Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 55, pp. 1–52.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

GLOBAL WELL-POSEDNESS OF SEMILINEAR HYPERBOLIC EQUATIONS, PARABOLIC EQUATIONS AND SCHRÖDINGER EQUATIONS

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Communicated by Binlin Zhang

Abstract. This article studies the existence and nonexistence of global solutions to the initial boundary value problems for semilinear wave and heat equation, and for Cauchy problem of nonlinear Schrödinger equation. This is done under three possible initial energy levels, except the NLS as it does not have comparison principle. The most important feature in this article is a new hypothesis on the nonlinear source terms which can include at least eight important and popular power-type nonlinearities as special cases. This article also finds some kinds of divisions for the initial data to guarantee the global existence or finite time blowup of the solution of the above three problems.

Contents

2010 *Mathematics Subject Classification.* 35L05, 35K05, 35Q55, 35A15.
Key words and phrases. Semilinear hyperbolic equation; semilinear parabolic equation; nonlinear Schrödinger equation; global solution; potential well.

Submitted December 5, 2017. Published February 23, 2018.

1. Introduction

We consider the following three problems: the initial boundary value problem of semilinear hyperbolic equation

$$u_{tt} - \Delta u = f(u), \quad x \in \Omega,\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0; \tag{1.3}$$

the initial boundary value problem of semilinear parabolic equation

$$u_t - \Delta u = f(u), \quad x \in \Omega,\ t > 0, \tag{1.4}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.5}$$

$$u(x,t) = 0, \quad x \in \partial\Omega,\ t \geq 0; \tag{1.6}$$

and the Cauchy problem of semilinear Schrödinger

$$iu_t + \Delta u = f(u), \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.7}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \tag{1.8}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. The motivation of this paper is try to extend the nonlinear term to a more general case as follows:

(A1) (i) $f \in C^1$, there exists a constant $p > 1$ such that

$$u\big(uf'(u) - pf(u)\big) \geq 0, \quad \forall u \in \mathbb{R};$$

(ii) there exist constants $q > 1$, $a_k > 0$ and $1 \leq k \leq l$ such that

$$|u|^q < |f(u)| \leq \sum_{k=1}^{l} a_k |u|^{p_k},$$

$$1 < p_l < p_{l-1} < \cdots < p_1 < \frac{n+2}{n-2} \quad \text{for } n \geq 3;$$

$$1 < p_l < p_{l-1} < \cdots < p_1 < \infty \quad \text{for } n = 1, 2.$$

The three model equations considered in the present paper are all the important well-known classical model equations. During these years, these model equations attract so many attentions and it is impossible to mention all of them. Especially, these established results for each of these three model equations seem to be "partitioned" into equivalence classes, as there are many different apparently unlinked methods for each of these three equations. In particular, we mention the potential well method introduced by Payne and Sattinger [20] and its applications on these three model equations in the present paper.

1.1. Wave equations.

Based on mountain pass theorem and the Nehari manifold, Sattinger [24] firstly studied problem (1.1)-(1.3) with nonlinear source $|u|^{p-2}u$ by introducing potential well method. Using the same method, Payne and Sattinger [20] extended the results to the following semilinear hyperbolic equation

$$u_{tt} - \Delta u = f(u) \tag{1.9}$$

with a general source $f(u)$, where $f(u)$ satisfies some assumptions, which will be discussed later. They studied a series of properties of energy functional and invariant sets, and also proved the finite time blow up of solutions. Under the same assumptions on $f(u)$ as in [20], Liu and Zhao [18] introduced a family of potential wells and obtained global existence and blow up of solutions for the initial boundary value problem of (1.9) with sub-critical initial energy, i.e. $E(0) < d$. They also proved the global existence of solutions with critical initial energy $E(0) = d$. After that, Liu and Xu [17] extended the results to the initial boundary value problem of (1.9) with combined nonlinear source terms of different sign $\sum_{k=1}^{l} a_k|u|^{p_k-1}u - \sum_{j=1}^{s} b_j|u|^{q_j-1}$, which can not be included by the assumptions of $f(u)$ in [20]. They obtained the global and blow-up solutions with sub-critical initial energy and proved the global existence of solution with critical initial energy. Subsequently, Xu [28] proved the blow up of solutions for the initial boundary value problem of (1.9) with critical initial energy and gave the sharp condition for global existence of solution. In [26], Wang considered the finite time blow up of solution for nonlinear Klein-Gordon equation with the same source $f(u)$ as in [20] with arbitrary high initial energy, i.e. $E(0) > 0$. Some others interesting results at positive initial energy can be found in [21, 22].

1.2. **Heat equations.** For problem (1.4)-(1.6) with nonlinear source term $|u|^{p-1}u$, Ikehata and Suzuki [7] investigated the parabolic equation

$$u_t - \Delta u = |u|^{p-1}u. \tag{1.10}$$

Depending on the initial datum u_0, it was shown that the problem admit both solutions which blow up in finite time and globally exist to converge to $u \equiv 0$ as time tends to infinity with sub-critical initial energy, i.e. $J(u_0) < d$. In [18], Liu and Zhao extended these results to a general source $f(u)$ in [20]

$$u_t - \Delta u = f(u). \tag{1.11}$$

By introducing a family of potential wells, they proved the finite time blow up of solution and gave a sharp condition of global existence of solution with sub-critical initial energy. Liu and Xu [17] considered problem (1.10) with combined nonlinear source terms of different sign $\sum_{k=1}^{l} a_k|u|^{p_k-1}u - \sum_{j=1}^{s} b_j|u|^{q_j-1}$, they showed that the global existence conclusions of wave equation with this nonlinearity also hold for reaction-diffusion equation, and they proved the blow up of solution with sub-critical initial energy, i.e. $E(0) < d$. Then Xu [28] continued to study problem (1.11) with critical initial energy, i.e. $J(u_0) = d$, he obtained the blow up of solution with critical initial data and also gave the sharp condition of global existence of solutions. Gazzola and Weth [11] investigated problem (1.10), they used comparison principle and variational methods to obtain the global solution and finite time blow up solutions in arbitrary high initial energy level, i.e. $J(u_0) > 0$. Later, these works attracted a lot of attentions [16, 4, 14].

1.3. **NLS equations.** In [12], Ginibre and Velo studied the nonlinear Schrödinger equation

$$\begin{gathered} iu_t + \Delta u = |u|^{p-1}u, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \end{gathered} \tag{1.12}$$

they established the local well-posedness of this Cauchy problem in the energy space $H^1_x(\mathbb{R}^n)$. After that, Zakharov [31], Glassey [13], Ogawa and Tsutsumi [19] proved that when $p \geq 1 + \frac{4}{n}$, the solution of problem (1.12) blows up in finite time for some initial data, especially for negative energy. Weinstein [27] gave a crucial criterion in terms of L^2-mass of the initial data for $p = 1 + \frac{4}{n}$. Zhang [32] investigated problem (1.12) and gave the sharp sufficient condition of blowup and global solutions in $\mathbb{R}^2$

and $\mathbb{R}^N$ separately. Tao, Visan and Zhang [25] systematically studied the following nonlinear Schrödinger equation with combined power-type nonlinearities

$$\begin{gathered} iu_t + \Delta u = \lambda_1 |u|^{p_1} u + \lambda_2 |u|^{p_2} u, \\ u(0,x) = u_0(x), \end{gathered} \tag{1.13}$$

they obtained local and global well-posedness, asymptotic behaviour (scattering), and finite time blow up of solutions. More precisely, they proved these phenomena under different conditions of parameters $\lambda_1, \lambda_2,\ p_1$ and p_2. We also recommend the readers [3] and the references therein to get more conclusions about the nonlinear Schrödinger equations.

As mentioned above, the established results not only extend the conclusions from negative energy blow up to positive energy blow up, from sub-critical initial energy to critical energy then to sup-critical energy, but also extend the nonlinear term to more general form. By observing the nonlinearities considered in the literatures we can list the following popular cases, which frequently appear in the physical or mathematical models:

- (i) $a|u|^{p-1}u$, $a > 0$, $p > 1$;
- (ii) $a|u|^p$, $a > 0$, $p > 1$;
- (iii) $-a|u|^p$, $a > 0$, $p > 1$;
- (iv) $\sum_{k=1}^{l} a_k |u|^{p_k - 1} u$, $a_k > 0$, $1 \le k \le l$, $1 < p_l < p_{l-1} < \dots < p_1$;
- (v) $\sum_{k=1}^{l} a_k |u|^{p_k - 1} u - \sum_{j=1}^{m} b_j |u|^{q_j - 1} u$, $a_k > 0$, $1 \le k \le l$, $b_j > 0$, $1 \le j \le m$, $1 < q_m < q_{m-1} < \dots < q_1 < p_l < p_{l-1} < \dots < p_1$;
- (vi) $a|u|^{p-1}u \pm b|u|^p$, $a > 0$, $b > 0$, $p > 1$;
- (vii) $\pm a|u|^p - b|u|^{p-1}u$, $a > 0$, $b > 0$, $p > 1$;
- (viii) $\sum_{k=1}^{l} a_k |u|^{p_k - 1} u \pm a|u|^p$, $a_k > 0$, $1 \le k \le l$, $a > 0$, $1 < p \le p_l < p_{l-1} < \dots < p_1$;
- (viiii) $\pm a|u|^p - \sum_{j=1}^{m} b_j |u|^{q_j - l} u$, $a > 0$, $b_j > 0$, $1 \le j \le m$, $1 < q_m < q_{m-1} < \dots < q_1 \le p$;
- (x) $\sum_{k=1}^{l} a_k |u|^{p_k - 1} u \pm a|u|^p - \sum_{j=1}^{m} b_j |u|^{q_j - 1} u$, $a_k > 0$, $1 \le k \le l$, $b_j > 0$, $1 \le j \le m$, $a > 0$, $1 < q_m < q_{m-1} < \dots < q_1 \le p \le p_l < p_{l-1} < \dots < p_1 < \frac{n+2}{n-2}$ for $n \ge 3$, $1 < q_m < q_{m-1} < \dots < q_1 \le p \le p_l < p_{l-1} < \dots < p_1 < \infty$ for $n = 1, 2$.

Clearly, a very general nonlinear term was introduced by the hypothesis (see [20, 18])

(A2) (i) $f \in C^1$, $f(0) = f'(0) = 0$;
(ii) (a) $f(u)$ is monotonic and is convex for $u > 0$, concave for $u < 0$, or
(b) $f(u)$ is convex for $-\infty < u < +\infty$;
(iii) $(p+1)F(u) \le uf(u)$, $|uf(u)| \le r|F(u)|$, where

$$2 < p + 1 \le r < \frac{n+2}{n-2} \quad \text{for } n \ge 3.$$

We also found that only (i), (ii) and (iv) can be included in (A2). So it is the right time to find a new assumptions system to define a much more general nonlinear term to include all these possible and important nonlinearities listed as above from (i) to (x). In the present paper, we introduce a new assumptions (A1) to take this task.

It is important to mention that the new assumptions (A1) further extend the former assumptions (A2) such that the general source $f(u)$ can include all nonlinearities listed above, which means that $f(u)$ in the present paper is a more general nonlinearity. And as far as we are concerned, this is the first work in the literature that consider wave equation, heat equation and NLS equation at the same time in a uniform frame.

In this article, for the wave equation, we introduce the potential well and some manifolds, and then we give a series of their properties. Through these properties, we not only prove the invariant property of these manifolds under the flow of (1.1)-(1.3), but also get the threshold condition of the global existence and nonexistence of solution under low initial energy level $E(0) < d$. At the critical energy level $E(0) = d$, combining the scaling method we obtain the global existence results, furthermore, by establishing a new invariant manifold, we obtain the global nonexistence of solution. Considering the idea in references [30, 26], we obtain the finite time blow up results at arbitrary positive initial energy level $E(0) > 0$. For the heat equation, we found that the properties of these manifolds also hold, and by the usage of the Galerkin method and concavity method, we prove the global existence and nonexistence for problem (1.4)-(1.6) under low initial energy level $E(0) < d$. Then we use the scaling method to extend the results about low initial energy to the critical initial energy level. When we discuss the arbitrary positive initial energy case $E(0) > 0$, inspired by the method in [29, 11], we construct the comparison principle corresponding to the steady state equation to problem (1.4)-(1.6), then we obtain both solution of problem (1.4)-(1.6) which blows up in finite time and global solution which converge to $u \equiv 0$ as time tends to infinity. Through the improved concavity argument in [15], we show the results of the finite time blow up of solution without help of the comparison principle. Finally, for the nonlinear Schrödinger equation, we reintroduce the potential well and prove the properties of the corresponding invariant manifolds, then we prove the global existence and nonexistence for problem (1.7)-(1.8) at only the low initial energy level $E(0) < d$ and leave other cases open as the failure of the comparison principle. The current main results of this paper can be summarized by the following table.

Table 1. Main results. ($\surd$) indicates result obtained here, (?) indicates open problem

		$E(0) < d$	$E(0) = d$	$E(0) > d$
Hyperbolic	Global existence	$\surd$	$\surd$	?
	Finite time blow up	$\surd$	$\surd$	$\surd$
Parabolic	Global existence	$\surd$	$\surd$	$\surd$
	Finite time blow up	$\surd$	$\surd$	$\surd$
NLS	Global existence	$\surd$	?	?
	Finite time blow up	$\surd$	?	?

1.4. **Open problems.**

- For problem (1.1)-(1.3) (semilinear hyperbolic equation), the existence of global solutions is still open at high energy level even for the classical nonlinear terms like u^p, $|u|^p$ and $|u|^{p-1}u$.

- For problem (1.7)-(1.8) (nonlinear Schrödinger equation), the question then arises as to what happens for large energy data $E(0) \geq d$. It is well-known that such results will be obtained if one could get the a priori bound (spacetime estimate) for all global Schwarz solutions u.

The outline of this article is as follows. In Section 2, we mainly consider the global well-posedness of the semilinear hyperbolic equation with general source term. Then in Section 3, we deal with the semilinear parabolic equation. In Section 4 the nonlinear Schrödinger equation is considered.

In this article $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $(u, v) = \int_\Omega uv\mathrm{d}x$, and $\langle \cdot, \cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

2. Semilinear hyperbolic equation

Before stating our results, we summarize here some definitions and auxiliary lemmas for problem (1.1)-(1.3) and problem (1.4)-(1.6). Then we prove the existence and nonexistence of solutions of the initial boundary value problem of the hyperbolic equation.

To deal with problem (1.1)-(1.3) and problem (1.4)-(1.6) let us introduced the potential energy functional

$$J(u) = \frac{1}{2}\|\nabla u\|^2 - \int_\Omega F(u)\mathrm{d}x, \quad F(u) = \int_0^u f(s)\mathrm{d}s,$$

the Nehari functional

$$I(u) = \|\nabla u\|^2 - \int_\Omega uf(u)\mathrm{d}x$$

and the depth of potential well mountain pass level

$$d = \inf_{u\in\mathcal{N}} J(u),$$

where

$$\mathcal{N} = \{u \in H_0^1(\Omega) : I(u) = 0,\ u \neq 0\}.$$

From (A1) we can derive the following lemma, which provide a connection between $J(u)$ and $I(u)$, further the depth of the potential well d.

Lemma 2.1. *Suppose that $f(u)$ satisfies* (A1). *Then it holds*

$$uf(u) \geq (p+1)F(u), \quad u \in \mathbb{R}. \tag{2.1}$$

Proof. We divide the proof into the following two cases:

(i) If $u \geq 0$, then (i) in (A1) yields

$$uf'(u) \geq pf(u)$$

and

$$\int_0^u sf'(s)\mathrm{d}s \geq p\int_0^u f(s)\mathrm{d}s = pF(u),\ u \geq 0,$$

which gives

$$uf(u) - \int_0^u f(s)\mathrm{d}s \geq pF(u)$$

and

$$(p+1)F(u) \leq uf(u),\ u \geq 0. \tag{2.2}$$

(ii) If $u < 0$, then from (i) in (A1) we obtain
$$uf'(u) \le pf(u)$$
and
$$\int_0^u sf'(s)\mathrm{d}s \ge p\int_0^u f(s)\mathrm{d}s = pF(u),\ u<0,$$
which gives
$$uf(u) - \int_0^u f(s)\mathrm{d}s \ge pF(u)$$
and
$$uf(u) \ge (p+1)F(u),\ u<0. \tag{2.3}$$
Inequality (2.1) follows from (2.2) and (2.3). □

Remark 2.2. We see that Lemma 2.1, i.e. (2.1), is essential in the proof of global existence and nonexistence of solution for nonlinear evolution equation by using potential well method since it reveals the relation between $f(u)$ and $F(u)$ and connects $J(u)$, $I(u)$ and d, which are very important to prove all of the following main results. In the previous work, (2.1) is often given as an additional independent assumption. In the present paper, we do it in a different way by taking out (2.1) from (A1), which helps us weaken the conditions on the nonlinearity $f(u)$.

Next we construct the relation between $\|\nabla u\|$ and $I(u)$ by the following lemma.

Lemma 2.3. *Suppose that $f(u)$ satisfies* (A1), $u \in H_0^1(\Omega)$. *Then*
- (i) *If $0 < \|\nabla u\| < r_0$, then $I(u) > 0$;*
- (ii) *If $I(u) < 0$, then $\|\nabla u\| > r_0$;*
- (iii) *If $I(u) = 0$ but $u \ne 0$, then $\|\nabla u\| \ge r_0$,*

where r_0 is the unique real root of equation $g(r) = 1$,
$$g(r) = \sum_{k=1}^{l} a_k C_k^{p_k+1} r^{p_k-1}, \quad \text{and} \quad C_k = \sup_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|_{p_k+1}}{\|\nabla u\|}.$$

Proof. (i) If $0 < \|\nabla u\| < r_0$, we can write
$$g(\|\nabla u\|) = \sum_{k=1}^{l} a_k C_k^{p_k+1}\|\nabla u\|^{p_k-1} < \sum_{k=1}^{l} a_k C_k^{p_k+1} r_0^{p_k-1} = 1. \tag{2.4}$$
Hence from (ii) in (A1), Sobolev inequality and (2.4) we obtain
$$\begin{aligned}
\int_\Omega uf(u)\mathrm{d}x &\le \sum_{k=1}^{l} a_k \int_\Omega |u|^{p_k+1}\mathrm{d}x \\
&= \sum_{k=1}^{l} a_k \|u\|_{p_k+1}^{p_k+1} \\
&\le \sum_{k=1}^{l} a_k C_k^{p_k+1}\|\nabla u\|^{p_k+1} \\
&= g(\|\nabla u\|)\|\nabla u\|^2 < \|\nabla u\|^2,
\end{aligned}$$
which implies $I(u) > 0$.

(ii) If $I(u) < 0$, then from the definition of $I(u)$ and (ii) in (A1) we can write

$$\|\nabla u\|^2 < \int_\Omega uf(u)\mathrm{d}x \le g(\|\nabla u\|)\|\nabla u\|^2,$$

which gives $g(\|\nabla u\|) > 1$. Then

$$g(\|\nabla u\|) = \sum_{k=1}^{l} a_k C_k^{p_k+1}\|\nabla u\|^{p_k-1} \ge \sum_{k=1}^{l} a_k C_k^{p_k+1} r_0^{p_k-1},$$

which implies $\|\nabla u\| > r_0$.

(iii) If $I(u) = 0$ but $u \neq 0$, same as (ii) we deduce

$$\|\nabla u\|^2 = \int_\Omega uf(u)\mathrm{d}x \le g(\|\nabla u\|)\|\nabla u\|^2,$$

which gives $g(\|\nabla u\|) \ge 1$. Then

$$g(\|\nabla u\|) = \sum_{k=1}^{l} a_k C_k^{p_k+1}\|\nabla u\|^{p_k-1} > \sum_{k=1}^{l} a_k C_k^{p_k+1} r_0^{p_k-1},$$

which ensures $\|\nabla u\| \ge r_0$. □

Here we estimate the depth of potential well.

Lemma 2.4. *Suppose that $f(u)$ satisfies* (A1). *Then*

$$d \ge d_0 = \frac{p-1}{2(p+1)} r_0^2, \tag{2.5}$$

where r_0 is defined in Lemma 2.3.

Proof. For all $u \in \mathcal{N}$, by (iii) in Lemma 2.3 we know $\|\nabla u\| \ge r_0$, then by Lemma 2.1 and $I(u)$ one gives

$$\begin{aligned} J(u) &= \frac{1}{2}\|\nabla u\|^2 - \int_\Omega F(u)\mathrm{d}x \\ &\ge \frac{1}{2}\|\nabla u\|^2 - \frac{1}{p+1}\int_\Omega uf(u)\mathrm{d}x \\ &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|\nabla u\|^2 + \frac{1}{p+1} I(u) \\ &= \frac{p-1}{2(p+1)}\|\nabla u\|^2 \\ &\ge \frac{p-1}{2(p+1)} r_0^2, \end{aligned}$$

which gives (2.5). □

For the sake of proving the blow up of solution, we introduce a scaling to $I(u)$.

Lemma 2.5. *Suppose that $f(u)$ satisfies* (A1), *$u \in H_0^1(\Omega)$ and $I(u) < 0$. Then there exists a $\lambda^* \in (0,1)$ such that $I(\lambda^* u) = 0$.*

Proof. Set

$$\varphi(\lambda) := \frac{1}{\lambda}\int_\Omega uf(\lambda u)\mathrm{d}x, \ \lambda > 0.$$

Then

$$\begin{aligned}I(\lambda u) =&\lambda^2\|\nabla u\|^2 - \int_\Omega \lambda u f(\lambda u)\mathrm{d}x\\ =&\lambda^2\Big(\|\nabla u\|^2 - \frac{1}{\lambda}\int_\Omega uf(\lambda u)\mathrm{d}x\Big)\\ =&\lambda^2\Big(\|\nabla u\|^2 - \varphi(\lambda)\Big).\end{aligned}$$

Applying $I(u) < 0$, we derive $\int_\Omega uf(u)\mathrm{d}x > \|\nabla u\|^2$, which combining with (ii) in Lemma 2.3 gives

$$\varphi(1) > \|\nabla u\|^2 > r_0^2.$$

On the other hand, by (ii) in (A1) we deduce

$$\begin{aligned}|\varphi(\lambda)| =&\frac{1}{\lambda^2}\int_\Omega |\lambda u f(\lambda u)|\mathrm{d}x\\ \le&\frac{1}{\lambda^2}\int_\Omega \sum_{k=1}^{l} a_k|\lambda u|^{p_k+1}\mathrm{d}x\\ =&\sum_{k=1}^{l} a_k\lambda^{p_k-1}\|u\|_{p_k+1}^{p_k+1},\end{aligned}$$

then we obtain that $\varphi(\lambda)\to 0$ as $\lambda\to 0$. Hence there exists a $\lambda^*\in(0,1)$ such that $\varphi(\lambda^*) = \|\nabla u\|^2$ and $I(\lambda^* u) = 0$. □

In the following lemma, we give a more precise estimate on $I(u)$.

Lemma 2.6. *Suppose that $f(u)$ satisfies* (A1)*, $u\in H_0^1(\Omega)$ and $I(u) < 0$. Then*

$$I(u) < (p+1)(J(u) - d). \tag{2.6}$$

Proof. Lemma 2.5 implies that there exists a $\lambda^*\in(0,1)$ such that $I(\lambda^* u) = 0$. Set

$$h(\lambda) := (p+1)J(\lambda u) - I(\lambda u),\ \ \lambda > 0.$$

Then by $J(u)$ and $I(u)$ we have

$$h(\lambda) = \frac{p-1}{2}\lambda^2\|\nabla u\|^2 + \int_\Omega \Big(\lambda u f(\lambda u) - (p+1)F(\lambda u)\Big)\mathrm{d}x,$$

combining (i) in (A1) with (ii) in Lemma 2.3 we derive

$$\begin{aligned}h'(\lambda) &= (p-1)\lambda\|\nabla u\|^2 + \int_\Omega \big(\lambda u^2 f'(\lambda u) + uf(\lambda u) - (p+1)uf(\lambda u)\big)\,\mathrm{d}x\\ &= (p-1)\lambda\|\nabla u\|^2 + \frac{1}{\lambda}\int_\Omega \lambda u\,(\lambda u f'(\lambda u) - pf(\lambda u))\,\mathrm{d}x\\ &\ge (p-1)\lambda\|\nabla u\|^2\\ &> (p-1)\lambda r_0^2 > 0.\end{aligned}$$

Hence $h(\lambda)$ is strictly increasing for $\lambda > 0$, which gives $h(1) > h(\lambda^*)$ for $1 > \lambda^* > 0$, namely

$$(p+1)J(u) - I(u) > (p+1)J(\lambda^* u) - I(\lambda^* u) = (p+1)J(\lambda^* u) \ge (p+1)d,$$

which gives (2.6) immediately. □

To deal with problem (1.1)-(1.3) let us introduce

$$W_H = \{u \in H_0^1(\Omega) : I(u) > 0\} \cup \{0\}, V_H = \{u \in H_0^1(\Omega) : I(u) < 0\}.$$

Definition 2.7. The function $u = u(x,t)$ is said to be a weak solution on $\Omega \times [0,T)$ for problem (1.1)-(1.3), if $u \in L^\infty(0,T;H_0^1(\Omega))$ and $u_t \in L^\infty(0,T;L^2(\Omega))$ satisfying

$$\begin{gathered}(u_t, v) + \int_0^t (\nabla u \nabla v) \mathrm{d}\tau = \int_0^t (f(u), v) \mathrm{d}\tau + (u_1, v), \\ \forall v \in H_0^1(\Omega),\ 0 \le t < T;\end{gathered} \tag{2.7}$$

$$u(x,0) = u_0(x) \quad \text{in } H_0^1(\Omega); \quad u_t(x,0) = u_1(x) \quad \text{in } L^2(\Omega); \tag{2.8}$$

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 - \int_\Omega F(u) \mathrm{d}x = E(0), \quad 0 \le t < T. \tag{2.9}$$

For convenience of the reader, we use the following common assumption in Subsection 2.1-2.3.

(A3) Let $f(u)$ satisfy (A1), $u_0(x) \in H_0^1(\Omega)$ and $u_1(x) \in L^2(\Omega)$.

Next we state a local existence theorem that can be established by combining the arguments of [10, Theorem 3.1] with slight modification.

Theorem 2.8 (Local existence)**.** *Let* (A3) *hold. Then there exist* $T > 0$ *and a unique solution of problem* (1.1)-(1.3) *over* $[0,T]$. *Moreover, if*

$$T = \sup\{T > 0 : u = u(t) \text{ exists on } [0,T]\} < \infty,$$

then $\lim_{t \to T} \|u(t)\|_q = \infty$ *for all* $q \ge 1$ *such that* $q > n(p-2)/2$.

2.1. **Low initial energy.** By using (2.9) and the similar arguments in [18] we can attain Theorem 2.9 and Corollary 2.10.

Theorem 2.9 (Invariant sets)**.** *Suppose that* $E(0) < d$. *Then both sets* W_H *and* V_H *are invariant along the flow of* (1.1)-(1.3) *respectively.*

The following corollary can help us derive the negative energy blowup without any cost after we have the supcritial energy blowup theory.

Corollary 2.10. *Suppose that* $E(0) < 0$ *or* $E(0) = 0$ *and* $u_0(x) \ne 0$. *Then all weak solutions of problem*(1.1)-(1.3) *belong to* V_H.

The global existence and nonexistence results for problem (1.1)-(1.3) under low initial energy $E(0) < d$ are listed as below.

Theorem 2.11. *Suppose that* $E(0) < d$, $u_0(x) \in W_H$. *Then there is a global weak solution to problem* (1.1)-(1.3) *satisfying* $u \in L^\infty(0,\infty;H_0^1(\Omega))$ *with* $u_t \in L^\infty(0,\infty;L^2(\Omega))$ *and* $u \in W_H$ *for* $0 \le t < \infty$.

Proof. We choose $\{w_j(x)\}_{j=1}^\infty$ as a system of basis in $H_0^1(\Omega)$. Construct the following approximate solutions $u_m(x,t)$ of problem (1.1)-(1.3) as

$$u_m(x,t) = \sum_{j=1}^m g_{jm}(t) w_j(x), \quad m = 1,2\dots$$

satisfying

$$(u_{mtt}, w_s) + (\nabla u_m, \nabla w_s) = (f(u_m), w_s), \quad s = 1,2\dots m, \tag{2.10}$$

$$u_m(x,0)=\sum_{j=1}^{m}g_{jm}(0)w_j(x)\to u_0(x)\quad \text{in } H_0^1(\Omega), \tag{2.11}$$

$$u_{mt}(x,0)=\sum_{j=1}^{m}g'_{jm}(0)w_j(x)\to u_1(x) \text{ in } L^2(\Omega). \tag{2.12}$$

Multiplying (2.10) by $g'_{sm}(t)$ and summing over $s=1,2,\dots,m$ yields $\frac{\mathrm{d}}{\mathrm{d}t}E_m(t)=0$, i.e.,

$$E_m(t)=E_m(0), \tag{2.13}$$

where

$$E_m(t)=\frac{1}{2}\|u_{mt}\|^2+J(u_m).$$

From $E(0)<d$, (2.11) and (2.12) we see that $E_m(0)<d$ for sufficiently large m. Combining (2.13) we have

$$\frac{1}{2}\|u_{mt}\|^2+J(u_m)<d,\quad 0\le t<\infty \tag{2.14}$$

for sufficiently large m. By $u_0(x)\in W_H$ and (2.11), we obtain $u_m(0)\in W_H$ for sufficiently large m. Furthermore by (2.14) we prove (see [18]) $u_m(t)\in W_H$ for $0\le t<\infty$ and sufficiently large m. From (2.14) we can obtain

$$\frac{1}{2}\|u_{mt}\|^2+\frac{p-1}{2(p+1)}\|\nabla u_m\|^2+\frac{1}{p+1}I(u_m)<d,\quad 0\le t<\infty.$$

Together with $u_m(t)\in W_H$ we obtain

$$\frac{1}{2}\|u_{mt}\|^2+\frac{p-1}{2(p+1)}\|\nabla u_m\|^2<d,\quad 0\le t<\infty, \tag{2.15}$$

$$\|\nabla u_m\|^2<\frac{2(p+1)}{p-1}d,\quad 0\le t<\infty, \tag{2.16}$$

$$\|u_{mt}\|^2<2d,\quad 0\le t<\infty, \tag{2.17}$$

$$\|f(u_m)\|_r\le\sum_{k=1}^{l}a_k\|u_m\|_{q_k}^{p_k}\le\sum_{k=1}^{l}a_kC_*^{p_k}\|\nabla u_m\|^{p_k}<C,\quad 0\le t<\infty, \tag{2.18}$$

where C_* appearing in (2.18) is the best embedding constant and

$$r=\frac{p_1+1}{p_1},\quad q_k=p_k\frac{p_1+1}{p_1}\le p_1+1.$$

Denote $\xrightarrow{w^*}$ as the weakly star convergence. Then from (2.16)-(2.18) we can find a χ and a convergent subsequence $\{u_\nu\}\subset\{u_m\}$ as $\nu\to\infty$ satisfying the following:

$$u_\nu\xrightarrow{w^*}u\quad \text{in } L^\infty(0,\infty;H_0^1(\Omega)) \text{ and a.e. in } Q=\Omega\times[0,\infty);$$
$$u_\nu\to u\quad \text{in } L^{p_1+1}(\Omega) \text{ strongly for } t>0;$$
$$u_{\nu t}\xrightarrow{w^*}u_t\quad \text{in } L^\infty(0,\infty;L^2(\Omega));$$
$$f(u_\nu)\xrightarrow{w^*}\chi=f(u)\quad \text{in } L^\infty(0,\infty;L^r(\Omega)).$$

Integrating (2.10) over $\tau\in[0,t]$ yields

$$(u_{mt},w_s)+\int_0^t(\nabla u_m,\nabla w_s)\mathrm{d}\tau=\int_0^t(f(u_m),w_s)\mathrm{d}\tau+(u_{mt}(0),w_s) \tag{2.19}$$

for all $0 \le t < \infty$. Let $m = \nu \to \infty$ in (2.19) we obtain

$$(u_t, w_s) + \int_0^t (\nabla u, \nabla w_s) \mathrm{d}\tau = \int_0^t (f(u), w_s) \mathrm{d}\tau + (u_1, w_s),$$

then

$$(u_t, v) + \int_0^t (\nabla u, \nabla v) \mathrm{d}\tau = \int_0^t (f(u), v) \mathrm{d}\tau + (u_1, v), \quad v \in H_0^1(\Omega),\ t > 0.$$

It follows easily from (2.11) and (2.12) that $u(x,0) = u_0(x)$ in $H_0^1(\Omega)$, $u_t(x,0) = u_1(x)$ in $L^2(\Omega)$.

Next we show that u satisfies (2.9) for $0 \le t < \infty$. First we prove that for the above subsequence $\{u_\nu\}$ it holds

$$\lim_{\nu \to \infty} \int_\Omega F(u_\nu) \mathrm{d}x = \int_\Omega F(u) \mathrm{d}x, \quad t > 0. \tag{2.20}$$

In fact we have

$$\begin{aligned}
\Big| \int_\Omega F(u_\nu) \mathrm{d}x - \int_\Omega F(u) \mathrm{d}x \Big| &\le \int_\Omega |F(u_\nu) - F(u)| \mathrm{d}x \\
&= \int_\Omega |f(\varphi_\nu)| |u_\nu - u| \mathrm{d}x \\
&\le \|f(\varphi_\nu)\|_r \|u_\nu - u\|_{p_1+1},
\end{aligned}$$

where $\varphi_\nu = u + \theta(u_v - u)$, $0 < \theta < 1$. From $\|u_\nu - u\|_{p_1+1} \to 0$ as $\nu \to \infty$ and $\|f(\varphi_\nu)\|_r \le C$ we obtain (2.20). Thus from (2.13) we have

$$\begin{aligned}
\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 &= \lim_{\nu \to \infty} \Big(\frac{1}{2}\|u_{\nu t}\|^2 + \frac{1}{2}\|\nabla u_\nu\|^2 \Big) \\
&= \lim_{\nu \to \infty} \Big(E_\nu(0) + \int_\Omega F(u_\nu) \mathrm{d}x \Big) \\
&= E(0) + \int_\Omega F(u) \mathrm{d}x.
\end{aligned}$$

Hence u satisfies (2.9) for $0 \le t < \infty$. Finally by Corollary 2.10 we obtain $u \in W_H$ for $0 \le t < \infty$. □

Now we are in a position to state the global nonexistence result for the solution of problem (1.1)-(1.3) under low initial energy $E(0) < d$.

Theorem 2.12 (Global nonexistence for $E(0) < d$)**.** *Suppose that $E(0) < d$ and $u_0(x) \in V_H$. Then problem* (1.1)-(1.3) *does not admit any global weak solution.*

Proof. For each weak solution $u \in L^\infty(0, T; H_0^1(\Omega))$ with $u_t \in L^\infty(0, T; L^2(\Omega))$ defined on maximal time interval $[0, T)$ for problem (1.1)-(1.3). Our goal is to prove $T < \infty$. Arguing by contradiction, we suppose that $T = +\infty$. Then $u \in L^\infty(0, \infty; H_0^1(\Omega))$ and $u_t \in L^\infty(0, \infty; L^2(\Omega))$. Set

$$M_H(t) := \|u\|^2,\ 0 \le t < \infty, \tag{2.21}$$

then

$$\dot{M}_H(t) = 2(u_t, u), \quad 0 \le t < \infty, \tag{2.22}$$

$$\dot{M}_H^2(t) \le 4\|u_t\|^2\|u\|^2 = 4M_H(t)\|u_t\|^2. \tag{2.23}$$

From (1.1) we have $u_{tt} \in L^\infty(0,\infty;H^{-1}(\Omega))$. Hence from (2.22) and (1.1) we obtain

$$\ddot{M}_H = 2\|u_t\|^2 + 2(u_{tt},u) = 2\|u_t\|^2 - 2I(u),\ 0 \le t < \infty \tag{2.24}$$

and

$$\begin{aligned} &M_H(t)\ddot{M}_H(t) - \frac{p+3}{4}\dot{M}_H^2(t) \\ &\ge M_H(t)\left(2\|u_t\|^2 - 2I(u) - (p+3)\|u_t\|^2\right) \\ &= M_H(t)\left(-(p+1)\|u_t\|^2 - 2I(u)\right),\ 0 \le t < \infty. \end{aligned}$$

From the energy inequality (2.9) we know that

$$E(0) \ge \frac{1}{2}\|u_t\|^2 + J(u), \quad 0 \le t < \infty,$$

which gives

$$-(p+1)\|u_t\|^2 \ge 2(p+1)\left(J(u) - E(0)\right)$$

and

$$\begin{aligned} M_H(t)\ddot{M}_H(t) - \frac{p+3}{4}\dot{M}_H^2(t) &\ge 2M_H(t)\left((p+1)(J(u)-E(0)) - I(u)\right) \\ &\ge 2M_H(t)\left((p+1)(J(u)-d) - I(u)\right). \end{aligned}$$

By Theorem 2.9 we have $u \in V_H$ and by (ii) in Lemma 2.3 it holds $\|\nabla u\| > r_0$ for $0 \le t < \infty$. Hence we have $M_H(t) > 0$ and from (2.6) in Lemma 2.6 we attain $(p+1)\left(J(u) - d\right) - I(u) > 0$, which gives

$$M_H(t)\ddot{M}_H(t) - \frac{p+3}{4}\dot{M}_H^2(t) > 0, \quad 0 \le t < \infty. \tag{2.25}$$

In addition, combining (2.24) and (2.6) we have

$$\begin{aligned} \ddot{M}_H &\ge -2I(u) \\ &> 2(p+1)(d - J(u)) \\ &> 2(p+1)(d - E(0)) \\ &:= C_0 > 0,\ 0 \le t < \infty \end{aligned}$$

and

$$\dot{M}_H > C_0 t + \dot{M}_H(0),\ 0 \le t < \infty.$$

Finally, there exists a large enough $t_0 \ge 0$ which ensures $\dot{M}_H(t_0) > 0$, together with $M_H(t_0) > 0$ and (2.25) gives that there exists a $T_1 > 0$ such that

$$\lim_{t\to T_1} M_H(t) = +\infty,$$

which contradicts $T = +\infty$. $\square$

From Theorem 2.12 and Theorem 2.13 a sharp condition for global well-posedness of solution can be shown for problem (1.1)-(1.3) as below.

Theorem 2.13 (Sharp conditions). *Suppose that $E(0) < d$. Then we have the following alternatives:*

(i) *If $I(u_0) > 0$, problem* (1.1)-(1.3) *possesses a global weak solution;*
(ii) *If $I(u_0) < 0$, problem* (1.1)-(1.3) *has no global weak solution.*

2.2. **Critical initial energy.** The global existence result for problem (1.1)-(1.3) under critical initial energy $E(0) = d$ is listed as below.

Theorem 2.14. *Suppose that $E(0) = d$, $u_0(x) \in W_H$. Then there is a global weak solution to problem (1.1)-(1.3) satisfying $u \in L^\infty(0,\infty; H_0^1(\Omega))$ with $u_t \in L^\infty(0,\infty; L^2(\Omega))$ and $u \in W_H$ for $0 \le t < \infty$.*

Proof. We prove this theorem by the following two cases (i) and (ii).

(i) $\|\nabla u_0\| \neq 0$. Let $\lambda_m = 1 - \frac{1}{m}$ and $u_{0m} = \lambda_m u_0,\ m = 2, 3, \dots$. Consider the initial data

$$u(x,0) = u_{0m}(x), \quad u_t(x,0) = u_1(x) \tag{2.26}$$

and corresponding problem (1.1)-(1.3). From $I(u_0) \ge 0$ and Lemma 2.5 we have $\lambda^* = \lambda^*(u_0) \ge 1$. Hence $I(u_{0m}) > 0$,

$$\begin{aligned} J(u_{0m}) &\ge \frac{1}{2}\|\nabla u_{0m}\|^2 - \frac{1}{p+1}\int_\Omega u_{0m} f(u_{0m})\mathrm{d}x \\ &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|\nabla u_{0m}\|^2 + \frac{1}{p+1} I(u_{0m}) > 0 \end{aligned}$$

and $J(u_{0m}) = J(\lambda_m u_0) < J(u_0)$. Also

$$0 < E_m(0) \equiv \frac{1}{2}\|u_1\|^2 + J(u_{0m}) < \frac{1}{2}\|u_1\|^2 + J(u_0) = E(0) = d.$$

So it follows from Theorem 2.11 that for each m problem (1.1), (2.26) and (1.3) admits a global weak solution $u_m(t) \in L^\infty(0,\infty; H_0^1(\Omega))$ with $u_{mt} \in L^\infty(0,\infty; L^2(\Omega))$ and $u_m(t) \in W_H$ for $0 \le t < \infty$ satisfying

$$\begin{aligned} &(u_{mt}, v) + \int_0^t (\nabla u_m, \nabla v)\mathrm{d}\tau \\ &= \int_0^t (f(u_m), v)\mathrm{d}\tau + (u_1, v),\ \forall v \in H_0^1(\Omega), \quad 0 \le t < \infty \end{aligned} \tag{2.27}$$

$$\frac{1}{2}\|u_{mt}\|^2 + J(u_m) = E_m(0) < d. \tag{2.28}$$

The remainder of proof is similar to that of Theorem 2.11.

(ii) $\|\nabla u_0\| = 0$. Note that $\|\nabla u_0\| = 0$ implies $J(u_0) = 0$ and $\frac{1}{2}\|u_1\|^2 = E(0) = d$. Let $\lambda_m = 1 - \frac{1}{m}$, $u_{1m}(x) = \lambda_m u_1(x), m = 2, 3, \dots$. Consider the initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_{1m}(x) \tag{2.29}$$

and corresponding problem (1.1),(1.3). From $\|\nabla u_0\| = 0$,

$$0 < E_m(0) = \frac{1}{2}\|u_{1m}\|^2 + J(u_0) = \frac{1}{2}\|\lambda_m u_1\|^2 < E(0) = d$$

and Theorem 2.11 it follows that for each m problem (1.1), (2.29) and (1.3) admits a global weak solution $u_m(t) \in L^\infty(0,\infty; H_0^1(\Omega))$ with $u_{mt} \in L^\infty(0,\infty; L^2(\Omega))$ and $u_m(t) \in W_H$ for $0 \le t < \infty$ satisfying (2.27) and (2.28). The remainder of proof is the same as that in the part (i) of proof of this theorem. □

Next we obtain the invariant set V_H along the flow of problem (1.1)-(1.3) with $E(0) = d$.

Theorem 2.15. *Suppose that $E(0) = d$ and $(u_0(x), u_1(x)) \ge 0$. Then all solutions of problem (1.1)-(1.3) belong to V_H, provided $u_0(x) \in V_H$.*

Proof. Let $u(x,t)$ be any weak solution of problem (1.1)-(1.3) with $E(0) = d$, $u_0 \in V_H$, and $(u_0(x), u_1(x)) \geq 0$, T be the maximum existence time of $u(x,t)$. Let us prove $u(x,t) \in V_H$ for $0 < t < T$. Arguing by contradiction, we suppose that there exists the first $t_0 \in (0,T)$ such that $I(u(t_0)) = 0$ and $I(u) < 0$ for $0 \leq t < t_0$. Then $\|\nabla u(t_0)\| \geq r_0 > 0$ and $\|\nabla u\| > r_0$ for $0 \leq t < t_0$. By the definition of d we obtain $J(u(t_0)) \geq d$. From Lemma 2.4 and

$$\frac{1}{2}\|u_t(t_0)\|^2 + J(u(t_0)) = E(t_0) \leq E(0) = d,$$

we obtain $J(u(t_0)) = d$ and $\|u_t(t_0)\|^2 = 0$. Recall the auxiliary function $M_H(t)$ defined as (2.21), then we have (2.22) with

$$\dot{M}_H(0 = 2(u_0(x), u_1(x)) > 0,$$
$$\ddot{M}_H(t) = 2\|u_t\|^2 + 2\langle u_{tt}, u\rangle = 2\|u_t\|^2 - 2I(u) > 0, \quad 0 \leq t < t_0.$$

Hence $\dot{M}_H(t)$ is strictly increasing with respect to $t \in [0, t_0]$, which together with $\dot{M}_H(0) = 2(u_0(x), u_1(x)) \geq 0$ gives

$$\dot{M}_H(t_0) = 2(u_t, u) > 0.$$

This contradicts $\|u_t(t_0)\|^2 = 0$. So this completes this proof. $\square$

Next we display a finite time blow up result at critical energy level $E(0) = d$.

Theorem 2.16 (Global nonexistence for $E(0) = d$)**.** *Suppose $E(0) = d$, $u_0(x) \in V_H$ and $(u_0(x), u_1(x)) \geq 0$. Then problem* (1.1)-(1.3) *does not admit any global weak solution.*

Proof. Recall the auxiliary function $M_H(t)$ defined as (2.21) and the proof of Theorem 2.11, we have

$$\begin{aligned} M_H(t)\ddot{M}_H(t) - \frac{p+3}{4}\dot{M}_H^2(t) \geq& 2M_H(t)\left((p+1)(J(u) - E(0)) - I(u)\right) \\ =& 2M_H(t)\left((p+1)(J(u) - d) - I(u)\right). \end{aligned}$$

As in Theorem 2.11, from (2.6) in Lemma 2.6 we attain $(p+1)\left(J(u) - d\right) - I(u) > 0$. Hence we obtain (2.25), by the concavity argument, we conclude the result. $\square$

2.3. **High initial energy.** In discussing the global nonexistence result for problem (1.1)-(1.3) at high energy level, we shall introduce some lemmas as follows.

Lemma 2.17. *Let u be a solution of problem* (1.1)-(1.3)*. If initial data $u_0(x)$ and $u_1(x)$ satisfy*

$$(u_0(x), u_1(x)) \geq 0, \tag{2.30}$$

then the mapping $\{t \to \|u(t)\|^2\}$ is strictly monotonically increasing with respect to t as long as $u(x,t) \in V_H$.

Proof. Recalling (2.24), since $u(t) \in V_H$, we attain that for any $t \in [0, T)$,

$$\ddot{M}_H(t) = 2\|u_t\|^2 - 2I(u) > 0. \tag{2.31}$$

Combining (2.30), we have $\dot{M}_H(0) = (u_0(x), u_1(x)) \geq 0$. Then, by (2.31), we have

$$\dot{M}_H(t) > \dot{M}_H(0) \geq 0,$$

which tells that the mapping $\{t \to \|u(t)\|^2\}$ is strictly monotonically increasing with respect to t. $\square$

Attention is now turned to the invariance of the unstable set V_H along the flow of problem (1.1)-(1.3) at high energy level.

Lemma 2.18. *Suppose that the initial data satisfy* (2.30) *and*

$$\|u_0\|^2 > \alpha E(0), \tag{2.32}$$

where $\alpha = 2C_{\rm poin}\big(1+\frac{2}{p-1}\big)$ *and* $C_{\rm poin}$ *is the coefficient of the Poincaré inequality* $C_{\rm poin}\|\nabla u\|^2 \geq \|u\|^2$. *Then the solution of problem* (1.1)-(1.3) *with* $E(0)>0$ *belongs to* V_H, *provided that* $u_0(x)\in V_H$.

Proof. To prove $u(t)\in V_H$ we argue by contradiction. By the continuity of $I(u(t))$, we suppose that $t_0\in(0,T)$ is the first time such that $I(u(t_0))=0$, and $I(u(t))<0$ for $t\in[0,t_0)$. Hence from Lemma 2.17, we obtain that $M_H(t)$ and $\dot M_H(t)$ are strictly increasing on the interval $[0,t_0)$. And then by (2.32), we have

$$M_H(t) > \|u_0\|^2 > \alpha E(0), \quad 0\leq t\leq t_0.$$

Moreover, from the continuity of $u(t)$ in t, we obtain

$$M_H(t_0) > \alpha E(0). \tag{2.33}$$

On the other hand, from (2.9) and the definition of $E(t)$ and $I(u)$, we obtain

$$\begin{aligned} E(0) &= E(t_0)\\ &\geq \frac12\|\nabla u(t_0)\|^2 - \int_\Omega F(u(t_0))\mathrm{d}x\\ &\geq \frac12\|\nabla u(t_0)\|^2 - \frac{1}{p+1}\int_\Omega u(t_0)f(u(t_0))\mathrm{d}x\\ &\geq \big(\frac12-\frac{1}{p+1}\big)\|\nabla u(t_0)\|^2 + \frac{1}{p+1}I(u(t_0)). \end{aligned}$$

Then the fact $I(u(t_0))=0$ directly gives

$$\|\nabla u(t_0)\|^2 \leq 2\big(1+\frac{2}{p-1}\big)E(0).$$

Combining this with Poincaré inequality, we have

$$M_H(t_0)\leq C_{\rm poin}\|\nabla u(t_0)\|^2 \leq 2C_{\rm poin}\big(1+\frac{2}{p-1}\big)E(0)\leq \alpha E(0),$$

which contradicts (2.33). Hence this lemma is proved. □

Theorem 2.19 (Global nonexistence for $E(0)>0$)**.** *Suppose* $E(0)>0$, $u_0(x)\in V_H$, (2.30) *and* (2.32) *hold. Then problem* (1.1)-(1.3) *does not admit any global weak solution.*

Proof. Let $u(x,t)$ be any weak solution of problem (1.1)-(1.3) with $E(0)>0$, $u_0\in V_H$ satisfying (2.30) and (2.32). Then from Lemma 2.18, we have $u(t)\in V_H$. Next let us prove that $u(x,t)$ blows up in finite time. Arguing by contradiction, we suppose that $u(x,t)$ exists globally. Recall the auxiliary function $\ddot M_H(t)$ defined as (2.24), where $t\in[0,T_0], T_0>0$. Obviously for any $t\in[0,T_0]$, we know $M_H(t)>0$. By the continuity of $M_H(t)$, there exists a constant $\rho>0$ independent of T_0 such that

$$M_H(t)\geq\rho,\ 0\leq t\leq T_0. \tag{2.34}$$

At the same time, (2.22) and (2.23) also hold for $t \in [0, T_0]$. Again from (2.24) and (2.23), we see

$$\begin{aligned} \ddot{M}_H(t)M_H(t) - \frac{p+3}{4}\dot{M}_H^2(t) \geq & M_H(t)(\ddot{M}_H(t) - (p+3)\|u_t\|^2) \\ = & M_H(t)(-2I(u) - (p+1)\|u_t\|^2). \end{aligned} \tag{2.35}$$

Let

$$\xi(t) := -2I(u) - (p+1)\|u_t\|^2.$$

Combining the energy $E(t)$, Lemma 2.1 and $I(u)$, we obtain

$$E(t) \geq \frac{1}{2}\|u_t\|^2 + \big(\frac{1}{2} - \frac{1}{p+1}\big)\|\nabla u\|^2 + \frac{1}{p+1}I(u(t)). \tag{2.36}$$

Making a simple transformation of the inequality (2.36), we have

$$-2I(u) \geq (p+1)\|u_t\|^2 + (p-1)\|\nabla u(t)\|^2 - 2(p+1)E(t). \tag{2.37}$$

From (2.9) and (2.37), we have

$$\xi(t) \geq (p-1)\|\nabla u(t)\|^2 - 2(p+1)E(0).$$

Let

$$\vartheta(t) := (p-1)\|\nabla u(t)\|^2 - 2(p+1)E(0),$$

then from (2.32), Lemma 2.17 and Poincaré inequality, we obtain

$$2C_{\rm poin}\big(1 + \frac{2}{p-1}\big)E(0) < \|u_0\|^2 < \|u\|^2 < C_{\rm poin}\|\nabla u\|^2,$$

which says that $\vartheta(t) > 0$. Then there exists a constant $\sigma > 0$ such that

$$\xi(t) > \sigma > 0.$$

Then

$$\ddot{M}_H(t)M_H(t) - \frac{p+3}{4}\dot{M}_H^2(t) \geq \rho\sigma > 0, \ 0 \leq t \leq T_0. \tag{2.38}$$

Substituting $Z_H(t) := \big(M_H(t)\big)^{-\frac{p-1}{4}}$ into (2.38) gives

$$Z_H(t) \leq -\frac{p-1}{4}\rho\sigma\big(M_H(t)\big)^{\frac{p+7}{p-1}}, \quad 0 \leq t \leq \infty,$$

which shows that $\lim_{t\to T^*} Z_H(t) = 0$, where T^* is independent of the choice of T_0. Then we choose $T^* < T_0$, such that

$$\lim_{t\to T^*} M_H(t) = +\infty.$$

This completes the proof. $\square$

3. Semilinear parabolic equation

This section states the existence and nonexistence of global solutions for problem (1.4)-(1.6). We denote the invariant sets for the solution of problem (1.4)-(1.6) by

$$W_P = \{u \in H_0^1(\Omega) : I(u) > 0\} \cup \{0\},$$
$$V_P = \{u \in H_0^1(\Omega) : I(u) < 0\},$$

where the definitions of J, I and d are the same as those in Section 2. To meet the need for high initial energy, we add the following definition, the unbounded sets separated by $\mathcal{N}$

$$\mathcal{N}_+ = \{u \in H_0^1(\Omega) : I(u) > 0\},$$
$$\mathcal{N}_- = \{u \in H_0^1(\Omega) : I(u) < 0\} := V_P.$$

We define the cone of nonnegative functions

$$\mathbb{K} = \{u \in H_0^1(\Omega) : u \geq 0 \text{ a.e. in } \Omega\}.$$

For any $u \in H_0^1(\Omega)$, its positive part and its negative part are

$$u^+ := \max\{u(x), 0\}, \quad u^- := \min\{u(x), 0\}.$$

First we claim that all the lemmas in Section 2 also hold in this section.

Definition 3.1 (Weak solution). *Function $u = u(x,t)$ is said to be a weak solution on $\Omega \times [0,T)$ for problem* (1.4)-(1.6)*, and $u \in L^\infty(0,T; H_0^1(\Omega))$ and $u_t \in L^2(0,T; L^2(\Omega))$ satisfying*

$$(u_t, v) + (\nabla u, \nabla v) = (f(u), v), \quad \forall v \in H_0^1(\Omega),\ 0 \leq t < T, \tag{3.1}$$
$$u(x,0) = u_0(x) \quad \text{in } H_0^1(\Omega), \tag{3.2}$$
$$\int_0^t \|u_\tau\|^2 \mathrm{d}\tau + J(u) = J(u_0), \quad 0 \leq t < T. \tag{3.3}$$

For later convenience, similarly as above Section 2, we use the following common assumption in Subsection 3.1-3.2.

(A4) Let $f(u)$ satisfy (A1), $u_0(x) \in H_0^1(\Omega)$.

Next we show the local existence theorem of problem (1.4)-(1.6), whose proof is similar to proof of [5, Theorem 1] with slight modifications.

Theorem 3.2. *Let* (A4) *hold. Then there exists $T \in [0,\infty)$ such that problem* (1.4)-(1.6) *possesses a unique solution $u \in C^0([0,T); H_0^1(\Omega)) \cap C^1((0,T); L^2(\Omega))$ which becomes a classical solution for $t > 0$.*

3.1. Low initial energy.

By using (3.3) and the similar arguments in [18] we can obtain the following result.

Theorem 3.3 (Invariant sets). *Suppose that $J(u_0) < d$. Then both W_P and V_P are invariant along the flow of* (1.4)-(1.6) *respectively.*

The global existence result for problem (1.4)-(1.6) under low initial energy $E(0) < d$ is listed as below.

Theorem 3.4 (Global existence for $J(u_0) < d$). *Suppose that $J(u_0) < d$ and $u_0(x) \in W_P$. Then there is a global weak solution to problem* (1.4)-(1.6) *satisfying $u \in L^\infty(0,\infty; H_0^1(\Omega))$ with $u_t \in L^2(0,\infty; L^2(\Omega))$ and $u \in W_P$ for $0 \leq t < \infty$.*

Proof. We choose $\{w_j(x)\}_{j=1}^{\infty}$ as a system of basis in $H_0^1(\Omega)$. Construct the following approximate solutions $u_m(x,t)$ of problem (1.4)-(1.6) as

$$u_m(x,t) = \sum_{j=1}^{m} g_{jm}(t) w_j(x), \quad m = 1, 2 \dots$$

satisfying

$$(u_{mt}, w_s) + (\nabla u_m, \nabla w_s) = (f(u_m), w_s), \quad s = 1, 2 \dots m; \tag{3.4}$$

$$u_m(x,0) = \sum_{j=1}^{m} g_{jm}(0) w_j(x) \to u_0(x) \quad \text{in } H_0^1(\Omega). \tag{3.5}$$

Multiplying (3.4) by $g'_{sm}(t)$ and summing over $s = 1, 2, \dots, m$ gives

$$\|u_{mt}\|^2 + \frac{\mathrm{d}}{\mathrm{d}t} J(u_m) = 0,$$

i.e.,

$$\int_0^t \|u_{m\tau}\|^2 \mathrm{d}\tau + J(u_m) = J(u_m(0)), \quad 0 \le t < \infty. \tag{3.6}$$

From $J(u_0) < d$ and (3.5) we obtain $J(u_{m0}) < d$ and

$$\int_0^t \|u_{m\tau}\|^2 \mathrm{d}\tau + J(u_m) < d, \ 0 \le t < \infty \tag{3.7}$$

for sufficiently large m. By $u_0(x) \in W_P$ and (3.5) we obtain $u_m(0) \in W_P$ for sufficiently large m. Furthermore By (3.7) we can attain $u_m(t) \in W_P$ for $0 \le t < \infty$ and sufficiently large m. From (3.7) and the definitions of $J(u)$ and $I(u)$ we obtain

$$\int_0^t \|u_{m\tau}\|^2 \mathrm{d}\tau + \frac{p-1}{2(p+1)} \|\nabla u_m\|^2 + \frac{1}{p+1} I(u_m) < d,$$

which together with $u_m(t) \in W_P$ gives

$$\int_0^t \|u_{m\tau}\|^2 \mathrm{d}\tau + \frac{p-1}{2(p+1)} \|\nabla u_m\|^2 < d. \tag{3.8}$$

From (3.8), (ii) in (A1) and Sobolev inequality we can get the following estimates

$$\|\nabla u_m\|^2 < \frac{2(p+1)}{p-1} d, \quad 0 \le t < \infty; \tag{3.9}$$

$$\int_0^t \|u_{m\tau}\|^2 \mathrm{d}\tau < d, \quad 0 \le t < \infty; \tag{3.10}$$

$$\|f(u_m)\|_r \le \sum_{j=1}^{l} a_k \|u_m\|_{q_k}^{p_k} \le \sum_{j=1}^{l} a_k C_*^{p_k} \|\nabla u_m\|^{p_k} \le C, \quad 0 \le t < \infty; \tag{3.11}$$

where C_* is the embedding constant and

$$r = \frac{p_1+1}{p_1}, \quad q_k = p_k \frac{p_1+1}{p_1} \le p_1 + 1.$$

Denote $\xrightarrow{w}$ and $\xrightarrow{w^*}$ as the weakly convergence and weakly star convergence respectively. From (3.9)-(3.11) we can find a χ and a convergent subsequence $\{u_\nu\} \subset \{u_m\}$ as $\nu \to \infty$ satisfying the following:

$$u_\nu \xrightarrow{w^*} u \quad \text{in } L^\infty(0,\infty; H_0^1(\Omega)) \text{ and a.e. in } Q = \Omega \times [0,\infty);$$

$$u_\nu \to u \quad \text{in } L^{p_1+1}(\Omega) \text{ strongly for } t>0;$$

$$u_{\nu t} \xrightarrow{w} u_t \quad \text{in } L^2(0,\infty;L^2(\Omega)); f(u_\nu) \xrightarrow{w^*} \chi = f(u) \quad \text{in } L^\infty(0,\infty;L^r(\Omega)).$$

Integrating (3.4) over $\tau \in [0,t]$ yields

$$(u_m, w_s) + \int_0^t (\nabla u_m, \nabla w_s)\mathrm{d}\tau = \int_0^t (f(u_m), w_s)\,\mathrm{d}\tau + (u_m(0), w_s)\,. \tag{3.12}$$

Let $m=\nu\to\infty$ in (3.12) we obtain

$$(u, w_s) + \int_0^t (\nabla u, \nabla w_s)\mathrm{d}\tau = \int_0^t (f(u), w_s)\mathrm{d}\tau + (u_0, w_s),$$

$$(u, v) + \int_0^t (\nabla u, \nabla v)\mathrm{d}\tau = \int_0^t (f(u), v)\mathrm{d}\tau + (u_0, v), \quad \forall v \in H_0^1(\Omega),\ 0\le t<\infty.$$

By (3.5) we obtain $u(x,0)=u_0(x)$ in $H_0^1(\Omega)$.

Now we turn to verify that u satisfies (3.3) for $0\le t<\infty$. In deed, as a consequence of Theorem 2.11 we have (2.20). Hence from the convergence of u_ν, $u_{\nu t}$, (3.6) and the definition of $J(u)$, we obtain

$$\begin{aligned}
\frac{1}{2}\|\nabla u\|^2 + \int_0^t \|u_\tau\|^2\mathrm{d}\tau &\le \lim_{\nu\to\infty}\inf \frac12\|\nabla u_\nu\|^2 + \lim_{\nu\to\infty}\inf\int_0^t \|u_{\nu\tau}\|^2\mathrm{d}\tau \\
&\le \lim_{\nu\to\infty}\inf\Big(\frac12\|\nabla u_\nu\|^2 + \int_0^t \|u_{\nu\tau}\|^2\mathrm{d}\tau\Big) \\
&\le \lim_{\nu\to\infty}\Big(J(u_\nu(0)) + \int_\Omega F(u_\nu)\mathrm{d}x\Big) \\
&= J(u_0) + \int_\Omega F(u)\mathrm{d}x,
\end{aligned}$$

from which we derive

$$\int_0^t \|u_\tau\|^2\mathrm{d}\tau + J(u) \le J(u_0), \quad 0\le t<\infty.$$

Consequently, according to Theorem 3.3 we can ensure $u\in W_P$ for $0\le t<\infty$. $\square$

Now we state the global nonexistence result for the solution of problem (1.4)-(1.6) under low initial energy $E(0)<d$.

Theorem 3.5. *Suppose that $J(u_0)<d$ and $u_0(x)\in V_P$. Then problem* (1.4)-(1.6) *does not admit any global weak solution.*

Proof. Let $u\in L^\infty(0,T;H_0^1(\Omega))$ be any weak solution defined on maximal time interval $[0,T)$ with $u_t\in L^2(0,T;L^2(\Omega))$ for problem (1.4)-(1.6). The key is to prove $T<\infty$. Arguing by contradiction, we suppose that $T=+\infty$, then $u\in L^\infty(0,\infty;H_0^1(\Omega))$ and $u_t\in L^2(0,\infty;L^2(\Omega))$. Set

$$M_P(t) := \int_0^t \|u\|^2\mathrm{d}\tau. \tag{3.13}$$

Then

$$\dot M_P(t) = \|u\|^2, \tag{3.14}$$

$$\ddot M_P(t) = 2(u_t,u) = -2I(u),\ 0\le t<\infty. \tag{3.15}$$

By (3.3), combining $I(u)$ and $J(u)$, one has

$$\int_0^t \|u_\tau\|^2 \mathrm{d}\tau + \frac{p-1}{2(p+1)}\|\nabla u\|^2 + \frac{1}{p+1}I(u) \ \le \int_0^t \|u_\tau\|^2 \mathrm{d}\tau + J(u) \le J(u_0),$$

hence

$$-2I(u) \ge 2(p+1)\int_0^t \|u_\tau\|^2 \mathrm{d}\tau + (p-1)\|\nabla u\|^2 - 2(p+1)J(u_0),$$

then

$$\begin{aligned} \ddot{M}_P(t) &\ge 2(p+1)\int_0^t \|u_\tau\|^2 \mathrm{d}\tau + (p-1)\|\nabla u\|^2 - 2(p+1)J(u_0) \\ &= 2(p+1)\int_0^t \|u_\tau\|^2 \mathrm{d}\tau + (p-1)\lambda_1 \dot{M}_P(t) - 2(p+1)J(u_0), \end{aligned} \tag{3.16}$$

denote by λ_1 the related first eigenvalue for $-\Delta\varphi = \lambda\varphi,\ x \in \Omega,\ \varphi|_{\partial\Omega} = 0$. In addition, from

$$\int_0^t (u_\tau, u)\mathrm{d}\tau = \frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\|u\|^2 \mathrm{d}\tau = \frac{1}{2}\left(\|u\|^2 - \|u_0\|^2\right),$$

we obtain

$$\begin{aligned} \Big(\int_0^t (u_\tau, u)\mathrm{d}\tau\Big)^2 &= \frac{1}{4}\left(\|u\|^4 - 2\|u_0\|^2\|u\|^2 + \|u_0\|^4\right) \\ &= \frac{1}{4}\big(\dot{M}_P^2(t) - 2\|u_0\|^2\dot{M}_P(t) + \|u_0\|^4\big). \end{aligned} \tag{3.17}$$

Hence by (3.16) and (3.17) we know that

$$\begin{aligned} & M_P(t)\ddot{M}_P(t) - \frac{p+1}{2}\dot{M}_P^2(t) \\ & \ge 2(p+1)\Big(\int_0^t \|u\|^2 \mathrm{d}\tau \int_0^t \|u_\tau\|^2 \mathrm{d}\tau - \Big(\int_0^t (u_\tau, u)\mathrm{d}\tau\Big)^2\Big) \\ & \quad + (p-1)\lambda_1 M_P(t)\dot{M}_P(t) - (p+1)\|u_0\|^2\dot{M}_P(t) \\ & \quad - 2(p+1)J(u_0)M_P(t) + \frac{p+1}{2}\|u_0\|^4, \end{aligned} \tag{3.18}$$

then by Schwartz inequality,

$$\int_0^t \|u\|^2 \mathrm{d}\tau \int_0^t \|u_\tau\|^2 \mathrm{d}\tau - \Big(\int_0^t (u_\tau, u)\mathrm{d}\tau\Big)^2 > 0,$$

combining this with (3.18) we obtain

$$\begin{aligned} & M_P(t)\ddot{M}_P(t) - \frac{p+1}{2}\dot{M}_P^2(t) \\ & \ge (p-1)\lambda_1 M_P(t)\dot{M}_P(t) - (p+1)\|u_0\|^2\dot{M}_P(t) - 2(p+1)J(u_0)M_P(t). \end{aligned} \tag{3.19}$$

From Theorem 3.3 we have $u \in V_P$ and $I(u) < 0$ for $0 \le t < \infty$. Thus from Lemma 2.6 one has

$$-2I(u) > 2(p+1)(d - J(u)), \quad 0 \le t < \infty\,.$$

By (3.15) and (3.3) we have

$$
\begin{aligned}
\ddot{M}_P(t) =& -2I(u)\\
>& 2(p+1)\left(d-J(u)\right)\\
\geq& 2(p+1)\left(d-J(u_0)\right)\\
:=& C_1>0,\quad 0\leq t<\infty,
\end{aligned} \tag{3.20}
$$

$$
\begin{gathered}
\dot{M}_P(t)\geq C_1t+\dot{M}_P(0)=C_1t+\|u_0\|^2>C_1t,\quad 0\leq t<\infty,\\
M_P(t)>\frac{C_1}{2}t^2+M_P(0)=\frac{C_1}{2}t^2,\quad 0\leq t<\infty.
\end{gathered}
$$

Therefore,

$$
\lim_{t\to\infty}M_P(t)=+\infty,\quad \lim_{t\to\infty}\dot{M}_P(t)=+\infty.
$$

Hence there exists a $t_0\geq 0$ such that

$$
\begin{gathered}
\frac{1}{2}(p-1)\lambda_1M_P(t)>(p+1)\|u_0\|^2,\quad t_0\leq t<\infty,\\
\frac{1}{2}(p-1)\lambda_1\dot{M}_P(t)>2(p+1)J(u_0),\ t_0\leq t<\infty,
\end{gathered}
$$

which combined with (3.19) give the inequality

$$
\begin{aligned}
&M_P(t)\ddot{M}_P(t)-\frac{p+1}{2}\dot{M}_P^2(t)\\
&\geq \big(\frac{1}{2}(p-1)\lambda_1M_P(t)-(p+1)\|u_0\|^2\big)\dot{M}_P(t)\\
&\quad+\big(\frac{1}{2}(p-1)\lambda_1\dot{M}_P(t)-2(p+1)J(u_0)\big)M_P(t)>0,\quad t_0\leq t<\infty.
\end{aligned}
$$

Then there exists a $T_1>0$ such that $\lim_{t\to T_1}M_P(t)=+\infty$, which contradicts $T=+\infty$. □

From Theorem 3.4 and Theorem 3.5 a sharp-like condition for global well posedness of solution will be shown for problem (1.4)-(1.6) as follows:

Theorem 3.6 (Sharp conditions)**.** *Suppose that $J(u_0)<d$. Then we have the following alternatives:*

(i) *If $I(u_0)>0$, problem* (1.4)- (1.6) *possesses a global weak solution;*
(ii) *If $I(u_0)<0$, problem* (1.4)- (1.6) *has no global weak solution.*

3.2. **Critical initial energy.** The global existence and nonexistence results for problem (1.4)-(1.6) under critical initial energy $E(0)=d$ are listed as follows.

Theorem 3.7. *Suppose that $J(u_0)=d$ and $u_0(x)\in W_P$. Then problem* (1.4)-(1.6) *possesses a global weak solution which satisfying $u\in L^\infty(0,\infty;H_0^1(\Omega))$ with $u_t\in L^2(0,\infty;L^2(\Omega))$ and $u(t)\in W_P$ for $0\leq t<\infty$.*

Proof. First $J(u_0)=d$ implies that $\|\nabla u\|\neq 0$. Pick a sequence $\{\lambda_m\}$ such that $0<\lambda<1, m=1,2,\dots$ and $\lambda_m\to 1$ as $m\to\infty$. Let $u_{0m}=\lambda_mu_0$ and consider the corresponding initial boundary value problem

$$
\begin{gathered}
u_t-\Delta u=f(u),\quad x\in\Omega,t>0,\\
u(x,0)=u_{0m}(x),\quad x\in\Omega,\\
u(x,t)=0,\quad x\in\partial\Omega,t\geq 0.
\end{gathered} \tag{3.21}
$$

From $I(u_0) \geq 0$ and Lemma 2.5, we have $\lambda^* = \lambda^*(u_0) \in (0, 1)$. Thus, we obtain $I(u_{0m}) = I(\lambda_m u_0) > 0$ and $J(u_{0m}) = J(\lambda_m u_0) < J(u_0) = d$. The remainder proof of global existence for the solution is similar to that in the proof of the low initial case, i.e. Theorem 3.4. □

Theorem 3.8 (Global nonexistence for $J(u_0) = d$)**.** *Suppose that* $J(u_0) = d$ *and* $u_0(x) \in V_P$*. Then problem* (1.4)-(1.6) *does not admit any global weak solution.*

Proof. Let u be a solution of (1.4)-(1.6) with $J(u_0) = d > 0$ and $I(u_0) < 0$, T be the maximum existence time of $u(t)$. We can deduce that $T < \infty$. From the continuities of $J(t) = J(u(t))$ and $I(t) = I(u(t))$ with respect to t, we know that there exists a sufficient small $t_1 > 0$ with $t_1 < T$ such that $J(u(t_1)) > 0$ and $I(u(t)) < 0$ for $t \in [0, t_1]$. Thus we have $(u_t, u) = -I(u) > 0$ and $\|u_t\| > 0$ for $t \in [0, t_1]$. From this and continuity of $\int_0^t \|u_\tau\|^2 \mathrm{d}\tau$, it follows that we can choose such t_1 that

$$0 < J(u(t_1)) = d - \int_0^{t_1} \|u_t\|^2 \mathrm{d}t = d_1 < d. \tag{3.22}$$

Testing (1.4) by u_t and integrating with respect to t from t_1 to t gives

$$J(u) + \int_{t_1}^{t} \|u_t\|^2 \mathrm{d}t = J(u(t_1)). \tag{3.23}$$

Taking $t = t_1$ as the initial time and by Theorem 3.3, we have $u(t) \in V_P$, for $t > t_1$. Thus from Lemma 2.6 we obtain

$$-2I(u) > 2(p+1)(d - J(u(t_1))), \quad t_1 < t < \infty,$$

then (3.20) turns into

$$\begin{aligned} \ddot{M}_P(t) &= -2I(u) \\ &> 2(p+1)(d - J(u)) \\ &\geq 2(p+1)(d - J(u(t_1))) \\ &:\equiv C_2 > 0, \quad t_1 < t < \infty, \end{aligned}$$

$$\dot{M}_P(t) \geq C_2 t + \dot{M}_P(t_1) \geq C_2 t, \quad t_1 < t < \infty, \tag{3.24}$$

$$M_P(t) > \frac{C_2}{2} t^2 + M_P(t_1) > \frac{C_2}{2} t^2, \quad t_1 < t < \infty. \tag{3.25}$$

From (3.24) and (3.25) it follows that for sufficiently large t we have

$$\begin{aligned} \frac{1}{2}(p-1)\lambda_1 M_P(t) &> (p+1)\|u_0\|^2, \quad t_1 < t < \infty, \\ \frac{1}{2}(p-1)\lambda_1 \dot{M}_P(t) &> 2(p+1)d, \quad t_1 < t < \infty. \end{aligned}$$

Thus (3.19) yields

$$M_P(t)\ddot{M}_P(t) - \frac{p+1}{2}\dot{M}_P^2(t) > 0.$$

The remainder proof of blow up for the solution is similar to that in the proof of the low initial energy case, i.e. Theorem 3.5. □

3.3. **High initial energy.** In fact, when the parameters of the equation are fixed, whether u global exists or blows up in finite time is just determined by the initial data $u_0(x)$. Following this consideration, let us introduce some sets, where $T^*(u_0)$ denotes the maximal existence time of the solution with initial datum $u_0(x) \in H_0^1(\Omega)$,

$$\mathcal{B}_P = \{u_0(x) \in H_0^1(\Omega) : \text{the solution } u(t) \text{ of } (1.4) \text{ blows up in finite time}\},$$
$$\mathcal{G}_P = \{u_0(x) \in H_0^1(\Omega) : T^*(u_0) = \infty\},$$
$$\mathcal{G}_{P,0} = \{u_0(x) \in \mathcal{G}_P : u(t) \mapsto 0 \text{ in } H_0^1(\Omega) \quad \text{as } t \to \infty\}.$$

Furthermore, we need to define the open sub-levels of J,

$$J^\hbar = \{u \in H_0^1(\Omega) : J(u) < \hbar\}.$$

Hence,

$$\mathcal{N}_\hbar := \mathcal{N} \cap J^\hbar \equiv \Big\{u \in \mathcal{N} \;\Big|\; \|\nabla u\|^2 < \frac{2\hbar(p+1)}{p-1}\Big\} \neq \emptyset \text{ for all } \hbar > d.$$

The above alternative characterization of d shows that

$$\operatorname{dist}(0, \mathcal{N}) = \min_{u \in \mathcal{N}} \|\nabla u\|^2 = \frac{2d(p+1)}{p-1} > 0.$$

We now define

$$\lambda_\hbar = \inf\{\|u\|^2 : u \in \mathcal{N}, J(u) < \hbar\},$$
$$\Lambda_\hbar = \sup\{\|u\|^2 : u \in \mathcal{N}, J(u) < \hbar\}$$

for all $\hbar > d$. Clearly we have the following monotonicity properties

$$\hbar \mapsto \lambda_\hbar \text{ is nonincreasing and}$$
$$\hbar \mapsto \Lambda_\hbar \text{ is nondecreasing.}$$

Firstly, let us discuss the stationary problem and comparison principle for problem (1.4)-(1.6):

$$\begin{gathered} -\Delta u = f(u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{3.26}$$

Lemma 3.9 ([1, 2]). *Suppose that $u_0(x) \in \mathcal{G}_P$. Then the solution $u(t) = S(t)u_0(x)$ of problem* (1.4)-(1.6) *converges to the solution of* (3.26) *as $t \to \infty$. Here, $S(t)$ denotes the corresponding nonlinear semigroup associated to* (1.4) *which maps an $H_0^1(\Omega)$ neighborhood of u_0 continuously into $C_0^1(\Omega)$ for all $t \in (0, T^*(u_0))$, where*

$$C_0^1(\Omega) := \{u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\} = C^1(\overline{\Omega}) \cap H_0^1(\Omega),$$

endowed with the standard norm $\|\cdot\|_{C^1}$ of $C^1(\overline{\Omega})$.

Furthermore, if $T^*(u_0) = \infty$, we denote by

$$\omega(u_0) := \cap_{t \geq 0} \overline{\{u(s) : s \geq t\}}$$

the ω-limit set of $u_0(x) \in H_0^1(\Omega)$.

Lemma 3.10 (Gronwall inequality). *Let $y(t) : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function, and assume that there is a constant $C > 0$, such that*

$$\int_s^{+\infty} u(t)\mathrm{d}t \leq Cy(s), \quad 0 \leq t < +\infty,$$

then for all $t \geq 0$, we have

$$y(t) \leq y(0)e^{1-\frac{t}{C}}.$$

We now prove the comparison principle.

Theorem 3.11. *Let $u_0(x),\ v_0(x) \in H_0^1(\Omega)\backslash\{0\}$ be such that $u_0(x) - v_0(x) \in \mathbb{K}$. Then $(S(t)u_0(x) - S(t)v_0(x)) \in \mathbb{K}$ for all $t > 0$. Moreover, if $u_0(x) \neq v_0(x)$, then, for $t > 0$ we obtain*

$$S(t)u_0(x) - S(t)v_0(x) > 0 \quad \text{in } \Omega. \tag{3.27}$$

Proof. Throughout this proof we put $u(t) := S(t)u_0(x)$ and $v(t) := S(t)v_0(x)$. $u,\ v \in C(\overline{\Omega} \times [0,T])$ for all $T < \overline{T} := \min\{T^*(u_0),\ T^*(v_0)\}$. By subtracting the two equations for u and v, we see that $z := u - v$ satisfies

$$\begin{gathered} z_t - \Delta z = H(t)z \quad \text{in } \Omega \times (0,\overline{T}), \\ z(0) = u_0(x) - v_0(x) \geq 0 \quad \text{in } \Omega, \\ z = 0 \quad \text{on } \partial\Omega \times (0,\overline{T}). \end{gathered} \tag{3.28}$$

Here $H(t) := H(\cdot,t)$ is given by

$$H(x,t) = \int_0^1 f(u(x,t) + sz(x,t))\mathrm{d}s \quad \text{for } x \in \Omega,\ t \geq 0,$$

where $s \in (0,1)$. Since $u,\ v$ are continuous functions, for all $T \in (0,\overline{T})$ we have

$$M_T := \sup_{\Omega\times(0,T)} H(x,\ t) < \infty.$$

Taking this into account, if we multiply (3.28) by z^- and integrating over Ω we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z^-(t)\|^2 = -\|\nabla z^-(t)\|^2 + \int_\Omega H(t)|z^-(t)|^2\mathrm{d}x \leq M_T\|z^-(t)\|^2$$

for all $t \in [0,T]$. By Lemma 3.10 and by the arbitrariness of T, this proves that $z^-(t) \equiv 0$. Since $z(t) = S(t)u_0(x) - S(t)v_0(x)$ satisfies the equation $z_t - \Delta z = H(t)z \geq 0$ on $[\delta,\overline{T})$ together with homogeneous Dirichlet boundary conditions, the strong parabolic maximum principle for initial data in $C_0^1(\Omega)$ implies that $z(t) > 0$ in Ω for $t \in (\delta,\overline{T})$. $\square$

To deduce the following lemma, we denote the corresponding Gâteaux derivative $J_u(u)\{h\}$ of $J(u)$ with respect to u at $u \in H_0^1(\Omega)$ in the direction $h \in H_0^1(\Omega)$ as follows

$$J_u(u)\{h\} := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(J(u+\varepsilon h) - J(u)\right).$$

If J has a continuous Gâteaux derivative on Ω, then $J \in C^1(\Omega)$. The second Gâteaux derivative at u is denoted by

$$J_{uu}(u)\{h,h\} := 2\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\left(J(h+\varepsilon h) - J(u)\right).$$

Further we show the Gâteaux of Taylor's theorem which will be used later.

Lemma 3.12 ([9])**.** *Suppose that the line segment between $u \in U \subset H_0^1(\Omega)$ and $u + \varepsilon h$ lies entirely within $U \subset H_0^1(\Omega)$. If F is C^k, then*

$$F(u+\varepsilon h) = F(u) + \varepsilon F_u(u)\{h\} + \frac{\varepsilon^2}{2!}F_{uu}(u)\{h,h\} + \dots$$

$$+\frac{\varepsilon^{k-1}}{(k-1)!}F_{u^{k-1}}(u)\{h^{k-1}\}+o(\varepsilon^{k-1}).$$

Lemma 3.13. *If u is a nontrivial solution of problem (3.26), then $J_u(u)\{u\}=0$, $J_{uu}(u)\{u,u\}<0$ and the first eigenvalue of the eigenvalue problem*

$$\begin{gathered} -\Delta\psi - f_u(u)\psi = \lambda\psi, \quad \text{in } \Omega, \\ \psi = 0, \quad \text{on } \partial\Omega \end{gathered} \tag{3.29}$$

is negative.

Proof. Let $u(x,t)$ be a nontrivial solution of (3.26). So it is easy to check that $\|\nabla u\|^2=\int_\Omega uf(u)\mathrm{d}x$, then

$$\begin{aligned} J_u(u)\{u\} &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(J((1+\varepsilon)u)-J(u)) \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\frac{1}{2}\int_\Omega(|\nabla(1+\varepsilon)u|^2-|\nabla u|^2)-\int_\Omega(F((1+\varepsilon)u)-F(u))\Big), \end{aligned}$$

recalling the definition of Gâteaux derivative and the integral mean value theorem, we obtain

$$\begin{aligned} F_u(u)\{u\} &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\big(F((1+\varepsilon)u)-F(u)\big) \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\int_0^{(1+\varepsilon)u}f(s)\mathrm{d}s-\int_0^u f(s)\mathrm{d}s\Big) \\ &= \lim_{\varepsilon\to 0}\frac{\int_u^{(1+\varepsilon)u}f(s)\mathrm{d}s}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{f(\xi)\varepsilon u}{\varepsilon}=f(u)u, \end{aligned} \tag{3.30}$$

where $u<\xi<(1+\varepsilon)u$, combining with (3.30) and the aid of Lemma 3.12 we can continue to get

$$\begin{aligned} J_u(u)\{u\} &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\frac{1}{2}\int_\Omega(|\nabla u|^2+2\varepsilon|\nabla u|^2+\varepsilon^2|\nabla u|^2-|\nabla u|^2) \\ &\quad -\int_\Omega(F(u)+\varepsilon uf(u)+o(\varepsilon)-F(u))\Big) \\ &\le \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\int_\Omega\varepsilon|\nabla u|^2-\int_\Omega\varepsilon uf(u)\Big) \\ &= \|\nabla u\|^2-\int_\Omega uf(u)\mathrm{d}x=0. \end{aligned}$$

As before, and using the condition (i) in (H1) we can write

$$\begin{aligned} & J_{uu}(u)\{u,u\} \\ &= 2\lim_{\varepsilon\to 0}\frac{J((1+\varepsilon)u)-J(u)}{\varepsilon^2} \\ &= 2\lim_{\varepsilon\to 0}\frac{\frac{1}{2}\big(\|\nabla(1+\varepsilon)u\|^2-\|\nabla u\|^2\big)-\big(\int_\Omega F((1+\varepsilon)u)\mathrm{d}x-\int_\Omega F(u)\mathrm{d}x\big)}{\varepsilon^2} \\ &= 2\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\Big(\frac{1}{2}(\|\nabla u\|^2+2\varepsilon\|\nabla u\|^2+\varepsilon^2\|\nabla u\|^2-\|\nabla u\|^2) \\ &\quad -\int_\Omega\Big(F((1+\varepsilon)u)-F(u)\Big)\mathrm{d}x\Big) \end{aligned}$$

$$\begin{aligned}
&= 2\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\Big(\frac{1}{2}(2\varepsilon\|\nabla u\|^2+\varepsilon^2\|\nabla u\|^2)-\int_\Omega\Big(F((1+\varepsilon)u)-F(u)\Big)\mathrm{d}x\Big)\\
&= 2\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\Big(\frac{1}{2}(2\varepsilon\|\nabla u\|^2+\varepsilon^2\|\nabla u\|^2)-\int_\Omega \varepsilon u f(u)+\frac{1}{2}(\varepsilon u)^2 f_u(u)\mathrm{d}x\Big)\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\Big(\varepsilon^2\|\nabla u\|^2-\int_\Omega(\varepsilon u)^2 f_u(u)\mathrm{d}x\Big)\\
&= \|\nabla u\|^2-\int_\Omega u^2 f_u(u)\mathrm{d}x\\
&\le \|\nabla u\|^2-p\int_\Omega u f(u)\mathrm{d}x<0.
\end{aligned}$$

By a simple computation, we have the corresponding eigenvalue of problem (3.29) as follows

$$\|\nabla u\|^2-\int_\Omega f_u(u)u^2=\lambda\|u\|^2.$$

Thanks to $J_{uu}(u)\{u,u\}<0$, then we assert that the eigenvalue λ is negative. □

Lemma 3.14. *Suppose that $u_1,\ u_2\in H_0^1(\Omega)\setminus\{0\}$ are solutions of (3.26) with $u_1\le u_2$. Then, either $u_1<0<u_2$ or $u_1\equiv u_2$.*

Proof. Assume that $u_1\ne u_2$. By comparison principle, we have $u_1<u_2$ in Ω. Considering the following eigenvalue problem

$$-\Delta\psi-f_u(u)\psi=\lambda\psi. \tag{3.31}$$

From Lemma 3.13, we know the first eigenvalues λ_{u_1} and λ_{u_2} are negative, and its corresponding positive first eigenfunctions ζ_1 and ζ_2 satisfying

$$\begin{gathered}
J_{uu}(u_1)\{\zeta_1,\zeta_1\}<0,\\
J_{uu}(u_2)\{\zeta_2,\zeta_2\}<0.
\end{gathered}$$

Because of the continuity of J_{uu}, taking $J(u_1+\delta\zeta_1)$ as a functional with a value of $u_1+\delta\zeta_1$, according to Lemma 3.12, we have

$$J(u_1+\delta\zeta_1)=J(u_1)+\frac{\delta^2}{2}J_{uu}(u_1)\{\zeta_1,\zeta_1\}+o(\delta^2)<J(u_1) \tag{3.32}$$

for sufficiently small $\delta>0$. Similarly, we also have

$$J(u_2-\delta\zeta_2)<J(u_2). \tag{3.33}$$

Now we define a closed set

$$\mathcal{Q}:=\{\mu\in H_0^1(\Omega):u_1\le\mu\le u_2\text{ a.e. in }\Omega\}$$

and

$$m_P:=\inf_{\mu\in\mathcal{Q}}J(\mu). \tag{3.34}$$

Find a small $\delta>0$ satisfying $u_1<u_1+\delta\zeta_1<u_2-\delta\zeta_2<u_2$, that is $u_1+\delta\zeta_1\in\mathcal{Q}$ and $u_2-\delta\zeta_2\in\mathcal{Q}$, thus (3.34) tells $m_P<\min\{J(u_1+\delta\zeta_1),J(u_2-\delta\zeta_2)\}$, further (3.32) and (3.33) help get

$$m_P<\min\{J(u_1),J(u_2)\}. \tag{3.35}$$

Next we prove that the minimum m_P is achieved by a function $\mu \in \mathcal{Q}$. Taking a minimizing sequence $\{\mu_n\}_n \subset \mathcal{Q}$ for $J|_{\mathcal{Q}} := J(\mu)|_{\mu\in\mathcal{Q}}$. As u_1 and u_2 solve problem (3.26), due to their existence and $\mu_n \in \mathcal{Q}$, we see

$$\|\nabla \mu_n\|^2 = 2J(\mu_n) + 2\int_\Omega F(\mu_n)\mathrm{d}x \le C,$$

where constant C does not depend on the choice of n. Selecting subsequences to make $\mu_n \rightharpoonup \mu \in H_0^1(\Omega)$ (weak convergence) and

$$\mu_n \to \mu \quad \text{a.e. in } \Omega, \qquad \int_\Omega F(\mu_n)\mathrm{d}x \to \int_\Omega F(\mu)\mathrm{d}x,$$

we can attain $\mu \in \mathcal{Q}$, and one infers from Fatou's lemma that

$$\begin{aligned} J(\mu) &= \frac{1}{2}\|\nabla \mu\|^2 - \int_\Omega F(\mu)\mathrm{d}x \\ &\le \frac{1}{2}\liminf_{n\to\infty}\|\nabla\mu_n\|^2 - \lim_{n\to\infty}\int_\Omega F(\mu_n)\mathrm{d}x \\ &= \liminf_{n\to\infty} J(\mu_n) = m_P. \end{aligned}$$

Hence we have

$$J(\mu) = m_P \tag{3.36}$$

and μ is minimizer of $J|_{\mathcal{Q}}$. Also (3.35) tells $\mu \not\equiv u_1$ and $\mu \not\equiv u_2$, which combining the comparison principle and $\mu \in \mathcal{Q}$ gives for any fixed $t = t_0$ that $S(t_0)u_1 \le S(t_0)\mu \le S(t_0)u_2$, i.e. $u_1 \le S(t_0)\mu \le u_2$, that is $S(t)\mu \in \mathcal{Q}$. According to the definition of m_P for any fixed $t = t_0$, $J(S(t)\mu) \ge m_P$. As t_0 is chosen arbitrarily, $S(t)\mu \in \mathcal{Q}$ and

$$J(S(t)\mu) \ge m_P \tag{3.37}$$

hold for any $t > 0$. On the other hand, testing (1.4) by u_t gives

$$\frac{\mathrm{d}}{\mathrm{d}t}J(u(t)) = -\|u_t\|^2, \tag{3.38}$$

which says that $t \mapsto J(S(t)\mu)$ is strictly decreasing along nonconstant trajectory, and from (3.38) we see that

$$J(S(t)\mu) \le m_P \quad \text{for } t \ge 0 \tag{3.39}$$

as the initial datum is μ. In combination with the above conclusions (3.36)-(3.39), one gets that

$$J(S(t)\mu) = J(\mu) = m_P \quad \text{for all } t \ge 0,$$

which implies that $S(t)\mu = \mu$ for all $t \ge 0$. Hence, μ is a solution of stationary problem (3.26) and by the comparison principle we have $u_1 < \mu < u_2$ in Ω. For sufficiently small $|\varepsilon|$, we have

$$(1+\varepsilon)\mu \in \mathcal{Q}.$$

Hence, from the minimum property of μ we obtain

$$J_{\mu\mu}(\mu)\{\mu,\mu\} = \lim_{\varepsilon\to 0}\frac{J((1+\varepsilon)\mu) - J(\mu)}{\varepsilon^2} \ge 0,$$

which combined with Lemma 3.13 imply $\mu \equiv 0$. $\square$

Before the following lemma, we define some sets

$$\mathcal{S}_{\pm} := \big\{u \in C_0^1(\Omega) : \pm u > 0 \text{ in } \Omega; \pm\frac{\partial u}{\partial \nu} < 0 \text{ on } \partial\Omega\big\}, \tag{3.40}$$

where ν is the exterior unit normal vector, and

$$\mathcal{S}_n := \{u \in C_0^1(\Omega) : u(x) < 0 < u(y) \text{ for some points } x, y \in \Omega\big\}, \tag{3.41}$$

which are both open and disjoint in $C_0^1(\Omega)$.

Lemma 3.15. *Let $u_1 \in \mathcal{G}_P \backslash \mathcal{G}_{P,0}$. It holds:*

(i) *if $\omega(u_1) \subset \mathcal{S}_+ \cup \mathcal{S}_n$, then $u_2 \in \mathcal{B}_P$ for every $u_2 \geq u_1$, $u_1 \not\equiv u_2$;*
(ii) *if $\omega(u_1) \subset \mathcal{S}_- \cup \mathcal{S}_n$, then $u_2 \in \mathcal{B}_P$ for every $u_2 \leq u_1$, $u_1 \not\equiv u_2$.*

Proof. From the Hopf boundary lemma, every nontrivial solution of (3.26) lies either in $\mathcal{S}_+, \mathcal{S}_-$ or in $\mathcal{S}_n$. We only prove (i); the proof of (ii) is similar. Let $u_1 \in \mathcal{G}_P \backslash \mathcal{G}_{P,0}, u_2 \geq u_1, u_1 \not\equiv u_2$. Denote

$$u(t) := S(t)u_1, \quad \hat{u}(t) := S(t)u_2.$$

From comparison principle and the definition of $\omega(u_1)$, we attain $\hat{u}(t) > u(t)$, i.e. $u_2 \notin \mathcal{G}_{P,0}$. Then we are going to prove that $u_2 \in \mathcal{B}_P$, considering $u_2 \notin \mathcal{G}_{P,0}$, arguing by contradiction, we suppose that $u_2 \in \mathcal{G}_P \backslash \mathcal{G}_{P,0}$ and distinguish the following two cases:

Case 1: There are an $\varepsilon > 0$ and a time sequence $t_n \to \infty$ such that $\|\hat{u}(x, t_n) - u(x, t_n)\|_{C^1} \geq \varepsilon$ for all n;

Case 2: $\|\hat{u}(x, t) - u(x, t)\|_{C^1} \to 0$ as $t \to \infty$.

If Case 1 happens, from compactness of $\omega(u_1)$ and $\omega(u_2)$, there exist subsequences such that $u(t_n) \to u^*$, $\hat{u}(t_n) \to \hat{u}^*$ in $C_0^1(\Omega)$, where u^* and $\hat{u}^*$ are nontrivial solutions of problem (3.26). By comparison principle, we have $\hat{u}^* \geq u^*$, where the solution u^* is not negative by the assumption $\omega(u_1) \subset \mathcal{S}_+ \cup \mathcal{S}_n$ of this lemma. Further by Lemma 3.14, we have $\hat{u}^* = u^*$. But this is impossible, since

$$\|\hat{u}^* - u^*\|_{C^1} = \lim_{n\to\infty} \|\hat{u}(t_n) - u(t_n)\|_{C^1} \geq \varepsilon.$$

Then Case 1 does not hold.

We now suppose that Case 2 happens. For every $\upsilon \in \omega(u_1)$, let λ_υ be the first eigenvalue of Dirichlet eigenvalue problem

$$\begin{gathered} -\Delta\theta - f_\upsilon(\upsilon)\theta = \lambda_\upsilon\theta \quad \text{in } \Omega, \\ \theta = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.42}$$

and let e_υ denote the unique positive L^∞ normalized eigenfunction corresponding to λ_υ. By Lemma 3.13 and the compactness of $\omega(u_1)$ in $C_0^1(\Omega)$, we have

$$\lambda_0 := \sup_{\upsilon \in \omega(u_1)} \lambda_\upsilon < 0. \tag{3.43}$$

Moreover, let $\chi \in C(\bar{\Omega})$ denote the distance function to the boundary $\partial\Omega$, that is, $\chi(x) = \text{dist}(x, \partial\Omega)$ for $x \in \Omega$. Then, again by compactness, there are $C_1, C_2 > 0$ such that

$$C_1\chi(x) \leq e_\upsilon(x) \leq C_2\chi(x) \quad \text{for all } \upsilon \in \omega(u_1), x \in \Omega. \tag{3.44}$$

Let $\eta(t) := \eta(x, t) = \hat{u}(t) - u(t)$, then in aid of comparison principle and the spirits of Theorem 3.11, $\eta(x, t) > 0$ for $x \in \Omega, t > 0$, and η solves the problem

$$\eta_t = \Delta\eta + H(t)\eta, \tag{3.45}$$

where $H(t) = \int_0^1 f_u(u(x,t) + s\eta(x,t))\mathrm{d}s$, for $x \in \Omega$, $t \geq 0$. Now fix $\tau > 0$ such that

$$C_2 \leq C_1 e^{\frac{|\lambda_0|}{2}\tau}, \tag{3.46}$$

which will be used later.

We claim that

$$\inf_{\upsilon \in \omega(u_1)} \sup_{t \leq s \leq t+\tau} \|H(s) - f_\upsilon(\upsilon)\|_\infty \to 0 \quad \text{as } t \to \infty. \tag{3.47}$$

Actually, arguing by contradiction we suppose that there exist some $\varepsilon > 0$ and a sequence t_n which converges to infinity, such that

$$\inf_{\upsilon \in \omega(u_1)} \sup_{t_n \leq s \leq t_n+\tau} \|H(s) - f_\upsilon(\upsilon)\|_\infty > \varepsilon \quad \text{for all } n. \tag{3.48}$$

From Lemma 3.9, there exist $\upsilon \in \omega(u_1)$ and a subsequence (still denote by t_n) such that

$$\sup_{t_n \leq s \leq t_n+\tau} \|u(s) - \upsilon\|_\infty \to 0 \quad \text{as } n \to \infty.$$

In addition, when $\|\eta(t)\|_{C^1} \to 0$ as $t \to \infty$ occurs (Case 2), we obtain

$$\sup_{t_n \leq s \leq t_n+\tau} \|H(s) - f_\upsilon(\upsilon)\|_\infty \to 0.$$

These contradict (3.48) and prove (3.47). We may therefore take $T_0 > 0$ such that

$$\inf_{\upsilon \in \omega(u_1)} \sup_{t \leq s \leq t+\tau} \|H(s) - f_\upsilon(\upsilon)\|_\infty \leq \frac{|\lambda_0|}{2} \tag{3.49}$$

for $t \geq T_0$, which will be used in the estimate of (3.52) later.

Next, we claim that

$$\int_\Omega \eta(t+\tau)\chi(x)\mathrm{d}x \geq \int_\Omega \eta(t)\chi(x)\mathrm{d}x \quad \text{for } t \geq T_0. \tag{3.50}$$

Indeed, by (3.49) and compactness, for all $t \geq T_0$ it is easy to find $\upsilon \in \omega(u_1)$ verifying

$$\|H(s) - f_\upsilon(\upsilon)\|_\infty \leq \frac{|\lambda_0|}{2}, \tag{3.51}$$

for all $s \in [t, t+\tau]$. Using (3.45), Green's Formula, (3.42), (3.43) and (3.51), for $\eta(x,t)$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\int_\Omega \eta(x,s)e_\upsilon \mathrm{d}x &= \int_\Omega \left(\Delta\eta(x,s) + H(x,s)\eta(x,s)\right) e_\upsilon \mathrm{d}x \\
&= \int_\Omega \eta(x,s)(\Delta e_\upsilon + H(x,s)e_\upsilon)\mathrm{d}x \\
&= \int_\Omega \eta(x,s)e_\upsilon(H(x,s) - f_\upsilon(\upsilon) - \lambda_\upsilon)\mathrm{d}x \\
&\geq \int_\Omega \eta(x,s)e_\upsilon(-\frac{|\lambda_0|}{2} - \lambda_\upsilon)\mathrm{d}x \\
&\geq \frac{|\lambda_0|}{2}\int_\Omega \eta(x,s)e_\upsilon \mathrm{d}x
\end{aligned} \tag{3.52}$$

for $s \in [t, t+\tau]$. Integrating (3.52) over $[t, t+\tau]$ gives

$$\ln\Big|\int_\Omega \eta(x,t+\tau)e_\upsilon \mathrm{d}x\Big| - \ln\Big|\int_\Omega \eta(x,t)e_\upsilon \mathrm{d}x\Big| \geq \frac{|\lambda_0|}{2}(t+\tau) - \frac{|\lambda_0|}{2}t;$$

that is,

$$\int_\Omega \eta(x,t+\tau)e_v \mathrm{d}x \geq e^{\frac{|\lambda_0|}{2}\tau}\int_\Omega \eta(x,t)e_v \mathrm{d}x.$$

Combining this with (3.44), one sees that

$$\begin{aligned}C_2\int_\Omega \eta(t+\tau)\chi(x)\mathrm{d}x &\geq \int_\Omega \eta(t+\tau)e_v\mathrm{d}x \\ &\geq e^{\frac{|\lambda_0|}{2}\tau}\int_\Omega \eta(t)e_v\mathrm{d}x \geq C_1 e^{\frac{|\lambda_0|}{2}\tau}\int_\Omega \eta(t)\chi(x)\mathrm{d}x.\end{aligned}$$

From the relationship between C_1 and C_2 we required in (3.46) and above estimate, we obtain (3.50), which easily indicate that

$$\int_\Omega \eta(T_0+\ell\tau)\chi(x)\mathrm{d}x \geq \int_\Omega \eta(T_0)\chi(x)\mathrm{d}x > 0, \tag{3.53}$$

for the fixed T_0 in (3.49) and every $\ell \in \mathbb{N}$. It is obvious that (3.53) contradict the hypothesis that $\|\eta(t)\|_{C^1} \to 0$ as $t \to \infty$. This completes the proof. □

Lemma 3.16. *Let $v \in H_0^1(\Omega)$ be a nontrivial solution of* (3.26)*, and $u_0(x) \in H_0^1(\Omega)$, $u_0(x) \not\equiv \pm v$.*

(i) *If $v^+ \not\equiv 0$ and $u_0(x) \geq v$, then $u_0(x) \in \mathcal{B}_P$;*
(ii) *If $v^- \not\equiv 0$ and $u_0(x) \leq v$, then $u_0(x) \in \mathcal{B}_P$;*
(iii) *If $0 \leq u_0(x) \leq v$, then $u_0(x) \in \mathcal{G}_{P,0}$.*

Proof. (i) Obviously, v is the nontrivial stationary solution of problem (1.4)-(1.6), i.e. $v \in \mathcal{G}_P\backslash\mathcal{G}_{P,0}$. If $v^+ \not\equiv 0$, considering (i) in Lemma 3.15, we have $u_0(x) \in \mathcal{B}_P$.

(ii) Analogously, if $v^- \not\equiv 0$, considering (ii) in Lemma 3.15, we have $u_0(x) \in \mathcal{B}_P$.

(iii) Since $0 \leq u_0(x) \leq v$ by comparison we have $u_0(x) \in \mathcal{G}_P$. Therefore, from Lemma 3.9, we obtain $S(t)u_0(x) \to v^\sharp$ in $H_0^1(\Omega)$ as $t \to \infty$. Suppose $v^\sharp$ is a nontrivial solution of (3.26) by contradiction. By comparison principle, we also know $0 \leq v^\sharp \leq v$. Due to $v^\sharp \not\equiv 0$ (nontrivial solution), combining $0 \leq u_0(x) \leq v$, $u_0(x) \not\equiv \pm v$ with Lemma 3.14, we derive the following two cases

(a) $v^\sharp < 0 < v$, or
(b) $v^\sharp \equiv v$.

As $u_0(x) \geq 0$ and $u_0(x) \not\equiv 0$, case (a) is impossible. Due to the fact that $S(t)u_0 \to v^\sharp$, $u_0(x) \leq v$, we deduce $v^\sharp \not\equiv v$ that kills case (b). Thus $v^\sharp$ is a trivial solution of (3.26), i.e., $\omega(u_0) = \{0\}$, that is $u_0(x) \in \mathcal{G}_{P,0}$. □

Theorem 3.17 (Global existence and nonexistence for $J(u_0) > 0$)**.** *For every positive M, there exist $u_P,\ v_P \in \mathcal{N}_+ \cap \mathbb{K} \cap C_0^1(\Omega)$ satisfying the following two conditions:*

(i) *$J(u_P) \geq M,\ J(v_P) \geq M$;*
(ii) *$u_P \in \mathcal{G}_{P,0},\ v_P \in \mathcal{B}_P$.*

Proof. Let $M > 0$ and v denote a positive solution of problem (3.26). Assume that

$$\Omega' = \{x \in \Omega : v \in H_0^1(\Omega), v > \epsilon\} \subset \Omega$$

is an open subset for a sufficiently small $\epsilon > 0$. Now, for any $h > 0$, choose a positive function $\phi_h \in C_0^1(\Omega')$ and make a continuous zero extension to $\Omega\backslash\Omega'$ such that

$$\|\nabla\phi_h\| \geq h \quad \text{and} \quad \|\phi_h\|_\infty \leq \epsilon.$$

For a fixed $h > 0$ we put $\varrho_+ := v + \phi_h$ and $\varrho_- := v - \phi_h$. Then $\varrho_\pm \in \mathbb{K}$, and (ii) in (A1) gives

$$\begin{aligned}\int_{\Omega'} \varrho_\pm f(\varrho_\pm)\mathrm{d}x &\le \sum_{k=1}^{l} a_k \int_{\Omega'} |\varrho_\pm|^{p_k+1}\mathrm{d}x \\ &\le \sum_{k=1}^{l} a_k\big(\|v\|^{p_k+1}_{L^{p_k+1}(\Omega')} + \|\phi_h\|^{p_k+1}_{L^{p_k+1}(\Omega')}\big) \\ &\le \sum_{k=1}^{l} a_k\big(\|v\|^{p_k+1}_{L^{p_k+1}(\Omega')} + \epsilon^{p_k+1}|\Omega'|\big),\end{aligned}$$

where $a_k,\ p_k$ are same in condition (ii) in (A1). From Lemma 2.1 and (ii) in (A1) we have

$$\begin{aligned}J(\varrho_\pm) =& \frac{1}{2}\|\nabla\varrho_\pm\|^2 - \int_\Omega F(\varrho_\pm)\mathrm{d}x \\ =& \frac{1}{2}\Big(\int_{\Omega'} |\nabla\varrho_\pm|^2\mathrm{d}x + \int_{\Omega\setminus\Omega'} |\nabla\varrho_\pm|^2\mathrm{d}x\Big) \\ &- \Big(\int_{\Omega'} F(\varrho_\pm)\mathrm{d}x + \int_{\Omega\setminus\Omega'} F(\varrho_\pm)\mathrm{d}x\Big) \\ \ge& \frac{1}{2}\int_{\Omega'} |\nabla\varrho_\pm|^2\mathrm{d}x - \frac{1}{p+1}\Big(\int_{\Omega'} \varrho_\pm f(\varrho_\pm)\mathrm{d}x + \int_{\Omega\setminus\Omega'} \varrho_\pm f(\varrho_\pm)\mathrm{d}x\Big).\end{aligned}$$

Obviously, consider that v is a positive solution of problem (3.26) and the continuous zero extension property of $\phi_h \in C_0^1(\Omega')$, we know that $\int_{\Omega\setminus\Omega'} \varrho_\pm f(\varrho_\pm)\mathrm{d}x = \int_{\Omega\setminus\Omega'} vf(v)\mathrm{d}x$ is bounded in $H_0^1(\Omega)$ and independent of t. Therefore,

$$\begin{aligned}J(\varrho_\pm) \ge& \frac{1}{2}\|\nabla\varrho_\pm\|^2_{L^2(\Omega')} - \frac{1}{p+1}\int_{\Omega'} \varrho_\pm f(\varrho_\pm)\mathrm{d}x - \frac{C(\epsilon)}{p+1} \\ \ge& \frac{1}{2}(h - \|\nabla v\|_{L^2(\Omega')})^2 - \sum_{k=1}^{l} \frac{a_k}{p+1}\left(\|v\|^{p_k+1}_{L^{p_k+1}(\Omega')} + \epsilon^{p_k+1}|\Omega'|\right) - \frac{C(\epsilon)}{p+1},\end{aligned}$$

where $C(\epsilon) = \int_{\Omega\setminus\Omega'} \varrho_\pm f(\varrho_\pm)\mathrm{d}x$. Similarly, we can also deduce

$$\begin{aligned}I(\varrho_\pm) =& \|\nabla\varrho_\pm\|^2 - \int_\Omega \varrho_\pm f(\varrho_\pm)\mathrm{d}x \\ =& \Big(\int_{\Omega'} |\nabla\varrho_\pm|^2\mathrm{d}x + \int_{\Omega\setminus\Omega'} |\nabla\varrho_\pm|^2\mathrm{d}x\Big) \\ &- \Big(\int_{\Omega'} \varrho_\pm f(\varrho_\pm)\mathrm{d}x + \int_{\Omega\setminus\Omega'} \varrho_\pm f(\varrho_\pm)\mathrm{d}x\Big) \\ \ge& \|\nabla\varrho_\pm\|^2_{L^2(\Omega')} - \int_{\Omega'} \varrho_\pm f(\varrho_\pm)\mathrm{d}x - C(\epsilon) \\ \ge& \big(h - \|\nabla v\|_{L^2(\Omega')}\big)^2 - \sum_{k=1}^{l} a_k\left(\|v\|^{p_k+1}_{L^{p_k+1}(\Omega')} + \epsilon^{p_k+1}|\Omega'|\right) - C(\epsilon).\end{aligned}$$

Hence, for h sufficiently large that both $J(\varrho_\pm) \ge M$ and $I(\varrho_\pm) > 0$ are satisfied, therefore $\varrho_\pm \in \mathcal{N}_+$ automatically holds. For such a number h, take $\varrho_- = u_P$ and

$\varrho_+ = v_P$. Since $0 \le u_P \le v$ we have $u_P \in \mathcal{G}_{P,0}$ by Lemma 3.16 (iii). On the other hand, by $0 \le v \le v_P$, we obtain $v_P \in \mathcal{B}_P$ by Lemma 3.16 (i). $\square$

Next we show a crucial condition for vanishing or blow-up of solution at arbitrarily high energy level to problem (1.4)-(1.6) as follows.

Lemma 3.18. *Suppose that $u \in H_0^1(\Omega)$, then*

(i) *For every $u \in \mathcal{N}_+$, we obtain $J(u) > 0$;*
(ii) *For all $u \in \mathcal{N}$, we show that $J(u) = \max_{\lambda \ge 0} J(\lambda u)$;*
(iii) *For each $\hbar > 0$, we assert that $J^{\hbar} \cap \mathcal{N}_+$ is bounded set in $H_0^1(\Omega)$.*

Proof. (i) For $u \in \mathcal{N}_+$, we have $I(u) > 0$, and make use of Lemma 2.1, we obtain

$$J(u) = \frac{1}{2}\|\nabla u\|^2 - \int_\Omega F(u)\mathrm{d}x \ge \frac{1}{p+1}I(u) + \frac{p-1}{2p+2}\|\nabla u\|^2 > 0.$$

(ii) For $u \in \mathcal{N}$, we can get $I(u) = 0$. Hence, combined with Lemma 2.5, we have

$$\frac{d}{d\lambda}J(\lambda u) = I(\lambda u) = 0,$$

which infers that $\lambda = 1$, and $J(u) = \max_{\lambda \ge 0} J(\lambda u)$ for $u \in \mathcal{N}$.

(iii) Since $J(u) < \hbar$ and $I(u) > 0$, we obtain

$$\begin{aligned} \hbar > J(u) &= \frac{1}{2}\|\nabla u\|^2 - \int_\Omega F(u)\mathrm{d}x \\ &\ge \frac{1}{p+1}I(u) + \frac{p-1}{2p+2}\|\nabla u\|^2 \\ &> \frac{p-1}{2p+2}\|\nabla u\|^2, \end{aligned}$$

which yields $\|\nabla u\|^2 < \hbar\frac{2p+2}{p-1}$. Then proof is complete. $\square$

Theorem 3.19. *Suppose that $u_0(x) \in \mathcal{N}_+$ and $\|u_0\|^2 \le \lambda_{J(u_0)}$. Then $u_0(x) \in \mathcal{G}_{P,0}$; and assume that $u_0(x) \in \mathcal{N}_-$ and $\|u_0\|^2 \ge \Lambda_{J(u_0)}$, then $u_0(x) \in \mathcal{B}_P$.*

Proof. Let $u(t) := S(t)u_0(x)$ for $t \in [0, T^*(u_0))$. Recalling the definition of J and I, testing (1.4) by u (respectively u_t) and straightforward computations give us

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 = -2I(u), \quad t \in [0, T^*(u_0)), \tag{3.54}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}J(u) = -\|u_t\|^2, \quad t \in [0, T^*(u_0)). \tag{3.55}$$

Firstly, if $u_0(x) \in \mathcal{N}_+$ satisfies $\|u_0\|^2 \le \lambda_{J(u_0)}$, we assert that $u(t) \in \mathcal{N}_+$ for any $t \in [0, T^*(u_0))$. Assume by contradiction that there exists the first $t_1 \in (0, T^*(u_0))$ such that $u(t) \in \mathcal{N}_+$ for $0 \le t < t_1$ and $u(t_1) \in \mathcal{N}$. with this, (3.54) and (3.55) one deduces that

$$\|u(t_1)\|^2 < \|u_0\|^2 \le \lambda_{J(u_0)}, \tag{3.56}$$

$$J(u(t_1)) < J(u_0). \tag{3.57}$$

As $u(t_1) \in \mathcal{N}$ and (3.57), the definition of $\lambda_{J(u_0)}$ gives $\|u(t_1)\|^2 \ge \lambda_{J(u_0)}$, which contradicts (3.56), hence $u(t) \in \mathcal{N}_+$. Combining (3.56) and (iii) of Lemma 3.18, $J^{J(u_0)} \cap \mathcal{N}_+$ is bounded in $H_0^1(\Omega)$ for $t \in [0, T^*(u_0))$ such that $T^*(u_0) = \infty$, i.e. $u_0(x) \in \mathcal{G}_P$.

Further, for any $w \in \omega(u_0)$, it follows from (3.54) and (3.55) that $\|w\|^2 < \lambda_{J(u_0)}$ and $J(w) \le J(u_0)$. And it was just proved above that $\omega(u_0) \subset \mathcal{N}_+$, which tell us that

$$\omega(u_0) \cap \mathcal{N} = \emptyset. \tag{3.58}$$

As Lemma 3.9 ensures that the solution $u(t) = S(t)u_0(x)$ of problem (1.4)-(1.6) trends to the so-called stationary solution of (3.26) as $t \to \infty$, and also $\mathcal{N}$ contains the nontrivial solutions of problem (3.26) except zero, (3.58) directly gives that $\omega(u_0) = \{0\}$, i.e. $u_0(x) \in \mathcal{G}_{P,0}$.

Finally, if $u_0(x) \in \mathcal{N}_-$ satisfies $\|u_0\|^2 \ge \Lambda_{J(u_0)}$. A similar contradiction as before indicates that $u(t) \in \mathcal{N}_-$ for all $t \in [0, T^*(u_0))$. Now suppose the contrary $T^*(u_0) = \infty$. Thus for any $w \in \omega(u_0)$ we derive that $\|w\|^2 > \Lambda_{J(u_0)}$ and $J(w) \le J(u_0)$ by (3.54) and (3.55). By definition of $\Lambda_{J(u_0)}$, similar to the above, we can deduce that $\omega(u_0) \subset \mathcal{N}_-$ and $\omega(u_0) \cap \mathcal{N} = \emptyset$. As $\mathcal{N}$ contains the nontrivial solutions of problem (3.26) and Lemma 3.9 tells that $S(t)u_0(x)$ converges to the solution of (3.26) as $t \to \infty$, the fact $\omega(u_0) \cap \mathcal{N} = \emptyset$ gives $\omega(u_0) = \{0\}$. However, as $\operatorname{dist}(0, \mathcal{N}_-) > 0$ and $\omega(u_0) \subset \mathcal{N}_-$, it can be seen that $0 \notin \omega(u_0)$. Consequently, we conclude $\omega(u_0) = \emptyset$. This contradicts the assumption that $u(t)$ is a global solution. So we assert that $T^*(u_0) < \infty$, this finishes the proof. $\square$

Lemma 3.20. *Let assumption* (A4) *hold. Suppose that $J(u_0) > 0$ and the initial datum satisfies*

$$\frac{p-1}{2C_{\text{poin}}(p+1)}\|u_0\|^2 > J(u_0), \tag{3.59}$$

where C_{poin} is the coefficient of Poincaré inequality

$$C_{\text{poin}}\|\nabla u\|^2 \ge \|u\|^2. \tag{3.60}$$

Then the map $t \mapsto \|u(t)\|^2$ is strictly increasing as long as $u(t) \in V_P$.

Proof. We introduce the following auxiliary function

$$F(t) := \|u(t)\|^2. \tag{3.61}$$

Then from Equation (1.4) it follows

$$F'(t) = 2(u_t, u) = -2I(u). \tag{3.62}$$

Hence by $u(t) \in V_P$ we arrive at

$$F'(t) > 0 \text{ for } t \in [0, T^*(u_0)). \tag{3.63}$$

Furthermore, from (3.59) and $J(u_0) > 0$ it implies that

$$F(0) = \|u_0\|^2 > \frac{2C_{\text{poin}}(p+1)}{p-1}J(u_0) > 0. \tag{3.64}$$

Therefore from (3.63) and (3.64) we can see that $F(t) > F(0) > 0$, which tells us that the map $t \mapsto \|u(t)\|^2$ is strictly increasing. $\square$

Remark 3.21. As the condition (3.59) of Lemma 3.20 is over strong and beyond what the conclusion of Lemma 3.20 needs, the condition of this lemma can be weaken, but we keep it to make this lemma work for the following Lemma 3.22 and Theorem 3.23, where (3.59) is necessary.

Next, we show the invariance of the unstable set V_P under the flow of problem (1.4)-(1.6) at arbitrarily positive initial energy level $J(u_0) > 0$.

Lemma 3.22 (Invariant set V_P)**.** *Let assumption* (A4) *hold and* $u(x,t)$ *be a weak solution of problem* (1.4)-(1.6) *with maximum existence time interval* $[0,T)$*,* $T \leq +\infty$*. Assume that the initial datum satisfies* (3.59)*. Then all solutions of problem* (1.4)-(1.6) *with* $J(u_0) > 0$ *belong to* V_P*, provided* $u_0(x) \in V_P$*.*

Proof. We prove $u(t) \in V_P$ for $t \in [0,T)$. Arguing by contradiction we assume that $t_0 \in (0,T)$ is the first time such that $I(u(t_0)) = 0$ and $I(u(t)) < 0$ for $t \in [0,t_0)$. From Lemma 3.20 it follows that the map $t \mapsto \|u(t)\|^2$ is strictly increasing on the interval $[0,t_0)$, which together with (3.59) gives

$$\|u(t)\|^2 > \|u_0\|^2 > \frac{2C_{\mathrm{poin}}(p+1)}{p-1}J(u_0),\ t \in (0,t_0). \tag{3.65}$$

Further, from the continuity of $u(t)$ in t, we obtain

$$\|u(t_0)\|^2 > \|u_0\|^2 > \frac{2C_{\mathrm{poin}}(p+1)}{p-1}J(u_0). \tag{3.66}$$

On the other hand, recalling the definition of J, (3.3) and Lemma 2.1, we see

$$\begin{aligned}
J(u_0) =& J(u(t_0)) + \int_0^{t_0} \|u_\tau\|^2 \mathrm{d}\tau \\
\geq& \frac{1}{2}\|\nabla u(t_0)\|^2 - \int_\Omega F(u(t_0))\mathrm{d}x \\
\geq& \frac{1}{2}\|\nabla u(t_0)\|^2 - \frac{1}{p+1}\int_\Omega u(t_0)f(u(t_0))\mathrm{d}x \\
=& (\frac{1}{2} - \frac{1}{p+1})\|\nabla u(t_0)\|^2 + \frac{1}{p+1}I(u(t_0)),
\end{aligned}$$

which together with $I(u(t_0)) = 0$ and Poincaré inequality shows that

$$J(u_0) \geq \frac{p-1}{2(p+1)}\|\nabla u(t_0)\|^2 \geq \frac{p-1}{2C_{\mathrm{poin}}(p+1)}\|u(t_0)\|^2, \tag{3.67}$$

which contradicts (3.66). So the proof is complete. □

Theorem 3.23 (Global nonexistence for $J(u_0) > 0$)**.** *Let* (A4) *hold, and suppose that* $J(u_0) > 0$ *and* $u_0(x) \in V_P$*. Then problem* (1.4)-(1.6) *does not admit any global weak solution provided that*

$$\|u_0\|^2 > \frac{2C_{\mathrm{poin}}(p+1)J(u_0)}{p-1}, \tag{3.68}$$

where C_{poin} *is the coefficient of the Poincaré inequality* (3.60)*.*

Proof. Arguing by contradiction we suppose that $u(x,t)$ exists globally. Testing (1.4) by u and from Lemma 2.1 we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 &= \int_\Omega uu_t \mathrm{d}x \\
&= -\|\nabla u\|^2 + \int_\Omega uf(u)\mathrm{d}x \\
&\geq -2J(u) + \frac{p-1}{p+1}\int_\Omega uf(u)\mathrm{d}x.
\end{aligned} \tag{3.69}$$

For the sake of clarity, the proof will be separated into two cases.

Case I: $J(u) \geq 0$ for $t > 0$. By considering (3.68), we take β such that

$$1 < \beta < \frac{(p-1)\|u_0\|^2}{2C_{\rm poin}(p+1)J(u_0)}. \tag{3.70}$$

Combining with (3.69), (3.3) and Lemma 3.22, we see

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 &\geq -2J(u) + \frac{p-1}{p+1}\int_\Omega uf(u)\mathrm{d}x \\
&= 2(\beta-1)J(u) - 2\beta J(u) + \frac{p-1}{p+1}\int_\Omega uf(u)\mathrm{d}x \\
&\geq -2\beta J(u_0) + 2\beta\int_0^t \|u_\tau\|^2\mathrm{d}\tau + \frac{p-1}{p+1}\int_\Omega uf(u)\mathrm{d}x \\
&\geq -2\beta J(u_0) + 2\beta\int_0^t \|u_\tau\|^2\mathrm{d}\tau - \frac{p-1}{p+1}I(u) + \frac{p-1}{p+1}\|\nabla u\|^2 \\
&> -2\beta J(u_0) + 2\beta\int_0^t \|u_\tau\|^2\mathrm{d}\tau + \frac{p-1}{p+1}\|\nabla u\|^2.
\end{aligned} \tag{3.71}$$

An application of Poincaré inequality leads to

$$\frac{p-1}{p+1}\|\nabla u\|^2 \geq \frac{p-1}{C_{\rm poin}(p+1)}\|u\|^2. \tag{3.72}$$

Substituting (3.72) into (3.71) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 > -4\beta J(u_0) + 4\beta\int_0^t \|u_\tau\|^2\mathrm{d}\tau + \frac{2(p-1)}{C_{\rm poin}(p+1)}\|u\|^2, \tag{3.73}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 - \frac{2(p-1)}{C_{\rm poin}(p+1)}\|u\|^2 > -4\beta J(u_0),$$

which yields

$$\|u\|^2 > \|u_0\|^2 e^{\frac{2(p-1)}{C_{\rm poin}(p+1)}t} + \frac{2\beta C_{\rm poin}(p+1)}{p-1}J(u_0)\Big(1 - e^{\frac{2(p-1)}{C_{\rm poin}(p+1)}t}\Big). \tag{3.74}$$

Substituting (3.74) into (3.73) and recalling the auxiliary function $M_P(t)$ in (3.13) yields

$$\ddot{M}_P(t) > 4\beta\int_0^t \|u_\tau\|^2\mathrm{d}\tau + \Big(\frac{2(p-1)}{C_{\rm poin}(p+1)}\|u_0\|^2 - 4\beta J(u_0)\Big)e^{\frac{2(p-1)}{C_{\rm poin}(p+1)}t}. \tag{3.75}$$

Now we take a small enough number $\varepsilon > 0$ and pick $c > 0$ large enough that

$$c > \frac{1}{4}\varepsilon^{-2}\|u_0\|^4. \tag{3.76}$$

We define a new auxiliary function

$$N_P(t) := M_P^2(t) + \varepsilon^{-1}\|u_0\|^2 M_P(t) + c.$$

Hence,

$$\begin{aligned}
\dot{N}_P(t) &= \big(2M_P(t) + \varepsilon^{-1}\|u_0\|^2\big)\dot{M}_P(t), \\
\ddot{N}_P(t) &= \big(2M_P(t) + \varepsilon^{-1}\|u_0\|^2\big)\ddot{M}_P(t) + 2\dot{M}_P^2(t).
\end{aligned} \tag{3.77}$$

Set $\delta := 4c - \varepsilon^{-2}\|u_0\|^4$, then (3.76) indicates $\delta > 0$. Hence we have

$$\dot{N}_P^2(t) = \big(4M_P^2(t) + 4\varepsilon^{-1}\|u_0\|^2 M_P(t) + \varepsilon^{-2}\|u_0\|^4\big)\dot{M}_P^2(t)$$

$$
\begin{aligned}
&= \left(4M_P^2(t) + 4\varepsilon^{-1}\|u_0\|^2 M_P(t) + 4c - \delta\right) \dot{M}_P^2(t) \\
&= \left(4N_P(t) - \delta\right) \dot{M}_P^2(t),
\end{aligned}
$$

which tells us that

$$
4N_P(t)\dot{M}_P^2(t) = \dot{N}_P^2(t) + \delta \dot{M}_P^2(t). \tag{3.78}
$$

By (3.69), Hölder and Young inequalities, we estimate the term $\dot{M}_P^2(t)$ as follows

$$
\begin{aligned}
\dot{M}_P^2(t) =&\|u\|^4 \\
=&\Big(\|u_0\|^2 + 2\int_0^t \int_\Omega u(\tau)u_\tau(\tau)\mathrm{d}x\mathrm{d}\tau\Big)^2 \\
\le&\Big(\|u_0\|^2 + 2\Big(\int_0^t \|u(\tau)\|^2\mathrm{d}\tau\Big)^{1/2}\Big(\int_0^t \|u_\tau(\tau)\|^2\mathrm{d}\tau\Big)^{1/2}\Big)^2 \\
=&\|u_0\|^4 + 4\|u_0\|^2\Big(\int_0^t \|u(\tau)\|^2\mathrm{d}\tau\Big)^{1/2}\Big(\int_0^t \|u_\tau(\tau)\|^2\mathrm{d}\tau\Big)^{1/2} \\
&+ 4M_P(t)\int_0^t \|u_\tau(\tau)\|^2\mathrm{d}\tau \\
\le&\|u_0\|^4 + 2\varepsilon\|u_0\|^2 M_P(t) + 2\varepsilon^{-1}\|u_0\|^2\int_0^t \|u_\tau(\tau)\|^2\mathrm{d}\tau \\
&+ 4M_P(t)\int_0^t \|u_\tau(\tau)\|^2\mathrm{d}\tau.
\end{aligned} \tag{3.79}
$$

Bearing in mind relation (3.78), we obtain

$$
\begin{aligned}
2N_P(t)\ddot{N}_P(t) &= 2\left(\left(2M_P(t) + \varepsilon^{-1}\|u_0\|^2\right)\ddot{M}_P(t) + 2\dot{M}_P^2(t)\right)N_P(t) \\
&= 2\left(2M_P(t) + \varepsilon^{-1}\|u_0\|^2\right)\ddot{M}_P(t)N_P(t) + 4N_P(t)\dot{M}_P^2(t) \\
&= 2\left(2M_P(t) + \varepsilon^{-1}\|u_0\|^2\right)\ddot{M}_P(t)N_P(t) + \dot{N}_P^2(t) + \delta\dot{M}_P^2(t).
\end{aligned} \tag{3.80}
$$

Now, from (3.78)-(3.80) and (3.75), we can write

$$
\begin{aligned}
&2\ddot{N}_P(t)N_P(t) - (1+\beta)\dot{N}_P^2(t) \\
&= 2\left(2M_P(t) + \varepsilon^{-1}\|u_0\|^2\right)\ddot{M}_P(t)N_P(t) + \delta\dot{M}_P^2(t) - \beta\dot{N}_P^2(t) \\
&= 2\left(2M_P(t) + \varepsilon^{-1}\|u_0\|^2\right)\ddot{M}_P(t)N_P(t) + \delta\dot{M}_P^2(t) - \beta(4N_P(t) - \delta)\dot{M}_P^2(t) \\
&= 2\left(2M_P(t) + \varepsilon^{-1}\|u_0\|^2\right)\ddot{M}_P(t)N_P(t) - 4\beta N_P(t)\dot{M}_P^2(t) + \delta(1+\beta)\dot{M}_P^2(t) \\
&> I_1 I_2 - I_3 I_4,
\end{aligned}
$$

where

$$
\begin{gathered}
I_1 := 2N_P(t)\left(2M_P(t) + \varepsilon^{-1}\|u_0\|^2\right), \\
I_2 := 4\beta\int_0^t \|u_\tau\|^2\mathrm{d}\tau + \Big(\frac{2(p-1)}{C_{\text{poin}}(p+1)}\|u_0\|^2 - 4\beta J(u_0)\Big)e^{\frac{2(p-1)}{C_{\text{poin}}(p+1)}t}, \\
I_3 := 4\beta N_P(t), \\
\begin{aligned}
I_4 :=&\|u_0\|^4 + 2\varepsilon\|u_0\|^2 M_p(t) + 2\varepsilon^{-1}\|u_0\|^2\int_0^t \|u_\tau(\tau)\|^2\mathrm{d}\tau \\
&+ 4M_P(t)\int_0^t \|u_\tau(\tau)\|^2\mathrm{d}\tau.
\end{aligned}
\end{gathered}
$$

Taking $\gamma = \frac{2(p-1)}{C_{\mathrm{poin}}(p+1)}\|u_0\|^2 - 4\beta J(u_0)$, then (3.70) ensures $\gamma > 0$. Choosing ε that

$$\varepsilon < \frac{\gamma e^{\frac{2(p-1)}{C_{\mathrm{poin}}(p+1)}t}}{2\beta\|u_0\|^2},$$

and with the facts that $e^{\frac{2(p-1)}{C_{\mathrm{poin}}(p+1)}t} > 1$ and $N_P(t) > 0$, we obtain

$$\begin{aligned}
&2\ddot{N}_P(t)N_P(t) - (1+\beta)\dot{N}_P^2(t)\\
>&I_1\Big(4\beta\int_0^t \|u_\tau\|^2 \mathrm{d}\tau + \gamma e^{\frac{2(p-1)}{C_{\mathrm{poin}}(p+1)}t}\Big) - I_3I_4\\
>&I_1\Big(4\beta\int_0^t \|u_\tau\|^2 \mathrm{d}\tau + 2\beta\varepsilon\|u_0\|^2\Big) - I_3I_4\\
=&4\beta N_P(t)\Big(2M_P(t) + \varepsilon^{-1}\|u_0\|^2\Big)\Big(2\int_0^t \|u_\tau\|^2 \mathrm{d}\tau + \varepsilon\|u_0\|^2\Big) - I_3I_4\\
=&I_3\Big(\big(2M_P(t) + \varepsilon^{-1}\|u_0\|^2\big)\,\Big(2\int_0^t \|u_\tau\|^2 \mathrm{d}\tau + \varepsilon\|u_0\|^2\Big) - I_4\Big) = 0.
\end{aligned}$$

Therefore

$$\ddot{N}_P(t)N_P(t) - \frac{1+\beta}{2}\dot{N}_P^2(t) > 0,$$

which implies that

$$\ddot{N}_P^{-\frac{\beta-1}{2}}(t) = -\frac{\beta-1}{2N_P^{\frac{\beta+3}{2}}(t)}\Big(\ddot{N}_P(t)N_P(t) - \frac{1+\beta}{2}\dot{N}_P^2(t)\Big) < 0.$$

Since $N_P(0) = c > \frac{1}{4}\varepsilon^{-2}\|u_0\|^4 > 0$ and $\dot{N}_P(0) = \varepsilon^{-1}\|u_0\|^4 > 0$, therefore, we can conclude that there exists some $T < \infty$ such that

$$\lim_{t\to T} N_P^{-\frac{\beta-1}{2}}(t) = 0;$$

that is,

$$\lim_{t\to T} N_P(t) = +\infty.$$

Now, by considering the continuity of both $N_P(t)$ and $M_P(t)$ with respect to t, we can conclude that

$$\lim_{t\to T} M_P(t) = +\infty.$$

Obviously, it contradicts $T = +\infty$.

Case II: $J(u) < 0$ for some $t > 0$. In this case, by considering (3.55) and the continuity of $J(u)$ in t, considering $J(u_0) > 0$, there exists $t_0 > 0$ such that $J(u(t_0)) = 0$ and $J(u(t)) < 0$ for $t > t_0$. According to Lemma 3.22, we shall deduce $u(t) \in V_P$. Then similar to the proof of Theorem 3.5, we can attain the results of blowup.

Thus, by considering the above two cases, the desired assertion immediately follows. □

Subsequently, according to Theorem 3.23, we will establish a criterion to guarantee the blowup of solutions in a finite time when the initial energy is arbitrarily high.

Theorem 3.24. *For every $M > 0$, there exists $u_M \in \mathcal{N}_-$ satisfies the following conditions:*

(i) $J(u_M) \geq M$;
(ii) $u_M \in \mathcal{B}_P$.

Proof. Let $M > 0$, and we take two disjoint open sets Ω_i $(i = 1, 2)$, which are arbitrary subdomains of Ω. Moreover, choosing $v \in H_0^1(\Omega_1) \subset H_0^1(\Omega)$ be an arbitrary nonzero function. Then it is easy to check that $\|\kappa v\|^2 \geq \frac{2C_{\rm poin}(p+1)}{p-1}M$ and $J(\kappa v) \leq 0$ for sufficiently large $\kappa > 0$. Fix such a real number $\kappa > 0$ and select a function $\tilde{v} \in H_0^1(\Omega_2)$ to ensure $J(\tilde{v}) = M - J(\kappa v)$. Then $u_M := \kappa v + \tilde{v}$ verifies

$$\begin{aligned} J(u_M) &= \frac{1}{2}\|\nabla \kappa v\|^2_{L^2(\Omega_1)} - \int_{\Omega_1} F(\kappa v)\mathrm{d}x + \frac{1}{2}\|\nabla \tilde{v}\|^2_{L^2(\Omega_2)} - \int_{\Omega_2} F(\tilde{v})\mathrm{d}x \\ &= J(\kappa v)|_{\kappa v \in H_0^1(\Omega_1)} + J(\tilde{v})|_{\tilde{v} \in H_0^1(\Omega_2)} = M \end{aligned}$$

and

$$\begin{aligned} \|\nabla u_M\|^2 &\geq \frac{1}{C_{\rm poin}}\|u_M\|^2 \\ &= \frac{1}{C_{\rm poin}}\Big(\|\kappa v\|^2_{L^2(\Omega_1)} + \|\tilde{v}\|^2_{L^2(\Omega_2)}\Big) \\ &\geq \frac{1}{C_{\rm poin}}\|\kappa v\|^2_{L^2(\Omega_1)} \\ &\geq \frac{2(p+1)}{p-1}J(u_M). \end{aligned} \tag{3.81}$$

On the other hand, by Lemma 2.1 and the definition of $I(u)$, we have

$$\begin{aligned} \frac{2(p+1)}{p-1}J(u_M) &= \frac{2(p+1)}{p-1}\Big(\frac{1}{2}\|\nabla u_M\|^2 - \int_\Omega F(u_M)\mathrm{d}x\Big) \\ &\geq \frac{2(p+1)}{p-1}\Big(\frac{1}{2}\|\nabla u_M\|^2 - \frac{1}{p+1}\int_\Omega u_M f(u_M)\mathrm{d}x\Big) \\ &\geq \frac{2(p+1)}{p-1}\Big(\frac{1}{2}\|\nabla u_M\|^2 - \frac{1}{p+1}\left(\|\nabla u_M\|^2 - I(u_M)\right)\Big) \\ &\geq \|\nabla u_M\|^2 + \frac{2}{p-1}I(u_M), \end{aligned}$$

combining with (3.81) it is sufficiently to obtain $I(u_M) < 0$. Hence, $u_M \in \mathcal{N}_- \cap \mathcal{B}_P$ by Theorem 3.23. □

4. Nonlinear Schrödinger equation

The main aim of this section is to consider problem (1.7)-(1.8), where $f(u)$ satisfies the Common assumption

(A5)

$$\begin{gathered} f(u) = -\sum_{k=1}^{l} a_k |u|^{p_k - 1}u; \\ 1 + \frac{4}{n} < p_l < p_{l-1} < \cdots < p_1 < \frac{n+2}{n-2} \quad \text{for } n \geq 3; \\ 1 + \frac{4}{n} < p_l < p_{l-1} < \cdots < p_1 < \infty \quad \text{for } n = 1, 2. \end{gathered}$$

By introducing a new potential well W_S and its corresponding outside set V_S, we attain some sharp conditions for global existence of the solution with the initial data satisfying $\mathbb{J}(u_0) < \mathbb{D}$.

In this section for problem (1.7)-(1.8), we define

$$
\begin{gathered}
H^1 = H^1(\mathbb{R}^n), \quad H = \{u \in H^1 : \|u\| = \|u_0\|\},\\
\Sigma = \{u \in H^1 : |x|u \in L^2(\mathbb{R}^n)\},\\
J_S(u) = \frac{1}{2}\|\nabla u\|^2 + \int_{\mathbb{R}^n} F(u)\mathrm{d}x, \quad F(u) = \int_0^u f(s)\mathrm{d}s,\\
\mathbb{J}(u) = \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^n} F(u)\mathrm{d}x = J_S(u) + \frac{1}{2}\|u\|^2,\\
Q(t) = \||x|u\|^2,
\end{gathered}
$$

where $\|\cdot\|_p = \|\cdot\|_{L^p(\mathbb{R}^n)}$, $\|\cdot\| = \|\cdot\|_2$.

In addition we redefine the Nehari functional $\mathbb{I}(u)$, the potential well depth d and the corresponding Nehari functional as follows

$$
\begin{gathered}
\mathbb{I}(u) = \|\nabla u\|^2 + \|u\|^2 + \sum_{k=1}^{l} \frac{n(p_k-1)}{2(p_k+1)} \int_{\mathbb{R}^n} uf(u)\mathrm{d}x,\\
\mathbb{D} = \inf_{u\in\mathbb{N}} \mathbb{J}(u), \quad \mathbb{N} = \{u \in H^1 : \mathbb{I}(u) = 0, u \neq 0\}.
\end{gathered}
$$

The following Proposition 4.1-4.3 are well known. Although Proposition 4.2 and Proposition 4.3 were widely used, it is not easy to find a literature to be cited. Especially the arguments will be very different for different nonlinear terms, hence in the present paper we give the proofs of these two propositions.

Proposition 4.1 (Local existence [3])**.** *Let assumption* (A5) *hold,* $u_0(x) \in H^1$*. Then problem* (1.7)-(1.8) *possesses a unique solution* $u \in C([0,T); H^1)$ *defined on maximum time-interval* $[0,T)$ *such that either*

- (i) $T = +\infty$*, or*
- (ii) $T < \infty$ *and* $\lim_{t\to T} \|u\|_{H^1} = +\infty$*.*

Proposition 4.2 (Conservation law)**.** *Let assumption* (A5) *hold,* $u_0(x) \in H^1$*,* $u \in C([0,T); H^1)$ *be a unique solution to problem* (1.7)-(1.8)*, then*

- (a) $\|u\| = \|u_0\|$*,* $t \in [0,T)$*;*
- (b) $J_S(u) = J_S(u_0)$*,* $t \in [0,T)$*.*

Proof. (a)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathbb{R}^n} |u|^2\mathrm{d}x\Big) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathbb{R}^n} u\bar{u}\mathrm{d}x\Big)\\
&= \int_{\mathbb{R}^n} (u\bar{u}_t + \bar{u}u_t)\mathrm{d}x\\
&= \int_{\mathbb{R}^n} (\overline{\bar{u}u_t} + \bar{u}u_t)\mathrm{d}x\\
&= 2\,\mathrm{Re}\int_{\mathbb{R}^n} \bar{u}u_t\mathrm{d}x.
\end{aligned} \tag{4.1}
$$

From (1.7) we have

$$
\bar{u}u_t = i\Big(\bar{u}\Delta u + \sum_{k=1}^{l} a_k|u|^{p_k-1}u\bar{u}\Big). \tag{4.2}
$$

Substituting (4.2) into (4.1) we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathbb{R}^n}|u|^2\mathrm{d}x\Big) &= 2\,\mathrm{Re}\int_{\mathbb{R}^n} i\Big(\bar{u}\Delta u+\sum_{k=1}^{l}a_k|u|^{p_k-1}u\bar{u}\Big)\mathrm{d}x\\
&= -2\,\mathrm{Im}\int_{\mathbb{R}^n}\Big(\bar{u}\Delta u+\sum_{k=1}^{l}a_k|u|^{p_k-1}u\bar{u}\Big)\mathrm{d}x\\
&= -2\,\mathrm{Im}\int_{\mathbb{R}^n}\Big(\bar{u}\Delta u+\sum_{k=1}^{l}a_k|u|^{p_k+1}\Big)\mathrm{d}x\\
&= -2\,\mathrm{Im}\int_{\mathbb{R}^n}\bar{u}\Delta u\mathrm{d}x\\
&= 2\,\mathrm{Im}\int_{\mathbb{R}^n}\nabla\bar{u}\nabla u\mathrm{d}x\\
&= 2\,\mathrm{Im}\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x=0.
\end{aligned}
\tag{4.3}
$$

(b)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(J_S(u)) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\int_{\mathbb{R}^n}\nabla u\nabla\bar{u}\mathrm{d}x+\int_{\mathbb{R}^n}F(u)\mathrm{d}x\Big)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\int_{\mathbb{R}^n}\nabla u\nabla\bar{u}\mathrm{d}x-\int_{\mathbb{R}^n}\int_0^u\sum_{k=1}^{l}a_k|s|^{p_k-1}s\mathrm{d}s\mathrm{d}x\Big)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\int_{\mathbb{R}^n}\nabla u\nabla\bar{u}\mathrm{d}x-\sum_{k=1}^{l}\frac{a_k}{p_k+1}\int_{\mathbb{R}^n}|u|^{p_k+1}\mathrm{d}x\Big)\\
&= \frac{1}{2}\int_{\mathbb{R}^n}(\nabla u_t\nabla\bar{u}+\nabla u\nabla\bar{u}_t)\mathrm{d}x-\sum_{k=1}^{l}\frac{a_k}{p_k+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}|u\bar{u}|^{\frac{p_k+1}{2}}\mathrm{d}x\\
&= \frac{1}{2}\int_{\mathbb{R}^n}(\nabla u_t\nabla\bar{u}+\nabla u\nabla\bar{u}_t)\mathrm{d}x-\sum_{k=1}^{l}\frac{a_k}{2}\int_{\mathbb{R}^n}|u\bar{u}|^{\frac{p_k-3}{2}}(u\bar{u})\frac{\partial(u\bar{u})}{\partial t}\mathrm{d}x\\
&= \frac{1}{2}\int_{\mathbb{R}^n}(\nabla u_t\nabla\bar{u}+\nabla u\nabla\bar{u}_t)\mathrm{d}x-\sum_{k=1}^{l}\frac{a_k}{2}\int_{\mathbb{R}^n}|u\bar{u}|^{\frac{p_k-3}{2}}|u\bar{u}|(u\bar{u}_t+u_t\bar{u})\mathrm{d}x\\
&= \frac{1}{2}\int_{\mathbb{R}^n}(\nabla u_t\nabla\bar{u}+\nabla u\nabla\bar{u}_t)\mathrm{d}x-\sum_{k=1}^{l}\frac{a_k}{2}\int_{\mathbb{R}^n}|u\bar{u}|^{\frac{p_k-1}{2}}(u\bar{u}_t+u_t\bar{u})\mathrm{d}x\\
&= \frac{1}{2}\int_{\mathbb{R}^n}(\overline{\nabla u\nabla\bar{u}_t}+\nabla u\nabla\bar{u}_t)\mathrm{d}x-\sum_{k=1}^{l}\frac{a_k}{2}\int_{\mathbb{R}^n}|u|^{p_k-1}(u\bar{u}_t+\overline{u\bar{u}_t})\mathrm{d}x\\
&= \mathrm{Re}\Big(\int_{\mathbb{R}^n}\nabla u\nabla\bar{u}_t\mathrm{d}x-\sum_{k=1}^{l}a_k\int_{\mathbb{R}^n}|u|^{p_k-1}u\bar{u}_t\mathrm{d}x\Big)\\
&= \mathrm{Re}\Big(-\int_{\mathbb{R}^n}\Delta u\bar{u}_t\mathrm{d}x-\sum_{k=1}^{l}a_k\int_{\mathbb{R}^n}|u|^{p_k-1}u\bar{u}_t\mathrm{d}x\Big).
\end{aligned}
\tag{4.4}
$$

Again from(1.7) we obtain

$$i|u_t|^2 = \Big(- \Delta u \bar{u}_t - \sum_{k=1}^{l} a_k |u|^{p_k - 1} u \bar{u}_t \Big). \tag{4.5}$$

Inserting (4.5) into (4.4) we can reach

$$\frac{\mathrm{d}}{\mathrm{d}t}(J_S(u)) = \mathrm{Re} \int_{\mathbb{R}^n} i|u_t|^2 \mathrm{d}x = 0. \tag{4.6}$$

Thus we conclude the claims (a) and (b). □

Proposition 4.3. *Suppose that $u_0(x) \in \Sigma$, then the solution $u(t)$ with initial data $u_0(x)$ for problem* (1.7)-(1.8) *belongs to Σ and satisfies*

$$Q''(t) \le 8 \Big(\int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x - \sum_{k=1}^{l} \frac{n(p_k - 1)}{2(p_k + 1)} \int_{\mathbb{R}^n} a_k |u|^{p_k + 1} \mathrm{d}x \Big).$$

Proof. From the definition of $Q(t)$, taking the first derivative of $Q(t)$, we have

$$\begin{aligned} Q'(t) &= \int_{\mathbb{R}^n} |x|^2 (u \bar{u}_t + \bar{u} u_t) \mathrm{d}x \\ &= \int_{\mathbb{R}^n} |x|^2 (\bar{u} u_t + \overline{u \bar{u}_t}) \mathrm{d}x \\ &= 2\mathrm{Re} \int_{\mathbb{R}^n} |x|^2 \bar{u} u_t \mathrm{d}x. \end{aligned} \tag{4.7}$$

From (4.2), (4.7) becomes

$$\begin{aligned} Q'(t) &= 2\mathrm{Re} \int_{\mathbb{R}^n} |x|^2 i \Big(\Delta u \bar{u} + \sum_{k=1}^{l} a_k |u|^{p_k - 1} u \bar{u} \Big) \mathrm{d}x \\ &= -2\mathrm{Im} \int_{\mathbb{R}^n} |x|^2 \Big(\Delta u \bar{u} + \sum_{k=1}^{l} a_k |u|^{p_k + 1} \Big) \mathrm{d}x \\ &= -2\mathrm{Im} \int_{\mathbb{R}^n} |x|^2 (\Delta u \bar{u}) \mathrm{d}x \\ &= 2\mathrm{Im} \int_{\mathbb{R}^n} |x|^2 (u \Delta \bar{u}) \mathrm{d}x. \end{aligned} \tag{4.8}$$

Furthermore, continuing to take the derivative of $Q'(t)$ and using Green's formula we obtain

$$\begin{aligned} Q''(t) &= 2\mathrm{Im} \int_{\mathbb{R}^n} |x|^2 (u_t \Delta \bar{u} + u \Delta \bar{u}_t) \mathrm{d}x \\ &= 2\mathrm{Im} \int_{\mathbb{R}^n} \big(|x|^2 u_t \Delta \bar{u} + \Delta(|x|^2 u) \bar{u}_t \big) \, \mathrm{d}x \\ &= 2\mathrm{Im} \int_{\mathbb{R}^n} \Big(|x|^2 u_t \Delta \bar{u} + \bar{u}_t \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} (|x|^2 u) \Big) \mathrm{d}x \\ &= 2\mathrm{Im} \int_{\mathbb{R}^n} \Big(|x|^2 u_t \Delta \bar{u} + \bar{u}_t \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \Big(\frac{\partial |x|^2}{\partial x_i} u + |x|^2 \frac{\partial u}{\partial x_i} \Big) \Big) \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= 2\mathrm{Im}\int_{\mathbb{R}^n}\Big(|x|^2u_t\Delta\bar{u}+\bar{u}_t\sum_{i=1}^n\frac{\partial}{\partial x_i}\Big(\frac{\partial\sum_{i=1}^n x_i^2}{\partial x_i}u+|x|^2\frac{\partial u}{\partial x_i}\Big)\Big)\mathrm{d}x\\
&= 2\mathrm{Im}\int_{\mathbb{R}^n}\Big(|x|^2u_t\Delta\bar{u}+\bar{u}_t\sum_{i=1}^n\frac{\partial}{\partial x_i}\Big(2x_iu+|x|^2\frac{\partial u}{\partial x_i}\Big)\Big)\mathrm{d}x\\
&= 2\mathrm{Im}\int_{\mathbb{R}^n}\Big(|x|^2u_t\Delta\bar{u}+\bar{u}_t\Big(2nu+4\sum_{i=1}^n x_i\frac{\partial u}{\partial x_i}+|x|^2\sum_{i=1}^n\frac{\partial^2 u}{\partial x_i^2}\Big)\Big)\mathrm{d}x\\
&= 2\mathrm{Im}\int_{\mathbb{R}^n}\big(|x|^2u_t\Delta\bar{u}+|x|^2\bar{u}_t\Delta u+\bar{u}_t(2nu+4x\nabla u)\big)\,\mathrm{d}x\\
&= 2\mathrm{Im}\int_{\mathbb{R}^n}\Big(|x|^2u_t\Delta\bar{u}+\overline{|x|^2u_t\Delta\bar{u}}+\bar{u}_t(2nu+4x\nabla u)\Big)\,\mathrm{d}x\\
&= 4\mathrm{Im}\int_{\mathbb{R}^n}\bar{u}_t(nu+2x\nabla u)\mathrm{d}x. \qquad (4.9)
\end{aligned}
$$

Here, replacing u_t by $\bar{u}_t$ in Eq.(1.7), we have

$$
\bar{u}_t=(-i)\Big(\Delta\bar{u}+\sum_{k=1}^l a_k|u|^{p_k-1}\bar{u}\Big), \tag{4.10}
$$

then (4.9) becomes

$$
\begin{aligned}
Q''(t) &= 4\mathrm{Im}\int_{\mathbb{R}^n}(-i)\Big(\Delta\bar{u}+\sum_{k=1}^l a_k|u|^{p_k-1}\bar{u}\Big)(nu+2x\nabla u)\mathrm{d}x\\
&= -4\mathrm{Re}\int_{\mathbb{R}^n}\Big(\Delta\bar{u}(nu+2x\nabla u)+\sum_{k=1}^l a_k|u|^{p_k-1}\bar{u}(nu+2x\nabla u)\Big)\mathrm{d}x\\
&= -4(I_1+I_2),
\end{aligned} \tag{4.11}
$$

where

$$
\begin{gathered}
I_1 := \mathrm{Re}\int_{R^n}\Delta\bar{u}(nu+2x\nabla u)\mathrm{d}x,\\
I_2 := \mathrm{Re}\int_{R^n}\sum_{k=1}^l a_k|u|^{p_k-1}\bar{u}(nu+2x\nabla u)\mathrm{d}x.
\end{gathered}
$$

Then we consider I_1 and I_2 separately. First, we calculate I_1 by using Green's formula as follows

$$
\begin{aligned}
I_1 &= \mathrm{Re}\int_{\mathbb{R}^n}\Delta\bar{u}(nu+2x\nabla u)\mathrm{d}x\\
&= \mathrm{Re}\int_{\mathbb{R}^n}\big(-n|\nabla u|^2-2\nabla(x\nabla u)\nabla\bar{u}\big)\,\mathrm{d}x\\
&= \mathrm{Re}\int_{\mathbb{R}^n}\Big(-n|\nabla u|^2-2\sum_{i=1}^n\frac{\partial}{\partial x_i}\Big(\sum_{j=1}^n x_j\frac{\partial u}{\partial x_j}\Big)\frac{\partial\bar{u}}{\partial x_i}\Big)\mathrm{d}x\\
&= \mathrm{Re}\int_{\mathbb{R}^n}\Big(-n|\nabla u|^2-2\sum_{i=1}^n\sum_{j=1}^n\frac{\partial}{\partial x_i}\big(x_j\frac{\partial u}{\partial x_j}\big)\frac{\partial\bar{u}}{\partial x_i}\Big)\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{Re}\int_{\mathbb{R}^n}\Big(-n|\nabla u|^2 - 2\sum_{i=1}^n\sum_{j=1}^n\big(\frac{\partial x_j}{\partial x_i}\frac{\partial u}{\partial x_j} + x_j\frac{\partial^2 u}{\partial x_i\partial x_j}\big)\frac{\partial \bar u}{\partial x_i}\Big)\mathrm{d}x\\
&= -n\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x - 2\mathrm{Re}\int_{\mathbb{R}^n}\Big(\sum_{i=1}^n\frac{\partial u}{\partial x_i}\frac{\partial \bar u}{\partial x_i} + \sum_{i=1}^n\sum_{j=1}^n x_j\frac{\partial^2 u}{\partial x_i x_j}\frac{\partial \bar u}{\partial x_i}\Big)\mathrm{d}x\\
&= -n\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x - 2\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x\\
&\quad - \mathrm{Re}\int_{\mathbb{R}^n}\sum_{i=1}^n\sum_{j=1}^n x_j\Big(\frac{\partial^2 u}{\partial x_i x_j}\frac{\partial \bar u}{\partial x_i} + \overline{\frac{\partial^2 u}{\partial x_i x_j}\frac{\partial \bar u}{\partial x_i}}\Big)\mathrm{d}x\\
&= -n\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x - 2\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x\\
&\quad - \mathrm{Re}\int_{\mathbb{R}^n}\sum_{i=1}^n\sum_{j=1}^n x_j\Big(\frac{\partial^2 u}{\partial x_i x_j}\frac{\partial \bar u}{\partial x_i} + \frac{\partial^2 \bar u}{\partial x_i x_j}\frac{\partial u}{\partial x_i}\Big)\mathrm{d}x\\
&= -n\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x - 2\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x - \mathrm{Re}\int_{\mathbb{R}^n}\sum_{i=1}^n\sum_{j=1}^n x_j\frac{\partial}{\partial x_j}\Big(\frac{\partial u}{\partial x_i}\frac{\partial \bar u}{\partial x_i}\Big)\mathrm{d}x\\
&= -n\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x - 2\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x - \mathrm{Re}\int_{\mathbb{R}^n}x\nabla(|\nabla u|^2)\mathrm{d}x\\
&= -n\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x - 2\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x + \mathrm{Re}\int_{\mathbb{R}^n}\nabla x(|\nabla u|^2)\mathrm{d}x\\
&= -n\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x - 2\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x + n\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x\\
&= -2\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x. \qquad (4.12)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_2 &= \mathrm{Re}\int_{\mathbb{R}^n}\sum_{k=1}^l a_k|u|^{p_k-1}\bar u n u + 2x\nabla u\sum_{k=1}^l a_k|u|^{p_k-1}\bar u\mathrm{d}x\\
&= n\mathrm{Re}\int_{\mathbb{R}^n}\sum_{k=1}^l a_k|u|^{p_k+1}\mathrm{d}x + \mathrm{Re}\int_{\mathbb{R}^n}2x\bar u\nabla u\sum_{k=1}^l a_k|u|^{p_k-1}\mathrm{d}x\\
&= n\mathrm{Re}\int_{\mathbb{R}^n}\sum_{k=1}^l a_k|u|^{p_k+1}\mathrm{d}x + \mathrm{Re}\int_{\mathbb{R}^n}x\left(\bar u\nabla u + \overline{\bar u\nabla u}\right)\sum_{k=1}^l a_k|u|^{p_k-1}\mathrm{d}x\\
&= n\mathrm{Re}\int_{\mathbb{R}^n}\sum_{k=1}^l a_k|u|^{p_k+1}\mathrm{d}x + \mathrm{Re}\int_{\mathbb{R}^n}x\left(\bar u\nabla u + u\nabla\bar u\right)\sum_{k=1}^l a_k|u|^{p_k-1}\mathrm{d}x\\
&= n\mathrm{Re}\int_{\mathbb{R}^n}\sum_{k=1}^l a_k|u|^{p_k+1}\mathrm{d}x + \mathrm{Re}\int_{\mathbb{R}^n}x\nabla(u\bar u)\sum_{k=1}^l a_k|u|^{p_k-1}\mathrm{d}x\\
&= n\mathrm{Re}\int_{\mathbb{R}^n}\sum_{k=1}^l a_k|u|^{p_k+1}\mathrm{d}x + \mathrm{Re}\int_{\mathbb{R}^n}x\nabla|u|^2\sum_{k=1}^l a_k\left(|u|^2\right)^{\frac{p_k-1}{2}}\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= n\mathrm{Re}\int_{\mathbb{R}^n}\sum_{k=1}^{l} a_k|u|^{p_k+1}\mathrm{d}x + \mathrm{Re}\int_{\mathbb{R}^n} x\sum_{k=1}^{l} a_k\Big(\left(|u|^2\right)^{\frac{p_k-1}{2}}\nabla|u|^2\Big)\mathrm{d}x \\
&= n\int_{\mathbb{R}^n}\sum_{k=1}^{l} a_k|u|^{p_k+1}\mathrm{d}x + 2\mathrm{Re}\sum_{k=1}^{l}\frac{a_k}{p_k+1}\int_{\mathbb{R}^n} x\nabla\left(|u|^2\right)^{\frac{p_k+1}{2}}\mathrm{d}x \\
&= n\int_{\mathbb{R}^n}\sum_{k=1}^{l} a_k|u|^{p_k+1}\mathrm{d}x - 2\mathrm{Re}\sum_{k=1}^{l}\frac{a_k}{p_k+1}\int_{\mathbb{R}^n} |u|^{p_k+1}\nabla x\mathrm{d}x \\
&= n\int_{\mathbb{R}^n}\sum_{k=1}^{l} a_k|u|^{p_k+1}\mathrm{d}x - 2\mathrm{Re}\sum_{k=1}^{l}\frac{na_k}{p_k+1}\int_{\mathbb{R}^n} |u|^{p_k+1}\mathrm{d}x \\
&= n\int_{\mathbb{R}^n}\sum_{k=1}^{l} a_k|u|^{p_k+1}\mathrm{d}x - \sum_{k=1}^{l}\frac{2n}{p_k+1}\int_{\mathbb{R}^n} a_k|u|^{p_k+1}\mathrm{d}x \qquad (4.13)\\
&= \sum_{k=1}^{l}\frac{n(p_k-1)}{p_k+1}\int_{\mathbb{R}^n} a_k|u|^{p_k+1}\mathrm{d}x. \qquad (4.14)
\end{aligned}
$$

Combining (4.12) and (4.14), we have

$$
Q''(t) \le 8\Big(\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x - \sum_{k=1}^{l}\frac{n(p_k-1)}{2(p_k+1)}\int_{\mathbb{R}^n} a_k|u|^{p_k+1}\mathrm{d}x\Big)\mathrm{d}x. \tag{4.15}
$$

□

Lemma 4.4. *Let assumption* (A5) *hold. Assume that* $u \in H^1$ *and* $0 < \|u\|_{H^1} < r_0$, *then* $\mathbb{I}(u) > 0$, *where*

$$
r_0 = \Big(\sum_{k=1}^{l}\frac{1}{aa_kC_*^{p_k+1}}\Big)^{\frac{1}{p_k-1}}, \quad C_* = \sup_{u\in H^1, u\neq 0}\frac{\|u\|_{p+1}}{\|u\|_{H^1}},\ a = \sum_{k=1}^{l}\frac{n(p_k-1)}{2(p_k+1)}.
$$

Proof. Using $0 < \|u\|_{H^1} < r_0$, we obtain

$$
0 < \sum_{k=1}^{l}\|u\|_{H^1}^{p_k-1} < \sum_{k=1}^{l}\frac{1}{aa_kC_*^{p_k+1}} = r_0^{p_k-1}.
$$

Then

$$
\begin{aligned}
a\int_{\mathbb{R}^n}|uf(u)|\mathrm{d}x &= a\int_{\mathbb{R}^n}\sum_{k=1}^{l} a_k|u|^{p_k+1}\mathrm{d}x \\
&= a\sum_{k=1}^{l} a_k\|u\|_{p_k+1}^{p_k+1} \\
&\le a\sum_{k=1}^{l} a_kC_*^{p_k+1}\|u\|_{H^1}^{p_k+1} \\
&= \sum_{k=1}^{l} aa_kC_*^{p_k+1}\|u\|_{H^1}^{p_k-1}\|u\|_{H^1}^2 < \|u\|_{H^1}^2,
\end{aligned}
$$

thus, we claim that $\mathbb{I}(u) > 0$. □

Lemma 4.5. *Let assumption* (A5) *hold. Assume that $u \in H^1$ and $\mathbb{I}(u) < 0$, then $\|u\|_{H^1} > r_0$.*

Proof. Obviously, $\mathbb{I}(u) < 0$ implies $\|u\| \neq 0$. Hence from

$$\begin{aligned}
\|u\|_{H^1}^2 &< a \int_{\mathbb{R}^n} |uf(u)| \mathrm{d}x \\
&= a \sum_{k=1}^{l} a_k \|u\|_{H^1}^{p_k+1} \\
&\le a \sum_{k=1}^{l} a_k C_*^{p_k+1} \|u\|_{H^1}^{p_k-1} \|u\|_{H^1}^2,
\end{aligned}$$

we obtain $\|u\|_{H^1} > r_0$. □

Lemma 4.6. *Let assumption* (A5) *hold. Assume that $u \in H^1 \backslash \{0\}$ and $\mathbb{I}(u) = 0$, then $\|u\|_{H^1} \ge r_0$.*

Proof. Utilizing Sobolev inequality and $\mathbb{I}(u) = 0$, we obtain

$$\begin{aligned}
\|u\|_{H^1}^2 &= a \int_{\mathbb{R}^n} |uf(u)| \mathrm{d}x \\
&= \sum_{k=1}^{l} a a_k \|u\|_{p_k}^{p_k} \\
&\le \sum_{k=1}^{l} a a_k C_*^{p_k+1} \|u\|_{H^1}^{p_k-1} \|u\|_{H^1}^2,
\end{aligned}$$

which together with $u \neq 0$, yields $\|u\|_{H^1} \ge r_0$. □

Lemma 4.7 (Depth of potential well)**.** *Let* (A5) *hold. Then*

$$\mathbb{D} \ge \mathbb{D}_0 = \Big(\frac{1}{2} - \sum_{k=1}^{l} \frac{2}{n(p_k - 1)}\Big)\Big(\sum_{k=1}^{l} \frac{1}{a a_k C_*^{p_k+1}}\Big)^{\frac{2}{p_k-1}}. \tag{4.16}$$

Proof. From $u \in \mathbb{N}$ we obtain $\|u\|_{H^1} \ge r_0$ and

$$\begin{aligned}
\mathbb{J}(u) &= \frac{1}{2}\|u\|_{H^1}^2 + \int_{\mathbb{R}^n} F(u) \mathrm{d}x \\
&= \frac{1}{2}\|u\|_{H^1}^2 - \int_{\mathbb{R}^n} \int_0^u \sum_{k=1}^{l} a_k |s|^{p_k-1} s \mathrm{d}s \mathrm{d}x \\
&= \frac{1}{2}\|u\|_{H^1}^2 - \sum_{k=1}^{l} \frac{a_k}{p_k+1} \int_{\mathbb{R}^n} |u|^{p_k+1} \mathrm{d}x \\
&= \frac{1}{2}\|u\|_{H^1}^2 + \sum_{k=1}^{l} \frac{1}{p_k+1} \int_{\mathbb{R}^n} uf(u) \mathrm{d}x \\
&= \frac{1}{2}\|u\|_{H^1}^2 + \sum_{k=1}^{l} \frac{2}{n(p_k-1)} \big(\mathbb{I}(u) - \|u\|_{H^1}^2\big) \\
&\ge \Big(\frac{1}{2} - \sum_{k=1}^{l} \frac{2}{n(p_k - 1)}\Big) r_0^2,
\end{aligned} \tag{4.17}$$

which gives (4.16). $\square$

For problem (1.7)-(1.8), let us denote

$$W_S = \{u \in H^1 : \mathbb{I}(u) > 0\},$$
$$V_S = \{u \in H^1 : \mathbb{I}(u) < 0\}.$$

Theorem 4.8 (Invariant sets)**.** *Let* (A5) *hold, and* $\mathbb{J}(u_0) < \mathbb{D}$*. Then the invariance of both sets* W_S *and* V_S *are ensured along the flow of problem* (1.7)-(1.8) *respectively.*

Proof. (i) Let u to be an any solution for problem (1.7)-(1.8) with $u_0 \in W_S$ and T be the maximum existence time of $u(t)$. Next we show that $u(t) \in W_S$ for $0 < t < T$. Arguing by contradiction, we assume that there exists a first $t_0 \in (0, T)$ such that $u(t) \in W_S$ for $t \in [0, t_0)$ and $u(t_0) \in \partial W_S$, i.e., $\mathbb{I}(u(t_0)) = 0$. From Proposition 4.2 (b) we know

$$\mathbb{J}(u) = \mathbb{J}(u_0) < \mathbb{D},\ \ 0 \le t < T. \tag{4.18}$$

By Proposition 4.2 (a), we obtain $u(t_0) \neq 0$. From the definition of $\mathbb{D}$ we see $\mathbb{J}(u(t_0)) \ge \mathbb{D}$, which contradicts (4.18).

(ii) By a similar argument above, we can guarantee that V_S is invariant under the flow of problem (1.7)-(1.8). $\square$

Next, we give the proofs of the well-posedness of solution and show the sharp conditions for global existence of the solution to problem (1.7)-(1.8).

Theorem 4.9 (Global existence)**.** *Let* (A5) *hold, and assume that* $\mathbb{J}(u_0) < \mathbb{D}$ *and* $u_0(x) \in W_S$*. Then the solution* $u(t)$ *of problem* (1.7)-(1.8) *globally exists and* $u(t) \in W_S$ *for* $0 \le t < \infty$*.*

Proof. Notice that Proposition 4.1 shows that the unique solution $u(t)$ defined on maximum time-interval $[0, T)$ exists locally in $C([0, T); H^1)$ for problem (1.7)-(1.8). It only remains to verify $T = +\infty$. Having Theorem 4.8 in mind, we ensure $u(t) \in W_S$ for $0 \le t < T$. First, (4.17) implies

$$\begin{aligned} \mathbb{D} > \mathbb{J}(u) &= \frac{1}{2}\|u\|_{H^1}^2 + \int_{\mathbb{R}^n} F(u)\mathrm{d}x \\ &\ge \Big(\frac{1}{2} - \sum_{k=1}^{l} \frac{2}{n(p_k - 1)}\Big)\|u\|_{H^1}^2 + \Big(\sum_{k=1}^{l} \frac{2}{n(p_k - 1)}\Big)\mathbb{I}(u), \quad 0 \le t < T. \end{aligned} \tag{4.19}$$

Since $\mathbb{I}(u) > 0$, (4.19) yields

$$\|u\|_{H^1}^2 < \Big(\frac{1}{2} - \sum_{k=1}^{l} \frac{2}{n(p_k - 1)}\Big)^{-1}\mathbb{D}, \quad 0 \le t < T,$$

then by Proposition 4.1 we have $T = +\infty$. Furthermore, Theorem 4.8 ensures $u(t) \in W_S$ for $0 \le t < T$. $\square$

Corollary 4.10. *Let assumption* (A5) *hold,* $\|u_0\| \in H^1$*,* $\mathbb{J}(u_0) < \mathbb{D}$ *and* $\|u_0\|_{H^1} < r_0$*. Then problem* (1.7)-(1.8) *possesses a unique global solution* $u(t) \in C([0, T); H)$ *and*

$$\|u\|_{H^1}^2 < \Big(\frac{1}{2} - \sum_{k=1}^{l} \frac{2}{n(p_k - 1)}\Big)^{-1}\mathbb{J}(u_0), \quad 0 \le t < \infty. \tag{4.20}$$

Proof. Notice that $\|u_0\|_{H^1} < r_0$ gives $\|u_0\|_{H^1} = 0$ or $0 < \|u_0\|_{H^1} < r_0$. Hence we shall complete this proof by dividing it into two cases:

(i) If $\|u_0\|_{H^1} = 0$, then $\|u_0\| = 0$. And we infer from $\|u\| = \|u_0\|$ that $\|u_0\|_{H^1} \equiv 0$ for $0 \le t < T$. Then Proposition 4.1 gives $T = +\infty$.

(ii) If $0 < \|u_0\|_{H^1} < r_0$, by Lemma 4.4 we have $\mathbb{I}(u_0) > 0$. While by Theorem 4.9 we know that problem (1.7) possesses a global unique solution $u(t) \in C([0,\infty); H)$ and $u(t) \in W_S$ for $0 \le t < \infty$. And (4.20) follows from Theorem 4.9 immediately. □

Theorem 4.11 (Finite time blow up)**.** *Let* (A5) *hold, and assume that* $\mathbb{J}(u_0) < \mathbb{D}$ *and* $u_0(x) \in \sum \cap V_S$. *Then the solution* $u(t)$ *to problem* (1.7)-(1.8) *blows up in finite time. More precisely, for some* $T < \infty$

$$\lim_{t\to T} \|u(t)\|_{H^1} = +\infty.$$

Proof. Since Proposition 4.1 shows that the unique solution $u(t)$ defined on maximum time-interval $[0, T)$ exists locally in $C([0, T); H^1)$ for problem (1.7)-(1.8). Our goal is to prove $T < \infty$. Arguing by contradiction, we suppose that $T = +\infty$ and define

$$Q(t) := \int_{\mathbb{R}^n} |x|^2 |u|^2 \mathrm{d}x.$$

Then from Proposition 4.3 we have

$$\begin{aligned} Q''(t) &\le 8\Big(\int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x - \sum_{k=1}^{l} \frac{n(p_k-1)}{2(p_k+1)} \int_{\mathbb{R}^n} a_k |u|^{p_k+1} \mathrm{d}x \Big) \\ &\le 8\mathbb{I}(u) - 8\|u\|^2, \quad 0 \le t < \infty. \end{aligned} \tag{4.21}$$

Theorem 4.8 ensures $u(t) \in V_S$ for $0 \le t < \infty$, which tells $\mathbb{I}(u) < 0$ for $0 \le t < \infty$. Hence from (4.21) we obtain

$$\begin{gathered} Q''(t) < -8\|u\|^2 = -8\|u_0\|^2 = -C_0, \quad 0 < t < \infty, \\ Q'(t) < -C_0 t + Q'(0), \quad 0 < t < \infty, \end{gathered}$$

where $C_0 > 0$ is a constant. Thus for sufficiently large t we have $Q'(t) < Q'(t_0) < 0$ for $t > t_0$ and

$$Q(t) < Q'(t_0)(t - t_0) + Q(t_0). \tag{4.22}$$

And also for sufficiently large t we have $Q(t) < 0$, as $Q(0) > 0$ by $\mathbb{I}(u_0) < 0$, there exists a $T_1 > 0$ such that

$$\lim_{t\to T_1} Q(t) = 0. \tag{4.23}$$

Note that

$$\begin{aligned} \operatorname{Re} \int_{\mathbb{R}^n} x\bar{u}\nabla u \mathrm{d}x &= -\operatorname{Re} \int_{\mathbb{R}^n} \nabla(x\bar{u}) u \mathrm{d}x \\ &= -\operatorname{Re} \int_{\mathbb{R}^n} (\nabla x \bar{u} + x\nabla\bar{u}) u \mathrm{d}x \\ &= -\operatorname{Re} \int_{\mathbb{R}^n} (n\bar{u}u + xu\nabla\bar{u}) \mathrm{d}x \\ &= -n \int_{\mathbb{R}^n} \bar{u} u \mathrm{d}x - \operatorname{Re} \int_{\mathbb{R}^n} x\overline{\bar{u}\nabla u} \mathrm{d}x, \end{aligned}$$

which implies $-n\int_{\mathbb{R}^n}|u|^2\mathrm{d}x = 2\operatorname{Re}\int_{\mathbb{R}^n} x\bar{u}\nabla u\mathrm{d}x$ and apply the Cauchy-Schwarz inequality, we obtain

$$\|u_0\|^2 = \|u\|^2 \le \frac{2}{n}\|\nabla u\|\||x|u\|. \tag{4.24}$$

From (4.23) and

$$\|u_0\|^2 = \|u\|^2 \le \frac{2}{n}\|\nabla u\|Q^{1/2}(t),$$

we realize that $\limsup_{t\to T_1}\|\nabla u\| = +\infty$, which contradicts $T = +\infty$. By combining $T < +\infty$ and Proposition 4.1, we achieve

$$\limsup_{t\to T}\|u(t)\|_{H^1} = +\infty.$$

□

From Theorems 4.9 and 4.11, the following theorems have its own interest about the global existence and finite time blow-up for the solution of problem (1.7)-(1.8) as follows.

Theorem 4.12 (Sharp conditions I). *Let* (A5) *hold and assume* $u_0(x) \in \Sigma$ *and* $\mathbb{J}(u_0) < \mathbb{D}$. *Then for problem* (1.7)-(1.8) *we have the following alternatives:*

- (i) *If* $\mathbb{I}(u_0) > 0$, *the solution* $u(t)$ *is a unique global solution in* $C([0,\infty); H\cap\Sigma)$;
- (ii) *If* $\mathbb{I}(u_0) < 0$, *the solution* $u(t)$ *blows up in finite time.*

Note that (4.16) gives

$$\begin{aligned}\mathbb{D} \ge \mathbb{D}_0 &= \Big(\frac{1}{2} - \sum_{k=1}^{l}\frac{2}{n(p_k-1)}\Big)\Big(\sum_{k=1}^{l}\frac{1}{aa_kC_*^{p_k+1}}\Big)^{\frac{2}{p_k-1}}\\ &= \Big(\frac{1}{2} - \sum_{k=1}^{l}\frac{2}{n(p_k-1)}\Big)r_0^2.\end{aligned}$$

Hence we have the following another sharp condition.

Theorem 4.13 (Sharp conditions II). *Let* (A5) *hold and assume that* $u_0(x) \in \Sigma$ *and* $\mathbb{J}(u_0) < \mathbb{D}_0$. *Then for problem* (1.7)-(1.8) *we have the following alternatives:*

- (i) *If* $\|u_0\|_{H^1} < r_0$, *the solution* $u(t)$ *is a unique global solution in* $C([0,\infty); H\cap\Sigma)$;
- (ii) *If* $\|u_0\|_{H^1} > r_0$, *the solution* $u(t)$ *blows up in finite time.*

Proof. If $\|u_0\|_{H^1} < r_0$, Corollary 4.10 gives the existence of the unique global solution $u(t) \in C([0,\infty); H\cap\Sigma)$. If $\|u_0\|_{H^1} > r_0$, then by

$$\begin{aligned}&\Big(\frac{1}{2} - \sum_{k=1}^{l}\frac{2}{n(p_k-1)}\Big)\|u_0\|_{H^1}^2 + \Big(\sum_{k=1}^{l}\frac{2}{n(p_k-1)}\Big)\mathbb{I}(u_0)\\ &= \mathbb{J}(u_0) < \mathbb{D}_0 = \Big(\frac{1}{2} - \sum_{k=1}^{l}\frac{2}{n(p_k-1)}\Big)r_0^2,\end{aligned} \tag{4.25}$$

we obtain $\mathbb{I}(u_0) < 0$. Hence by Theorem 4.12, the solution of problem (1.7)-(1.8) blows up in finite time. □

Noting that $\mathbb{J}(u_0) < \frac{1}{2}\|u_0\|_{H^1}^2$ for $u_0 \neq 0$, we obtain the following corollary.

Corollary 4.14. *Let assumption* (A5) *hold. Assume that* $u_0(x) \in H^1$ *and*

$$\|u_0\|_{H^1}^2 \le \Big(\frac{1}{2} - \sum_{k=1}^{l} \frac{2}{n(p_k-1)}\Big)\Big(\sum_{k=1}^{l} \frac{1}{aa_k C_*^{p_k+1}}\Big)^{\frac{2}{p_k-1}}. \tag{4.26}$$

Then it possesses a global unique solution $u(t) \in C([0,\infty); H)$ *for problem* (1.7)-(1.8).

Proof. If $\|u_0\|_{H^1} = 0$, then from Theorem 4.9 we know that the unique global solution $u(t) \equiv 0$. If $\|u_0\|_{H^1} \neq 0$, then (4.26) gives $\mathbb{J}(u_0) < \mathbb{D}_0$ and $\|u_0\|_{H^1} < r_0$. Again by Theorem 4.13, problem (1.7)-(1.8) thus has a unique global solution $u(t) \in C([0,\infty); H)$. $\square$

Authors' contributions. The work presented here was carried out in collaboration between all authors. Runzhang Xu suggested this research and gave the outline of the proofs. He also led the team to write the first draft and revise the paper. Yuxuan Chen proved all the theories of wave equation and heat equation for the critical and supcritical initial energy and also revised the NLS part. Shaohua Chen and Jihong Shen revised the whole paper. Yanbing Yang and Tao Yu revised the computations in NLS part. Zhengsheng Xu completed and revised the most part of arguments of wave equation and heat equation with subcritical initial energy. All authors have contributed to read and approved the manuscript.

Acknowledgement. We appreciate all those referees who contributed to give insightful criticism, comments and wise advice. This work was supported by the National Natural Science Foundation of China (11471087), the China Postdoctoral Science Foundation, the Fundamental Research Funds for the Central Universities. This research was first motivated by the discussions with Professor Yacheng Liu, the supervisor of Runzhang Xu. We also express our memories and loves to dear Professor Yacheng Liu at this moment, seven years after he left.

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