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EXISTENCE OF POSITIVE SOLUTIONS TO KIRCHHOFF TYPE PROBLEMS INVOLVING SINGULAR AND CRITICAL NONLINEARITIES

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ABSTRACT. In this study, we study a Kirchhoff type problem involving singular and critical nonlinearities. With aid of variational methods and concentration compactness principle, we prove that the problem admits a weak solution.

1. Introduction and statement of main result

We are interested in the Kirchhoff type problem

$$\begin{gathered} -\Big(a+b\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = f(x,u), \quad \text{in } \Omega,\\ u=0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^3$, $0\in\Omega$, $a>0, b\geq 0$.

Existence and multiplicity of solutions to (1.1) have been studied intensively by many researchers. There are lots of works in the literature not only on the subcritical cases such as [2, 5, 11, 12, 20, 24, 26, 27, 30], but also on the critical cases like [3, 8, 9, 10, 13, 15, 17, 18, 19, 21, 23, 28, 29]. In particular, Naimen [22] investigated the kirchhoff type equation

$$\begin{gathered} -\Big(1+b\int_\Omega |\nabla u|^2\Big)\Delta u = \beta u + u^5, \quad \text{in } \Omega,\\ u=0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

here Ω is a 3 dimensional open ball. For the reader's convenience, we report here one of the main results of [22].

Theorem 1.1 ([22, Theorem 1.1]). *Let $\beta\in\mathbb{R}$ be a given constant. Then the following assertions hold.*

- (i) *If $\beta<\beta_1/4$ (β_1 is the principal eigenvalue of $-\Delta$ on the open ball), problem (1.2) has no solution for all $b\geq 0$.*
- (ii) *If $\beta_1/4<\beta<\beta_1$, there exists a constant $A_1=A_1(\beta)>0$ such that (1.2) has a solution for all $0<b<A_1$.*

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(iii) *If $\beta = \beta_1$, there exists a constant $A_2 = A_2(\beta) > 0$ such that* (1.2) *has a solution for all $0 < b < A_2$ and* (1.2) *has no solution for $b = 0$.*

In (1.2), if $b = 0$, Brezis-Nirenberg [4] found a solution provided $\beta_1/4 < \beta < \beta_1$, thereby, Theorem 1.1 (ii) extends one of the main results of Brezis-Nirenberg [4] to the Kirchhoff type problem. When $N = 3$, we see that it is not easy to establish a solution in the case of $0 < \beta < \beta_1$, the reason is that, it is difficult to estimate the critical value level for this case. However, for 4-dimensional case, Brezis and Nirenberg [4] obtained a positive solution provided $0 < \beta < \beta_1$. Therefore, we also see that dimensions of space make an effect on parameter β.

Recently, Perera et al. [25] considered the problem

$$\begin{gathered} -\Delta u = \beta u + u^{2^*-1} - \mu, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 4$). They obtained a ground state solution when $0 < \beta < \beta_1$ and $\mu > 0$ enough small. It remains open to extend this study for the case $N = 3$ (see [25, Remark 1.4]).

Based on the above work, in this article we consider the case that problem has a combination of a critical Sobolev exponent term and a singular term. More precisely, we study the Kirchhoff type equation of the form

$$\begin{gathered} -\Big(a + b\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = \lambda\frac{u}{|x|^{2-s}} + u^5 - \mu, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

where $0 < s < 1$, λ, μ are two positive real numbers, and $0 < \lambda < a\lambda_1$, here λ_1 is the first eigenvalue for eigenvalue problem

$$\begin{gathered} -\Delta u = \lambda|x|^{s-2}u \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

where $0 < s < 2$, Chaudhuri et al. in [7] proved that problem (1.4) has a sequence of eigenvalues

$$0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \leq \dots \to +\infty.$$

Moreover, the first eigenvalue is characterized by

$$\lambda_1 := \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2|x|^{s-2}}.$$

Our main result reads as follows.

Theorem 1.2. *Assume $a > 0, b \geq 0$ and $0 < \lambda < a\lambda_1$. Then there exists $\mu_* > 0$ such that* (1.3) *has at least a nontrivial solution for every $\mu \in (0, \mu_*)$. Moreover, if $\mu = 0$, then* (1.3) *admits a positive solution.*

Remark 1.3. On the one hand, compared with Theorem 1.1, we see that the coefficient b is restrained in (ii) and (iii). Moreover, in Theorem 1.2, we also see that the singular term $1/|x|^{2-s}$ can release the restriction on $\beta_1/4 < \beta < \beta_1$. On the other hand, the problem mentioned in [25, Remark 1.4] is hard to tackle, however, if we add a singular term, the problem can be solved. So our results can be regarded as partial solution to that problem.

In the next section we present some lemmas and the proof of Theorem 1.2.

2. Proof of main results

Let us give the following some notation:

- The space $H_0^1(\Omega)$ is equipped with the norm $\|u\|^2 = \int_\Omega |\nabla u|^2\,dx$, the norm in $L^p(\Omega)$ is denoted by $|\cdot|_p$;
- $u_n^+(x) = \max\{u_n(x), 0\}$, $u_n^-(x) = \max\{-u_n(x), 0\}$; $C, C_1, C_2, \dots,$ denote various positive constants, which may vary from line to line;
- Let S be the best Sobolev constant, namely

$$S := \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx}{\left(\int_\Omega |u|^6\,dx\right)^{1/3}}. \tag{2.1}$$

Existence of a positive solution. Consider the energy functional $I_\mu : H_0^1(\Omega) \to \mathbb{R}$ given by

$$I_\mu(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{2}\int_\Omega \frac{(u^+)^2}{|x|^{2-s}}\,dx - \frac{1}{6}\int_\Omega (u^+)^6\,dx + \mu\int_\Omega u\,dx.$$

Lemma 2.1. *There exist $\alpha, \rho, \Lambda_0 > 0$ such that the functional I_μ satisfies the following conditions for each $\mu \in [0, \Lambda_0)$:*

(i) *$I_\mu(u) > \alpha$ if $\|u\| = \rho$;*
(ii) *There exists $e \in H_0^1(\Omega)$ such that $I_\mu(e) < 0$.*

Proof. (i) For $u \in H_0^1(\Omega)$, by Sobolev and Young inequalities, it holds that

$$\mu\int_\Omega u^-\,dx \le \frac{5}{6}|\Omega|\mu^{6/5} + \frac{1}{6S^3}\|u\|^6.$$

Then

$$\begin{aligned} I_\mu(u) &\ge \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{2}\int_\Omega \frac{(u^+)^2}{|x|^{2-s}}\,dx - \frac{1}{6}\int_\Omega (u^+)^6\,dx - \mu\int_\Omega u^-\,dx \\ &\ge \frac{a\lambda_1 - \lambda}{2\lambda_1}\|u\|^2 - \frac{1}{3S^3}\|u\|^6 - \frac{5}{6}|\Omega|\mu^{6/5}. \end{aligned}$$

Set $\rho = [\frac{(a\lambda_1-\lambda)S^3}{2\lambda_1}]^{1/4}$, $\Lambda_0 = [\frac{2}{5}(\frac{S(a\lambda_1-\lambda)}{2\lambda_1})^{3/2}|\Omega|^{-1}]^{5/6}$, we have

$$I_\mu\big|_{\|u\|=\rho} \ge \frac{1}{3}\Big[\frac{(a\lambda_1-\lambda)S}{2\lambda_1}\Big]^{3/2} =: \alpha$$

provided $\mu \in [0, \Lambda_0)$.

(ii) For $u \in H_0^1(\Omega)\setminus\{0\}$, $t > 0$, it holds that

$$I_\mu(tu) \le \frac{at^2}{2}\|u\|^2 + \frac{bt^4}{4}\|u\|^4 - \frac{t^6}{6}\int_\Omega (u^+)^6\,dx + \mu t\int_\Omega u\,dx \to -\infty$$

as $t \to \infty$. So we can easily find $e \in H_0^1(\Omega)$ with $\|e\| > \rho$, such that $I_\mu(e) < 0$. The proof is complete. $\square$

To use variational methods, we firstly derive some results related to the Palais-Smale compactness condition. We say that I_μ satisfies the (PS) condition at the level $c \in \mathbb{R}$ ($(PS)_c$ condition for short) if any sequence $\{u_n\} \subset H_0^1(\Omega)$ along with

$$I_\mu(u_n) \to c, \quad I'_\mu(u_n) \to 0 \quad \text{in } (H_0^1(\Omega))^*$$

as $n \to \infty$ possesses a convergent subsequence. If I_μ satisfies $(PS)_c$ condition for each $c \in \mathbb{R}$, then we say that I_μ satisfies the (PS) condition. Define

$$\Lambda = \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(b^2S^4 + 4aS)^{3/2}}{24}.$$

Lemma 2.2. *Assume $0 < \lambda < a\lambda_1$, then I_μ satisfies the $(PS)_c$ condition for $c < \Lambda - D\mu^{6/5}$, where $D = \frac{5}{6}(9|\Omega|^{5/6}6^{-\frac{1}{6}})^{6/5}$.*

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ be a sequence satisfying

$$I_\mu(u_n) \to c, \quad I'_\mu(u_n) \to 0, \quad \text{as } n \to \infty. \tag{2.2}$$

on the contrary assume $\{u_n\}$ is unbounded, then

$$\begin{aligned} 1 + c + o(1)\|u_n\| &\geq I_\mu(u_n) - \frac{1}{6}\langle I'_\mu(u_n), u_n\rangle \\ &\geq \frac{a}{3}\|u_n\|^2 - \frac{\lambda}{3}\int_\Omega \frac{(u^+)^2}{|x|^{2-s}}\,dx - \frac{5\mu}{6}\int_\Omega u_n^-\,dx \\ &\geq \frac{a\lambda_1 - \lambda}{3\lambda_1}\|u_n\|^2 - C\|u_n\|, \end{aligned}$$

which implies that the last inequality is an absurd. So $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Based on the concentration compactness principle (see [16]), there exist a subsequence, still denoted by $\{u_n\}$, such that

$$\begin{gathered} |\nabla u_n|^2 \rightharpoonup d\eta \geq |\nabla u|^2 + \sum_{j\in J}\eta_j\delta_j, \\ |u_n|^6 \rightharpoonup d\gamma = |u|^6 + \sum_{j\in J}\gamma_j\delta_j, \end{gathered}$$

where J is an at most countable index set, δ_{x_j} is the Dirac mass at x_j, and let $x_j \in \Omega$ in the support of η, γ. Moreover, it holds

$$\eta_j \geq S\gamma_j^{1/3} \quad textforallj \in J. \tag{2.3}$$

For $\varepsilon > 0$, let $\phi_{\varepsilon,j}(x)$ be a smooth cut-off function centered at x_j such that $0 \leq \phi_{\varepsilon,j}(x) \leq 1$, and

$$\phi_{\varepsilon,j}(x) = \begin{cases} 1 & \text{in } B(x_j, \\ = 0 & \text{in } \Omega\backslash B(x_j, 2\varepsilon), \end{cases} \qquad |\nabla\phi_{\varepsilon,j}(x)| \leq \frac{2}{\varepsilon}.$$

By Hölder's inequality and (2.1),

$$\begin{aligned} \Big|\int_\Omega \frac{u_n^+}{|x|^{2-s}}\phi_{\varepsilon,j}u_n\,dx\Big| &\leq \Big(\int_{B(x_j,2\varepsilon)}|u_n|^6\,dx\Big)^{1/3}\Big(\int_{B(x_j,2\varepsilon)}\frac{dx}{|x|^{\frac{3(2-s)}{2}}}\Big)^{2/3} \\ &\leq C\|u_n\|^2\varepsilon^s. \end{aligned}$$

Note that $\{u_n\}$ is bounded in $H_0^1(\Omega)$, then

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \frac{u_n^+}{|x|^{2-s}}\phi_{\varepsilon,j}u_n\,dx = 0.$$

Similarly, we have

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \phi_{\varepsilon,j}u_n\,dx = 0.$$

As $\phi_{\varepsilon,j}u_n$ is bounded in $H_0^1(\Omega)$, taking the test function $\varphi = \phi_{\varepsilon,j}u_n$ in (2.2), it holds

$$\begin{aligned}
0 &= \lim_{\varepsilon\to 0}\lim_{n\to\infty}\langle I'_\mu(u_n), \phi_{\varepsilon,j}u_n\rangle \\
&= \lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big\{(a+b\|u_n\|^2)\int_\Omega (\nabla u_n, \nabla(\phi_{\varepsilon,j}u_n))\,dx \\
&\quad - \lambda\int_\Omega \frac{(u_n^+)^2}{|x|^{2-s}}\phi_{\varepsilon,j}\,dx - \int_\Omega (u_n^+)^6\phi_{\varepsilon,j}\,dx + \mu\int_\Omega u_n\phi_{\varepsilon,j}\,dx\Big\} \\
&= \lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big\{(a+b\|u_n\|^2)\int_\Omega \big(|\nabla u_n|^2\phi_{\varepsilon,j} + u_n\nabla u_n\nabla\phi_{\varepsilon,j}\big)\,dx - \int_\Omega |u_n|^6\phi_{\varepsilon,j}\,dx\Big\} \\
&\geq (a+b\eta_j)\eta_j - \gamma_j,
\end{aligned}$$

so that $\gamma_j \geq (a+b\eta_j)\eta_j$. Applying (2.3), we deduce that

$$\gamma_j \geq aS\gamma_j^{1/3} + bS^2\gamma_j^{2/3}, \quad \text{or} \quad \gamma_j = 0. \tag{2.4}$$

Set $X = \nu_j^{1/3}$, it follows from (2.4) that $X^2 \geq aS + bS^2X$; that is,

$$X \geq \frac{bS^2 + \sqrt{b^2S^4 + 4aS}}{2},$$

using (2.3) again, consequently

$$\eta_j \geq SX \geq \frac{bS^3 + \sqrt{b^2S^6 + 4aS^3}}{2} =: K.$$

Next we show that

$$\eta_j \geq \frac{bS^3 + \sqrt{b^2S^6 + 4aS^3}}{2}$$

is impossible. To obtain a contradiction assume that there exists $j_0 \in J$ such that $\eta_{j_0} \geq \frac{bS^3+\sqrt{b^2S^6+4aS^3}}{2}$. By (2.2) and Young inequality,

$$\begin{aligned}
c &= \lim_{n\to\infty}\big\{I_\mu(u_n) - \frac{1}{4}\langle I'_\mu(u_n), u_n\rangle\big\} \\
&= \lim_{n\to\infty}\Big\{(\frac{1}{2}-\frac{1}{4})a\|u_n\|^2 + b(\frac{1}{4}-\frac{1}{4})\|u_n\|^4 + (\frac{1}{4}-\frac{1}{6})\int_\Omega |u_n|^6\,dx \\
&\quad + \frac{3\mu}{4}\int_\Omega u_n\,dx - \frac{\lambda}{4}\int_\Omega |u_n|^2|x|^{s-2}\,dx\Big\} \\
&\geq \frac{a}{4}\Big(\|u\|^2 + \sum_{j\in J}\eta_j\Big) + \frac{1}{12}\Big(\int_\Omega |u|^6\,dx + \sum_{j\in J}\gamma_j\Big) \\
&\quad - \frac{\lambda}{4}\int_\Omega |u|^2|x|^{s-2}\,dx - \frac{3\mu}{4}\int_\Omega u^-\,dx \\
&\geq \frac{a}{4}\eta_{j_0} + \frac{1}{12}\gamma_{j_0} + \frac{a}{4}\|u\|^2 - \frac{\lambda}{4}\int_\Omega |u|^2|x|^{s-2}\,dx \\
&\quad + \frac{1}{12}\int_\Omega |u|^6\,dx - \frac{3\mu}{4}\int_\Omega |u|\,dx \\
&\geq \frac{aK}{2} + \frac{b}{4}K^2 - \frac{K^3}{6S^3} - \frac{1}{4}\Big(aK + bK^2 - \frac{K^3}{S^3}\Big) - D\mu^{6/5},
\end{aligned}$$

where $D = \frac{5}{6}(9|\Omega|^{5/6}6^{-\frac{1}{6}})^{6/5}$. Easy computations show that

$$\frac{aK}{2} + \frac{b}{4}K^2 - \frac{K^3}{6S^3} = \Lambda,$$
$$aK + bK^2 - K^3S^{-3} = 0.$$

Applying the result, we get $\Lambda - D\mu^{6/5} \le c < \Lambda - D\mu^{6/5}$. This is a contradiction. It indicates that J is empty, which implies that

$$\int_\Omega (u_n^+)^6\,dx \to \int_\Omega (u^+)^6\,dx.$$

Now, set $\lim_{n\to\infty}\|u_n\| = l$, by (2.2), we have

$$(a + b\|u_n\|^2)\|u_n\|^2 - \lambda\int_\Omega (u_n^+)^2|x|^{s-2}\,dx - \int_\Omega (u_n^+)^6\,dx + \mu\int_\Omega u_n\,dx = o(1), \tag{2.5}$$

and

$$\begin{aligned}&(a + b\|u_n\|^2)\int_\Omega (\nabla u_n, \nabla\varphi)\,dx\\ &= \lambda\int_\Omega u_n^+\varphi|x|^{s-2}\,dx + \int_\Omega (u_n^+)^5\varphi\,dx - \mu\int_\Omega \varphi\,dx + o(1)\end{aligned} \tag{2.6}$$

for any $\varphi \in H_0^1(\Omega)$. Let $n \to \infty$, then from (2.5), one gets

$$(a + bl^2)l^2 - \lambda\int_\Omega (u^+)^2|x|^{s-2}\,dx - \int_\Omega (u^+)^6\,dx + \mu\int_\Omega u\,dx = 0.$$

Similarly, from (2.6),

$$\begin{aligned}&(a + bl^2)\int_\Omega (\nabla u, \nabla\varphi)\,dx\\ &= \lambda\int_\Omega u^+\varphi|x|^{s-2}\,dx + \int_\Omega (u^+)^5\varphi\,dx - \mu\int_\Omega \varphi\,dx.\end{aligned} \tag{2.7}$$

Taking the test function $\varphi = u$ in (2.7), we have

$$(a + bl^2)\|u\|^2 - \lambda\int_\Omega (u^+)^2|x|^{s-2}\,dx - \int_\Omega (u^+)^6\,dx + \mu\int_\Omega u\,dx = 0.$$

So we obtain $l = \|u\|$, consequently $u_n \to u$ in $H_0^1(\Omega)$. The proof is complete. $\square$

From [4], it is well known that the function

$$U_\varepsilon(x) = \frac{(3\varepsilon)^{1/4}}{(\varepsilon + |x|^2)^{1/2}}, \quad x \in \mathbb{R}^3,\ \varepsilon > 0$$

satisfies

$$-\Delta U_\varepsilon = U_\varepsilon^5 \quad \text{in } \mathbb{R}^3,$$
$$\int_{\mathbb{R}^3} |U_\varepsilon|^6 = \int_{\mathbb{R}^3} |\nabla U_\varepsilon|^2 = S^{3/2}.$$

Let $\eta \in C_0^\infty(\Omega)$ be a cut-off function such that $0 \le \eta \le 1$, $|\nabla\eta| \le C$ and $\eta(x) = 1$ for $|x| < R_0$ and $\eta(x) = 0$ for $|x| > 2R_0$, we set $u_\varepsilon(x) = \eta(x)U_\varepsilon(x)$. Then it holds

$$\|u_\varepsilon\|^2 = S^{3/2} + O(\varepsilon^{1/2}),$$
$$|u_\varepsilon|_6^6 = S^{3/2} + O(\varepsilon^{3/2}).$$

Lemma 2.3. *Assume* $0 < s < 1$, *then* $\sup_{t\geq 0} I_\mu(tu_\varepsilon) < \Lambda - D\mu^{6/5}$ *for some* $\varepsilon = \varepsilon(\mu) > 0$ *small enough.*

Proof. Since $\lim_{t\to\infty} I_\mu(tu_\varepsilon) = -\infty$, which suggests that $\sup_{t\geq 0} I_\mu(tu_\varepsilon)$ attained at $t_\varepsilon > 0$, i.e.,

$$at_\varepsilon\|u_\varepsilon\|^2 + bt_\varepsilon^3\|u_\varepsilon\|^4 - \lambda t_\varepsilon \int_\Omega \frac{u_\varepsilon^2}{|x|^{2-s}}\,dx - t_\varepsilon^5 \int_\Omega u_\varepsilon^6\,dx + \mu \int_\Omega u_\varepsilon\,dx = 0,$$

so that

$$t_\varepsilon^4 \int_\Omega u_\varepsilon^6\,dx \geq a\|u_\varepsilon\|^2 + bt_\varepsilon^2\|u_\varepsilon\|^4 - \lambda \int_\Omega \frac{u_\varepsilon^2}{|x|^{2-s}}\,dx. \tag{2.8}$$

It follows from (2.8) that t_ε is bounded below, i.e., there exists a positive constant $t_0 > 0$ (independently of ε) such that $0 < t_0 \leq t_\varepsilon$. Besides, it holds

$$t_\varepsilon^2 \int_\Omega u_\varepsilon^6\,dx = \frac{a\|u_\varepsilon\|^2 - \lambda\int_\Omega u_\varepsilon^2|x|^{s-2}\,dx}{t_\varepsilon^2} + b\|u_\varepsilon\|^4 + \frac{\mu}{t_\varepsilon^3}\int_\Omega u_\varepsilon\,dx,$$

which implies that t_ε is bounded above for all $\varepsilon > 0$; that is, there exists a positive real number $t_1 > 0$ (independently of ε), such that $t_\varepsilon \leq t_1 < +\infty$. Set

$$J(t) = \frac{at^2}{2}\|u_\varepsilon\|^2 + \frac{bt^4}{4}\|u_\varepsilon\|^4 - \frac{t^6}{6}\int_\Omega u_\varepsilon^6\,dx.$$

As in [14] we have

$$\sup_{t\geq 0} J(t) \leq \Lambda + O(\varepsilon^{1/2}).$$

According to the definition u_ε, for $0 < \alpha < 1$, it holds

$$\begin{aligned}
\int_\Omega u_\varepsilon\,dx &\leq C\varepsilon^{1/4}\int_{|x|\leq R_0} \frac{1}{(\varepsilon + |x|^2)^{1/2}}\,dx \\
&= C\varepsilon^{1/4}\int_0^{R_0} \frac{r^2}{(\varepsilon + r^2)^{1/2}}\,dr \\
&\leq C\varepsilon^{1/4}\int_0^{R_0} r\,dr = C\varepsilon^{1/4}.
\end{aligned}$$

From [6, Proposition 2.4], for some $K > 0$, we have

$$\int_\Omega u_\varepsilon^2|x|^{s-2}\,dx = K\varepsilon^{\frac{s}{2}} + O(\varepsilon^{1/2}).$$

Consequently,

$$\begin{aligned}
\sup_{t\geq 0} I_\mu(tu_\varepsilon) &\leq \sup_{t\geq 0} J(t) - \frac{t_0^2\lambda}{2}\int_\Omega \frac{u_\varepsilon^2}{|x|^{2-s}}\,dx + t_1\mu\int_\Omega u_\varepsilon\,dx \\
&\leq \Lambda + C_1\varepsilon^{1/2} - C_2\varepsilon^{\frac{s}{2}} + C_3\mu\varepsilon^{1/4},
\end{aligned}$$

here C_i ($i = 1, 2, 3$) (independently of ε, μ) are there positive constants. Since $0 < s < 1$, let $\varepsilon = \mu^{\frac{12}{5}}$, $\mu < \Lambda_1 = \left[\frac{C_2}{C_1 + C_3 + D}\right]^{\frac{5}{6(1-s)}}$, then

$$\begin{aligned}
C_1\varepsilon^{1/2} - C_2\varepsilon^{\frac{s}{2}} + C_3\mu\varepsilon^{1/4} &= C_1\mu^{6/5} - C_2\mu^{6s/5} + C_3\mu^{8/5} \\
&\leq (C_1 + C_3)\mu^{6/5} - C_2\mu^{6s/5} \\
&< -D\mu^{6/5},
\end{aligned}$$

so that

$$\sup_{t\geq 0} I_\mu(tu_\varepsilon) \leq \Lambda - D\mu^{6/5},$$

provided $\mu < \Lambda_1$ sufficiently small. The proof is complete. $\square$

Proof of Theorem 1.2. Let $\mu_* = \min\{\Lambda_0, \Lambda_1\}$, then Lemmas 2.1–2.3, for all $0 \leq \mu < \mu_*$. Assume $\mu \neq 0$. Then applying the mountain-pass lemma [1], there exists a sequence $\{v_n\} \subset H_0^1(\Omega)$ such that

$$I_\mu(v_n) \to c_\mu > 0, \quad \text{and} \quad I'_\mu(v_n) \to 0, \tag{2.9}$$

where

$$c_\mu = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\mu(\gamma(t)),$$

$$\Gamma = \big\{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = e\big\}.$$

By Lemmas 2.2 and 2.3, $\{v_n\} \subset H_0^1(\Omega)$ has a convergent subsequence, say $\{v_n\}$, we may assume that $v_n \to v_\mu$ in $H_0^1(\Omega)$ as $n \to \infty$. Hence, from (2.9), it holds

$$I_\mu(v_\mu) = \lim_{n\to\infty} I_\mu(v_n) = c_\mu > 0,$$

which implies that $v_\mu \not\equiv 0$. Furthermore, from the continuity of I'_μ, we obtain that v_μ is a nontrivial solution of (1.1).

If $\mu = 0$, applying the mountain-pass lemma, there is a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that

$$I_0(u_n) \to c_0 \in (0, \Lambda), \quad \text{and} \quad I'_0(u_n) \to 0.$$

Arguing as in the previous proof, $\{u_n\}$ has a subsequence strongly convergent in $H_0^1(\Omega)$ to a critical point v_0 of I_0. Moreover, for every $\phi \in H_0^1(\Omega)$, we have

$$(a + b\|v_0\|^2) \int_\Omega (\nabla v_0, \nabla\phi) - \lambda \int_\Omega v_0^+ \phi|x|^{s-2}\,dx - \int_\Omega (v_0^+)^5 \phi\,dx = 0. \tag{2.10}$$

Taking the test $\phi = v_0^-$ in (2.10), it follows that

$$\|v_0^-\| = 0,$$

which implies that $v_0 \geq 0$ in Ω and $-(a + b\|v_0\|^2)\Delta v_0 \geq 0$. Note that $I_0(v_0) = \lim_{n\to\infty} I_0(v_n) = c_0 > 0$, which means that $v_0 \not\equiv 0$ in Ω. Therefore, by the strong maximum principle, we have $v_0 > 0$ in Ω. The proof is complete. $\square$

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