Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 38, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE AND UNIQUENESS OF SOLUTIONS TO SINGULAR QUASILINEAR SCHRÖDINGER EQUATIONS

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Communicated by Vicentiu D. Radulescu

Abstract. In this article we study a quasilinear Schrödinger equations with singularity. We obtain a unique and positive solution by using the minimax method and some analysis techniques.

1. Introduction and statement of main results

This article concerns the singular quasilinear Schrödinger equation with the Dirichlet boundary value condition

$$\begin{gathered} -\Delta u-\Delta(u^2)u=g(x)u^{-r}-u^{p-1} \quad \text{in } \Omega, \\ u>0 \quad \text{in } \Omega, \\ u=0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a bounded smooth domain with boundary $\partial\Omega$, $r\in(0,1)$ and $p\in[2,22^*]$ are constants. The coefficient $g\in L^{\frac{22^*}{22^*-1+r}}(\Omega)$ with $g(x)>0$ for almost every $x\in\Omega$ and $2^*=\frac{2N}{N-2}$ denotes the critical Sobolev exponent for the embedding $H_0^1(\Omega)\hookrightarrow L^q(\Omega)$ for every $q\in[1,2^*]$.

Solutions of (1.1) are related to standing wave solutions for the quasilinear Schrödinger equations

$$i\partial_t\psi=-\Delta\psi+\psi+\eta(|\psi|^2)\psi-k\Delta\rho(|\psi|^2)\rho'(|\psi|^2)\psi, \tag{1.2}$$

where $\psi=\psi(t,x)$, $\psi:\mathbb{R}\times\Omega\to\mathbb{C}$, $k>0$ is a constant. The quasilinear equations of the form (1.2) play an important role in several areas of physics in correspondence to different type of functions ρ. For example, it models the superfluid film equation in plasma physics for $\rho(s)=s$ (see [14]), while for $\rho(s)=(1+s)^{1/2}$ it models the self-channeling of a high-power ultra short laser pulse in matter (see [2, 6, 23]). For further physical motivations and developing the physical aspects we refer to [13, 15, 16, 21] and the references therein.

Motivated by the above mentioned physical aspects, equation (1.2) has received a lot of attention. Indeed, up to our knowledge, the first existence results for the subcritical quasilinear equations have been discussed in [21] using constraint minimization arguments. Subsequently, many authors in [4, 18, 19] were interested

2010 *Mathematics Subject Classification.* 35J20, 35A15, 35J75, 35J62.
Key words and phrases. Quasilinear Schrödinger equation; singularity; uniqueness.

Submitted June 23, 2017. Published January 30, 2018.

in the existence results of standing wave solutions for (1.2) by using a change of variable and reducing the quasilinear equations into the semilinear ones in an appropriate Orlicz space. For critical case, we can refer to [26, 10, 9, 19]. It is worth noticing that up to now there are only one paper [8] investigating the singular case, where they established the singular quasilinear Schrödinger equation

$$-\Delta u-\frac{1}{2}\Delta(u^2)u=\lambda u^3-u-u^{-\alpha},\quad u>0,\ x\in\Omega,$$

where Ω is a ball in $\mathbb{R}^N$ ($N\geq 2$) centered at the origin, $0<\alpha<1$. And they proved the existence of radially symmetric positive solutions by employing Nehari manifold and some techniques related to implicit function theorem when λ belongs to a certain neighborhood of the first eigenvalue λ_1 of the eigenvalue problem

$$-\Delta u-\frac{1}{2}\Delta(u^2)u=\lambda u^3.$$

The singular problems are much more complicated than the regular one and they require some hard analysis. For singular elliptic problems, there are many authors (see e.g. [11, 5, 3, 27, 7, 12, 22]) have studied. Especially, Ghergu and Rădulescu in [11] established several existence and nonexistence results for the boundary value problem

$$\begin{gathered}-\Delta u+K(x)g(u)=\lambda f(x,u)+\mu h(x)\quad\text{in }\Omega,\\ u>0\quad\text{in }\Omega,\\ u=0\quad\text{on }\partial\Omega,\end{gathered}\tag{1.3}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N\geq 2$), λ and μ are positive parameters, h is a positive function, f has a sublinear growth and the function g satisfies the condition

$$\lim_{s\to\infty}g(s)=+\infty.$$

Obviously, $g(s)=s^{-r}, r\in(0,1)$ satisfies the above assumption. When $K(x)\equiv -1$, $f(x,u)=u^p$ and $g(s)=s^{-r}$ in (1.3), where $r\in(0,1), p\geq 0$, Coclite and Palmieri in [3] proved that there is at least one solution for all $\lambda\geq 0$ if $0<p<1$, moreover, there exists a solution for small $\lambda>0$ and no solution for large $\lambda>0$ if $p\geq 1$. For Second-Order Differential Equations, such as Sturm-Liouville operator, Dirac Operators etc., there are many authors being interested, we can refer to [20, 17] and the references therein.

The main purpose of this article is to study the singular quasilinear Schrödinger equation (1.1) and introduce a uniqueness result of solutions for (1.1), which is the first work on this subject up to our knowledge.

Notation. C is a positive constant whose value can be different. The domain of an integral is Ω unless otherwise indicated. $\int f(x)dx$ is abbreviated to $\int f(x)$. $L^p(\Omega)$, $1\leq p\leq\infty$, denotes the Lebesgue space with the norms $\|u\|_p=(\int|u|^p)^{\frac{1}{p}}$, for $1\leq p<\infty$, $\|u\|_\infty=\inf\{C>0:|u(x)|\leq C$ almost everywhere in $\Omega\}$. $X=H_0^1(\Omega)$ denotes the Hilbert space equipped with the norm $\|u\|=(\int|\nabla u|^2)^{1/2}$. The main result is described as follows.

Theorem 1.1. *Suppose that $r\in(0,1)$, $p\in[2,22^*]$ and $g\in L^{\frac{22^*}{22^*-1+r}}(\Omega)$ with $g(x)>0$ for almost every $x\in\Omega$. Then problem (1.1) has a unique positive solution in X. Moreover, this solution is the global minimizer solution.*

The classic semilinear singular equation

$$
\begin{gathered}
-\Delta u = g(x)u^{-r} + \lambda u^{p-1}, \quad \text{in } \Omega, \\
u = 0, \quad \text{on } \partial\Omega,
\end{gathered}
$$

where $p = 2^*$, has been studied for $\lambda > 0$ in [27] and also in [7] for $\lambda = 0$ under the condition $g(x) \in L^\infty(\Omega)$. We point out that the condition $g \in L^{\frac{22^*}{22^*-1+r}}(\Omega)$ is more general than the condition $g(x) \in L^\infty(\Omega)$. To the best of our knowledge, the existence and uniqueness of solutions for the quasilinear Schrödinger equation (1.1) has not been discussed up to now.

This article is organized as follows: Some preliminaries are given in the next section. In Section 3, we give the proof of Theorem 1.1.

2. Preliminary results

We observe that the energy functional corresponding to (1.1) given by

$$
J(u) := \frac{1}{2}\int (1+2u^2)|\nabla u|^2 - \frac{1}{1-r}\int g(x)|u|^{1-r} + \frac{1}{p}\int |u|^p
$$

is not well defined in X. To overcome this problem, we use the change of variable $v := f^{-1}(u)$ introduced in [18], where f is defined by

$$
f'(t) = \frac{1}{\sqrt{1+2f^2(t)}} \text{ on } [0,+\infty), \quad \text{and} \quad f(t) = -f(-t) \text{ on } (-\infty, 0].
$$

We list some properties of f, whose proofs can be found in [4, 25].

Lemma 2.1. *The function f satisfies the following properties:*

(1) *f is uniquely defined, C^∞ and invertible;*
(2) *$|f'(t)| \le 1$ for all $t \in \mathbb{R}$;*
(3) *$|f(t)| \le |t|$ for all $t \in \mathbb{R}$;*
(4) *$f(t)/t \to 1$ as $t \to 0$;*
(5) *$|f(t)f'(t)| < 1/\sqrt{2}$, $\forall t \in \mathbb{R}$;*
(6) *$f(t)/2 \le tf'(t) \le f(t)$ for all $t \ge 0$;*
(7) *$|f(t)| \le 2^{1/4}|t|^{1/2}$ for all $t \in \mathbb{R}$;*
(8) *the function $f^{-r}(t)f'(t)$ is decreasing for all $t > 0$;*
(9) *the function $f^{p-1}(t)f'(t)$ is increasing for all $t > 0$.*

Proof. We only prove (8) and (9). By $f''(t) = -2f(t)[f'(t)]^4$, for all $t \in \mathbb{R}$, $p \ge 2$ and (5), with simple computation we obtain

$$
\frac{d[f^{-r}(t)f'(t)]}{dt} = -rf^{-r-1}(t)[f'(t)]^2 - 2f^{1-r}(t)[f'(t)]^4 < 0, \quad \forall t > 0
$$

and

$$
\frac{d[f^{p-1}(t)f'(t)]}{dt} = f^{p-2}(t)[f'(t)]^2[p-1-2f^2(t)[f'(t)]^2] > 0, \quad \forall t > 0,
$$

which imply that $f^{-r}(t)f'(t)$ is decreasing and $f^{p-1}(t)f'(t)$ is increasing for all $t > 0$. □

By exploiting the change of variable, we can rewrite the functional in the form

$$
I(v) := \frac{1}{2}\int |\nabla v|^2 - \frac{1}{1-r}\int g(x)|f(v)|^{1-r} + \frac{1}{p}\int |f(v)|^p, \quad v \in X.
$$

By Lemma 2.1-(7), the Hölder inequality and the Sobolev inequality we have

$$\int g(x)|f(v)|^{1-r} \le C\|g\|_{\frac{22^*}{22^*-1+r}} \|v\|^{\frac{1-r}{2}}. \tag{2.1}$$

Then I is well-defined but only continuous on X. Also equation (1.1) can be rewritten as

$$-\Delta v = g(x)f^{-r}(v)f'(v) - f^{p-1}(v)f'(v),\ v > 0,\ x \in \Omega. \tag{2.2}$$

In general, a function $v \in X$ is called a weak solution of (2.2) with $v > 0$ in Ω if it holds

$$\int \nabla v \nabla w - g(x)f^{-r}(v)f'(v)w + f^{p-1}(v)f'(v)w = 0, \quad \forall w \in X. \tag{2.3}$$

We observe that if $v \in X$ is a weak solution of (2.2), the function $u = f(v) \in X$ is a solution of (1.1) (cf:[4]).

3. Proof of Theorem 1.1

In this section, we shall show that there exists a unique positive solution v_0 of (2.2), which is the global minimizer of the functional I in X, and then $u_0 = f(v_0) \in X$ is the unique positive solution of (1.1).

Lemma 3.1. *The functional I attains the global minimizer in X; that is, there exists $v_0 \in X \setminus \{0\}$ such that $I(v_0) = m := \inf_X I < 0$.*

Proof. For $v \in X$, from (2.1) it follows that

$$I(v) \ge \frac{1}{2}\|v\|^2 - \frac{C}{1-r}\|g\|_{\frac{22^*}{22^*-1+r}} \|v\|^{\frac{1-r}{2}}. \tag{3.1}$$

Since $r \in (0,1)$, I is coercive and bounded from below on X. Thus $m := \inf_X I$ is well defined. For $t > 0$ and given $v \in X \setminus \{0\}$ by Lemma 2.1-(7) one gets

$$\begin{aligned} I(tv) &= \frac{t^2}{2}\|v\|^2 - \frac{1}{1-r}\int g(x)|f(tv)|^{1-r} + \frac{1}{p}\int |f(tv)|^p \\ &\le \frac{t^2}{2}\|v\|^2 - \frac{1}{1-r}\int g(x)|f(tv)|^{1-r} + \frac{C}{p}t^{\frac{p}{2}}\int |v|^{\frac{p}{2}}. \end{aligned}$$

Note that the function $|\frac{f(tv)}{tv}|^{1-r}$ is non-increasing for $t > 0$. By Lemma 2.1-(4) and Beppo-Levi Monotone Convergence Theorem, we can see

$$\lim_{t\to 0^+} \frac{I(tv)}{t^{1-r}} = -\frac{1}{1-r}\int g(x)|v|^{1-r} < 0.$$

So we have $I(tv) < 0$ for all $v \not\equiv 0$ and $t > 0$ small enough. Hence, we obtain $m < 0$.

According to the definition of m, there exists a minimizing sequence $\{v_n\} \subset X$ such that $\lim_{n\to\infty} I(v_n) = m < 0$. Since $I(v_n) = I(|v_n|)$, we may assume that $v_n \ge 0$. It follows from (3.1) that there exists a constant $C > 0$ such that $\|v_n\| \le C$. Passing if necessary to a subsequence, we can assume that there exists $v_0 \in X$ such that

$$\begin{gathered} v_n \rightharpoonup v_0 \quad \text{in } X, \\ v_n \to v_0 \quad \text{in } L^p(\Omega),\ p \in [1, 2^*), \\ v_n(x) \to v_0(x) \quad \text{a.e. in } \Omega, \end{gathered}$$

there exists a function $k \in L^p(\Omega)$, $p \in [1, 2^*)$, such that

$$|u_n(x)| \le k(x) \quad \text{a.e. in } \Omega. \tag{3.2}$$

By Vitali's theorem (see [24]), we claim that

$$\lim_{n\to\infty} \int g(x) f^{1-r}(v_n) = \int g(x) f^{1-r}(v_0). \tag{3.3}$$

Indeed, we only need prove that $\{\int g(x) f^{1-r}(v_n),\ n \in \mathbb{N}\}$ is equi-absolutely-continuous. For all $\varepsilon > 0$, by the absolutely-continuity of $\int |g(x)|^{\frac{22^*}{22^*-1+r}}$, there exists $\delta > 0$ such that $\int_E |g(x)|^{\frac{22^*}{22^*-1+r}} < \varepsilon^{\frac{22^*}{22^*-1+r}}$ for all $E \subset \Omega$ with meas $E < \delta$. Consequently, by (2.1) and the fact that $\|v_n\| \le C$, we have

$$\int_E g(x) f^{1-r}(v_n) \le C\|v_n\|^{\frac{1-r}{2}} \Big(\int_E |g(x)|^{\frac{22^*}{22^*-1+r}} \Big)^{\frac{22^*-1+r}{22^*}} < C\varepsilon.$$

Thus, (3.3) is valid. In the case that $p \in [2, 22^*)$, by Lemma 2.1-(7) and (3.2) we see

$$|f(v_n)|^p \le C|v_n|^{\frac{p}{2}} \le Ck^{\frac{p}{2}} \in L^1(\Omega),$$

then the Lebesgue Dominated Convergence Theorem implies

$$\int f^p(v_n) = \int f^p(v_0) + o(1).$$

Combining the above equality, the weakly lower semi-continuity of the norm, and (3.3), we have

$$m \le I(v_0) = \frac{1}{2}\|v_0\|^2 - \frac{1}{1-r}\int g(x) f^{1-r}(v_0) + \frac{1}{p}\int f^p(v_0) \le \liminf_{n\to\infty} I(v_n) = m,$$

which yields that $I(v_0) = m < 0$ and $v_0 \not\equiv 0$. In the case that $p = 22^*$, by Brézis-Lieb's Lemma (see [1]) and Lemma 2.1-(7), one obtains

$$\int f^{22^*}(v_n) = \int f^{22^*}(v_0) + \int f^{22^*}(v_n - v_0) + o(1),$$

which together with the weakly lower semi-continuity of the norm and (3.3), we have

$$\begin{aligned} m \le I(v_0) &= \frac{1}{2}\|v_0\|^2 - \frac{1}{1-r}\int g(x) f^{1-r}(v_0) + \frac{1}{p}\int f^p(v_0) \\ &\le \liminf_{n\to\infty} I(v_n) - \lim_{n\to\infty} \frac{1}{22^*}\int f^{22^*}(v_n - v_0) \le m, \end{aligned}$$

which also implies that $I(v_0) = m < 0$ and $v_0 \not\equiv 0$. □

Proof of Theorem 1.1. Since $I(v_0) = m < 0$, we obtain that $v_0 \ge 0$ and $v_0 \not\equiv 0$. Now, we divide the proof in three steps:

First, we claim that $v_0 > 0$ in Ω. Fix $\phi \in X$ with $\phi \ge 0$, let $t > 0$, one has

$$\begin{aligned} 0 &\le I(v_0 + t\phi) - I(v_0) \\ &= \frac{1}{2}\|v_0 + t\phi\|^2 - \frac{1}{2}\|v_0\|^2 - \frac{1}{1-r}\int g(x)[f^{1-r}(v_0 + t\phi) - f^{1-r}(v_0)] \\ &\quad + \frac{1}{p}\int f^p(v_0 + t\phi) - f^p(v_0). \end{aligned}$$

Dividing by $t > 0$ and passing to the limit as $t \to 0^+$ in the above inequality, we have

$$\begin{aligned} &\frac{1}{1-r} \liminf_{t \to 0^+} \int g(x) \frac{f^{1-r}(v_0 + t\phi) - f^{1-r}(v_0)}{t} \\ &\le \int \nabla v_0 \nabla \phi + f^{p-1}(v_0) f'(v_0) \phi. \end{aligned} \tag{3.4}$$

Note that

$$\int g(x) \frac{f^{1-r}(v_0 + t\phi) - f^{1-r}(v_0)}{t} = (1-r) \int g(x) f^{-r}(v_0 + t\theta\phi) f'(v_0 + t\theta\phi)\phi,$$

where $\theta(x) \in (0, 1)$. For any $x \in \Omega$, we denote

$$h(t) = g(x) f^{-r}(v_0 + t\theta\phi) f'(v_0 + t\theta\phi)\phi, \quad t > 0.$$

It follows from $g(x) > 0$ a.e. $x \in \Omega$ and Lemma 2.1-(8) that $h(t)$ is non-increasing for $t > 0$. Moreover,

$$\lim_{t \to 0^+} h(t) = g(x) f^{-r}(v_0(x)) f'(v_0(x)) \phi(x)$$

for every $x \in \Omega$, which may be $+\infty$ when $v_0(x) = 0$. Consequently, by the Beppo-Levi Monotone Convergence Theorem, we obtain

$$\liminf_{t \to 0^+} \frac{1}{1-r} \int g(x) \frac{f^{1-r}(v_0 + t\phi) - f^{1-r}(v_0)}{t} = \int g(x) f^{-r}(v_0) f'(v_0)\phi,$$

which together with (3.4) implies that

$$\int \nabla v_0 \nabla \phi - g(x) f^{-r}(v_0) f'(v_0)\phi + f^{p-1}(v_0) f'(v_0)\phi \ge 0, \quad \phi \in X, \ \phi \ge 0. \tag{3.5}$$

Therefore,

$$-\Delta v_0 + f^{p-1}(v_0) f'(v_0) \ge 0$$

in the weak sense. Hence the maximum principle implies that $v_0 > 0$ in Ω.

Secondly, we show that v_0 is a solution of (2.2), that is, we prove v_0 satisfies (2.3). For given $\delta > 0$, define $H : [-\delta, \delta] \to \mathbb{R}$ by $H(t) = I((1 + t)v_0)$, then H attains its minimum at $t = 0$ by Lemma 3.1, namely

$$H'(0) = \|v_0\|^2 - \int g(x) f^{-r}(v_0) f'(v_0) v_0 - f^{p-1}(v_0) f'(v_0) v_0 = 0. \tag{3.6}$$

Choose $\varphi \in X \setminus \{0\}, \varepsilon > 0$. Define $\phi \in X$ by $\phi = (v_0 + \varepsilon\varphi)^+$. Let

$$\Omega_1 = \{x \in \Omega : v_0(x) + \varepsilon\varphi(x) > 0\}, \quad \Omega_2 = \{x \in \Omega : v_0(x) + \varepsilon\varphi(x) \le 0\}.$$

Easily, we see $\phi|_{\Omega_1} = v_0 + \varepsilon\varphi$ and $\phi|_{\Omega_2} = 0$. Inserting ϕ into (3.5) and applying with (3.6), one obtains

$$\begin{aligned}
0 &\le \int \nabla v_0 \nabla \phi - g(x) f^{-r}(v_0) f'(v_0)\phi + f^{p-1}(v_0) f'(v_0)\phi \\
&= \int_{\Omega_1} \nabla v_0 \nabla (v_0 + \varepsilon\varphi) - g(x) f^{-r}(v_0) f'(v_0)(v_0 + \varepsilon\varphi) \\
&\quad + f^{p-1}(v_0) f'(v_0)(v_0 + \varepsilon\varphi) \\
&= \int_{\Omega\setminus\Omega_2} \nabla v_0 \nabla (v_0 + \varepsilon\varphi) - g(x) f^{-r}(v_0) f'(v_0)(v_0 + \varepsilon\varphi) \\
&\quad + f^{p-1}(v_0) f'(v_0)(v_0 + \varepsilon\varphi) \\
&= \varepsilon \int \nabla v_0 \nabla \varphi - g(x) f^{-r}(v_0) f'(v_0)\varphi + f^{p-1}(v_0) f'(v_0)\varphi \\
&\quad - \int_{\Omega_2} \nabla v_0 \nabla (v_0 + \varepsilon\varphi) - g(x) f^{-r}(v_0) f'(v_0)(v_0 + \varepsilon\varphi) \\
&\quad + f^{p-1}(v_0) f'(v_0)(v_0 + \varepsilon\varphi) \\
&\le \varepsilon \int \nabla v_0 \nabla \varphi - g(x) f^{-r}(v_0) f'(v_0)\varphi + f^{p-1}(v_0) f'(v_0)\varphi \\
&\quad - \varepsilon \int_{\Omega_2} \nabla v_0 \nabla \varphi + f^{p-1}(v_0) f'(v_0)\varphi.
\end{aligned} \tag{3.7}$$

From $\operatorname{meas}\Omega_2 \to 0$ as $\varepsilon \to 0$, it follows that

$$\int_{\Omega_2} \nabla u_0 \nabla \varphi + f^{p-1}(v_0) f'(v_0)\varphi \to 0 \quad \text{as } \varepsilon \to 0.$$

Then dividing by $\varepsilon > 0$ and letting $\varepsilon \to 0$ in (3.7), we conclude that

$$\int \nabla v_0 \nabla \varphi - g(x) f^{-r}(v_0) f'(v_0)\varphi + f^{p-1}(v_0) f'(v_0)\varphi \ge 0.$$

By the arbitrariness of φ, the above inequality also holds for $-\varphi$, so we get that v_0 solves (2.3). Hence $v_0 \in X$ is a positive solution of (2.2) with $I(v_0) = m < 0$, that is, v_0 is the global minimizer solution.

Finally, we show that $v_0 \in X$ is the unique solution of (2.2). Assume that $v \in X$ is also a solution of (2.2), it follows from (2.3) that

$$\int \nabla v_0 \nabla (v_0 - v) - g(x) f^{-r}(v_0) f'(v_0)(v_0 - v) + f^{p-1}(v_0) f'(v_0)(v_0 - v) = 0 \tag{3.8}$$

and

$$\int \nabla v \nabla (v_0 - v) - g(x) f^{-r}(v) f'(v)(v_0 - v) + f^{p-1}(v) f'(v)(v_0 - v) = 0. \tag{3.9}$$

Subtracting (3.8) and (3.9), since $g(x) > 0$ a.e. $x \in \Omega$, by Lemma 2.1-(8), (9) we get

$$\begin{aligned}
\|v_0 - v\|^2 &= \int g(x)[f^{-r}(v_0) f'(v_0) - f^{-r}(v) f'(v)](v_0 - v) \\
&\quad - \int [f^{p-1}(v_0) f'(v_0) - f^{p-1}(v) f'(v)](v_0 - v) \le 0,
\end{aligned}$$

which implies that $\|v_0 - v\| = 0$, that is $v_0 = v$. Therefore, $v_0 \in X$ is the unique solution of (2.2), and then $u_0 = f(v_0) \in X$ is the unique solution of (1.1). We complete the proof of Theorem 1.1. $\square$

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