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# PROPERTIES ON MEASURE PSEUDO ALMOST AUTOMORPHIC FUNCTIONS AND APPLICATIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. In this article, we establish some new composition theorems on measure pseudo almost automorphic functions via measure theory. The obtained compositions theorems generalize those established under the well-known Lipschitz conditions or the classical uniformly continuous conditions. Then using the theories of resolvent operators and fixed point theorem, we investigate the existence and uniqueness of measure pseudo almost automorphic solutions to a fractional differential equation in Banach spaces.

## 1. Introduction

The almost automporphic function introduced by Bochner [6] is seen as a significant generalization of the classical almost periodic function. Since then, almost automorphic functions have been considerably investigated and undergone some interesting, natural and powerful generalizations. The concept of asymptotically almost automorphic functions was introduced by N'Guérékata [22]. Liang, Xiao and Zhang [15, 29] further developed the theory of pseudo almost automorphic functions suggested by N'Guérékata in [21]. Blot et al. [4] introduced the notion of weighted pseudo almost automorphic functions with values in a Banach space, which seems to be more general and complicated than pseudo-almost automorphic functions. One can refer to [1, 2, 10, 11, 12, 20, 21, 23, 27, 30] and references therenin for more results on above mentioned functions and their applications in differential equations. In 2012, Blot, Cieutat and Ezzinbi [5] applied the abstract measure theory to define an ergodic function and established fundamental properties of measure pseudo almost automorphic functions, and thus the classical theories of pseudo almost automorphic functions and weighted pseudo almost automorphic functions become particular cases of this approach. After that, the measure pseudo almost automorphic function has been developed in different ways, see for instance [8, 13, 28] and references therein.

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Fractional calculus can be seen a generalization of the ordinary differentiation and integration to arbitrary non-integer order, and has been recognized as one of the most powerful tools to describe long-memory processes in the last decades. Many phenomena from physics, chemistry, mechanics, electricity can be modeled by ordinary and partial differential equations involving fractional derivatives, we refer to [3, 14, 25, 26, 31, 32] and references therein for more developments on this topic.

Inspired by above mentioned works [5, 23], the aim of this work is first to establish some new composition theorems on measure pseudo almost automorphic functions via measure theory. The obtained compositions theorems generalize those based upon the well-known Lipschitz conditions or the classical uniformly continuous conditions. These composition theorems are new even for (weighted-) pseudo almost automorphic functions. Then using the theories of resolvent operators and fixed point theorem, we investigate the existence and uniqueness of measure pseudo almost automorphic solutions to the following fractional differential equation

$$D^\alpha u(t) = \mathscr{A} u(t) + \int_{-\infty}^{t} a(t-s)\mathscr{A} u(s)ds + f(t, u(\gamma(t))),\ t \in \mathbb{R}, \tag{1.1}$$

where $(\mathbb{X}, \|\cdot\|)$ is a Banach space, $\mathscr{A}$ is a closed linear operator defined on Banach space $\mathbb{X}$, $a \in L^1_{\rm loc}(\mathbb{R}_+)$ is a scalar-valued kernel, $f, \gamma$ are appropriate functions satisfying some properties specified later, and for $\alpha > 0$, the fractional derivative $D^\alpha$ is understood in the sense of Weyl.

The rest of this article is organized as follows. In Section 2, we introduce some basic definitions, lemmas, and preliminary results which will be used throughout this paper. In Section 3, we first establish new composition theorems of measure pseudo almost automorphic functions, and then we investigate the existence and uniqueness of measure pseudo almost automorphic mild solutions to equation (1.1).

## 2. Preliminaries

This section presents some preliminary results needed in the sequel. Throughout this article, $(\mathbb{X}, \|\cdot\|)$ denotes a Banach space and $BC(\mathbb{R}, \mathbb{X})$ denotes the Banach space of bounded continuous functions from $\mathbb{R}$ to $\mathbb{X}$, equipped with the supremum norm $\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|$. We also denote by $\mathfrak{B}(\mathbb{X})$ the space of bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$ endowed uniform operator topology.

**Definition 2.1** ([6])**.** A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$. The collection of all such functions will be denoted by $AA(\mathbb{R}, \mathbb{X})$.

**Definition 2.2** ([16, 30])**.** A continuous function $f : \mathbb{R} \to \mathbb{X}$ (resp. $\mathbb{R} \times \mathbb{X} \to \mathbb{X}$) is called pseudo-almost automorphic if it can be decomposed as $f = g + \phi$, where $g \in AA(\mathbb{R}, \mathbb{X})$ (resp. $AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$) and $\phi \in PAA_0(\mathbb{R}, \mathbb{X})$ (resp. $PAA_0(\mathbb{R} \times \mathbb{X}, \mathbb{X})$).

Denote by $PAA(\mathbb{R},\mathbb{X})$ (resp. $PAA(\mathbb{R}\times\mathbb{X},\mathbb{X})$) the set of all such functions, where

$$PAA_0(\mathbb{R},\mathbb{X}) := \Big\{\phi \in BC(\mathbb{R},\mathbb{X}) : \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \|\phi(t)\|dt = 0\Big\},$$

$$PAA_0(\mathbb{R}\times\mathbb{X},\mathbb{X}) := \Big\{\phi \in BC(\mathbb{R}\times\mathbb{X},\mathbb{X}) : \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \|\phi(t,x)\|dt = 0 \text{ uniformly for } x \text{ in any bounded subset of } \mathbb{X}\Big\}.$$

Let $\mathbb{U}$ denote the set of all functions (weights) $\rho : \mathbb{R} \to (0,\infty)$, which are locally integrable over $\mathbb{R}$ such that $\rho > 0$ almost everywhere. For a given $r > 0$ and for each $\rho \in \mathbb{U}$, we set

$$m(r,\rho) = \int_{-r}^{r} \rho(t)dt.$$

We denote by $\mathbb{U}_\infty$ the set of all $\rho \in \mathbb{U}$ with $\lim_{r\to\infty} m(r,\rho) = \infty$.

**Definition 2.3** ([4]). Let $\rho \in \mathbb{U}_\infty$. A bounded continuous function $f : \mathbb{R} \to \mathbb{X}$(resp. $\mathbb{R}\times\mathbb{X} \to \mathbb{X}$) is called weighted pseudo almost automorphic if it can be decomposed as $f = g + \phi$, where $g \in AA(\mathbb{R},\mathbb{X})$ (resp. $AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$) and $\phi \in PAA_0(\mathbb{R},\mathbb{X},\rho)$ (resp. $PAA_0(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho)$). The class of all such functions will be denoted by $WPAA(\mathbb{R},\mathbb{X},\rho)$ (resp. $WPAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho)$), where

$$PAA_0(\mathbb{R},\mathbb{X},\rho) := \Big\{\phi \in BC(\mathbb{R},\mathbb{X}) : \lim_{r\to\infty} \frac{1}{m(r,\rho)}\int_{-r}^{r} \|\phi(t)\|\rho(t)dt = 0\Big\};$$

$$PAA_0(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho) := \Big\{\phi \in BC(\mathbb{R}\times\mathbb{X},\mathbb{X}) : \lim_{r\to\infty} \frac{1}{m(r,\rho)}\int_{-r}^{r} |\phi(t,x)|\rho(t)dt = 0 \text{ uniformly for } x \text{ in any bounded subset of } \mathbb{X}\Big\}.$$

Let $\mathcal{B}$ denote the Lebesgue $\sigma$-field of $\mathbb{R}$ and $\mathcal{M}$ be the set of all positive measures $\mu$ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < +\infty$, for all $a,b \in \mathbb{R}$ with $a < b$. For $\mu \in \mathcal{M}$ and $\tau \in \mathbb{R}$, let $\mu_\tau$ denote the positive measures on $\mathcal{B}$ defined by

$$\mu_\tau(\mathbb{A}) = \mu(\{a+\tau : a \in \mathbb{A}\}), \quad \mathbb{A} \in \mathcal{B}.$$

For $\mu \in \mathcal{M}$, we always assume that the following hypothesis holds throughout this paper:

(A1) For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and bounded interval $I$ such that

$$\mu_\tau(\mathbb{A}) \le \beta\mu(\mathbb{A}),$$

when $\mathbb{A} \in \mathcal{B}$ satisfies $\mathbb{A} \cap I = \emptyset$.

**Definition 2.4** ([5]). Let $\mu \in \mathcal{M}$. A bounded continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be $\mu$-ergodic if

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])}\int_{[-r,r]} \|f(t)\|d\mu(t) = 0.$$

We denote the space of all such functions by $\varepsilon(\mathbb{R},\mathbb{X},\mu)$.

**Definition 2.5** ([5]). Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be measure pseudo almost automorphic if $f$ is written in the form: $f = g + \phi$, where $g \in AA(\mathbb{R},\mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$. We denote the space of all such functions by $PAA(\mathbb{R},\mathbb{X},\mu)$.

**Definition 2.6** ([5]). Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is said to be $\mu$-pseudo almost automorphic if $f$ is written in the form: $f = g + \phi$, where $g \in AA(\mathbb{R}\times\mathbb{X}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}\times\mathbb{X}, \mathbb{X}, \mu)$. We denote the space of all such functions by $PAA(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$, where

$$\varepsilon(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu) := \Big\{\phi \in BC(\mathbb{R} \times \mathbb{X}, \mathbb{X}) : \lim_{r\to\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \|\phi(t, x)\| d\mu(t) = 0$$
$$\text{uniformly for } x \text{ in any bounded subset of } \mathbb{X}.$$

**Definition 2.7** ([5]). Let $\mu_1$ and $\mu_2 \in \mathcal{M}$. $\mu_1$ is said to be equivalent to $\mu_2(\mu_1 \sim \mu_2)$ if there exist constants $\alpha$ and $\beta > 0$ and a bounded interval $I$(eventually $I = \emptyset$) such that

$$\alpha\mu_1(\mathbb{A}) \le \mu_2(\mathbb{A}) \le \beta\mu_1(\mathbb{A}),$$

for $\mathbb{A} \in \mathcal{B}$ satisfying $\mathbb{A} \cap I = \emptyset$.

Now we recall some basic facts on $\mu$-ergodicity and $\mu$-pseudo almost automorphy.

**Lemma 2.8** ([5, Lemma 3.2]). *Let $\mu \in \mathcal{M}$. Then $\mu$ satisfies (A1) if and only if $\mu$ and $\mu_\tau$ are equivalent for all $\tau \in \mathbb{R}$.*

**Lemma 2.9** ([5, Theorem 3.5]). *Let $\mu \in \mathcal{M}$ satisfy (A1). Then $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant, therefore $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is also translation invariant.*

**Lemma 2.10** ([5, Theorem 2.14]). *Let $\mu \in \mathcal{M}$ and $I$ be the bounded interval (eventually $I = \emptyset$). Assume that $f \in BC(\mathbb{R}, \mathbb{X})$. Then the following assertions are equivalent.*

(I) $f \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$;
(II) $\lim_{r\to+\infty} \frac{1}{\mu([-r,r]\setminus I)} \int_{[-r,r]\setminus I} \|f(t)\| d\mu(t) = 0$;
(III) *For any* $\varepsilon > 0, \lim_{r\to+\infty} \frac{\mu(\{t\in[-r,r]\setminus I : \|f(t)\|>\varepsilon\})}{\mu([-r,r]\setminus I)} = 0$.

**Lemma 2.11** ([5, Theorem 4.1]). *Let $\mu \in \mathcal{M}$ and $f \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ be such that $f = g + \phi$, where $g \in AA(\mathbb{R}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. If $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant, then $\{g(t) : t \in \mathbb{R}\} \subset \overline{\{f(t) : t \in \mathbb{R}\}}$, (the closure of the range of $f$).*

**Lemma 2.12** ([5, Theorem 4.7]). *Let $\mu \in \mathcal{M}$. Assume that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then the decomposition of a $\mu$-pseudo almost automorphic function in the form $f = g + \phi$ where $g \in AA(\mathbb{R}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ is unique.*

**Lemma 2.13** ([5, Theorem 4.9]). *Let $\mu \in \mathcal{M}$. Assume that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then $(PAA(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.*

**Definition 2.14** ([23]). Given a function $f : \mathbb{R} \to \mathbb{X}$, the Wely fractional integral of order $\alpha > 0$ is defined by

$$D^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t - s)^{\alpha-1} f(s) ds, \quad t \in \mathbb{R}$$

when this integral is convergent. The Wely fractional derivative $D^\alpha f$ of order $\alpha > 0$ is defined by

$$D^\alpha f(t) := \frac{d^n}{dt^n} D^{-(n-\alpha)} f(t), \quad t \in \mathbb{R}$$

where $n = [\alpha] + 1$.

**Definition 2.15.** [23] Let $\mathscr{A}$ be a closed and linear operator with domain $D(\mathscr{A})$ defined on a Banach space $\mathbb{X}$, and $\alpha > 0$. Given $a \in L^1_{\rm loc}(\mathbb{R}_+)$, the operator $\mathscr{A}$ is called the generator of an $\alpha$-resolvent family, if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : [0,\infty) \to \mathfrak{B}(\mathbb{X})$ such that $\{\frac{\lambda^\alpha}{1+\hat{a}(\lambda)} : \mathrm{Re}\lambda > \omega\} \subset \bar{\rho}(\mathscr{A})$ and for all $x \in \mathbb{X}$,

$$\begin{aligned}\big(\lambda^\alpha - (1+\hat{a}(\lambda))\mathscr{A}\big)^{-1}x &= \frac{1}{1+\hat{a}(\lambda)}\Big(\frac{\lambda^\alpha}{1+\hat{a}(\lambda)} - \mathscr{A}\Big)^{-1}x \\ &= \int_0^\infty e^{-\lambda t} S_\alpha(t)x\,dt, \quad Re\lambda > 0,\end{aligned}$$

where $\hat{a}$ denotes the Laplace transform of $a$, $\bar{\rho}(\mathscr{A})$ denotes the resolvent set of $\mathscr{A}$. In this case, $S_\alpha(t)_{t\geq 0}$ is called the $\alpha$-resolvent family generated by $\mathscr{A}$.

Sufficient conditions for $\{S_\alpha(t)\}_{t\geq 0} \subset \mathfrak{B}(\mathbb{X})$ to be a resolvent family can be found in [9, 17, 19].

## 3. Main results

This section first shows new composition theorems for $\mu$-pseudo almost automorphic functions, and then the theorems obtained are applied to existence and uniqueness of $\mu$-pseudo almost automorphic solutions to the problem (1.1).

Let $\mu \in \mathcal{M}$ and the set $\mathscr{B}(r,\mu)$ be defined as

$$\mathscr{B}(r,\mu) := \Big\{\nu : \mathbb{R} \to \mathbb{R}_+ : \lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \nu(t) d\mu(t) < \infty\Big\}.$$

### 3.1. Composition theorems of $\mu$-pseudo almost automorphic functions.

**Theorem 3.1.** *Let $\mu \in \mathcal{M}$ and $f = g + h \in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$ with $g \in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$, $h \in \varepsilon(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$. Assume that the following condition are satisfied:*

- (A2) *There exists a function $\mathcal{L}(\cdot) \in \mathscr{B}(r,\mu)$ such that*
$$\|f(t,x) - f(t,y)\| \leq \mathcal{L}(t)\|x-y\|$$
*for all $x$, $y \in \mathbb{X}$ and $t \in \mathbb{R}$;*
- (A3) *$g(t,x)$ is uniformly continuous in any bounded subset $K' \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$.*

*If $u = u_1 + u_2 \in PAA(\mathbb{R},\mathbb{X},\mu)$ with $u_1 \in AA(\mathbb{R},\mathbb{X})$, $u_2 \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$. Then the function $f(\cdot,u(\cdot))$ belongs to $PAA(\mathbb{R},\mathbb{X},\mu)$.*

*Proof.* Since $f \in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$ and $u \in PAA(\mathbb{R},\mathbb{X},\mu)$, we have by definition that $f = g + h$ and $u = u_1 + u_2$ where $g \in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$, $h \in \varepsilon(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$, $u_1 \in AA(\mathbb{R},\mathbb{X})$ and $u_2 \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$. The function $f$ can be decomposed as

$$\begin{aligned}f(t,u(t)) &= g(t,u_1(t)) + f(t,u(t)) - g(t,u_1(t)) \\ &= g(t,u_1(t)) + f(t,u(t)) - f(t,u_1(t)) + h(t,u_1(t)).\end{aligned}$$

Define

$$G(t) = g(t,u_1(t)), \quad F(t) = f(t,u(t)) - f(t,u_1(t)), \quad H(t) = h(t,u_1(t)).$$

Then $f(t,u(t)) = G(t) + F(t) + H(t)$. Since the function $g$ satisfies condition (A3), it follows [15, Lemma 2.2] that the function $g(\cdot,u_1(\cdot)) \in AA(\mathbb{R},\mathbb{X})$. To show that $f(\cdot,u(\cdot)) \in PAA(\mathbb{R},\mathbb{X},\mu)$, it is sufficient to show that $F + H \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$.

Initially, we prove that $F \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$. Clearly, $f(t,u(t))-f(t,u_1(t)) \in BC(\mathbb{R},\mathbb{X})$, without loss of generality, we assume that $\|f(t,u(t))-f(t,u_1(t))\| \le \mathcal{C}$. Owing to the fact that $u_2 \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$ and Lemma 2.10 (III), for any $\varepsilon > 0$, we get

$$\lim_{r\to\infty} \frac{\mu(\{t \in [-r,r] : \|u_2(t)\| > \varepsilon\})}{\mu([-r,r])} = 0.$$

Therefore,

$$\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]} \|F(t)\| d\mu(t)\\
&= \frac{1}{\mu([-r,r])}\int_{[-r,r]} \|f(t,u(t))-f(t,u_1(t))\| d\mu(t)\\
&= \frac{1}{\mu([-r,r])}\int_{\{t\in[-r,r]:\|u_2(t)\|>\varepsilon\}} \|f(t,u(t))-f(t,u_1(t))\| d\mu(t)\\
&\quad + \frac{1}{\mu([-r,r])}\int_{[-r,r]\setminus\{t\in[-r,r]:\|u_2(t)\|>\varepsilon\}} \|f(t,u(t))-f(t,u_1(t))\| d\mu(t)\\
&\le \mathcal{C}\frac{\mu(\{t \in [-r,r] : \|u_2(t)\| > \varepsilon\})}{\mu([-r,r])}\\
&\quad + \frac{1}{\mu([-r,r])}\int_{[-r,r]\setminus\{t\in[-r,r]:\|u_2(t)\|>\varepsilon\}} \mathcal{L}(t)\|u_2(t)\| d\mu(t)\\
&\le \mathcal{C}\frac{\mu(\{t \in [-r,r] : \|u_2(t)\| > \varepsilon\})}{\mu([-r,r])} + \varepsilon\frac{1}{\mu([-r,r])}\int_{[-r,r]} \mathcal{L}(t) d\mu(t).
\end{aligned}$$

Taking into account that $\mathcal{L}(\cdot) \in \mathscr{B}(r,\mu)$, we obtain

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])}\int_{[-r,r]} \|F(t)\| d\mu(t) = 0,$$

which shows that $F(\cdot) \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$.

Next, we show that $H \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$. Since $u(t)$, $u_1(t)$ are bounded, we can choose a bounded subset $\mathbb{B} \subset \mathbb{X}$ such that $u(\mathbb{R}), u_1(\mathbb{R}) \subset \mathbb{B}$. Since $g$ satisfies the condition (A3), then for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that $x, y \in \mathbb{B}$ and $\|x-y\| \le \delta$ imply that $\|g(t,x)-g(t,y)\| \le \varepsilon$ for all $t \in \mathbb{R}$. Put $\delta_0 = \min\{\varepsilon,\delta\}$, then

$$\|h(t,x)-h(t,y)\| \le \|f(t,x)-f(t,y)\| + \|g(t,x)-g(t,y)\| \le (\mathcal{L}(t)+1)\varepsilon.$$

for all $x, y \in \mathbb{B}$ with $\|x-y\| \le \delta_0$.

Set $\mathbb{I} = u_1([-r,r])$. Then $\mathbb{I}$ is compact in $\mathbb{R}$ since the image of a compact set under a continuous mapping is compact. So we can find finite open balls $O_k$, $(k = 1,2,\dots,m)$ with center $x_k \in \mathbb{I}$ and radius $\delta$ small enough such that $\mathbb{I} \subset \cup_{k=1}^m O_k$ and

$$\|h(t,u_1(t))-h(t,x_k)\| \le (\mathcal{L}(t)+1)\varepsilon, \quad u_1(t) \in O_k,\ t \in [-r,r].$$

Suppose $\|h(t,x_q)\| = \max_{1\le k\le m}\|h(t,x_k)\|$, where $q$ is an index number among $\{1,2,\dots,m\}$. The set $B_k = \{t \in [-r,r] : u_1(t) \in O_k\}$ is open in $[-r,r]$ and $[-r,r] = \cup_{k=1}^m B_k$. Let

$$E_1 = B_1, \quad E_k = B_k \setminus \cup_{j=1}^{k-1} B_j \quad (2 \le k \le m).$$

Then $E_i \cap E_j = \emptyset$ when $i \neq j$, $1 \leq i, j \leq m$. Observing that

$$\begin{aligned}
&\frac{1}{\mu([-r,r])} \int_{[-r,r]} \|h(t,u_1(t))\| d\mu(t) \\
&= \frac{1}{\mu([-r,r])} \int_{\cup_{k=1}^m E_k} \|h(t,u_1(t))\| d\mu(t) \\
&\leq \frac{1}{\mu([-r,r])} \sum_{k=1}^m \int_{E_k} (\|h(t,u_1(t)) - h(t,x_k)\| + \|h(t,x_k)\|) d\mu(t) \\
&\leq \frac{1}{\mu([-r,r])} \sum_{k=1}^m \int_{E_k} (\mathcal{L}(t)+1)\varepsilon d\mu(t) + \frac{1}{\mu([-r,r])} \sum_{k=1}^m \int_{E_k} \|h(t,x_k)\| d\mu(t) \\
&\leq \varepsilon \Big[1 + \frac{1}{\mu([-r,r])} \int_{[-r,r]} \mathcal{L}(t) d\mu(t)\Big] + \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|h(t,x_q)\| d\mu(t).
\end{aligned}$$

Taking into account $\mathcal{L}(\cdot) \in \mathscr{B}(r,\mu)$ and $h \in \varepsilon(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$, we obtain

$$\lim_{r \to +\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|h(t,u_1(t))\| d\mu(t) = 0.$$

That is, $h(\cdot, u_1(\cdot)) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. Hence $f(\cdot, u(\cdot)) \in PAA(\mathbb{R}, \mathbb{X}, \mu)$, which completes of the proof. $\square$

**Remark 3.2.** (1) Condition (A2) covers the classical Lipschitz condition as a special case. In fact, let $\mathcal{L}(t) \equiv \mathcal{L} > 0$, then

$$\lim_{r \to \infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \mathcal{L} d\mu(t) = \mathcal{L} \lim_{r \to \infty} \frac{1}{\mu([-r,r])} \mu([-r,r]) < \infty.$$

(2) For $1 < p < \infty$, if $\nu^p(\cdot) \in \mathscr{B}(r,\mu)$, then $\nu(\cdot) \in \mathscr{B}(r,\mu)$. In fact, by Hölder inequality,

$$\begin{aligned}
\frac{1}{\mu([-r,r])} \int_{[-r,r]} \nu(t) d\mu(t) &\leq \frac{1}{\mu([-r,r])} \Big[\int_{[-r,r]} \nu^p(t) d\mu(t)\Big]^{1/p} \Big[\int_{[-r,r]} d\mu(t)\Big]^{1-\frac{1}{p}} \\
&\leq \frac{\big[\int_{[-r,r]} \nu^p(t) d\mu(t)\big]^{1/p}}{[\mu([-r,r])]^{1/p}} \\
&= \Big[\frac{1}{\mu([-r,r])} \int_{[-r,r]} \nu^p(t) d\mu(t)\Big]^{1/p}.
\end{aligned}$$

Obviously, $\nu^p(\cdot) \in \mathscr{B}(r,\mu)$ implies $\nu(\cdot) \in \mathscr{B}(r,\mu)$.

(3) Considering $\mu(\mathbb{R}) = +\infty$, if $\nu : \mathbb{R} \to \mathbb{R}_+$ satisfies $\int_{\mathbb{R}} \nu(t) d\mu(t) < \infty$, then $\nu(\cdot) \in \mathscr{B}(r,\mu)$. If $p > 1$ and $\int_{\mathbb{R}} \nu^p(t) d\mu(t) < \infty$, then

$$\frac{1}{\mu([-r,r])} \int_{[-r,r]} \nu(t) d\mu(t) \leq \frac{\big[\int_{[-r,r]} \nu^p(t) d\mu(t)\big]^{1/p}}{[\mu([-r,r])]^{1/p}} \leq \frac{\big[\int_{\mathbb{R}} \nu^p(t) d\mu(t)\big]^{1/p}}{[\mu([-r,r])]^{1/p}} \to 0.$$

Thus, for $1 \leq p < \infty$, if $\int_{\mathbb{R}} \nu^p(t) d\mu(t) < \infty$, then $\nu(\cdot) \in \mathscr{B}(r,\mu)$.

(4) For pseudo almost automorphy, i.e. the measure $\mu$ is the Lebesgue measure, then $\mathcal{L}(\cdot) \in \mathscr{B}(r,\mu)$ is reduced to

$$\lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \mathcal{L}(t) dt < \infty. \tag{3.1}$$

Besides $\mathcal{L}(t) \equiv \mathcal{L} > 0$, from the above arguments (3), for any $\mathcal{L}(\cdot) \in L^p(\mathbb{R}, \mathbb{R}_+), p \geq 1$, the condition (3.1) is true. At this time, Theorem 3.1 is just as [16, Theorem 2.4].

(5) For weighted pseudo almost automorphy, i.e. the measure $\mu$ is absolutely continuous with respect to the Lebesgue measure with a Radon Nikodym derivative $\rho$, then $\mathcal{L}(\cdot) \in \mathscr{B}(r, \mu)$ is reduced to

$$\lim_{r\to\infty} \frac{1}{\int_{-r}^{r} \rho(t)dt} \int_{-r}^{r} \mathcal{L}(t)\rho(t)dt < \infty. \tag{3.2}$$

Also $\mathcal{L}(t) \equiv \mathcal{L} > 0$, owing to $\int_{-\infty}^{\infty} \rho(t) = +\infty$ and the above arguments (3), for any $\mathcal{L}(\cdot) : \mathbb{R} \to \mathbb{R}_+$ satisfying $\mathcal{L}^p(t)\rho(t) \in L^1(-\infty, +\infty)$(abbr. $\mathcal{L} \in L^p(\mathbb{R}, \rho)$),$1 \leq p < \infty$, the condition (3.2) holds true. On the other hand, if $\mathcal{L}(\cdot)\rho(\cdot) \in L^p(-\infty, +\infty)$, $p > 1$, then by Hölder inequality,

$$\frac{1}{\int_{-r}^{r} \rho(t)dt} \int_{-r}^{r} \mathcal{L}(t)\rho(t)dt \leq \frac{1}{\int_{-r}^{r} \rho(t)dt} \Big[\int_{-r}^{r} (\mathcal{L}(t)\rho(t))^p dt\Big]^{1/p} 2r.$$

Hence, if $\mathcal{L}(\cdot)\rho(\cdot) \in L^p(-\infty, +\infty),\ p > 1$ and

$$\lim_{r\to\infty} \frac{r}{\int_{-r}^{r} \rho(t)dt} < \infty,$$

then the condition (3.2) may be true.

Next, we consider a more general case in the following theorem.

**Theorem 3.3.** *Let $\mu \in \mathcal{M}$ and $f = g + h \in PAA(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$. Assume that*

(A4) *There exists a function $\mathcal{L}(\cdot) \in \mathscr{B}(r, \mu)$ such that for any bounded subset $Q \subset \mathbb{X}$ and for each $\varepsilon > 0$, there exists a constant $\delta > 0$ satisfying*

$$\|f(t, x) - f(t, y)\| \leq \mathcal{L}(t)\varepsilon$$

*for all $x,\ y \in Q$ with $\|x - y\| \leq \delta$ and $t \in \mathbb{R}$;*

(A5) *$g(t, x)$ is uniformly continuous on any bounded subset $Q \subset \mathbb{X}$ uniformly in $t \in \mathbb{R}$.*

*Then $f(\cdot, \phi(\cdot)) \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ for $\forall\phi \in PAA(\mathbb{R}, \mathbb{X}, \mu)$.*

*Proof.* Let $f = g + h$ with $g \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, $h \in \varepsilon(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$, and $\phi = u + v$, with $u \in AA(\mathbb{R}, \mathbb{X})$, and $v \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. Now we define

$$\begin{aligned} f(t, \phi(t)) &= g(t, u(t)) + f(t, \phi(t)) - g(t, u(t)) \\ &= g(t, u(t)) + f(t, \phi(t)) - f(t, u(t)) + h(t, u(t)). \end{aligned}$$

Let us rewrite

$$G(t) = g(t, u(t)), \Phi(t) = f(t, \phi(t)) - f(t, u(t)), H(t) = h(t, u(t)).$$

Thus, we have $F(t) = G(t) + \Phi(t) + H(t)$. In view of [15, Lemma 2.2], $G(t) \in AA(\mathbb{R}, \mathbb{X})$. Next we prove that $\Phi(t) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. Clearly , $\Phi(t) \in BC(\mathbb{R}, \mathbb{X})$, and we can assume that $\|\Phi(t)\| \leq \mathcal{C}$. For $\Phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$, it is enough to show that

$$\lim_{r\to\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \|\Phi(t)\| d\mu(t) = 0.$$

By Lemma 2.11, $u(\mathbb{R}) \subset \overline{\phi(\mathbb{R})}$ is a bounded set. From assumption (A4) with $Q = \overline{\phi(\mathbb{R})}$, we conclude that for each $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for all $t \in \mathbb{R}$,

$$\|\phi - u\| \leq \delta \;\Rightarrow\; \|f(t, \phi(t)) - f(t, u(t))\| \leq \mathcal{L}(t)\varepsilon.$$

Denote by the following set $A_{r,\varepsilon}(v) = \{t \in [-r, r] : \|v(t)\| > \varepsilon\}$. Thus we obtain

$$\begin{aligned} A_{r,\mathcal{L}(t)\varepsilon}(\Phi) &= A_{r,\mathcal{L}(t)\varepsilon}(f(t, \phi(t)) - f(t, u(t))) \\ &\subseteq A_{r,\delta}(\phi(t) - u(t)) = A_{r,\delta}(v). \end{aligned}$$

Therefore

$$\frac{\mu\{A_{r,\mathcal{L}(t)\varepsilon}(\Phi)\}}{\mu([-r, r])} \leq \frac{\mu\{A_{r,\delta}(v)\}}{\mu([-r, r])}.$$

Since $\phi(t) = u(t) + v(t)$ and $v \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$, Lemma 2.10 (III) yields that for the above-mentioned $\delta$ we have

$$\lim_{r\to\infty} \frac{\mu\{t \in [-r, r] : \|\phi(t) - u(t)\| > \delta\}}{\mu([-r, r])} = 0,$$

and then we obtain

$$\lim_{r\to\infty} \frac{\mu\{A_{r,\mathcal{L}(t)\varepsilon}(\Phi)\}}{\mu([-r, r])} = 0. \tag{3.3}$$

Thus

$$\begin{aligned} &\frac{1}{\mu([-r, r])} \int_{[-r,r]} \|\Phi(t)\| d\mu(t) \\ &= \frac{1}{\mu([-r, r])} \int_{A_{r,\mathcal{L}(t)\varepsilon}} \|\Phi(t)\| d\mu(t) + \frac{1}{\mu([-r, r])} \int_{[-r,r]\setminus A_{r,\mathcal{L}(t)\varepsilon}} \|\Phi(t)\| d\mu(t) \\ &\leq \mathcal{C} \frac{\mu\{A_{r,\mathcal{L}(t)\varepsilon}(\Phi)\}}{\mu([-r, r])} + \varepsilon \frac{1}{\mu([-r, r])} \int_{[-r,r]} \mathcal{L}(t) d\mu(t). \end{aligned}$$

From relation (3.3) and the fact $\mathcal{L}(\cdot) \in \mathscr{B}(r, \mu)$, we can see that

$$\lim_{r\to\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \|\Phi(t)\| d\mu(t) = 0,$$

i.e. $\Phi(t) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$.

Finally, it is only to show that $H(t) = h(t, u(t)) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. We have the set $u([-r, r])$ is compact since $u$ is continuous on $\mathbb{R}$ as almost automorphic functions. So $g$ is uniformly continuous on $\mathbb{R} \times u([-r, r])$. Then it follows from (A4) that for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for $x_1,\ x_2 \in u([-r, r])$ with $\|x_1 - x_2\| < \delta$ we have

$$\begin{aligned} \|h(t, x_1) - h(t, x_2)\| &= \|[f(t, x_1) - f(t, x_2)] + [g(t, x_2) - g(t, x_1)]\| \\ &\leq \|f(t, x_1) - f(t, x_2)\| + \|g(t, x_2) - g(t, x_1)\| \\ &\leq (\mathcal{L}(t) + 1)\varepsilon, \forall t \in [-r, r]. \end{aligned}$$

The remainder of the proof is similar to that of Theorem 3.1, we can also show that $H(t) = h(t, u(t)) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. This completes the proof. □

**Remark 3.4.** (1) Condition (A4) covers the following Hölder type condition as an special case:

(A4') There exists a function $\mathcal{L}(\cdot) \in \mathscr{B}(r,\mu)$ such that for any bounded subset $Q \subset \mathbb{X}$ satisfying

$$\|f(t,x) - f(t,y)\| \le \mathcal{L}(t)\|x-y\|^{\eta}, \quad 0 < \eta < 1,$$

for all $x,\, y \in Q$ and $t \in \mathbb{R}$.

In fact, if condition (A4') holds, then for any bounded subset $Q \subset \mathbb{X}$ and for each $\varepsilon > 0$, there exists a constant $\delta = (\varepsilon)^{\frac{1}{\eta}}$ such that for all $x,\, y \in Q$ with $\|x-y\| \le \delta = (\varepsilon)^{\frac{1}{\eta}}$ and $t \in \mathbb{R}$ satisfying

$$\|f(t,x) - f(t,y)\| \le \mathcal{L}(t)\|x-y\|^{\eta} \le \mathcal{L}(t)[(\varepsilon)^{\frac{1}{\eta}}]^{\eta} < \mathcal{L}(t)\varepsilon.$$

(2) Take $\mathcal{L}(t) \equiv \mathcal{L}$, then the condition (A4) is reduced to the following well-known uniformly continuous condition

(A4") $f(t,x)$ is uniformly continuous on any bounded subset $Q \subset \mathbb{X}$ uniformly in $t \in \mathbb{R}$.

From the proofs of Theorems 3.1-3.3, we can conclude the following corollary.

**Corollary 3.5.** *Let $\mu \in \mathcal{M}$ and $h \in \varepsilon(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$. Assume that*

(A6) *There exists a function $\mathcal{L}(\cdot) \in \mathscr{B}(r,\mu)$ such that for any bounded subset $Q \subset \mathbb{X}$ and for each $\varepsilon > 0$, there exists a constant $\delta > 0$ satisfying*

$$\|h(t,x) - h(t,y)\| \le \mathcal{L}(t)\varepsilon$$

*for all $x,\, y \in Q$ with $\|x-y\| \le \delta$ and $t \in \mathbb{R}$.*

*Then $h(\cdot,\phi(\cdot)) \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$ for all $\phi \in AA(\mathbb{R},\mathbb{X})$.*

### 3.2. Existence of $\mu$-pseudo almost automorphic solutions to (1.1).

**Definition 3.6** ([23])**.** Let $\alpha > 0$ and $\mathscr{A}$ be the generator of an $\alpha$-resolvet family $\{S_\alpha(t)\}_{t\ge 0}$. A function $u \in C(\mathbb{R},\mathbb{X})$ is called a mild solution to (1.1) if the function $s \mapsto S_\alpha(t-s)f(s,u(\gamma(s)))$ is integrable on $(-\infty,t)$ for each $t \in \mathbb{R}$ and

$$u(t) = \int_{-\infty}^{t} S_\alpha(t-s)f(s,u(\gamma(s)))ds.$$

The following lemma can be similarly derived as [24, Lemma7].

**Lemma 3.7.** *Suppose $\mu \in \mathcal{M}$ and the following condition is satisfied*

(A7) *$\gamma : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function, and there exists a continuous function $\lambda : \mathbb{R} \to \mathbb{R}_+$ such that*

$$d\mu_\gamma(t) \le \lambda(t)d\mu(t), \qquad \sup_{t\in[-r^*,r^*]} \lambda(t) = M_{r^*}, \qquad \lim_{r\to\infty} \sup\Big(\frac{M_{r^*}\mu([-r^*,r^*])}{\mu([-r,r])}\Big) < \infty,$$

*where $\mu_\gamma(\mathbb{A}) = \mu(\gamma^{-1}(\mathbb{A}))$ for all $\mathbb{A} \in \mathcal{B}(\mathbb{R})$, $r^* = |\gamma(-r)| + |\gamma(r)|$, and for each $u(\cdot) \in AA(\mathbb{R},\mathbb{X})$, $u(\gamma(\cdot)) \in AA(\mathbb{R},\mathbb{X})$.*

*If $u(\cdot) \in PAA(\mathbb{R},\mathbb{X},\mu)$, then $u(\gamma(\cdot)) \in PAA(\mathbb{R},\mathbb{X},\mu)$.*

Let us list the some assumptions to be used later.

(A8) Assume that the operator $\mathscr{A}$ generates an $\alpha$-resolvent family $\{S_\alpha(t)\}_{t\ge 0}$ on a Banach space $\mathbb{X}$, and there exist constants $C > 0, \omega > 0$ such that $\|S_\alpha(t)\| \le Ce^{-\omega t}$ for all $t \ge 0$;

(A9) there exists a nonegative function $l \in L^p(\mathbb{R}) \cap \mathscr{B}(r,\mu)(1 \le p < \infty)$ such that $f = g + h \in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$ satisfies conditions (A2) and (A3) in Theorem 3.1.

The following lemma can be derived from [5, Theorem 3.9].

**Lemma 3.8.** Let $\mu \in \mathcal{M}$. Assume that the operator $\mathscr{A}$ generates an $\alpha$-resolvent family $\{S_\alpha(t)\}_{t\ge 0}$ satisfying the condition (A8). If $f \in PAA(\mathbb{R},\mathbb{X},\mu)$, then

$$F(t) = \int_{-\infty}^{t} S_\alpha(t-s)f(s)ds \in PAA(\mathbb{R},\mathbb{X},\mu), \quad t \in \mathbb{R}.$$

**Theorem 3.9.** *Assume that conditions* (A7)–(A9) *are satisfied. Then* (1.1) *admits a unique mild solution* $u \in PAA(\mathbb{R},\mathbb{X},\mu)$.

*Proof.* Let the operator $\Upsilon$ be defined as

$$(\Upsilon u)(t) := \int_{-\infty}^{t} S_\alpha(t-s)f(s,u(\gamma(s)))ds.$$

For $u \in PAA(\mathbb{R},\mathbb{X},\mu)$, by Lemma 3.7 and Theorem 3.1, it follows that the function $s \to f(s,u(\gamma(s)))$ is in $PAA(\mathbb{R},\mathbb{X},\mu)$. Moreover, from Lemma 3.8 we infer that $\Upsilon u \in PAA(\mathbb{R},\mathbb{X},\mu)$, that is, $\Upsilon$ maps $PAA(\mathbb{R},\mathbb{X},\mu)$ into itself.

Since $l \in L^p(\mathbb{R}), 1 < p < \infty$, let $\tau(t) = \int_{-\infty}^{t} l^p(s)ds$. Now we define an equivalent norm over $PAA(\mathbb{R},\mathbb{X},\mu)$ as

$$\|f\|_\tau = \sup_{t\in\mathbb{R}}\{e^{-\theta\tau(t)}\|f\|\}, \quad f \in PAA(\mathbb{R},\mathbb{X},\mu),$$

where $\theta > 0$, is a sufficiently large constant. Then, for each $u$, $v \in PAA(\mathbb{R},\mathbb{X},\mu)$, we have

$$\begin{aligned}
\|(\Upsilon u)(t) - (\Upsilon v)(t)\| &\le \int_{-\infty}^{t} \|S_\alpha(t-s)[f(s,u(s)) - f(s,v(s))]\|ds \\
&\le C\int_{-\infty}^{t} e^{-\omega(t-s)}l(s)\|u(s)-v(s)\|ds \\
&\le C\int_{-\infty}^{t} e^{-\omega(t-s)}l(s)e^{\theta\tau(s)}\|u-v\|_\tau ds \\
&\le C\Big[\int_{-\infty}^{t} e^{\theta p\tau(s)}l^p(s)ds\Big]^{1/p}\Big[\int_{-\infty}^{t} e^{-\omega q(t-s)}ds\Big]^{1/q}\|u-v\|_\tau \\
&\le C(\omega q)^{-1/q}\Big[\int_{-\infty}^{t} e^{\theta p\tau(s)}d\tau(s)\Big]^{1/p}\|u-v\|_\tau \\
&\le C(\omega q)^{-1/q}(p\theta)^{-1/p}e^{\theta\tau(t)}\|u-v\|_\tau.
\end{aligned}$$

Consequently,

$$\|\Upsilon u - \Upsilon v\|_\tau \le C(\omega q)^{-1/q}(p\theta)^{-1/p}\|u-v\|_\tau,$$

which implies that $\Upsilon$ is a contraction for sufficiently large $\theta$.

On the other hand, for $p = 1$, we have

$$\begin{aligned}
\|(\Upsilon u)(t) - (\Upsilon v)(t)\| &\le \int_{-\infty}^{t} \|S_\alpha(t-s)[f(s,u(s)) - f(s,v(s))]\|ds \\
&\le C\int_{-\infty}^{t} l(s)\|u(s)-v(s)\|ds
\end{aligned}$$

$$\le C\|u-v\|_\infty \int_{-\infty}^t l(s)ds,$$

and

$$\begin{aligned}\|(\Upsilon^2 u)(t)-(\Upsilon^2 u)(t)\| &\le C\int_{-\infty}^t l(s)\|(\Upsilon u)(s)-(\Upsilon v)(s)\|ds\\ &\le C^2\|u-v\|_\infty \int_{-\infty}^t l(s)\int_{-\infty}^s l(\sigma)d\sigma ds\\ &\le \frac{C^2}{2}\|u-v\|_\infty\Big(\int_{-\infty}^t l(s)ds\Big)^2.\end{aligned}$$

Using induction on $n$ ,in the same way, we obtain

$$\begin{aligned}\|(\Upsilon^n u)(t)-(\Upsilon^n v)(t)\| &\le \frac{C^n}{(n-1)!}\|u-v\|_\infty\Big[\int_{-\infty}^t l(s)\Big(\int_{-\infty}^s l(\sigma)d\sigma\Big)^{n-1}ds\Big]\\ &\le \frac{C^n}{n!}\|u-v\|_\infty\Big(\int_{-\infty}^t l(s)ds\Big)^n.\end{aligned}$$

Thus,

$$\|\Upsilon^n u-\Upsilon^n v\|_\infty \le \frac{(C\|l\|_{L^1(\mathbb{R})})^n}{n!}\|u-v\|_\infty.$$

Since $\frac{(C\|l\|_{L^1(\mathbb{R})})^n}{n!}<1$ for $n$ sufficiently large, $\Upsilon$ is still a contraction.

From the above arguments, we can show $\Upsilon$ is a contraction for $p\ge 1$. We can complete the whole proof via Banach contraction mapping principle. □

Finally, we consider some special results on pseudo almost typed automorphic solutions to the equation (1.1).

**Lemma 3.10** ([7, Lemma 3.1]). *Assume that the following condition is satisfied*

(A7') $\gamma:\mathbb{R}\to\mathbb{R}$ *is continuously differentiable on* $\mathbb{R}$, *and* $\gamma'(t)>0$ *is nondecreasing with*

$$\limsup_{r\to\infty}\Big(\frac{|\gamma(-r)|+|\gamma(r)|}{r\gamma'(-r)}\Big)<\infty,$$

*and for each* $u(\cdot)\in AA(\mathbb{R},\mathbb{X})$, $u(\gamma(\cdot))\in AA(\mathbb{R},\mathbb{X})$.

*If* $u(\cdot)\in PAA(\mathbb{R},\mathbb{X})$, *then* $u(\gamma(\cdot))\in PAA(\mathbb{R},\mathbb{X})$.

**Lemma 3.11** ([7, Lemma 3.2]). Assume that the following condition holds

(A7") $\gamma:\mathbb{R}\to\mathbb{R}$ is continuously differentiable on $\mathbb{R}$, and $\gamma'(t)>0$ is nondecreasing with

$$\limsup_{r\to\infty}\Big(\frac{m(r^*,\rho)}{m(r,\rho)\gamma'(-r)}\Big)<\infty,\quad\text{and}\quad 0<\sup_{t\in\mathbb{R}}\frac{\rho(t)}{\rho(\gamma(t))}<\infty,$$

and for each $u(\cdot)\in AA(\mathbb{R},\mathbb{X})$, $u(\gamma(\cdot))\in AA(\mathbb{R},\mathbb{X})$, where $\gamma^*=|\gamma(-r)|+|\gamma(r)|$.

If $u(\cdot)\in WPAA(\mathbb{R},\mathbb{X},\rho)$, then $u(\gamma(\cdot))\in WPAA(\mathbb{R},\mathbb{X},\rho)$.

The following result is based on Theorem 3.9 and Lemma 3.10.

**Corollary 3.12.** *Let* $f=g+h\in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X})$ *with* $g\in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$ *satisfying (A3) in Theorem 3.1,* $h\in PAA_0(\mathbb{R}\times\mathbb{X},\mathbb{X})$. *Assume that (A7')–(A8) and the following conditions hold*

(A2') *There exists a function* $l \in L^p(\mathbb{R})(1 \le p < \infty)$ *such that*

$$\|f(t,x) - f(t,y)\| \le l(t)\|x - y\|$$

*for all* $x$, $y \in \mathbb{X}$ *and* $t \in \mathbb{R}$.

*Then* (1.1) *has a unique mild solution in* $PAA(\mathbb{R}, \mathbb{X})$.

Theorem 3.9 and Lemma 3.11 imply the following result.

**Corollary 3.13.** *Let* $\rho \in \mathbb{U}$, *and* $f = g + h \in WPAA(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \rho)$ *with* $g \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ *satisfying (A3) in Theorem 3.1,* $h \in PAA_0(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \rho)$. *Assume that* (A7")-(A8) *and the following conditions hold*

(A2") *There exists a function* $l \in L^p(\mathbb{R}) \cap L^p(\mathbb{R}, \rho)(1 \le p < \infty)$ *such that*

$$\|f(t,x) - f(t,y)\| \le l(t)\|x - y\|$$

*for all* $x$, $y \in \mathbb{X}$ *and* $t \in \mathbb{R}$.

*Then* (1.1) *admits a unique mild solution in* $WPAA(\mathbb{R}, \mathbb{X}, \rho)$.

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