\}, \quad c_j = \inf_{A \in \Sigma_j} \sup_{u \in A} g(u).$$

(i) If $\Sigma_j \neq \emptyset$ and $c_j \in \mathbb{R}$, then c_j is a critical value of g.

(ii) If there exists $r \in \mathbb{N}$ such that $c_j = c_{j+1} = \cdots = c_{j+r} = c \in \mathbb{R}$ and $c \neq g(0)$, then $\gamma(K_c) \geq r + 1$.

Lemma 3.10. *Assume that* (A1) *and* (A4) *hold. Then Φ is bounded from below and satisfies the Palais-Smale condition.*

Proof. By a method similar to the one in [25, Lemma 3.3], for all $\varepsilon > 0$, it follows from (A4) that

$$|F(t, u(t))| \leq \varepsilon |u(t)|^2, \quad \text{for all } \ t \in \mathbb{R}. \tag{3.18}$$

For any $u \in W^{\alpha,2}_\lambda(\mathbb{R})$, by (3.18), we get

$$\begin{aligned} \Phi(u) &\geq \frac{1}{2}\widehat{M}(\|u\|^2_{\alpha,\lambda}) - \int_{\mathbb{R}} F(t, u(t))dt \\ &\geq \frac{1}{2}\widehat{M}(\|u\|^2_{\alpha,\lambda}) - \varepsilon \int_{\mathbb{R}} |u(t)|^2 dt \\ &\geq \frac{1}{2}\widehat{M}(\|u\|^2_{\alpha,\lambda}) - \varepsilon \|u\|^2_{\alpha,\lambda}. \end{aligned}$$

If $\|u\|_{\alpha,\lambda} \leq 1$, then by (3.7), we have

$$\Phi(u) \geq \frac{\widehat{M}(1)}{2} \|u\|^{2\Upsilon}_{\alpha,\lambda} - \varepsilon \|u\|^2_{\alpha,\lambda} \geq -\varepsilon. \tag{3.19}$$

If $\|u\|_{\alpha,\lambda} > 1$, then by (A1) and $\varepsilon = \frac{\varrho}{4}$, we get

$$\Phi(u) \geq \frac{\varrho}{2} \|u\|^2_{\alpha,\lambda} - \varepsilon \|u\|^2_{\alpha,\lambda} \geq \frac{\varrho}{4} \|u\|^2_{\alpha,\lambda}. \tag{3.20}$$

Combining (3.19) and (3.20), one can infer that Φ is coercive. Thus Φ is bounded from below and satisfies the Palais-Smale condition. $\square$

Theorem 3.11. *Assume that* (A1), (A4), (A7), (A8) *hold. Then problem* (1.1) *has infinitely many nontrivial solutions.*

Proof. The assumption (A8) implies that Φ is even and $\Phi(0) = 0$, and by Lemma 3.10, $J \in C^1(X^\alpha, \mathbb{R})$ is bounded from below and satisfies the Palais-Smale condition. We make use of Theorem 3.9 to complete the proof. First, we show that there exists $\varepsilon > 0$ such that

$$\gamma(\Phi^{-\varepsilon}) \geq n \quad \text{for any } n \in \mathbb{N}. \tag{3.21}$$

For each k, we take k disjoint open sets K_i such that $\cup_{i=1}^k K_i \subset \mathbb{R}$. For $i = 1, \dots, k$, letting $u_i \in (W^{\alpha,2}_\lambda(\mathbb{R}) \cap C_0^\infty(K_i)) \setminus \{0\}$ with $\|u_i\|_{\alpha,\lambda} = 1$, we set

$$X^\alpha_n = \operatorname{span}\{u_1, \dots, u_n\}, \quad S_n = \{u \in X^\alpha_n : \|u\|_{\alpha,\lambda} = 1\}. \tag{3.22}$$

For an $u \in X_n^\alpha$, we can write

$$u(t) = \sum_{i=1}^{n} \lambda_i u_i(t) \quad \text{for } t \in \mathbb{R}, \tag{3.23}$$

for some $\lambda_i \in \mathbb{R}$, $i = 1, 2, \dots, n$. So

$$\|u\|_{L^{\gamma_0}} = \Big(\int_{\mathbb{R}} |u(t)|^{\gamma_0} dt \Big)^{1/\gamma_0} = \Big(\sum_{i=1}^{n} |\lambda_i|^{\gamma_0} \int_{\mathbb{R}} |u_i(t)|^{\gamma_0} dt \Big)^{1/\gamma_0}, \tag{3.24}$$

and

$$\begin{aligned} \|u\|_{\alpha,\lambda}^2 &= \int_{\mathbb{R}} |\mathbb{D}_+^{\alpha,\lambda} u(t)|^2 dt = \sum_{i=1}^{n} \lambda_i^2 \int_{\mathbb{R}} |\mathbb{D}_+^{\alpha,\lambda} u_i(t)|^2 dt \\ &= \sum_{i=1}^{n} \lambda_i^2 \|u_i\|_{\alpha,\lambda}^2 = \sum_{i=1}^{n} \lambda_i^2. \end{aligned} \tag{3.25}$$

Since all norms of a finite dimensional normed space are equivalent, there exists a constant $\Theta > 0$ such that

$$\Theta \|u\|_{\alpha,\lambda} \le \|u\|_{L^{\gamma_0}} \quad \text{for } u \in X_n^\alpha. \tag{3.26}$$

From (A7), for $u \in S_n$, we can take some Λ_0 such that

$$\int_{\mathbb{R}} F(t, u(t)) dt = \int_{\mathbb{R}} F\Big(t, \sum_{i=1}^{n} \lambda_i u_i(t)\Big) dt \ge b_1 \int_{\Lambda_0} \Big| \sum_{i=1}^{n} \lambda_i u_i(t) \Big|^{\gamma_0} dt := \varrho. \tag{3.27}$$

We claim that $\varrho > 0$. To this end, suppose otherwise, for any bounded open set $\Lambda \subset \mathbb{R}$, there exists $\{u_k\}_{k\in\mathbb{N}} \in S_n$ such that

$$\int_{\Lambda} |u_k(t)|^{\gamma_0} dt = b_1 \int_{\Lambda} \Big| \sum_{i=1}^{n} \lambda_{ik} u_i(t) \Big|^{\gamma_0} dt \to 0,$$

as $k \to +\infty$, where $u_k = \sum_{i=1}^{n} \lambda_{ik} u_i(t)$ with $\sum_{i=1}^{n} \lambda_{ik}^2 = 1$. Then we have

$$\lim_{k\to+\infty} \lambda_{ik} := \lambda_{i0} \in [0,1] \quad \text{and} \quad \sum_{i=1}^{n} \lambda_{i0}^2 = 1.$$

Thus, for any bounded open set $\Lambda \subset \mathbb{R}$, we get

$$\int_{\Lambda} \Big| \sum_{i=1}^{n} \lambda_{i0} u_i(t) \Big|^{\gamma_0} dt = 0.$$

Since Λ is arbitrary, therefore $u_0 = \sum_{i=1}^{n} \lambda_{i0} u_i(t) = 0$ a.e. on $\mathbb{R}$, which contradicts that $\|u_0\|_{X^\alpha} = 1$. Hence

$$\int_{\mathbb{R}} F(t, u(t)) dt = \int_{\mathbb{R}} F\Big(t, \sum_{i=1}^{n} \lambda_i u_i(t)\Big) dt \ge b_1 \int_{\Lambda_0} \Big| \sum_{i=1}^{n} \lambda_i u_i(t) \Big|^{\gamma_0} dt = \varrho > 0. \tag{3.28}$$

From (A7), (3.24)-(3.26) and (3.28), we have

$$\begin{aligned}\Phi(su) &= \frac{1}{2}\widehat{M}(\|su\|^2_{\alpha,\lambda}) - \int_{\mathbb{R}} F(t,su(t))dt \\ &\le \frac{1}{2}\max_{0\le l\le 1} M(l)s^2 - \sum_{i=1}^n \int_{K_i} F(t,su_i(t))dt \\ &\le \frac{1}{2}\max_{0\le l\le 1} M(l)s^2 - b_1 s^{\gamma_0}\sum_{i=1}^n |\lambda_i|^{\gamma_0}\int_{I_0} |u_i(t)|^{\gamma_0}dt \\ &\le \frac{1}{2}\max_{0\le l\le 1} M(l)s^2 - b_1 s^{\gamma_0}\|u\|^{\gamma_0}_{L^{\gamma_0}} \\ &\le \frac{1}{2}\max_{0\le l\le 1} M(l)s^2 - b_1(\Theta s)^{\gamma_0}\|u\|^{\gamma_0}_{\alpha,\lambda} \\ &\le \frac{1}{2}\max_{0\le l\le 1} M(l)s^2 - b_1(\Theta s)^{\gamma_0}, \quad \forall u\in S_n,\ 0<s<\delta,\end{aligned} \tag{3.29}$$

which implies that there exist $\varepsilon > 0$ and $\sigma > 0$ such that

$$\Phi(\sigma u) < -\varepsilon \quad \forall u \in S_n. \tag{3.30}$$

Let

$$S_n^\sigma = \{\sigma u : u \in S_n\}, \quad \Omega = \big\{(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n : \sum_{i=1}^n \lambda_i^2 < \sigma^2\big\}.$$

Then it follows from (3.30) that

$$\Phi(u) < -\varepsilon \quad \forall u \in S_n^\sigma. \tag{3.31}$$

So, by (3.31) and the fact that $\Phi \in C^1(W_\lambda^{\alpha,2}(\mathbb{R}),\mathbb{R})$ and is even, we get

$$S_n^\sigma \subset \Phi^{-\varepsilon} \in \Sigma. \tag{3.32}$$

On the other hand, in view of (3.23) and (3.25), there exists an odd homeomorphism mapping $\Psi \in C(S_n^\sigma, \partial\Omega)$. Using properties of the genus (see [18, 3° of Propositions 7.5 and 7.7]), one can obtain

$$\gamma(\Phi^{-\varepsilon}) \ge \gamma(S_n^\sigma) = n. \tag{3.33}$$

Hence (3.21) is obtained. Set

$$c_n = \inf_{A\in\Sigma_n}\sup_{u\in A}\Phi(u).$$

As Φ is bounded from below on X^α and (3.33) implies that $-\infty < c_n \le -\varepsilon < 0$, therefore c_n (for all $n \in \mathbb{N}$) is a real negative number. Thus it follows from Theorem 3.9 that Φ has infinitely many nontrivial critical points, which correspond to infinitely many nontrivial solutions to system (1.1). The proof is complete. □

Theorem 3.12. *Assume that* (A2), (A4), (A7), (A8) *hold. Then, for* $2 < q < 2\beta$ *and* $1 < \gamma_0 < 2\beta$, *problem* (1.1) *has infinitely many nontrivial solutions.*

Proof. In view of (A2) and (A4) with $2 < q < 2\beta$, one can show that the functional Φ is bounded from below and satisfies the Palais-Smale condition. The rest of the proof is similar to that of Theorem 3.11, so we omit it. □

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References

[1] M. Agranovih; *Sobolev spaces, their generalizations, and elliptic problems in smooth and Lipschitz domains*, Springer, Switzerland, (2015).

[2] B. Ahmad, A. Alsaedi, M. Kirane; *Nonexistence results for the Cauchy problem of time fractional nonlinear systems of thermoelasticity*, Math. Methods. Appl. Sci., **40** (2017), 4272-4279.

[3] B. Ahmad, A. Alsaedi, M. Kirane, et al.; *An inverse problem for space and time fractional evolution equations with an involution perturbation*, Quaest. Math., **40** (2017), 151-160.

[4] A. Cartea, D. del-Castillo-Negrete; *Fluid limit of the continuous-time random walk with general Lévy jump distribution functions*, Phys. Rev. E, **76** (2007), 041105.

[5] A. Cartea, D. del-Castillo-Negrete; *Fractional diffusion models of option prices in markets with jumps*, Phys. A, **374** (2007), 749-763.

[6] F. Colasuonno, P. Pucci; *Multiplicity of solutions for $p(x)$-polyharmonic elliptic Kirchhoff equation*, Nonlinear Anal., **74** (2011), 5962-5974.

[7] F. Jiao, Y.Zhou; *Existence of solutions for a class of fractional boundary value problems via critical point theory*, Comput. Math. Appl., **62** (2011), 1181-1199.

[8] F. Jiao, Y. Zhou; *Existence results for fractional boundary value problem via critical point theory*, Internat. J. Bifur. Chaos, **22** (2012), 1-17.

[9] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; *Theory and Applications of Fractional Differential Equations*, Elsevier, Netherlands, (2006).

[10] J. Mawhinm; *Critical point theory and Hamiltonian systems*, Springer, Science & Business Media, (2013).

[11] M. M. Meerschaert; *Fractional calculus, anomalous diffusion, and probability, in: R. Metzler, J. Klafter (Eds.), Fractional Dynamics*, World Scientific, Singapore, (2012), 265-284.

[12] M. M. Meerschaert, F. Sabzikar; *Stochastic integration for tempered fractional Brownian motion*, Stochastic Proc. Appl., **124** (2014), 2363-2387.

[13] M. M. Meerschaert, F. Sabzikar, M. Phanikumar, et al.; *Tempered fractional time series model for turbulence in geophysical flows*, J. Stat. Mech., (2014), P09023.

[14] M. M. Meerschaert, A. Sikorskii; *Stochastic Models for Fractional Calculus*, De Gruyter, Berlin, (2012).

[15] N. Nyamoradi, Y. Zhou; *Bifurcation results for a class of fractional Hamiltonian systems with Liouville-Weyl fractional derivatives*, J. Vibration Control, **22** (2016), 1969-1971.

[16] N. Nyamoradi, Y. Zhou; *Homoclinic orbits for a class of fractional Hamiltonian systems via variational methods*, J. Optim. Theory Appl., **174** (2017), 210-222.

[17] I. Podlubny; *Fractional Differential Equations, Mathematics in Science and Engineering*, Academic Press, New York, (1999).

[18] P. Rabinowitz; *Minimax method in critical point theory with applications to differential equations*, CBMS Amer. Math. Soc., **65**, (1986).

[19] F. Sabzikar, M. Meerschaert, J. Chen; *Tempered fractional calculus*, J. Comput. Phys., **293** (2015), 14-28.

[20] F. Sabzikar, M. M. Meerschaert, J. Chen; *Tempered fractional calculus*, J. Comput. Phys., **293** (2015), 14-28.

[21] C. Stuart; *Bifurcation into spectral gaps*, Société Mathématique de Belgique, (1995).

[22] H. R. Sun, Q. G. Zhang; *Existence of solutions for fractional boundary value problem via the Mountain Pass method and an iterative technique*, Comput. Math. Appl., **64** (2012), 3436-3443.

[23] C. Torres; *Ground state solution for differential equations with left and right fractional derivatives*, Math. Methods Appl. Sci., **38** (2015), 5063-5073.

[24] C. Torres; *Mountain pass solution for a fractional boundary value problem*, J. Fract. Calc. Appl, **5** (2014), 1-10.

[25] C. Torres; *Tempered fractional differential equation: variational approach*, Math. Methods Appl. Sci., **40** (2017), 4962-4973.

[26] J. R. Wang, M. Feckan, Y. Zhou; *A survey on impulsive fractional differential equations*, Fract. Calcu. Appl. Anal., **19** (2016), 806-831.
[27] J. R. Wang, M. Feckan, Y. Zhou; *Center stable manifold for planar fractional damped equations*, Appl. Math. Comput., **296** (2017), 257-269.
[28] Z. Zhang, R. Yuan; *Solutions for subquadratic fractional Hamiltonian systems without coercive conditions*, Math. Methods Appl. Sci., **37** (2014), 2934-2945.
[29] Z. Zhang, R. Yuan; *Variational approach to solutions for a class of fractional Hamiltonian systems*, Math. Methods Appl. Sci., **37** (2014), 1873-1883.
[30] Y. Zhou; *Attractivity for fractional differential equations*, Appl. Math. Letters, **75** (2018), 1-6.
[31] Y. Zhou, B. Ahmad, A. Alsaedi; *Existence of nonoscillatory solutions for fractional neutral differential equations,* Appl. Math. Letters, **72** (2017), 70-74.
[32] Y. Zhou, L. Peng; *Weak solution of the time-fractional Navier-Stokes equations and optimal control,* Comput. Math. Appl., **73** (2017), 1016-1027.
[33] Y. Zhou, L. Peng; *On the time-fractional Navier-Stokes equations,* Comput. Math. Appl., **73** (2017), 874-891.
[34] Y. Zhou, L. Peng, B. Ahmad, et al.; *Topological properties of solution sets of fractional stochastic evolution inclusions*, Adv. Difference Equ., **2017** (2017), 90, 20pp.
[35] Y. Zhou, V. Vijayakumar, R. Murugesu; *Controllability for fractional evolution inclusions without compactness,* Evol. Equ. Control Theory, **4** (2015), 507-524.
[36] Y. Zhou, L. Zhang; *Existence and multiplicity results of homoclinic solutions for fractional Hamiltonian systems,* Comput. Math. Appl., **73.6** (2017), 1325-1345.

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