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CRITICAL SECOND-ORDER ELLIPTIC EQUATION WITH ZERO DIRICHLET BOUNDARY CONDITION IN FOUR DIMENSIONS

ZAKARIA BOUCHECHE, HICHEM CHTIOUI, HICHEM HAJAIEJ

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Abstract. We are concerned with the nonlinear critical problem $-\Delta u = K(x)u^3$, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded domain of $\mathbb{R}^4$. Under the assumption that K is strictly decreasing in the outward normal direction on $\partial\Omega$ and degenerate at its critical points for an order $\beta \in (1, 4)$, we provide a complete description of the lack of compactness of the associated variational problem and we prove an existence result of Bahri-Coron type.

1. Introduction and statement of main results

In this article, we study the following nonlinear elliptic partial differential equation with zero Dirichlet boundary condition

$$\begin{gathered} -\Delta u = K(x)u^{\frac{n+2}{n-2}} \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where K is a given function on a bounded domain Ω of $\mathbb{R}^n, n \geq 3$. Our goal is to establish optimal conditions on the function K ensuring existence of solutions of (1.1).

In some sense, (1.1) is related to the well-known scalar curvature problem on an n-dimensional closed manifold (M^n, g_0), $n \geq 3$. The latter consists in finding a new metric g conformally equivalent to g_0 with prescribed scalar curvature $K(x)$ on M^n. See, for example, [1, 4, 6, 9, 10, 14, 15, 16, 17, 25].

Equation (1.1) has a underlying variational problem whose solutions correspond to the positive critical points of the Euler-Lagrange functional J (defined in section 2). Since the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ is not compact, the functional J violates the Palais-Smale condition in the sense that there exist non-compact sequences along which the functional is bounded while its gradient goes to zero. This fact generates loss of compactness and blow-up phenomenon, [3].

By a direct integration, we can see that $\max_{x\in\Omega} K(x) > 0$ is a a necessary condition to solve problem (1.1). When $K \equiv 1$, the problem is called the Yamabe problem. In this case Pohozaev proved that (1.1) has no solution if Ω is star-shaped,

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see [23]. In contrast, Bahri-Coron [5] established the existence of solution if Ω has non trivial topology. Dancer [18], gave examples of contractible domains on which a solution of the Yamabe problem exists. When $K \not\equiv 1$, there are a few results concerning (1.1). For example, in [8, 12, 27] existence results were provided for (1.1) in dimensions $n \geq 4$, under the following two conditions:

(A1) $\frac{\partial K}{\partial \nu}(x) < 0$ for all $x \in \partial\Omega$. Here, ν is the outward normal vector on $\partial\Omega$.

(A2) K is a $\mathcal{C}^2$-positive Morse function such that

$$\begin{gathered} -\frac{\Delta K(y)}{3\,K(y)} + 8H(y,y) \neq 0, \quad \text{when } \nabla K(y) = 0, \text{ if } n = 4 \\ \Delta K(y) \neq 0, \quad \text{when } \nabla K(y) = 0, \text{ if } n \geq 5. \end{gathered} \tag{1.2}$$

Here, $H(\cdot,\cdot)$ is the regular part of the Green function of the Laplacian with Dirichlet boundary condition, that is for each $x \in \Omega$,

$$\begin{gathered} G(x,y) = |x-y|^{2-n} - H(x,y) \quad \text{in } \Omega \\ \Delta H(x,\cdot) = 0 \quad \text{in } \Omega \\ G(x,\cdot) = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

Noting that the non-degeneracy condition (A2) excludes some interesting class of functions K, we will assume here a more general hypothesis, namely the β-flatness condition:

(A3) K is a $\mathcal{C}^1$-positive function such that, for each critical point y of K, there exists a real number β so that

$$K(x) = K(y) + \sum_{k=1}^{n} b_k |(x-y)_k|^\beta + R(x-y), \quad \text{for } x \text{ close to } y,$$

where $b_k =: b_k(y) \neq 0$ for $k = 1,\dots,n$, and $R(z)$ is $\mathcal{C}^1$ near 0 with

$$\lim_{z\to 0} |R(z)||z|^{-\beta} = 0, \quad \lim_{z\to 0} |\nabla R(z)||z|^{1-\beta} = 0.$$

Let us point out that the non-degeneracy condition (A2) is a particular case of the β-flatness condition (A3) (in suitable coordinates), when $\beta = 2$.

Remark 1.1. As an example of function K satisfying condition (A1) and the non-degeneracy condition, or more generally, the β-flatness condition (A3), we have $K : \mathbb{B}^4 \to \mathbb{R}$, $K(x) = 1 - \|x\|^2$, where $\mathbb{B}^4$ is the unit ball of $\mathbb{R}^4$. We can see that the unit outward normal vector ν_x at any $x \in \partial\mathbb{B}^4$ is equal to x. Therefore, $\frac{\partial}{\partial \nu}K(x) = -2 < x, x >= -2 < 0$ on $\partial\mathbb{B}^4$. Moreover, $0_{\mathbb{R}^4}$ is a non-degenerate critical point of K. To obtain a more general situation, let for $\gamma > 0$ and small, $\psi(t)$ a cut-off function defined by $\psi(t) = 1$ if $|t| < \gamma$, $\psi(t) = 0$ if $|t| > 2\gamma$ and $\psi'(t) < 0$ if $\gamma < |t| < 2\gamma$. Define for $\beta > 1$, $K : \mathbb{B}^4 \to \mathbb{R}$ as

$$K(x) = \psi(\|x\|)(1 - \sum_{k=1}^{4} |x_k|^\beta) + (1 - \psi(\|x\|))(1 - \|x\|^2).$$

It satisfies conditions (A1) and (A3). Observe that for $\beta \neq 2$, K does not satisfies (A2).

Recall that (A3) was used widely as a standard assumption to guarantee the existence of solution to the scalar curvature problem on closed manifolds; see, for example, [1, 2, 13, 19, 20, 21]. However all the existence results (in the non-perturbative setting), concern the case where the β-flatness orders of K at all its critical points are in $(1, n-2]$ or in $[n-2, n)$. The aim of this paper is to consider the refined flatness-condition (A3) in the mixed case; that is when the order of flatness at some critical points of K lie in $(1, n-2]$ and at other critical points lie in $[n-2, n)$. For the sake of clarity, we consider the four dimensional case in the current paper.

To state our main result, we need to introduce some notation, and state the assumptions that we will use. Let

$$\mathcal{K} := \{y \in \Omega : \nabla K(y) = 0\}$$

the set of the critical points of K in Ω. To say y satisfies the condition (A3), we adopt the notation $y \in (\mathbf{f})_\beta$. Let

$$\begin{aligned} \mathcal{K}_2 &:= \{y \in (\mathbf{f})_\beta : \beta(y) = 2\}, \\ \mathcal{K}_{<2} &:= \{y \in (\mathbf{f})_\beta : \beta(y) < 2\}, \\ \mathcal{K}_{>2} &:= \{y \in (\mathbf{f})_\beta : \beta(y) > 2\}. \end{aligned}$$

We will assume the following:

(A5) For each $y \in \mathcal{K}_{<2}$, $\sum_{k=1}^4 b_k(y) \neq 0$.

(A6) For each $y \in \mathcal{K}_2$,

$$-\frac{1}{12}\frac{\sum_{k=1}^4 b_k(y)}{K(y)} + H(y,y) \neq 0.$$

Let

$$\mathcal{K}^+_{<2} := \{y \in \mathcal{K}_{<2} : -\sum_{k=1}^4 b_k(y) > 0\},$$

$$\mathcal{K}^+_2 := \{y \in \mathcal{K}_2 : -\frac{1}{12}\frac{\sum_{k=1}^4 b_k(y)}{K(y)} + H(y,y) > 0\}.$$

For each p-tuple, $p \geq 1$, of distinct points $\tau_p := (z_{i_1}, \dots, z_{i_s}, y_{i_{s+1}}, \dots, y_{i_p})$, $0 \leq s \leq p$, such that $z_{i_j} \in \mathcal{K}^+_2, y_{i_k} \in \mathcal{K}_{>2}$ for all $j = 1, \dots, s$ for all $k = s+1, \dots, p$, we define a $p \times p$ symmetric matrix $M(\tau_p) = (m_{ij})$ by

$$m_{jj} := \begin{cases} -\frac{1}{12}\frac{\sum_{k=1}^4 b_k(z_{i_j})}{\big(K(z_{i_j})\big)^2} + \frac{H(z_{i_j}, z_{i_j})}{K(z_{i_j})}, & \text{if } 1 \leq j \leq s, \\ \frac{H(y_{i_j}, y_{i_j})}{K(z_{i_j})}, & \text{if } s+1 \leq j \leq p, \end{cases} \tag{1.4}$$

and, for $k \neq j$,

$$m_{jk} := \begin{cases} -\frac{G(z_{i_j}, z_{i_k})}{\big(K(z_{i_j})K(z_{i_k})\big)^{1/2}}, & if\, 1 \leq k, j \leq s, \\ -\frac{G(y_{i_j}, y_{i_k})}{\big(K(y_{i_j})K(y_{i_k})\big)^{1/2}}, & if\, s+1 \leq k, j \leq p, \\ -\frac{G(z_{i_j}, y_{i_k})}{\big(K(z_{i_j})K(y_{i_k})\big)^{1/2}}, & if\, 1 \leq j \leq s,\ s+1 \leq k \leq p. \end{cases} \tag{1.5}$$

Let $\rho(\tau_p)$ be the least eigenvalue of $M(\tau_p), \forall p \geq 1$.

(A7) Assume that $\rho(\tau_p) \neq 0$ for each distinct points $z_{i_1}, \dots, z_{i_s} \in \mathcal{K}^+_2$, $y_{i_{s+1}}, \dots, y_{i_p} \in \mathcal{K}_{>2}$.

We denote the following two sets

$$\begin{aligned}\mathcal{C}_{\geq 2} := \Big\{ &\tau_p := (z_{i_1}, \dots, z_{i_s}, y_{i_{s+1}}, \dots, y_{i_p}), p \geq 1 \text{ and } 0 \leq s \leq p : z_{i_j} \in \mathcal{K}_2^+, \\ &y_{i_k} \in \mathcal{K}_{>2}\, \forall j = 1, \dots, s,\ \forall k = s+1, \dots, p,\ z_{i_j} \neq z_{i_k}, y_{i_j} \neq y_{i_k}, \forall j \neq k \\ &\text{and } \rho(\tau_p) > 0 \Big\};\end{aligned}$$

$$\begin{aligned}\mathcal{C}_{\infty} := \Big\{ &\tau_p := (z_{i_1}, \dots, z_{i_s}, y_{i_{s+1}}, \dots, y_{i_p}),\ p \geq 1 \text{ and } 0 \leq s \leq p \text{ s.t} \\ &z_{i_j} \in \mathcal{K}_{<2}^+, \forall j = 1, \dots, s, \text{ and } (y_{i_{s+1}}, \dots, y_{i_p}) \in \mathcal{C}_{\geq 2} \Big\}.\end{aligned}$$

In the above definitions, it is to be understood that if $s = 0$, we omit the points z_{i_j}; and if $s = p$, we omit the points y_{i_k}.

We define an index $i : \mathcal{C}_\infty \to \mathbb{Z}$ by

$$i(y_{i_1}, \dots, y_{i_p}) = 5p - 1 - \sum_{j=1}^{p} \widetilde{i}(y_{i_j}),$$

where $\widetilde{i}(y_{i_j}) := \sharp\{1 \leq k \leq 4 : b_k(y_{i_j}) < 0\}$. Now, let us state our main result.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^4$, be a smooth bounded domain, and $0 < K \in \mathcal{C}^1(\bar{\Omega})$ satisfying assumptions* (A1), (A4)–(A8) *and* (S3) *with* $\beta \in (1, 4)$. *If*

$$\sum_{\tau_p \in \mathcal{C}_\infty} (-1)^{i(\tau_p)} \neq 1,$$

then problem (1.1) *has a solution.*

Our method is based on a revisited version of the celebrated critical points at infinity theory which goes back to Bahri [3]. To be able to prove Theorem 1.2, we will present, in section 2 of this paper, some preliminary results that prepare the field to apply Bahri's approach. In section 3, we will provide a complete description of the loss of compactness of the problem. We will first prove that under the assumption (A1), the boundary $\partial\Omega$ does not effect the existence of critical points at infinity. We will then show that under condition (A3), $\beta \in (1, 4)$, the critical points at infinity of the associated variational problem correspond to the element of $\mathcal{C}_\infty$.

In the previous contributions, two cases were addressed. In the first situation, the strong interaction of the bubbles forces all blow up points to be single (this appears in the case $n - 2 < \beta < n$). In the second case, the interaction of two different bubbles are negligible with respect to the self-interactions (this appears in the case $1 < \beta < n - 2$). The main novelty of this current study, is that we develop a self-contained approach enabling us to establish existence results when both phenomenons occur $1 < \beta < n$. Lastly in section 4, we will prove the existence result of this paper.

2. Variational structure and lack of compactness

Problem (1.1) enjoys a variational structure. Indeed, solutions of (1.1) correspond to positive critical points of the functional

$$I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{4} \int_\Omega K|u|^4 \tag{2.1}$$

defined on $H_0^1(\Omega)$. Let

$$\Sigma := \{u \in H_0^1(\Omega), \text{ s.t } \|u\|^2 = \int_\Omega |\nabla u|^2 = 1\}, \quad \Sigma^+ := \{u \in \Sigma, u \geq 0\}.$$

Instead of working with the functional I defined above, it is more convenient here to work with the functional

$$J(u) = \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega K|u|^4\right)^{1/2}} \tag{2.2}$$

defined on Σ. One can easily verify that if u is a critical point of J in Σ^+, then $J(u)u$ is a solution of (1.1).

As mentioned previously, the variational viewpoint is delicate since the functional J does not satisfy the Palais-Smale condition (P-S) in short). This means that there exist sequences along which J is bounded, its gradient goes to zero and which are not convergent. The analysis of the sequences failing (P-S) condition can be realized following the work [3]. For $a \in \Omega, \lambda > 0$, let

$$\delta_{a,\lambda}(x) = \sqrt{8}\frac{\lambda}{1 + \lambda^2|x-a|^2} \tag{2.3}$$

the family of solutions of the following problem

$$-\Delta u = u^3, \quad u > 0 \quad \text{in } \mathbb{R}^4. \tag{2.4}$$

Let P be the projection from $H^1(\Omega)$ onto $H_0^1(\Omega)$; that is, $u := Pf$ is the unique solution of

$$\Delta u = \Delta f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{2.5}$$

Now we define the set of potential critical points at infinity associated to the functional J. Let, for $\varepsilon > 0$, $p \in \mathbb{N}^*$,

$$\begin{aligned} V(p,\varepsilon) = \Big\{ & u \in \Sigma^+ : \exists a_i \in \Omega, \lambda_i > 1/\varepsilon, \ \alpha_i > 0 \text{ for } 1 \leq i \leq p, \\ & \text{with } \|u - \sum_{i=1}^p \alpha_i P\delta_{a_i,\lambda_i}\| < \varepsilon, \ \varepsilon_{ij} < \varepsilon, \ \forall i \neq j, \\ & \lambda_i d_i > \frac{1}{\varepsilon}, \Big|\frac{\alpha_i^2 K(a_i)}{\alpha_j^2 K(a_j)} - 1\Big| < \varepsilon \ \forall i,j = 1,\dots,p \Big\}, \end{aligned}$$

where $d_i = \mathrm{d}(a_i, \partial\Omega)$ and $\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2\right)^{-1}$. If u is a function in $V(p,\varepsilon)$, one can find an optimal representation of u following [3]; namely we have the following result.

Proposition 2.1. *For any $p \in \mathbb{N}^*$, there is $\varepsilon_p > 0$ such that if $\varepsilon < \varepsilon_p$ and $u \in V(p,\varepsilon)$, then the minimization problem*

$$\min\Big\{\|u - \sum_{i=1}^p \alpha_i P\delta_{a_i,\lambda_i}\|, \alpha_i > 0, \lambda_i > 0, a_i \in \Omega\Big\} \tag{2.6}$$

has a unique solution $(\bar\alpha, \bar a, \bar\lambda)$ (up to permutation). Thus, we can write u uniquely as follows (we drop the bar):

$$u = \sum_{i=1}^p \alpha_i P\delta_{a_i,\lambda_i} + v, \tag{2.7}$$

where v satisfies

$$\langle v, \phi_i \rangle = 0, \quad \text{for } i = 1, \dots, p, \text{ where } \phi_i := P\delta_i, \frac{\partial P\delta_i}{\partial \lambda_i}, \frac{\partial P\delta_i}{\partial a_i}. \tag{2.8}$$

Here, $P\delta_i := P\delta_{a_i,\lambda_i}$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product defined on $H_0^1(\Omega)$ by

$$\langle u, v \rangle = \int_\Omega \nabla u \nabla v.$$

From now on, we will say that $v \in (V_0)$ if v satisfies (2.8).

The failure of the (P-S) condition can be described following the ideas developed in [3, 22, 26]. Such a description is by now standard and reads as follows: let ∂J be the gradient of J.

Proposition 2.2. *Let $(u_j)_j \subset \Sigma^+$ be a sequence such that ∂J tends to zero and $J(u_j)$ is bounded. Then there exists an integer $p \in \mathbb{N}^*$, a sequence $\varepsilon_j > 0$, $\varepsilon_j \to 0$, and an extracted subsequence of u_j's, again denoted by u_j, such that $u_j \in V(p, \varepsilon_j)$.*

Now arguing as in [4], we have the following Morse lemma which permits us to get the v-contribution by showing that it can be neglected with respect to the concentration phenomenon.

Proposition 2.3. *There is a $\mathcal{C}^1$-map which to each $(\alpha_i, a_i, \lambda_i)$ such that $\sum_{i=1}^p \alpha_i P\delta_{a_i,\lambda_i} \in V(p, \varepsilon)$ associates $\bar{v} := \bar{v}(\alpha_i, a_i, \lambda_i)$ such that $\bar{v}$ is unique and satisfies*

$$J\big(\sum_{i=1}^p \alpha_i P\delta_{a_i,\lambda_i} + \bar{v}\big) = \min_{v \in (V_0)} \Big\{ J\big(\sum_{i=1}^p \alpha_i P\delta_{a_i,\lambda_i} + v\big) \Big\}.$$

Moreover, there exists a change of variables $v - \bar{v} \mapsto V$, such that J reads in $V(p, \varepsilon)$ as

$$J\Big(\sum_{i=1}^p \alpha_i P\delta_{a_i,\lambda_i} + v\Big) = J\Big(\sum_{i=1}^p \alpha_i P\delta_{a_i,\lambda_i} + \bar{v}\Big) + \|V\|^2.$$

The following proposition gives precise estimate of $\bar{v}$.

Proposition 2.4 ([11])**.** *Let $u = \sum_{i=1}^p \alpha_i P\delta_{a_i,\lambda_i} \in V(p, \varepsilon)$, and let $\bar{v}$ be defined in proposition 2.3. Then we have the following estimate: there exists $c > 0$ independent of u such that*

$$\begin{aligned} \|\bar{v}\| = O\Big(& \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^{\beta_i}} + \frac{(\log \lambda_i)^{3/4}}{\lambda_i^3} + \sum_{i \neq j} \varepsilon_{ij} \big(\log \varepsilon_{ij}^{-1}\big)^{1/2} \\ &+ \sum_i \frac{1}{(\lambda_i\, d_i)^2}\Big). \end{aligned} \tag{2.9}$$

Following Bahri [3], we introduce the following definition.

Definition 2.5. A critical point at infinity of J in Σ^+ is a limit of a flow-line $u(s)$ of the equation

$$\begin{gathered} \frac{\partial u}{\partial s} = -\partial J(u) \\ u(0) = u_0 \in \Sigma^+ \end{gathered}$$

such that $u(s)$ remains in $V(p, \varepsilon(s))$, for $s \geq s_0$.

Note that $\varepsilon(s)$ tends to zero when $s \to +\infty$. Using proposition 2.1, $u(s)$ can be written as

$$u(s) = \sum_{i=1}^{p} \alpha_i(s) P\delta_{a_i(s),\lambda_i(s)} + v(s).$$

Denoting by $a_i := \lim a_i(s)$ and $\alpha_i := \lim \alpha_i(s)$, we denote by

$$(a_1, \dots, a_p)_\infty \quad \text{or} \quad \sum_{i=1}^{p} \alpha_i P\delta_{a_i,\infty}$$

such a critical point at infinity.

3. Characterization of the critical points at infinity

In this section, we study the concentration phenomenon of the variational structure of the problem through the flow-lines of a suitable decreasing pseudo gradient of J. This leads to the characterization of the critical points at infinity of the problem. To reach this goal, we need first to study the asymptotic behavior of the gradient of J.

3.1. **Expansion of the gradient of the functional.**

Proposition 3.1. *For ε small enough and $u = \sum_{i=1}^{p} \alpha_i P\delta_i \in V(p,\varepsilon)$, we have the following expansion: (1)*

$$\begin{aligned}\langle \partial J(u), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\rangle_{H_0^1} &= 64\pi^2 J(u)\Big[-\alpha_i \frac{H(a_i,a_i)}{\lambda_i^2} - \sum_{j\neq i} \alpha_j \big(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{H(a_i,a_j)}{\lambda_i \lambda_j}\big)\Big] \\ &\quad + o\Big(\frac{1}{\lambda_i} + \sum_{i\neq j} \varepsilon_{ij} + \sum_{k\neq j} \varepsilon_{kj}^{3/2} + \sum_{k=1}^{p} \frac{1}{(\lambda_k d_k)^3}\Big).\end{aligned}$$

(2) If $a_i \in B\big(z_{j_i}, \rho\big)$, with $z_{j_i} \in (\mathbf{f})_\beta$, *and ρ is a positive constant small enough so that (A3) holds in $B\big(z_{j_i}, 4\rho\big)$, then the above estimate can be improved. Let $C > 0$ and $\delta > 0$ two positive constants large enough and small enough, respectively.*

(a) *If $\beta > 2$, then*

$$\begin{aligned}\langle \partial J(u), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\rangle_{H_0^1} &= 64\pi^2 J(u)\Big[-\alpha_i \frac{H(a_i,a_i)}{\lambda_i^2} - \sum_{j\neq i} \alpha_j \big(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{H(a_i,a_j)}{\lambda_i \lambda_j}\Big] \\ &\quad \times \big(1 + o(1)\big) + \big(\textit{if } \lambda_i |a_i - z_{j_i}| \geq C\big) o\big(\frac{|\nabla K(a_i)|}{\lambda_i}\big) \\ &\quad + o\Big(\sum_{i\neq j} \varepsilon_{ij} + \sum_{k\neq j} \varepsilon_{kj}^{3/2} + \sum_{k=1}^{p} \frac{1}{(\lambda_k d_k)^3}\Big).\end{aligned}$$

(b) *If $\beta = 2$, then*

$$\begin{aligned}&\langle \partial J(u), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\rangle_{H_0^1} \\ &= 64\pi^2 J(u)\Big[-\alpha_i \frac{H(a_i,a_i)}{\lambda_i^2} + \alpha_i \frac{1}{12} \frac{\sum_{j=1}^{4} b_j}{K(a_i)\lambda_i^2}\Big]\big(1 + o(1)\big) \\ &\quad + o\Big(\sum_{i\neq j} \varepsilon_{ij} + \sum_{k\neq j} \varepsilon_{kj}^{3/2} + \sum_{k=1}^{p} \frac{1}{(\lambda_k d_k)^3}\Big),\end{aligned}$$

(c) *If $\beta < 2$, then*

$$\begin{aligned}
\langle \partial J(u), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\rangle_{H_0^1} &= 64\pi^2 J(u)\Big[-\alpha_i \frac{H(a_i,a_i)}{\lambda_i^2} - \sum_{j\neq i}\alpha_j\big(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{H(a_i,a_j)}{\lambda_i\lambda_j}\big) \\
&\quad + \big(if \lambda_i|a_i - z_{j_i}| \leq \delta\big) c_3 \alpha_i \frac{\sum_{j=1}^4 b_j}{K(a_i)\lambda_i^\beta}\Big]\big(1+o(1)\big) \\
&\quad + \big(if \delta \leq \lambda_i|a_i - z_{j_i}| \leq C\big) O\big(\frac{1}{\lambda_i^{\beta_i}}\big) \\
&\quad + \big(if \lambda_i|a_i - z_{j_i}| \geq C\big)\, o\big(\frac{|\nabla K(a_i)|}{\lambda_i}\big) \\
&\quad + o\Big(\sum_{i\neq j}\varepsilon_{ij} + \sum_{k\neq j}\varepsilon_{kj}^{3/2} + \sum_{k=1}^p \frac{1}{(\lambda_k d_k)^3}\Big),
\end{aligned}$$

where

$$c_3 = \frac{2}{\pi^2}\int_{\mathbb{R}^4} \frac{|y_1|^\beta\big(|y|^2-1\big)}{\big(|y|^2+1\big)^5}\, dy.$$

Proof. Claim (1) is immediate from [12] (see [12, Proposition 3.4]). Claim (2)(a) is proved in [11] (see [11, Proposition 3.1]). Concerning claim (2)(b) and (2)(c), regarding the estimates used to prove claim (1), we need to estimate the quantity

$$\int_\Omega K(x)\delta_i^3 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}\, dx. \tag{3.1}$$

If $\beta < 2$, let $\delta > 0$ a fixed constant small enough.
If $\lambda_i|a_i - z_{j_i}| \leq \delta$, let $B_i := B\big(a_i, \rho\big)$, then, by the condition (A3), we obtain

$$\begin{aligned}
&\int_\Omega K(x)\delta_i^3\lambda_i \frac{\partial \delta_i}{\partial \lambda_i}\, dx \\
&= \int_{B_i} \big[K(x) - K(z_{j_i})\big]\delta_i^3 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}\, dx + o\big(\frac{1}{\lambda_i^\beta}\big) \\
&= \frac{64}{\lambda_i^\beta}\sum_{j=1}^4 b_j \int_{B(o,\rho\lambda_i)} |y_j + \lambda_i(a_i - z_{j_i})_j|^\beta \frac{\big(1-|y|^2\big)}{\big(|y|^2+1\big)^5}\, dy + o\big(\frac{1}{\lambda_i^\beta}\big).
\end{aligned} \tag{3.2}$$

Now, by elementary calculations, for δ small enough, we obtain

$$\int_{B(o,\rho\lambda_i)} |y_j + \lambda_i(a_i - z_{j_i})_j|^\beta \frac{\big(1-|y|^2\big)}{\big(|y|^2+1\big)^5}\, dy = -\frac{\pi^2}{2}c_3 + o\big(1\big). \tag{3.3}$$

Combining (3.2) and (3.3), we obtain

$$\int_\Omega K(x)\delta_i^3\lambda_i\frac{\partial \delta_i}{\partial \lambda_i}\, dx = -32\pi^2 c_3 \frac{\sum_{j=1}^4 b_j}{\lambda_i^\beta}\big(1+o(1)\big).$$

If $\lambda_i|a_i - z_{j_i}| \geq \delta$, let $M > 0$ a fixed constant large enough, $B_{i,k} := B\big(a_i, \frac{|(a_i - z_{j_i})_k|}{2M}\big)$ for $1 \leq k \leq n$, and $B_{z_{j_i}} := B\big(z_{j_i}, 2\rho\big)$, then

$$\int_\Omega K(x)\delta_i^3\lambda_i\frac{\partial \delta_i}{\partial \lambda_i}\, dx = \int_{B_{z_{j_i}}} \big[K(x) - K(a_i)\big]\delta_i^3\lambda_i \frac{\partial \delta_i}{\partial \lambda_i}\, dx + o\big(\frac{1}{\lambda_i^\beta}\big). \tag{3.4}$$

By (A3), we obtain

$$\begin{aligned}
&\int_{B_{z_{j_i}}} \big[K(x)-K(a_i)\big]\delta_i^3\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\,dx\\
&=\sum_{k=1}^4 b_k\int_{B_{z_{j_i}}}|(a_i-z_{j_i})_k-(a_i-x)_k|^\beta\delta_i^3\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\,dx\\
&\quad-|(a_i-z_{j_i})_k|^\beta\int_{B_{z_{j_i}}}\delta_i^3\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\,dx\ +o\Big(\frac{1}{\lambda_i^\beta}+\frac{|a_i-z_{j_i}|^{\beta-1}}{\lambda_i}\Big).
\end{aligned}\tag{3.5}$$

Now, observe that

$$\begin{aligned}
&\int_{B_{z_{j_i}}\setminus B_{i,k}}|(a_i-z_{j_i})_k-(a_i-x)_k|^\beta\delta_i^3\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\,dx\\
&=O\Big(\int_{B_{z_{j_i}}\setminus B_{i,k}}(|(a_i-z_{j_i})_k|^\beta+|(a_i-x)_k|^\beta)\delta_i^4\,dx\Big)=O\Big(\frac{1}{\lambda_i^\beta}\Big).
\end{aligned}\tag{3.6}$$

In addition, by elementary calculations, we obtain

$$\begin{aligned}
&\int_{B_{i,k}}|(a_i-z_{j_i})_k-(a_i-x)_k|^\beta\delta_i^3\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\,dx-|(a_i-z_{j_i})_k|^\beta\int_{B_{i,k}}\delta_i^3\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\,dx\\
&=o\Big(\frac{|a_i-z_{j_i}|^{\beta-1}}{\lambda_i}\Big).
\end{aligned}\tag{3.7}$$

Combining (3.4), (3.5), (3.6) and (3.7), we have

$$\int_\Omega K(x)\delta_i^3\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\,dx=o\Big(\frac{|a_i-z_{j_i}|^{\beta-1}}{\lambda_i}\Big)+(\text{ if }\delta\le\lambda_i|a_i-z_{j_i}|\le C)\,O\Big(\frac{1}{\lambda_i^\beta}\Big).$$

We notice from condition (A3) that, for ρ small enough,

$$\frac12|x-z_{j_i}|^{\beta-1}\le|\nabla K(x)|\le 2|x-z_{j_i}|^{\beta-1},\quad \forall x\in B_{z_{j_i}}.$$

Then we can write

$$\int_\Omega K(x)\delta_i^3\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\,dx=o\Big(\frac{|\nabla K(x)|}{\lambda_i}\Big)+(\text{ if }\delta\le\lambda_i|a_i-z_{j_i}|\le C)\,O\Big(\frac{1}{\lambda_i^\beta}\Big).$$

This completes the proof of claim (2)(c).

$\beta=2$, then the estimate of (3.1) is immediate from (3.2), and we obtain

$$\int_\Omega K(x)\delta_i^3\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\,dx=-\frac83\pi^2\frac{\sum_{j=1}^4 b_j}{\lambda_i^2}\big(1+o(1)\big).$$

This completes the proof of claim (2)(b). □

Proposition 3.2. *For ε small enough and $u=\sum_{i=1}^p\alpha_iP\delta_i\in V(p,\varepsilon)$, we have: (1)*

$$\begin{aligned}
\langle\partial J(u),\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\rangle_{H_0^1}&=64\pi^2J(u)\Big[-\frac{1}{12}\alpha_i^3J^2\frac{\nabla K(a_i)}{\lambda_i}+\frac{\alpha_i}{\lambda_i^3}\frac{\partial H(a_i,a_i)}{\partial a_i}\\
&\quad-\sum_{j\ne i}\alpha_j\Big(\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i}-\frac{1}{\lambda_j\lambda_i^2}\frac{\partial H(a_i,a_j)}{\partial a_i}\Big)\Big](1+o(1))
\end{aligned}$$

$$+ o\Big(\sum_{k\neq j}\varepsilon_{kj}^{3/2} + \sum_{k=1}^{p}\frac{1}{(\lambda_k d_k)^3}\Big) + O\Big(\lambda_j |a_i - a_j| \varepsilon_{ij}^{5/2}\Big).$$

(2) If $a_i \in B\big(z_{j_i}, \rho\big)$, where ρ is a positive constant small enough so that (A3) holds in $B\big(z_{j_i}, 4\rho\big)$, then the above estimate can be improved. Let C a positive constant large enough.

(a) *If $\beta \neq 2$. We distinguish two cases. If $\lambda_i|a_i - z_{j_i}| \le C$, we obtain*

$$\begin{aligned}\langle \partial J(u), \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial (a_i)_k}\rangle_{H_0^1} &= -256 J^3(u)\alpha_i^3 \frac{b_k}{\lambda_i^\beta}\int_{\mathbb{R}^4} |y_k + \lambda_i(a_i - z_{j_i})_k|^\beta \frac{y_k}{\big(|y|^2+1\big)^5}\,dy \\ &\quad + \big(\text{if } \beta < 2\big)\, o\big(\frac{1}{\lambda_i^\beta}\big) + \big(\text{if } \beta > 2\big) o\big(\frac{1}{\lambda_i^2}\big) \\ &\quad + o\big(\sum_{k\neq j}\varepsilon_{kj}^{3/2} + \sum_{k=1}^{p}\frac{1}{(\lambda_k d_k)^3}\big) + O\Big(\sum_{j\neq i}\big|\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i}\big|\Big).\end{aligned}$$

If $\lambda_i|(a_i - z_{j_i})_k| \ge C$, we obtain

$$\begin{aligned}\langle \partial J(u), \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial (a_i)_k}\rangle_{H_0^1} &= -\frac{16\pi^2}{3} J^3(u)\alpha_i^3 b_k \beta \operatorname{sgn}\big[(a_i - z_{j_i})_k\big]\frac{|(a_i - z_{j_i})_k|^{\beta-1}}{\lambda_i} \\ &\quad + \big(\text{if } \beta > 2\big) o\big(\frac{1}{\lambda_i^2}\big) + o\big(\frac{|a_i - z_{j_i}|^{\beta-1}}{\lambda_i}\big) \\ &\quad + o\big(\sum_{k\neq j}\varepsilon_{kj}^{3/2} + \sum_{k=1}^{p}\frac{1}{(\lambda_k d_k)^3}\big) + O\Big(\sum_{j\neq i}\big|\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i}\big|\Big).\end{aligned}$$

Here, $(a_i)_k, k \in \{1,\dots,4\}$, denotes the k^{th} component of a_i in some local coordinates system.

(b) *If $\beta = 2$. Then*

$$\begin{aligned}\langle \partial J(u), \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\rangle_{H_0^1} &= -\frac{16\pi^2}{3} J^3(u)\alpha_i^3 \frac{\nabla K(a_i)}{\lambda_i}\big(1 + o(1)\big) + o\big(\frac{1}{\lambda_i^2}\big) \\ &\quad + o\big(\sum_{k\neq j}\varepsilon_{kj}^{3/2} + \sum_{k=1}^{p}\frac{1}{(\lambda_k d_k)^3}\big) + O\Big(\sum_{j\neq i}\big|\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i}\big|\Big).\end{aligned}$$

Proof. Claim (1) is immediate from [12]. Concerning claim (2)(a), arguing as in the proof of [11, proposition 3.2], claim (2)(a) is proved under the following estimates: let C a positive constant large enough and $a_i \in B\big(z_{j_i}, \rho\big)$, where ρ is a positive constant small enough so that (A3) holds in $B\big(z_{j_i}, 4\rho\big)$.

If $\lambda_i|(a_i - z_{j_i})_k| \ge C$, then

$$\begin{aligned}&\int_\Omega K(x)\delta_i^3 \frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial (a_i)_k}\,dx \\ &= \frac{8\pi^2}{3}\operatorname{sgn}\big[(a_i - z_{j_i})_k\big]\frac{|(a_i - z_{j_i})_k|^{\beta-1}}{\lambda_i} b_k + o\big(\frac{1}{\lambda_i^\beta} + \frac{|a_i - z_{j_i}|^{\beta-1}}{\lambda_i}\big).\end{aligned} \tag{3.8}$$

If $\lambda_i|a_i - z_{j_i}| \le C$, we obtain

$$\int_\Omega K(x)\delta_i^3 \frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial (a_i)_k}\,dx$$

$$= 64\frac{b_k}{\lambda_i^\beta}\int_{\mathbb{R}^n}|y_k+\lambda_i(a_i-z_{j_i})_k|^\beta\frac{y_k}{\big(|y|^2+1\big)^5}\,dy+o\Big(\frac{1}{\lambda_i^\beta}+\frac{|a_i-z_{j_i}|^{\beta-1}}{\lambda_i}\Big).$$

The proof of claim (2)(b) is immediate from estimate (3.8). The proof is complete. □

3.2. **Critical points at infinity.** This subsection is devoted to the characterization of the critical points at infinity , associated to the problem (1.1), in $V(p,\varepsilon)$, $p\geq 1$. This characterization is obtained through the construction of a suitable pseudo-gradient at infinity for which the Palais-Smale condition is satisfied along the decreasing flow lines as long as these flow lines do not enter in the neighborhood of finite number of critical points y_{i_j}, $j=1,\dots,p$, of K such that $(y_{i_1},\dots,y_{i_p})\in\mathcal{C}_\infty$. Now, we introduce the following main result.

Theorem 3.3. *There exists a pseudo-gradient W so that the following holds. There is a constant $c>0$ independent of $u=\sum_{i=1}^p\alpha_iP\delta_{a_i\lambda_i}\in V(p,\varepsilon)$ so that (i)*

$$\Big\langle\partial J(u),W(u)\Big\rangle\leq -c\Big(\sum_i^p\Big[\frac{|\nabla K(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^2}+\frac{1}{(\lambda_id_i)^3}\Big]+\sum_{i\neq j}\varepsilon_{ij}^{3/2}\Big).$$

(ii)

$$\langle\partial J(u+\overline{v}),W(u)+\frac{\partial\overline{v}}{\partial(\alpha,a,\lambda)}(W)\rangle\leq -c\Big(\sum_i^p\Big[\frac{|\nabla K(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^2}+\frac{1}{(\lambda_id_i)^3}\Big]+\sum_{i\neq j}\varepsilon_{ij}^{3/2}\Big).$$

(iii) The minimal distance to the boundary, $d_i(t):=\mathrm{d}(a_i(t),\partial\Omega)$, only increases if it is small enough.
(iv) $|W|$ is bounded. Furthermore, the only case where the maximum of the λ_i's is not bounded is when each point a_j is close to a critical point y_{i_j} with $y_{i_j}\neq y_{i_k}$, for each $j\neq k$, and $(y_{i_1},\dots,y_{i_p})\in\mathcal{C}_\infty$.

Before giving the proof of theorem 3.3, we need to state three results which deal with three specific cases of theorem 3.3. The proof of these results will be given later. Let $d_0>0$ and $r_0>0$ be two constants small enough such that

$$\frac{\partial K}{\partial\nu}(x)<-c_0,\quad \forall x\in\Omega_{d_0}:=\{x\in\Omega: d(x,\partial\Omega)\leq 2d_0\},$$

where $c_0>0$ is a fixed constant, for all $y\in\mathcal{K}_{<2}$ all $z\in\mathcal{K}_2\cup\mathcal{K}_{>2}$, and $z\notin B(y,2r_0)$. Then, we have the following propositions.

Proposition 3.4. [11] *In the set*

$$V_1(p,\varepsilon):=\big\{u=\sum_{i=1}^p\alpha_iP\delta_i\in V(p,\varepsilon): d(a_i,\partial\Omega)\leq 2d_0,\forall i=1,\dots,p\big\},$$

there exists a pseudo-gradient W_1 so that the following holds: There is a constant $c>0$ independent of $u\in V_1(p,\varepsilon)$ so that

$$\langle\partial J(u),W_1(u)\rangle\leq -c\Big(\sum_i^p\Big[\frac{1}{\lambda_i}+\frac{1}{(\lambda_id_i)^3}\Big]+\sum_{i\neq j}\varepsilon_{ij}^{3/2}\Big).$$

Proposition 3.5. *Let $\beta:=\max\{\beta(z)\,/\,z\in\mathcal{K}_{<2}\}$. In the set*

$$\widetilde{V}_2(p,\varepsilon):=\Big\{u=\sum_{i=1}^p\alpha_iP\delta_i\in V(p,\varepsilon),\ d(a_i,\partial\Omega)\geq d_0,\ \text{and}$$

$$a_i \in \cup_{z\in\mathcal{K}_{<2}} B(z,r_0),\, \forall i=1,\dots,p\Big\},$$

there exists a pseudo-gradient W_2 so that the following holds: There is a constant $c>0$ independent of $u \in \widetilde{V}_2(p,\varepsilon)$ so that

$$\langle \partial J(u), W_2(u)\rangle \leq -c\Big(\sum_i^p \big[\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^\beta}\big] + \sum_{i\neq j}\varepsilon_{ij}\Big).$$

Proposition 3.6. *In the set*

$$\widetilde{V}_3(p,\varepsilon) := \Big\{u=\sum_{i=1}^p \alpha_i P\delta_i \in V(p,\varepsilon),\ d(a_i,\partial\Omega)\geq d_0,\ \text{and}$$
$$a_i \notin \cup_{z\in\mathcal{K}_{<2}} B(z,2r_0),\, \forall\, i=1,\dots,p\Big\},$$

there exists a pseudo-gradient W_3 so that the following holds: There is a constant $c>0$ independent of $u \in \widetilde{V}_3(p,\varepsilon)$ so that

$$\langle \partial J(u), W_3(u)\rangle \leq -c\Big(\sum_i^p \big[\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\big] + \sum_{i\neq j}\varepsilon_{ij}\Big).$$

Proof of Theorem 3.3. We divide the set $\{1,\dots,p\}$ into three subsets. The first contains the indices of the points near the boundary $\partial\Omega$, the second contains the indices of the points near the critical points that belong to $\mathcal{K}_{<2}$, and the third contains the indices of the points far away both $\partial\Omega$ and $\mathcal{K}_{<2}$. Let $u=\sum_{i=1}^p \alpha_i P\delta_i \in V(p,\varepsilon)$. Let us define

$$\begin{gathered}
B := \{1\leq i\leq p : d_i \geq 2d_0\},\\
B_1 := B \cup \Big\{i\notin B : \exists (i_1,\dots,i_r) \text{ with } i_1=i,\, i_r\in B\\
\text{and } |a_{i_{k-1}} - a_{i_k}| < \frac{d_0}{p}, \forall k\leq r\Big\},\\
B_2(u) := \{1,\dots,p\}\backslash B_1 =: B_2,\\
B' := \{i\in B_1 : a_i \in \cup_{z\in\mathcal{K}_{<2}} B(z,r_0)\},\\
B_1'(u) := B' \cup \Big\{i\in B_1\backslash B' : \exists (i_1,\dots,i_r) \text{ with } i_1=i,\, i_r\in B'\\
\text{and} \quad |a_{i_{k-1}} - a_{i_k}| < \frac{r_0}{p}, \forall k\leq r\Big\} =: B_1' \quad ,\\
B_2'(u) := B_1\backslash B_1' =: B_2'.
\end{gathered}$$

We have the following two observations:

1. $d_i := d(a_i,\partial\Omega)\leq 2d_0$ for all $i\in B_2$.
2. The advantage of B_1' and B_2' is that if $i\in B_1', j\in B_2'$, and $k\in B_2$, then

$$|a_i - a_k| \geq \frac{d_0}{p}, |a_i-a_k|\geq \frac{d_0}{p}, \quad |a_i-a_j|\geq\frac{r_0}{p}.$$

Thanks to propositions 3.4, 3.5 and 3.6, and in order to complete the construction of the pseudo-gradient W suggested in theorem 3.3, it only remains to focus attention at the two following subsets of $V(p,\varepsilon)$.

Subset 1.

$$V_b(p,\varepsilon) := \{u = \sum_{i=1}^{p} \alpha_i P\delta_i \in V(p,\varepsilon) : B_1'(u) \neq \emptyset, B_2'(u) \neq \emptyset \text{ and } B_2(u) = \emptyset\}.$$

Let

$$u = \sum_{i=1}^{p} \alpha_i P\delta_i =: u_1 + u_2 \in V_b(p,\varepsilon),$$

where $u_k := \sum_{i\in B_k'(u)} \alpha_i P\delta_i$, $1 \leq k \leq 2$. Observe that

$$u_1 \in \widetilde{V}_2(\operatorname{card}(B_1'),\varepsilon) \text{ and } u_2 \in \widetilde{V}_3(\operatorname{card}(B_2'),\varepsilon).$$

We distinguish three cases.

Case 1.1: $u_1 \in \widetilde{V}_2^1(\operatorname{card}(B_1'),\varepsilon)$ and $u_2 \in \widetilde{V}_3^1(\operatorname{card}(B_2'),\varepsilon)$. In this case, we let $V_1 := \widetilde{W}_2^1(u_1)$ and $V_2 := \widetilde{W}_3^1(u_2)$ the pseudo gradients defined in lemma 3.7 and lemma 3.11, respectively. Define

$$W_{23}^1(u) := V_1 + V_2.$$

From the observation (2), we obtain, for each $i \in B_1'$ and $j \in B_2'$,

$$\varepsilon_{ij} = o\Big(\frac{1}{\lambda_i^{\beta_{j_i}}} + \frac{1}{\lambda_j^2}\Big). \tag{3.9}$$

Thus, by using lemmas 3.7 and 3.11, we obtain

$$\langle \partial J(u), \widetilde{W}_{23}^1(u)\rangle \leq -c\Big(\sum_{i=1}^{p}\Big[\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\Big] + \sum_{i\neq j}\varepsilon_{ij}\Big).$$

Notice that in this case all the λ_i's $, 1 \leq i \leq p$, increase and go to $+\infty$ along the flow-lines generated by $\widetilde{W}_{23}^1$.

Case 1.2: $u_1 \notin \widetilde{V}_2^1(\operatorname{card}(B_1'),\varepsilon)$. Without loss of generality, we assume $\lambda_1 \leq \cdots \leq \lambda_p$. u_1 has to satisfy $u_1 \in \widetilde{V}_2^i(\operatorname{card}(B_1'),\varepsilon), i = 2 \,\text{or}\, 3$. Thus we define $Z_1 := \widetilde{W}_2^i(u_1), i = 2 \,\text{or}\, 3$, the corresponding vector field. Using (3.9), we derive

$$\begin{aligned}\langle \partial J(u), Z_1\rangle \leq &- c\Big(\sum_{i\in B_1'}\Big[\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^{\beta_{j_i}}}\Big] + \sum_{i\neq j, i,j\in B_1'}\varepsilon_{ij}\Big)\\ &+ o\Big(\sum_{i\in B_2'}\frac{1}{\lambda_i^2}\Big).\end{aligned} \tag{3.10}$$

Now, let $i_1 := \min(B_1')$, and we denote

$$J_1 := \{i_1\} \cup \{i < i_1 : \lambda_j \leq M\lambda_{j-1}, \forall i < j \leq i_1\} =: \{i_0, \dots, i_1\}.$$

Observe that

$$\lambda_i = o(\lambda_j), \forall i < i_0, \forall j \geq i_0. \tag{3.11}$$

Let

$$\widetilde{u} := \sum_{i<i_0} \alpha_i P\delta_i.$$

Observe that $\widetilde{u} \in \widetilde{V}_3(i_0-1,\varepsilon)$. Thus we define $Z_2 := W_3(\widetilde{u})$ the corresponding vector field. Using proposition 3.6, we derive from the observations (3.11) and (3.9)

$$\begin{aligned}\langle \partial J(u), Z_2\rangle \leq & -c\Big(\sum_{i<i_0}\Big[\frac{|\nabla K(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^2}\Big]+\sum_{i\neq j,i,j<i_0}\varepsilon_{ij}\Big)\\ & + o\big(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}\big)+O\big(\sum_{i\neq j,i<i_0,j\geq i_0,j\in B_2'}\varepsilon_{ij}\big).\end{aligned} \tag{3.12}$$

Combining (3.10) and (3.12), we obtain

$$\begin{aligned}\langle \partial J(u), Z_1+Z_2\rangle \leq & -c\Big(\sum_{i\in B_1'}\Big[\frac{|\nabla K(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^{\beta_{j_i}}}\Big]+\sum_{i<i_0}\Big[\frac{|\nabla K(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^2}\Big]\\ & + \sum_{i\neq j,i,j<i_0}\varepsilon_{ij}+\sum_{i\neq j,i,j\in B_1'}\varepsilon_{ij}\Big)\\ & + o\big(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}\big)+O\big(\sum_{i\neq j,i<i_0,j\geq i_0,j\in B_2'}\varepsilon_{ij}\big).\end{aligned} \tag{3.13}$$

Observe that $\frac{1}{\lambda_{i_1}^2}$ appears in the upper bound (3.13), and then we can make appear all the $\frac{1}{\lambda_i^2}$'s, $i\in B_2', i\geq i_0$. We need to add some other terms. For this, we define

$$Z_3 := -\sum_{i\in B_2',i\geq i_0} 2^i\lambda_i\frac{\partial P\delta_{a_i,\lambda_i}}{\partial\lambda_i}.$$

Arguing as in the proof of lemma 3.9 (see the estimate of Z_3' in lemma 3.9), we derive under the observation (3.9) and (3.11)

$$\begin{aligned}\langle \partial J(u), Z_3\rangle \leq & -c\Big(\sum_{i\in B_2',i\geq i_0} O\big(\frac{1}{\lambda_i^2}\big)+\sum_{j\in B_2',i\neq j}\varepsilon_{ij}\Big)\\ & + o\Big(\sum_{i\in B_1'}\frac{1}{\lambda_i^{\beta_{j_i}}}+\sum_{i\in B_2'}\frac{1}{\lambda_i^2}+\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{k\neq r}\varepsilon_{kr}\Big).\end{aligned} \tag{3.14}$$

From (3.11), (3.14) becomes

$$\begin{aligned}\langle \partial J(u), Z_3\rangle \leq & -c\Big(\sum_{i\in B_2',i\geq i_0}\sum_{j\in B_2',i\neq j}\varepsilon_{ij}\Big)\\ & + o\Big(\sum_{i\in B_1'}\frac{1}{\lambda_i^{\beta_{j_i}}}+\sum_{i\in B_2'}\frac{1}{\lambda_i^2}+\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{k\neq r}\varepsilon_{kr}\Big).\end{aligned} \tag{3.15}$$

Combining (3.9), (3.11) and (3.15), we derive, for $M_1>0$ a fixed constant large enough,

$$\begin{aligned}& \langle \partial J(u), Z_1+Z_2+M_1Z_3\rangle\\ & \leq -c\Big(\sum_{i\in B_2'}\frac{1}{\lambda_i^2}+\sum_{i\in B_1'}\frac{1}{\lambda_i^{\beta_{j_i}}}+\sum_{i\in\{1,i_0-1\}\cup B_1'}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{k\neq r}\varepsilon_{kr}\Big)\\ & \quad + o\Big(\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i}\Big).\end{aligned} \tag{3.16}$$

To finish the construction in this case, we need to add another vector field. For this, let $\overline{X}_i$, $i \in B'_2, i \geq i_0$, the vector field defined in lemma 3.10. Let

$$\widetilde{W}^2_{23} := Z_1 + Z_2 + M_1 Z_3 + \sum_{i\in B'_2, i\geq i_0} \overline{X}_i.$$

Arguing as in the proof of lemma 3.10, we obtain

$$\langle \partial J(u), \widetilde{W}^2_{23}\rangle \leq -c\Big(\sum_{i\in B'_2} \frac{1}{\lambda_i^2} + \sum_{i\in B'_1} \frac{1}{\lambda_i^{\beta_{j_i}}} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{k\neq r} \varepsilon_{kr}\Big). \tag{3.17}$$

Case 1.3: $u_2 \notin \widetilde{V}^1_3(\mathrm{card}(B'_2), \varepsilon)$. Arguing as in the proof in the **case 1.2**, with the simple change of the role: we consider B'_2 instead of B'_1. We construct a vector field $\widetilde{W}^3_{23}$ with the same properties as that of $\widetilde{W}^2_{23}$; that is $\widetilde{W}^3_{23}$ does not increase the maximum of the λ'_is and we have the following: there is a constant $c > 0$ independent of $u \in V_b(p, \varepsilon)$ so that

$$\langle \partial J(u), \widetilde{W}^3_{23}\rangle \leq -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{k\neq r} \varepsilon_{kr}\Big). \tag{3.18}$$

The vector field S_1 required in the subset $V_b(p, \varepsilon)$ will be defined by convex combination of $\widetilde{W}^1_{23}$, $\widetilde{W}^2_{23}$ and $\widetilde{W}^3_{23}$.

Subset 2:

$$V_c(p, \varepsilon) := \{u = \sum_{i=1}^{p} \alpha_i P\delta_i \in V(p, \varepsilon), \text{ s.t } B_2(u) \neq \emptyset\}.$$

Arguing as in the proof in case 1.2, with the simple change of the role: we consider B_2 instead of B'_1. We construct a vector field S_2 with the same properties as that of $\widetilde{W}^2_{23}$; that is S_2 does not increase the maximum of the λ'_is and we have the following: there is a constant $c > 0$ independent of $u \in V_c(p, \varepsilon)$ so that

$$\langle \partial J(u), S_2\rangle \leq -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \sum_{i\in B_2} \frac{1}{(\lambda_i d_i)^3} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{k\neq r} \varepsilon_{kr}^{3/2}\Big). \tag{3.19}$$

Now, we define the pseudo-gradient W as a convex combination of W_2, W_3, S_1 and S_2. The construction of W is completed, and it satisfies claim (i) of theorem 3.3. Arguing as in [4, appendix 2], claim (ii) follows from (i) and proposition 2.4. The conditions (iii) and (iv) are satisfied by the definition of the vector field W. □

Proposition 3.5. Let $\eta > 0$ a fixed constant small enough such that $|y_i - y_j| > \eta$, $\forall\, i \neq j$. We divide the set $\widetilde{V}_2(p, \varepsilon)$ into three sets:

$$\widetilde{V}^1_2(p, \varepsilon) := \Big\{u = \sum_{i=1}^{p} \alpha_i P\delta_i \in \widetilde{V}_2(p, \varepsilon), a_i \in B(y_{j_i}, \eta) \text{ with}$$
$$y_{j_i} \neq y_{j_k}\ \forall i \neq k, \text{ and } -\sum_{k=1}^{4} b_k(y_{j_i}) > 0, \forall i = 1, \dots, p\Big\},$$

$$\widetilde{V}_2^2(p,\varepsilon) := \Big\{u = \sum_{i=1}^{p} \alpha_i P\delta_i \in \widetilde{V}_2(p,\varepsilon) : a_i \in B(y_{j_i},\eta), \forall i = 1,\dots,p,$$
$$y_{j_i} \neq y_{j_k}\ \forall i \neq k \text{ and } \exists\, i_1,\dots,i_q : -\sum_{k=1}^{4} b_k(y_{i_k}) < 0, \forall k = 1,\dots,q\Big\},$$
$$\widetilde{V}_2^3(p,\varepsilon) := \Big\{u = \sum_{i=1}^{p} \alpha_i P\delta_i \in \widetilde{V}_2(p,\varepsilon), a_i \in B(y_{j_i},\eta), \forall\, i = 1,\dots,p,$$
$$\text{and } \exists i \neq k : y_{j_i} = y_{j_k}\Big\}.$$

□

We will define the pseudo-gradient depending on the sets $\widetilde{V}_2^i(p,\varepsilon), i = 1-3$, to which u belongs.

Lemma 3.7. *In $\widetilde{V}_2^1(p,\varepsilon)$, there exists a pseudo-gradient $\widetilde{W}_2^1$ so that the following holds: There is a constant $c > 0$ independent of $u \in \widetilde{V}_2^1(p,\varepsilon)$ so that*

$$\langle \partial J(u), \widetilde{W}_2^1(u)\rangle \leq -c\Big(\sum_{i=1}^{p}\Big[\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^{\beta}}\Big] + \sum_{i\neq j}\varepsilon_{ij}\Big).$$

Proof. In this region, We have, for $i \neq k$, $|a_i - a_k| \geq c$, then

$$\begin{aligned} \varepsilon_{ik} &= o\big(\frac{1}{\lambda_i^{\beta_{j_i}}} + \frac{1}{\lambda_k^{\beta_{j_k}}}\big),\\ \lambda_i \frac{\partial \varepsilon_{ik}}{\partial \lambda_i} &= O(\varepsilon_{ik}) = o\big(\frac{1}{\lambda_i^{\beta_{j_i}}} + \frac{1}{\lambda_k^{\beta_{j_k}}}\big),\\ \frac{1}{\lambda_i}\frac{\partial \varepsilon_{ik}}{\partial a_i} &= O(\varepsilon_{ik}) = o\big(\frac{1}{\lambda_i^{\beta_{j_i}}} + \frac{1}{\lambda_k^{\beta_{j_k}}}\big), \quad \text{since } \beta_{j_k}, \beta_{j_i} < 2. \end{aligned} \tag{3.20}$$

Let Ψ be a positive cut-off function defined by

$$\Psi(t) = \begin{cases} 1 & \text{if } t \leq C,\\ 0, & \text{if } t \geq 2C \end{cases}$$

where C is a positive constant large enough. We define, for each $i = 1,\dots,p$,

$$\begin{aligned} \overline{T}_i =& \sum_{k=1}^{4}\Big[1 - \Psi\big(\lambda_i|(a_i - y_{j_i})_k|\big)\Big] b_k \operatorname{sgn}\big[(a_i - y_{j_i})_k\big]\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial (a_i)_k}\\ &+ \sum_{k=1}^{4}\Psi\big(\lambda_i|(a_i - y_{j_i})_k|\big) b_k \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial (a_i)_k}\\ &\times \int_{\mathbb{R}^4} |y_k + \lambda_i(a_i - y_{j_i})_k|^{\beta_{j_i}} \frac{y_k}{\big(|y|^2+1\big)^5}\, dy. \end{aligned} \tag{3.21}$$

Using proposition 3.2 and (3.33), we obtain

$$
\begin{aligned}
&\langle \partial J(u), \overline{T}_i\rangle \\
&= -c\Big(\sum_{k=1}^{4}\Big[1-\Psi\big(\lambda_i|(a_i-y_{j_i})_k|\big)\Big] b_k^2 \frac{|(a_i-y_{j_i})_k|^{\beta_{j_i}-1}}{\lambda_i} \\
&\quad + \frac{1}{\lambda_{j_i}^{\beta_{j_i}}}\sum_{k=1}^{4}\Psi\big(\lambda_i|(a_i-y_{j_i})_k|\big) b_k^2\Big(\int_{\mathbb{R}^4}|y_k+\lambda_i(a_i-y_{j_i})_k|^{\beta_{j_i}}\frac{y_k}{\big(|y|^2+1\big)^5}\,dy\Big)^2\Big) \\
&\quad + o\big(\sum_{\ell=1}^{p}\frac{|\nabla K(a_\ell)|}{\lambda_\ell}+\frac{1}{\lambda_\ell^{\beta_{j_\ell}}}+\sum_{j\neq\ell}\varepsilon_{j\ell}\big) \\
&\leq -c\Big(\Big[1-\Psi\big(\lambda_i|(a_i-y_{j_i})_{k_i}|\big)\Big]\frac{|\nabla K(a_i)|}{\lambda_i} \\
&\quad + \frac{1}{\lambda_{j_i}^{\beta_{j_i}}}\Psi\big(\lambda_i|(a_i-y_{j_i})_k|\big)\Big(\int_{\mathbb{R}^4}|y_k+\lambda_i(a_i-y_{j_i})_k|^{\beta_{j_i}}\frac{y_k}{\big(|y|^2+1\big)^5}\,dy\Big)^2\Big) \\
&\quad + o\big(\sum_{\ell=1}^{p}\frac{|\nabla K(a_\ell)|}{\lambda_\ell}+\frac{1}{\lambda_\ell^{\beta_{j_\ell}}}+\sum_{j\neq\ell}\varepsilon_{j\ell}\big),
\end{aligned}
\tag{3.22}
$$

where k_i denotes the index such that $|(a_i-y_{j_i})_{k_i}| = \max_{1\leq k\leq 4}|(a_i-y_{j_i})_k|$.

If $\Psi\big(\lambda_i|(a_i-y_{j_i})_{k_i}| \leq 1/2$, then $|\nabla K(a_i)|/\lambda_i$ appears in the upper bound of (3.22), and so $1/\lambda_i^{\beta_{j_i}}$, since $1/\lambda_i^{\beta_{j_i}} = o\big(|\nabla K(a_i)|/\lambda_i\big)$.

If $\Psi\big(\lambda_i|(a_i-y_{j_i})_{k_i}| \geq 1/2$. Let $\delta>0$ a fixed constant small enough, and Φ a positive cut-off function defined by $\Phi(t)=1$ if $t\leq\delta$ and $\Phi(t)=0$ if $t\geq 2\delta$. We distinguish two subcases:

Subcase 1: $\Phi\big(\lambda_i|a_i-y_{j_i}|\big)\leq 1/2$. Observe that , in this subcase, we have

$$
\Big(\int_{\mathbb{R}^4}|y_k+\lambda_i(a_i-y_{j_i})_k|^{\beta_{j_i}}\frac{y_k}{\big(|y|^2+1\big)^5}\,dy\Big)^2 \geq c_\delta,
$$

where $c_\delta>0$ is a fixed constant depend only on δ. Thus, we can make appear $\frac{1}{\lambda_i^{\beta_{j_i}}}$ in the upper bound of (3.22), and so $\frac{|\nabla K(a_i)|}{\lambda_i}$, since $\frac{1}{\lambda_i^{\beta_{j_i}}}\sim\frac{|\nabla K(a_i)|}{\lambda_i}$,

Subcase 2: $\Phi\big(\lambda_i|a_i-y_{j_i}|\big)\geq 1/2$. For each $i=1,\dots,p$ let

$$
Y_i := \alpha_i\Big(-\sum_{k=1}^{4}b_k(y_{j_i})\Big)\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}.
$$

Using proposition 3.1, we obtain

$$
\langle\partial J(u), Y_i\rangle \leq -c\frac{1}{\lambda_i^{\beta_{j_i}}} + o\Big(\sum_{\ell=1}^{p}\frac{|\nabla K(a_\ell)|}{\lambda_\ell}+\frac{1}{\lambda_\ell^{\beta_{j_\ell}}}+\sum_{j\neq\ell}\varepsilon_{j\ell}Big). \tag{3.23}
$$

Observe that, in this subcase, we have $\frac{|\nabla K(a_i)|}{\lambda_i} = o\big(\frac{1}{\lambda_i^{\beta_{j_i}}}\big)$. Thus, we can make appear $\frac{|\nabla K(a_i)|}{\lambda_i}$ in the upper bound of (3.23). We define

$$
\widetilde{W}_2^1 := \sum_{i=1}^{p}\overline{T}_i + \sum_{i=1}^{p}\Phi\big(\lambda_i|a_i-y_{j_i}|\big)Y_i.
$$

Combining (3.20), (3.22) and (3.23), we obtain

$$\langle \partial J(u), \widetilde{W}_2^1(u)\rangle \leq -c\Big(\sum_{i=1}^{p}\Big[\frac{|\nabla K(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^{\beta_{j_i}}}\Big]+\sum_{i\neq j}\varepsilon_{ij}\Big).$$

This completes the proof. □

Observe that the variation of the λ_i's occurs only under the condition $\lambda_i|a_i-y_{j_i}|\leq \delta$ for $1\leq i\leq p$, for $\delta>0$ a fixed constant small enough. In this case all the λ_i's increase and go to $+\infty$ along the flow-lines generated by $\widetilde{W}_2^1$.

Lemma 3.8. *In $\widetilde{V}_2^2(p,\varepsilon)$, there exists a pseudo-gradient $\widetilde{W}_2^2$ so that the following holds: There is a constant $c>0$ independent of $u\in \widetilde{V}_2^2(p,\varepsilon)$ so that*

$$\langle \partial J(u), \widetilde{W}_2^2\rangle \leq -c\Big(\sum_{i=1}^{p}\Big[\frac{|\nabla K(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^{\beta}}\Big]+\sum_{i\neq j}\varepsilon_{ij}\Big).$$

Proof. Without loss of generality, we assume $a_i\in B(y_{j_i},\eta)$, with $-\sum_{k=1}^4 b_k(y_{j_i})<0$ for $i=1,\dots,q$. We define

$$Z_1 := \sum_{i=1}^{q}\overline{T}_i+\sum_{i=1}^{q}\Phi\big(\lambda_i|a_i-y_{j_i}|\big)Y_i,$$

where Φ is the cut-off function defined above. Arguing as in the proof of lemma 3.7, we obtain

$$\langle \partial J(u), Z_1\rangle \leq -c\Big(\sum_{i=1}^{q}\frac{|\nabla K(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^{\beta_{j_i}}}\Big)+o\Big(\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^{\beta_{j_i}}}\Big). \tag{3.24}$$

Let $M>0$ be a fixed constant large enough. We define

$$I:=\big\{1\leq i\leq p, \text{ s.t } \lambda_i^{\beta_{j_i}}<\frac{1}{M}\min\{\lambda_k^{\beta_{j_k}},\, k=1,\dots,q\}\big\}.$$

Observe that all the $\frac{1}{\lambda_i^{\beta_{j_i}}}{}'$s, $i\notin I$, appear in the upper bound (3.24). We need to add some other terms. For this, let $\widetilde{u}=\sum_{i\in I}\alpha_i P\delta_i$. Observe that $\widetilde{u}\in \widetilde{V}_2^1(\ell,\varepsilon)$, where $\ell:=\operatorname{card}(I)$. Define

$$Z_2:=\widetilde{W}_2^1(\widetilde{u}),$$

where $\widetilde{W}_2^1$ is the pseudo-gradient defined in lemma 3.7. From (3.20) and lemma 3.7, we obtain

$$\langle \partial J(u), Z_2\rangle \leq -c\Big(\sum_{i\in I}\frac{|\nabla K(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^{\beta_{j_i}}}\Big)+o\Big(\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^{\beta_{j_i}}}\Big). \tag{3.25}$$

Now, we define

$$\widetilde{W}_2^2:=Z_1+Z_2+\sum_{i>q,i\notin I}\overline{T}_i.$$

Combining (3.20), (3.22), (3.24) and (3.25),we obtain

$$\langle \partial J(u), \widetilde{W}_2^2(u)\rangle \leq -c\Big(\sum_{i=1}^{p}\Big[\frac{|\nabla K(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^{\beta_{j_i}}}\Big]+\sum_{i\neq j}\varepsilon_{ij}\Big).$$

This completes the proof. □

Lemma 3.9. *In $\widetilde{V}_2^3(p,\varepsilon)$, there exists a pseudo-gradient $\widetilde{W}_2^3$ so that the following holds: There is a constant $c>0$ independent of $u\in\widetilde{V}_2^3(p,\varepsilon)$ so that*

$$\langle \partial J(u), \widetilde{W}_2^3(u)\rangle \le -c\Big(\sum_{i=1}^p\Big[\frac{|\nabla K(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^{\beta}}\Big]+\sum_{i\neq j}\varepsilon_{ij}\Big).$$

Proof. We order the concentrations λ_i's in such a way: $\lambda_1^{\beta_{j_1}}\le\dots\le\lambda_p^{\beta_{j_p}}$. Let $M_1>0$ and $C>0$ two fixed constants large enough. We set

$$\begin{gathered}
I_1:=\{1\le i\le p:\lambda_i|a_i-y_{j_i}|\ge C\},\\
I_2:=\{1\}\cup\{2\le i\le p:\lambda_i^{\beta_{j_i}}\le M_1\lambda_k^{\beta_{j_k}},\ \forall\, 1\le k\le i\}.
\end{gathered}$$

We define

$$\begin{gathered}
Z_3=\sum_{i\in I_1} b_k\cdot\operatorname{sgn}[(a_i-y_{j_i})_{k_i}]\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial (a_i)_{k_i}},\\
Z_3'=-M_1\sum_{i\notin I_2}2^i\lambda_i\frac{\partial P\delta_{a_i,\lambda_i}}{\partial\lambda_i}-m_1\sum_{i\in I_2}\lambda_i\frac{\partial P\delta_{a_i,\lambda_i}}{\partial\lambda_i},
\end{gathered}$$

where $m_1>0$ is a fixed constant small enough and $|(a_i-z_{j_i})_{k_i}|=\max_{1\le k\le 4}|(a_i-z_{j_i})_k|$. From proposition 3.2, we obtain

$$\begin{aligned}
\langle\partial J(u),Z_3\rangle &\le -c\sum_{i\in I_1}\frac{|(a_i-y_{j_i})_{k_i}|^{\beta_{j_i}-1}}{\lambda_i}+o\Big(\sum_{k=1}^p\frac{1}{\lambda_k^{\beta_{j_k}}}\Big)\\
&\quad+O\Big(\sum_{i\neq j,\, i\in I_1}\frac{1}{\lambda_i}\Big|\frac{\partial\varepsilon_{ij}}{\partial a_i}\Big|\Big)+o\Big(\sum_{i\neq j}\varepsilon_{ij}\Big).
\end{aligned}$$

Observe that

$$\begin{gathered}
\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i}=o(\varepsilon_{ij}),\quad \forall 1\le i\le p,\ \forall j\in I_2,\\
|\nabla K(a_i)|\sim|a_i-y_{j_i}|^{\beta_{j_i}-1},\quad \forall i\in I_1.
\end{gathered}$$

Then

$$\langle\partial J(u),Z_3(u)\rangle\le -c\sum_{i\in I_1}\frac{|\nabla K(a_i)|}{\lambda_i}+O\Big(\sum_{i\neq j, i\in I_1, j\notin I_2}\varepsilon_{ij}\Big)+o\Big(\sum_{k=1}^p\frac{1}{\lambda_k^{\beta_{j_k}}}+\sum_{i\neq j}\varepsilon_{ij}\Big).$$

From the definition of I_1, we can make appear the quantity $\sum_{i\in I_1}\frac{1}{\lambda_i^{\beta_{j_i}}}$ in the last upper bound, and we obtain

$$\begin{aligned}
\langle\partial J(u),Z_3(u)\rangle &\le -c\Big(\sum_{i\in I_1}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\in I_1}\frac{1}{\lambda_i^{\beta_{j_i}}}\Big)\\
&\quad+O\Big(\sum_{i\neq j, i\in I_1, j\notin I_2}\varepsilon_{ij}\Big)+o\Big(\sum_{k=1}^p\frac{1}{\lambda_k^{\beta_{j_k}}}+\sum_{i\neq j}\varepsilon_{ij}\Big).
\end{aligned}\tag{3.26}$$

Now, from proposition 3.1, we obtain

$$\begin{aligned}\langle\partial J(u), Z_3'(u)\rangle \le & -cM_1\Big(\sum_{i\notin I_2} O\big(\frac{1}{\lambda_i^{\beta_{j_i}}}\big) - \sum_{i\notin I_2, i\neq j} 2^i\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}\Big)\\ & - m_1\Big(\sum_{i\in I_2} O\big(\frac{1}{\lambda_i^{\beta_{j_i}}}\big) - \sum_{i\in I_2, i\neq j} \lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}\Big)\\ & + o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_{j_i}}} + \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{k\neq r}\varepsilon_{kr}\Big).\end{aligned} \tag{3.27}$$

Observe that

$$\begin{gathered}2^i\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + 2^j\lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j} \le -c\varepsilon_{ij}, \quad \forall i\neq j,\\ \lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} \le -c\varepsilon_{ij}, \forall 1\le i\le p, \quad \forall j\in I_2, i\neq j,\\ \frac{1}{\lambda_i^{\beta_{j_i}}} = o\big(\frac{1}{\lambda_1^{\beta_{j_1}}}\big), \quad \forall i\notin I_2.\end{gathered}$$

Thus, for m_1 small enough, the estimate (3.27) becomes

$$\begin{aligned}\langle\partial J(u), Z_3'(u)\rangle \le & -c\Big(M_1\sum_{i\notin I_2, i\neq j}\varepsilon_{ij} + m_1\sum_{i,j\in I_2, i\neq j}\varepsilon_{ij}\Big)\\ & + o\Big(\frac{1}{\lambda_1^{\beta_{j_1}}} + \sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i}\Big).\end{aligned} \tag{3.28}$$

Combining (3.26) and (3.28), we obtain

$$\begin{aligned}\langle\partial J(u), Z_3(u)+Z_3'(u)\rangle \le & -c\Big(\sum_{i\in I_1}\big[\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^{\beta_{j_i}}}\big] + \sum_{k\neq j}\varepsilon_{kj}\Big)\\ & + o\Big(\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i}\Big).\end{aligned} \tag{3.29}$$

We need to add other terms. For this, we distinguish two cases:

Case 1: $I_1\cap I_2\neq\emptyset$. In this case, we can make appear $\frac{1}{\lambda_1^{\beta_{j_1}}}$ in the last upper bound, and so all the $\frac{1}{\lambda_i^{\beta_{j_i}}}{}'$s, $1\le i\le p$, and the $\frac{|\nabla K(a_i)|}{\lambda_i}{}'$s, $i\notin I_1$. We obtain

$$\langle\partial J(u), Z_3(u)+Z_3'(u)\rangle \le -c\Big(\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_{j_i}}} + \sum_{k\neq j}\varepsilon_{kj}\Big).$$

The vector field $\widetilde{W}_2^{33} := Z_3 + Z_3'$ satisfies lemma 3.9.

Case 2: $I_1\cap I_2 = \emptyset$. In this case, we recall that, for each $i\in I_2$, the point a_i is close to a critical point y_{j_i} of K. If we suppose that there exist $i,k\in I_2$ such that $a_i, a_k\in B(y,\eta)$, for $\eta>0$ small enough and y a critical point of K, then $\lambda_i|a_i-a_k|\le 2C$ (we assume that $\lambda_i\le\lambda_k$). This implies $\varepsilon_{ik}\ge c\lambda_i/\lambda_k$, which is a contradiction with the fact that λ_i and λ_k are of the same order. Thus, for $\widetilde{u} = \sum_{i\in I_2}\alpha_i P\delta_i$, we have $a_i\in B(y_{j_i},\eta)$ and $a_k\in B(y_{j_k},\eta)$ with $y_{j_i}\neq y_{j_k}$

for$i,k \in I_2, i \neq k$. Therefore $\widetilde{u} \in \widetilde{V}_2^i(\ell,\varepsilon)$, where $i = 1$ or 2 and $\ell = \mathrm{card}(I_2)$. Let Z_3'' the corresponding vector field in $\widetilde{V}_2^i(\ell,\varepsilon)$ ($i = 1$ or 2). We have

$$\begin{aligned}
&\langle \partial J(u), Z_3''(\widetilde{u})\rangle \\
&\leq -c\Big(\sum_{i\neq j, i,j\in I_2} \varepsilon_{ij} + \sum_{i\in I_2} \frac{1}{\lambda_i^{\beta_{j_i}}}\Big) + O\Big(\sum_{i\in I_2, j\notin I_2} \varepsilon_{ij}\Big) + o\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta_{j_i}}}\Big).
\end{aligned} \tag{3.30}$$

Observe that $1/\lambda_1^{\beta_{j_1}}$ appears in the upper bound 3.30, then we can make appear all the terms $\frac{1}{\lambda_i^{\beta_{j_i}}}$, $1 \leq i \leq p$, and $\sum_{i\notin I_1} \frac{|\nabla K(a_i)|}{\lambda_i}$. We define

$$\widetilde{W}_2^{333} = Z_3 + Z_3{}' + Z_3{}''.$$

Combining (3.29) and (3.30), we obtain

$$\langle \partial J(u), \widetilde{W}_2^{333}(u)\rangle \leq -c\Big(\sum_{i=1}^{p}[\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^{\beta_{j_i}}}] + \sum_{i\neq j}\varepsilon_{ij}\Big).$$

The vector field $\widetilde{W}_2^3$, required in lemma 3.9 will be a convex combination of $\widetilde{W}_2^{33}$ and $\widetilde{W}_2^{333}$. The vector field W_2, required in proposition 3.5 will be defined by a convex combination of the vector fields $\widetilde{W}_2^1(u)$, $\widetilde{W}_2^2(u)$ and $\widetilde{W}_2^3(u)$. □

Proof of Proposition 3.6. Let $\eta > 0$ a fixed constant small enough with $|y_i - y_j| > 2\eta$ for $i \neq j$. We divide the set $\widetilde{V}_3(p,\varepsilon)$ into five sets:

$$\begin{aligned}
\widetilde{V}_3^1(p,\varepsilon) := \Big\{ &u = \sum_{i=1}^{p} \alpha_i P\delta_i \in \widetilde{V}_3(p,\varepsilon) : a_i \in B(y_{j_i},\eta), y_{j_i} \in \mathcal{K}_2^+ \cup \mathcal{K}_{>2}, \\
&\forall i = 1,\dots,p, \text{ with } y_{j_i} \neq y_{j_k}, \forall i\neq k,\ \rho(y_{j_1},\dots,y_{j_p}) > 0\Big\},\\
\widetilde{V}_3^2(p,\varepsilon) := \Big\{ &u = \sum_{i=1}^{p} \alpha_i P\delta_i \in \widetilde{V}_3(p,\varepsilon) : a_i \in B(y_{j_i},\eta), y_{j_i} \in \mathcal{K}_2^+ \cup \mathcal{K}_{>2}, \\
&\forall\, i = 1,\dots,p, \text{ with } y_{j_i} \neq y_{j_k}, \forall i\neq k,\ \rho(y_{j_1},\dots,y_{j_p}) < 0\Big\},\\
\widetilde{V}_3^3(p,\varepsilon) := \Big\{ &u = \sum_{i=1}^{p} \alpha_i P\delta_i \in \widetilde{V}_3(p,\varepsilon) : a_i \in B(y_{j_i},\eta), y_{j_i} \in \mathcal{K}_2 \cup \mathcal{K}_{>2}, \\
&\forall i = 1,\dots,p, \text{ with } y_{j_i} \neq y_{j_k}, \forall i\neq k,\ \exists y_{j_k} \in \mathcal{K}_2, \\
&\text{s.t. } -\frac{1}{12}\frac{\sum_{\ell=1}^{4} b_\ell(y_{j_k})}{K(y_{j_k})} + H(y_{j_k},y_{j_k}) < 0\Big\},\\
\widetilde{V}_3^4(p,\varepsilon) := \Big\{ &u = \sum_{i=1}^{p} \alpha_i P\delta_i \in \widetilde{V}_3(p,\varepsilon) : a_i \in B(y_{j_i},\eta), y_{j_i} \in \mathcal{K}_2 \cup \mathcal{K}_{>2}, \\
&\forall i = 1,\dots,p \text{ and } \exists i\neq k : y_{j_i} = y_{j_k}\Big\},\\
\widetilde{V}_3^5(p,\varepsilon) := \Big\{ &u = \sum_{i=1}^{p} \alpha_i P\delta_i \in \widetilde{V}_3(p,\varepsilon) : \exists a_i \notin \cup_{y\in\mathcal{K}_2\cup\mathcal{K}_{>2}} B(y,\eta)\Big\}.
\end{aligned}$$

We will define the pseudo-gradient depending on the sets $V_i(p,\varepsilon), i = 1-5$, to which u belongs. □

Lemma 3.10. *In $\widetilde{V}_3^2(p,\varepsilon)$, there exists a pseudo-gradient $\widetilde{W}_3^2$ so that the following holds: There is a constant $c>0$ independent of $u\in\widetilde{V}_3^2(p,\varepsilon)$ so that*

$$\langle \partial J(u), \widetilde{W}_3^2\rangle \le -c\Big(\sum_{i=1}^p \frac{1}{\lambda_i^2} + \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j}\varepsilon_{ij}\Big).$$

Proof. Let ρ be the least eigenvalue of M. Then there exists an eigenvector $e=(e_1,\dots,e_p)$ associated to ρ such that $\|e\|=1$ with $e_i>0, \forall i=1,\dots,p$. Indeed, let $e=(e_1,\dots,e_p)$ an eigenvector associated to ρ, with $\|e\|=1$. By elementary calculation, we obtain, since $m_{ij}<0$ for all $i\neq j$, we have

$$e_i>0, \forall i=1,\dots,p, \quad \text{or} \quad e_i<0, \forall i=1,\dots,p. \tag{3.31}$$

Let $\gamma>0$ such that for any $x\in B(e,\gamma) := \{y\in S^{p-1} \,/\, \|y-e\|\le\gamma\}$ we have ${}^T xMx < (1/2)\rho$. Two cases may occur:

Case 1: $|\Lambda|^{-1}\Lambda\in B(e,\gamma)$. In this region, we have for any $i\neq j$, $|a_i-a_j|\ge c$, and therefore

$$\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} = -\varepsilon_{ij}(1+o(1)) = -\frac{1}{\lambda_i\lambda_j|a_i-a_j|^2}(1+o(1)), \tag{3.32}$$

$$\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i} = o\big(\varepsilon_{ij}\big). \tag{3.33}$$

We define

$$\widetilde{W}_3^{22} = -\sum_{i=1}^p \alpha_i\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}.$$

From proposition 3.1 and (3.32), we obtain

$$\begin{aligned}
&\langle \partial J(u), \widetilde{W}_3^{22}\rangle\\
&= -64\pi^2 J(u)\Big[\sum_{i=1}^p \alpha_i^2\frac{H(a_i,a_i)}{\lambda_i^2} - \sum_{i\in\mathcal{K}_2}\alpha_i^2\frac{1}{12}\frac{\sum_{k=1}^4 b_k(y_{j_i})}{K(a_i)\lambda_i^2} - \sum_{j\neq i}\alpha_j\alpha_i\lambda_i\frac{G(a_i,a_j)}{\lambda_i\lambda_j}\Big]\\
&\quad + o\Big(\sum_{i=1}^p\frac{1}{\lambda_i^2}\Big) + \sum_{i=1}^p\big(\text{if } \lambda_i|a_i-y_{j_i}|\ge C\,\big)o\Big(\frac{|\nabla K(a_i)|}{\lambda_i}\Big).
\end{aligned}$$

Observe that, since $u\in V(p,\varepsilon)$, we have $J(u)\alpha_i K(a_i)^{1/2} = (1+o(1))$. Thus we derive that

$$\begin{aligned}
&\langle \partial J(u), \widetilde{W}_3^{22}\rangle\\
&= -c\Big[{}^T\Lambda M\Lambda\Big]\big(1+o(1)\big) + \sum_{i=1}^p\big(\text{if } \lambda_i|a_i-y_{j_i}|\ge C\big)o\Big(\sum_{i=1}^p\frac{|\nabla K(a_i)|}{\lambda_i}\Big)\\
&\le -c\Big(\sum_{i=1}^p\frac{1}{\lambda_i^2}\Big) + \sum_{i=1}^p\big(\text{if } \lambda_i|a_i-y_{j_i}|\ge C\,\big)o\Big(\frac{|\nabla K(a_i)|}{\lambda_i}\Big)\\
&\le -c\Big(\sum_{i=1}^p\frac{1}{\lambda_i^2} + \sum_{i\neq j}\varepsilon_{ij}\Big) + \sum_{i=1}^p\big(\text{if } \lambda_i|a_i-y_{j_i}|\ge C\,\big)o\Big(\sum_{i=1}^p\frac{|\nabla K(a_i)|}{\lambda_i}\Big),
\end{aligned} \tag{3.34}$$

where M is the matrix defined by (1.4) and (1.5), and $\Lambda :={}^T (\frac{1}{\lambda_1},\dots,\frac{1}{\lambda_p})$.

Case 2: $|\Lambda|^{-1}\Lambda \notin B(e,\gamma)$. In this case, we define

$$\widetilde{W}_3^{222} = -\sum_{i=1}^{p} |\Lambda|\alpha_i\gamma_i\lambda_i^2 \frac{\partial P\delta_i}{\partial\lambda_i},$$

where

$$\gamma_i = \frac{|\Lambda|e_i - \Lambda_i}{\|y(0)\|} - \frac{y_i(0)}{\|y(0)\|^3}(y(0), |\Lambda|e - \Lambda)$$

and $y(t) = (1-t)\Lambda + t|\Lambda|e$. Define $\Lambda(t) = y(t)/\|y(t)\|$. Using proposition 3.1 we derive

$$\begin{aligned}
&\langle \partial J(u), \widetilde{W}_3^{222}\rangle \\
&= cJ(u)|\Lambda|\Big[\sum_{i=1}^{p}\alpha_i^2\gamma_i\frac{H(a_i,a_i)}{\lambda_i} - \sum_{i\in\mathcal{K}_2}\alpha_i^2\gamma_i\frac{1}{12}\frac{\sum_{k=1}^4 b_k(y_{j_i})}{K(a_i)\lambda_i} - \sum_{j\neq i}\alpha_j\alpha_i\gamma_i\frac{G(a_i,a_j)}{\lambda_j}\Big] \\
&\quad + o\Big(\sum_{i\neq j}\varepsilon_{ij} + \sum_{i=1}^{p}\frac{1}{\lambda_i^2}\Big) + \sum_{i=1}^{p}\big(\text{if } \lambda_i|a_i - y_{j_i}| \geq C\big)o\Big(\frac{|\nabla K(a_i)|}{\lambda_i}\Big) \\
&= \frac{c}{J(u)}|\Lambda|^2\Big[{}^T\Lambda'(0)M\Lambda(0)\Big] \\
&\quad + o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}\Big) + \sum_{i=1}^{p}\big(\text{if } \lambda_i|a_i - y_{j_i}| \geq C\big)o\Big(\frac{|\nabla K(a_i)|}{\lambda_i}\Big),
\end{aligned} \tag{3.35}$$

since $|a_i - a_j| \geq c$ for $i \neq j$. We claim that

$$\frac{\partial}{\partial t}({}^T\Lambda(t)M\Lambda(t)) < -c, \text{for } t \text{ near } 0, \tag{3.36}$$

where $c > 0$ is a constant independent of $|\Lambda|^{-1}\Lambda \in \big(B(e,\gamma)\big)^c$. Indeed,

$${}^T\Lambda(t)M\Lambda(t) = \rho + \frac{(1-t)^2}{\|y(t)\|^2}\big[{}^T\Lambda M\Lambda - \rho|\Lambda|^2\big]. \tag{3.37}$$

Equation (3.37) implies

$$\begin{aligned}
&\frac{\partial}{\partial t}({}^T\Lambda(t)M\Lambda(t)) \\
&= \frac{2(1-t)}{\|y(t)\|^4}\big[{}^T\Lambda M\Lambda - \rho|\Lambda|^2\big]\big[-(1-t)|\Lambda|(e,\Lambda) - t|\Lambda|^2\big] \\
&= \frac{2(1-t)|\Lambda|^4}{\|y(t)\|^4}\Big[\frac{1}{|\Lambda|^4}\big({}^T\Lambda M\Lambda - \rho|\Lambda|^2\big)\big(-|\Lambda|(e,\Lambda)\big) + o(1)\Big]
\end{aligned} \tag{3.38}$$

By using the observation (3.31), we derive that there exists $c > 0$ (c independent of $|\Lambda|^{-1}\Lambda \in \big(B(e,\gamma)\big)^c$) such that

$${}^T\Lambda M\Lambda - \rho|\Lambda|^2 \geq c|\Lambda|^2. \tag{3.39}$$

Also, observe that

$$|\Lambda|(e,\Lambda) \geq \alpha|\Lambda|^2, \quad \text{where } \alpha := \inf\{e_i,\, 1 \leq i \leq p\}. \tag{3.40}$$

Combining (3.38), (3.39) and (3.40), the claim (3.36) follows.

Now, by combining (3.35) and (3.36), we obtain

$$\begin{aligned}&\langle \partial J(u), \widetilde{W}_3^{222}\rangle\\ &\le -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}+\sum_{i\neq j}\varepsilon_{ij}\Big)+\sum_{i=1}^{p}\big(\text{if } \lambda_i|a_i-y_{j_i}|\ge C\,\big)o\Big(\frac{|\nabla K(a_i)|}{\lambda_i}\Big).\end{aligned}\tag{3.41}$$

We define $\widetilde{W}_3^{2222}$ as a convex combination of $\widetilde{W}_3^{22}$ and $\widetilde{W}_3^{222}$. Combining (3.34) and (3.41), we obtain

$$\begin{aligned}&\langle \partial J(u), \widetilde{W}_3^{2222}\rangle\\ &\le -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}+\sum_{i\neq j}\varepsilon_{ij}\Big)+\sum_{i=1}^{p}\big(\text{if } \lambda_i|a_i-y_{j_i}|\ge C\big)o\Big(\frac{|\nabla K(a_i)|}{\lambda_i}\Big).\end{aligned}\tag{3.42}$$

Let Ψ a positive cut-off function defined by $\Psi(t)=1$, if $t\le C$ and $\Psi(t)=0$, if $t\ge 2C$, where C is a positive constant large enough. To make appear $\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i}$, we define, for each $i=1,\dots,p$,

$$\overline{X}_i=\sum_{k=1}^{4}\Big[1-\Psi\big(\lambda_i|(a_i-y_{j_i})_k|\big)\Big]b_k\cdot\operatorname{sgn}\big[(a_i-y_{j_i})_k\big]\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial (a_i)_k}.$$

Using proposition 3.2 and (3.33), we obtain

$$\begin{aligned}\langle \partial J(u), \overline{X}_i\rangle &= -c\sum_{k=1}^{4}\Big[1-\Psi\big(\lambda_i|(a_i-y_{j_i})_k|\big)\Big]b_k^2\frac{|(a_i-y_{j_i})_k|^{\beta_{j_i}-1}}{\lambda_i}\\ &\quad + o\Big(\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^2}\Big)\\ &\le -c\Big[1-\Psi\big(\lambda_i|(a_i-y_{j_i})_{k_i}|\big)\Big]\frac{|\nabla K(a_i)|}{\lambda_i}\\ &\quad + o\Big(\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^2}\Big),\end{aligned}\tag{3.43}$$

where k_i denotes the index such that $|(a_i-y_{j_i})_{k_i}|=\max_{1\le k\le 4}|(a_i-y_{j_i})_k|$. Combining (3.42) and (3.43), we obtain

$$\begin{aligned}&\langle \partial J(u), \widetilde{W}_3^{2222}+\sum_{i=1}^{p}\overline{X}_i\,\rangle\\ &\le -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}+\sum_{i\neq j}\varepsilon_{ij}+\big[1-\Psi\big(\lambda_i|(a_i-y_{j_i})_{k_i}|\big)\big]\frac{|\nabla K(a_i)|}{\lambda_i}\Big)\\ &\quad + o\Big(\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i}\Big).\end{aligned}\tag{3.44}$$

If $\Psi\le 1/2$, then $|\nabla K(a_i)|/\lambda_i$ appears in the upper bound of (3.44). However, if $\Psi>1/2$, then we have $\frac{|\nabla K(a_i)|}{\lambda_i}\le c\frac{1}{\lambda_i^2}$, and so we can make appear $|\nabla K(a_i)|/\lambda_i$ from $1/\lambda_i^2$. The vector field required in lemma 3.10 will be defined by $\widetilde{W}_3^2 := \widetilde{W}_3^{2222}+\sum_{i=1}^{p}\overline{X}_i$. $\square$

Lemma 3.11. *In $\widetilde{V}_3^1(p,\varepsilon)$, there exists a pseudo-gradient $\widetilde{W}_3^1$ so that the following holds: There is a constant $c>0$ independent of $u\in\widetilde{V}_3^1(p,\varepsilon)$ so that*

$$\langle \partial J(u), \widetilde{W}_3^1(u)\rangle \le -c\Big(\sum_{i=1}^p \frac{1}{\lambda_i^2}+\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\Big).$$

Proof. Let $\delta>0$ a fixed constant small enough, and denote $\alpha_{j_i}:=\frac{1}{\beta_{j_i}-1}, \forall \beta_{j_i}\ge 2$. We distinguish two cases:

Case 1: $\max_{1\le i\le p}\{\lambda_i^{\alpha_{j_i}}|a_i-y_{j_i}|\}\le\delta$. In this case, we define

$$Y_1:=\sum_{i=1}^p \alpha_i\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}.$$

Arguing as in the proof of the estimate (3.34), and using the fact $\rho(y_{j_1},\dots,y_{j_p})>0$, we obtain

$$\begin{aligned} &\langle \partial J(u), Y_1\rangle\\ &\le -c\Big(\sum_{i=1}^p\frac{1}{\lambda_i^2}+\sum_{i\neq j}\varepsilon_{ij}\Big)+\sum_{i=1}^p\big(\text{if } \lambda_i|a_i-y_{j_i}|\ge C\,\big)o\Big(\sum_{i=1}^p\frac{|\nabla K(a_i)|}{\lambda_i}\Big). \end{aligned}\tag{3.45}$$

Observe that $|\nabla K(a_i)|/\lambda_i=o\big(1/\lambda_i^2\big)$ for $1\le i\le p$. Thus, from (3.45), we obtain

$$\langle \partial J(u), Y_1\rangle \le -c\Big(\sum_{i=1}^p \frac{1}{\lambda_i^2}+\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\Big).\tag{3.46}$$

Case 2: $\max_{1\le i\le p}\{\lambda_i^{\alpha_{j_i}}|a_i-y_{j_i}|\}>\delta$. Let

$$i_1:=\min\{1\le i\le p:\lambda_i^{\alpha_{j_i}}|a_i-y_{j_i}|\}>\delta\}.$$

Without loss of generality, we suppose $\lambda_1\le\dots\le\lambda_p$. Let $M>0$ a fixed constant large enough. We set

$$I:=\{i_1\}\cup\{i<i_1:\lambda_{j-1}\ge\frac{1}{M}\lambda_j,\forall i<j\le i_1\}=:\{i_0,\dots,i_1\}.\tag{3.47}$$

Let

$$\widetilde{u}:=\sum_{i<i_0}\alpha_iP\delta_i.$$

Then $\widetilde{u}$ has to satisfy the case 1, or $\widetilde{u}\in\widetilde{V}_3^2(i_0-1,\varepsilon)$. Then, we define $Z_1(\widetilde{u})$ the corresponding vector field, and we obtain

$$\langle \partial J(u), Z_1\rangle \le -c\Big(\sum_{i<i_0}\frac{1}{\lambda_i^2}+\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\neq j,i,<i_0}\varepsilon_{ij}\Big)+O\Big(\sum_{i<i_0,j\ge i_0}\frac{1}{\lambda_i\lambda_j}\Big).\tag{3.48}$$

Observe that $\lambda_i=o\big(\lambda_j\big)$ for all $i<i_0$ and all $j\ge i_0$. Thus (3.48) becomes

$$\langle \partial J(u), Z_1\rangle \le -c\Big(\sum_{i<i_0}\frac{1}{\lambda_i^2}+\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\neq j,i,<i_0}\varepsilon_{ij}\Big).\tag{3.49}$$

Now, let $\overline{T}_{i_1}$ the vector field defined in (3.21). By the same argument used in the proof of lemma 3.7, we obtain

$$\langle \partial J(u), \overline{T}_{i_1}\rangle \le -c\Big(\frac{1}{\lambda_{i_1}^2}+\frac{|\nabla K(a_{i_1})|}{\lambda_{i_1}}\Big)+o\Big(\sum_{i=1}^p\frac{1}{\lambda_i^2}\Big).\tag{3.50}$$

Observe that all the terms $\frac{1}{\lambda_i^2}$, $i_0 \leq i \leq p$, appear in the upper bound (3.50). Combining (3.49) and (3.50), we obtain

$$\begin{aligned}
&\langle \partial J(u), Z_1 + \overline{T}_{i_1} \rangle \\
&\leq -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \sum_{i<i_0} \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{|\nabla K(a_{i_1})|}{\lambda_{i_1}} + \sum_{i \neq j, i<i_0} \varepsilon_{ij}\Big) \\
&\leq -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \sum_{i<i_0} \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{|\nabla K(a_{i_1})|}{\lambda_{i_1}} + \sum_{i \neq j, 1 \leq i,j \leq p} \varepsilon_{ij}\Big).
\end{aligned} \tag{3.51}$$

Now, arguing as in the proof of lemma 3.10, we obtain

$$\langle \partial J(u), Z_1 + \overline{T}_{i_1} + \sum_{i \geq i_0} \overline{X}_i \rangle \leq -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j, 1 \leq i,j \leq p} \varepsilon_{ij}\Big). \tag{3.52}$$

The vector field $\widetilde{W}_3^{11} := Z_1 + \overline{T}_{i_1} + \sum_{i \geq i_0} \overline{X}_i$ satisfies lemma 3.11. The vector field $\widetilde{W}_3^1$ required in lemma 3.11 will be defined by convex combination of Y_1 and $\widetilde{W}_3^{11}$. $\square$

Observe that the variation of the maximum of the λ_i's occurs only under the condition $\lambda_i^{\alpha_{j_i}} |a_i - y_{j_i}| \leq \delta$ for $1 \leq i \leq p$ and $\delta > 0$ a fixed constant small enough. In this case all the λ_i's increase and go to $+\infty$ along the flow-lines generated by $\widetilde{W}_3^1$.

Lemma 3.12. *In $\widetilde{V}_3^3(p, \varepsilon)$, there exists a pseudo-gradient $\widetilde{W}_3^3$ so that the following holds: There is a constant $c > 0$ independent of $u \in \widetilde{V}_3^3(p, \varepsilon)$ so that*

$$\langle \partial J(u), \widetilde{W}_3^3(u) \rangle \leq -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij}\Big).$$

Proof. Without loss of generality, we can assume that $a_j \in B(y_{i_j}, \eta)$, $y_{i_j} \in \mathcal{K}_2$ with

$$-\frac{1}{12} \frac{\sum_{k=1}^{4} b_k(y_{i_j})}{K(y_{i_j})} + H\big(y_{i_j}, y_{i_j}\big) < 0, \forall\, j = 1, \dots, q.$$

We define

$$Z_1' := -\sum_{i=1}^{q} \alpha_i \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}.$$

Using proposition 3.1 and the fact $|a_i - a_j| \geq c$ for all $i \neq j$, we obtain

$$\begin{aligned}
\langle \partial J(u), Z_1' \rangle &\leq -c\Big(\sum_{i=1}^{q} \frac{1}{\lambda_i^2} + \sum_{j \neq i} \frac{G(a_i, a_j)}{\lambda_i \lambda_j}\Big) + o\big(\sum_{i=1}^{p} \frac{1}{\lambda_i^2}\big) \\
&\leq -c\Big(\sum_{i=1}^{q} \frac{1}{\lambda_i^2} + \sum_{j \neq i} \varepsilon_{ij}\Big) + o\big(\sum_{i=1}^{p} \frac{1}{\lambda_i^2}\big).
\end{aligned} \tag{3.53}$$

Let $M > 0$ a fixed constant large enough. We define

$$I' := \Big\{1 \leq i \leq p : \lambda_i \leq \frac{1}{M} \min\{\lambda_k,\, k = 1, \dots, q\}\Big\}.$$

Observe that all the $\frac{1}{\lambda_i^2}$'s, $i \notin I'$, appear in the upper bound (3.53). We need to add some other terms. For this, let $\widetilde{u} = \sum_{i\in I'} \alpha_i P\delta_i$. $\widetilde{u}$ has to satisfy $\widetilde{u} \in \widetilde{V}_3^i(\ell,\varepsilon)$ for $i = 1$ or 2, where $\ell := \mathrm{card}(I')$. Thus, we define the corresponding vector field

$$Z_2' := \widetilde{W}_3^i(\widetilde{u}), \quad \text{for } i = 1 \text{ or } 2.$$

We obtain

$$\begin{aligned}\langle \partial J(u), Z_2'\rangle \leq & -c\Big(\sum_{i\in I'} \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{j\neq i, j\in I'} \varepsilon_{ij}\Big) \\ & + o\big(\sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\big) + O\big(\sum_{j\neq i, j\in I', i\notin I'} \varepsilon_{ij}\big).\end{aligned} \tag{3.54}$$

Now, we define

$$\widetilde{W}_3^3 := M_1 Z_1' + Z_2' + \sum_{i\notin I'} \overline{X}_i,$$

where $M_1 > 0$ is a fixed constant large enough. Combining (3.43), (3.53)and (3.54), and the fact $|a_i - a_j| \geq c, \forall i \neq j$, we obtain

$$\langle \partial J(u), \widetilde{W}_3^3(u)\rangle \leq -c\Big(\sum_{i=1}^p \Big[\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^{\beta_{j_i}}}\Big] + \sum_{i\neq j} \varepsilon_{ij}\Big).$$

This completes the proof. □

Lemma 3.13. *In $\widetilde{V}_3^4(p,\varepsilon)$, there exists a pseudo-gradient $\widetilde{W}_3^4$ so that the following holds: There is a constant $c > 0$ independent of $u \in \widetilde{V}_3^4(p,\varepsilon)$ so that*

$$\langle \partial J(u), \widetilde{W}_3^4(u)\rangle \leq -c\Big(\sum_{i=1}^p \frac{1}{\lambda_i^2} + \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j} \varepsilon_{ij}\Big).$$

Proof. For each critical point y_k of K, we set $B_k := \{1 \leq j \leq p : a_j \in B(y_k,\eta)\}$. Without loss of generality, we can assume $y_1,\dots,y_q$ are the critical points such that $\mathrm{card}(B_k) \geq 2$ for all $k = 1,\dots,q$. Let χ be a smooth cut-off function such that $\chi \geq 0, \chi = 0$ if $t \leq \gamma$, and $\chi = 1$ if $t \geq 1$, where γ is a small constant. Set $\overline{\chi}(\lambda_j) = \sum_{i\neq j, i\in B_k} \chi(\lambda_j/\lambda_i)$. Define

$$\widetilde{W}_3^{44} = -\sum_{k=1}^q \sum_{j\in B_k} \alpha_j \overline{\chi}(\lambda_j)\lambda_j \frac{\partial P\delta_j}{\partial \lambda_j}.$$

Using proposition 3.1, we derive that

$$\begin{aligned}&\langle \partial J(u), \widetilde{W}_3^{44}\rangle \\ &= cJ(u)\sum_{k=1}^q \sum_{j\in B_k} \alpha_j \overline{\chi}(\lambda_j)\Big[\alpha_j \frac{H(a_j,a_j)}{\lambda_j^2} - (\text{ if } y_k \in \mathcal{K}_2)\alpha_j \frac{1}{12} \frac{\sum_{i=1}^4 b_i(y_k)}{K(a_j)\lambda_i^2} \\ &\quad + \sum_{i\neq j} \alpha_i\big(\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + \frac{H(a_i,a_j)}{\lambda_i\lambda_j}\big)\Big] + o\Big(\sum_{i\neq k} \varepsilon_{ik} + \sum_{i=1}^p \frac{1}{\lambda_i^2}\Big) \\ &\quad + \sum_{k=1}^q \sum_{i\in B_k, \overline{\chi}(\lambda_i)\neq 0} \big(\text{if } \lambda_i|a_i - y_{j_i}| \geq C\big) o\big(\frac{|\nabla K(a_i)|}{\lambda_i}\big).\end{aligned}$$

For $j \in B_k$, with $k \leq q$, if $\overline{\chi}(\lambda_j) \neq 0$, then there exists $i \in B_k$ such that $\lambda_j^{-1} = o(\lambda_i^{-1})$ or $\lambda_j^{-2} = o(\varepsilon_{ij})$ (for η small enough). Furthermore, for $j \in B_k$, if $i \notin B_k$ or $i \in B_k$, with $\gamma < \lambda_i/\lambda_j < 1/\gamma$, then we have $\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} = -\varepsilon_{ij}(1 + o(1))$. In the case where $i \in B_k$ and $\lambda_i/\lambda_j \leq \gamma$, we have $\overline{\chi}(\lambda_j) - \overline{\chi}(\lambda_i) \geq 1$ and $\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \leq -\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j}$. Thus

$$\overline{\chi}(\lambda_j)\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + \overline{\chi}(\lambda_i)\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \leq \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} = -\varepsilon_{ij}(1 + o(1))$$

Thus we derive that

$$\begin{aligned}
\langle \partial J(u), \widetilde{W}_3^{44} \rangle &\leq -c \sum_{k=1}^{q} \sum_{j \in B_k, \overline{\chi}(\lambda_j) \neq 0} (\sum_{i \neq j} \varepsilon_{ij}) + o\Big(\sum_{i \neq k} \varepsilon_{ik} + \sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \frac{|\nabla K(a_i)|}{\lambda_i} \Big) \\
&\leq -c \sum_{k=1}^{q} \sum_{j \in B_k, \overline{\chi}(\lambda_j) \neq 0} (\frac{1}{\lambda_j^2} + \sum_{i \neq j} \varepsilon_{ij}) + o\Big(\sum_{i \neq k} \varepsilon_{ik} + \sum_{i=1}^{p} \frac{1}{\lambda_i^2} \Big) \\
&\quad + \sum_{k=1}^{q} \sum_{i \in B_k, \overline{\chi}(\lambda_i) \neq 0} \big(\text{if } \lambda_i |a_i - y_{j_i}| \geq C \big) o\big(\frac{|\nabla K(a_i)|}{\lambda_i} \big).
\end{aligned}$$

Observe that $\{j \in B_k, \overline{\chi}(\lambda_j) = 0\}$ contains at most one index. Thus we obtain

$$\begin{aligned}
\langle \partial J(u), \widetilde{W}_3^{44} \rangle &\leq - c \big(\sum_{k=1}^{q} \sum_{j \in B_k, \overline{\chi}(\lambda_j) \neq 0} \frac{1}{\lambda_j^2} + \sum_{i \neq j, j \in \cup_{k=1}^{q} B_k} \varepsilon_{ij} \big) + o\Big(\sum_{i \neq k} \varepsilon_{ik} + \sum_{i=1}^{p} \frac{1}{\lambda_i} \Big) \\
&\quad + \sum_{k=1}^{q} \sum_{i \in B_k, \overline{\chi}(\lambda_i) \neq 0} \big(\text{if } \lambda_i |a_i - y_{j_i}| \geq C \big) o\big(\frac{|\nabla K(a_i)|}{\lambda_i} \big).
\end{aligned} \tag{3.55}$$

This upper bound does not contains all the indices. We need to add some terms. Let

$$\lambda_{i_0} = \inf\{\lambda_i, i = 1, \dots, p\}.$$

Two cases may occur:

Case 1: If $i_0 \in \cup_{k=1}^{q} B_k$ with $\overline{\chi}(\lambda_{i_0}) \neq 0$, then we can make appear in the last upper bound $\frac{1}{\lambda_{i_0}^2}$, and therefore all the $\frac{1}{\lambda_i^2}$, and so ε_{ik}, $1 \leq i, k \leq p$.

Case 2: $i_0 \in \{i \in \cup_{k=1}^{q} B_k, \overline{\chi}(\lambda_i) = 0\} \cup (\cup_{k=1}^{q} B_k)^c$. In this case, we define

$$D = (\{i \in \cup_{k=1}^{q} B_k, \overline{\chi}(\lambda_j) = 0\} \cup (\cup_{k=1}^{q} B_k)^c) \cap \{1 \leq i \leq p, \lambda_i/\lambda_{i_0} < 1/\gamma\}.$$

It is easy to see that for $i, j \in D, i \neq j$, we have $a_i \in B(y_{k_i}, \eta)$ and $a_j \in B(y_{k_j}, \eta)4$ with $k_i \neq k_j$. Let

$$u_1 = \sum_{i \in D} \alpha_i P\delta_{(a_i, \lambda_i)}.$$

Then u_1 has to satisfy one of the three above cases, that is, $u_1 \in \widetilde{V}_3^i(\text{card}(D), \varepsilon)$ for $i = 1 - 3$. Thus we can apply the associated vector field which we will denote $\widetilde{W}_3^{444}$, and we have the following estimate:

$$\begin{aligned}
\langle \partial J(u), \widetilde{W}_3^{444} \rangle &\leq -c \big(\sum_{i \in D} \frac{1}{\lambda_i^2} + \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j, i, j \in D} \varepsilon_{ij} \big) \\
&\quad + O\Big(\sum_{r \notin D, k \in D} \varepsilon_{rk} + \sum_{i \notin D} \frac{1}{\lambda_i^2} \Big).
\end{aligned} \tag{3.56}$$

Observe that for $k \in D$ and $r \notin D$, we have either $r \in \cup_{k=1}^{q} B_k, \overline{\chi}(\lambda_r) \neq 0$ (in this case we have ε_{kr} in the upper bound (3.55)) or no, and in this last case we observe that $a_i \in B(y_{j_i}, \eta)$ for $i = r, k$ with $j_r \neq j_k$. Thus

$$\varepsilon_{kr} \leq \frac{c}{\lambda_k \lambda_r} \leq \frac{c\gamma}{\lambda_{i_0}^2} = o(\frac{1}{\lambda_{i_0}^2}).$$

We get the same observation for $\lambda_i^{-2}, i \notin D$. Now we define

$$Y_2 = C\widetilde{W}_3^{44} + \widetilde{W}_3^{444},$$

where C is a positive constant large enough. Combining (3.55) and (3.56), we obtain

$$\langle \partial J(u), Y_2 \rangle \leq -c\big(\sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij}\big) + o\big(\sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i}\big).$$

We define $\widetilde{W}_3^{4444}$ as a convex combination of $\widetilde{W}_3^{44}$ and Y_2. Then the vector field

$$\widetilde{W}_3^{4} := \widetilde{W}_3^{4444} + \sum_{i=1}^{p} \overline{X}_i$$

satisfies the claim of lemma 3.13. □

Lemma 3.14. *In $\widetilde{V}_3^5(p, \varepsilon)$, there exists a pseudo-gradient $\widetilde{W}_3^5$ so that the following holds: There is a constant $c > 0$ independent of $u \in \widetilde{V}_3^5(p, \varepsilon)$ so that*

$$\langle \partial J(u), \widetilde{W}_3^5(u) \rangle \leq -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij}\Big).$$

Proof. Without loss of generality, we suppose $\lambda_1 \leq \cdots \leq \lambda_p$. We denote by i_1 the index satisfying $a_{i_1} \notin \cup_{\nabla K(y)=0} B(y, \eta)$ and $a_i \in B(y_{j_i}, \eta), \forall i < i_1$. Let

$$\widetilde{u} = \sum_{i < i_1} \alpha_i P\delta_i.$$

Observe that $\widetilde{u} \in \widetilde{V}_3^i(i_1 - 1, \varepsilon)$ for $i = 1-4$. Then we define $Z_4'(\widetilde{u})$ the corresponding vector field and we have

$$\langle \partial J(u), {Z_4}'(\widetilde{u}) \rangle \leq -c\Big(\sum_{i<i_1} \frac{1}{\lambda_i^2} + \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j, i,j < i_1} \varepsilon_{ij}\Big) + O\Big(\sum_{i<i_1, j \geq i_1} \varepsilon_{ij} + \sum_{i \geq i_1} \frac{1}{\lambda_i^2}\Big).$$

Let now

$$Z_4 = \frac{1}{\lambda_{i_1}} \frac{\nabla K(a_{i_1})}{|\nabla K(a_{i_1})|} \frac{\partial P\delta_{a_{i_1}, \lambda_{i_1}}}{\partial a_{i_1}} - M_3 \sum_{i \geq i_1} 2^i \lambda_i \frac{\partial P\delta_{a_i, \lambda_i}}{\partial \lambda_i},$$

where $M_3 > 0$ is a fixed constant large enough. From propositions 3.1 and 3.2, we obtain, since $\nabla K(a_{i_1}) \geq c > 0$,

$$\langle \partial J(u), Z_4(u) \rangle \leq \frac{-c}{\lambda_{i_1}} + O\Big(\sum_{j \neq i_1} \lambda_j |a_{i_1} - a_j| \varepsilon_{i_1 j}^2\Big) - M_3 c \sum_{i \geq i_1, j \neq i} \varepsilon_{ij}.$$

Observe that $\lambda_j |a_{i_1} - a_j| \varepsilon_{i_1 j}^2 = O(\varepsilon_{i_1 j}), \forall j \neq i_1$. Thus

$$\langle \partial J(u), Z_4(u) \rangle \leq \frac{-c}{\lambda_{i_1}} + O\Big(\sum_{j \neq i_1} \varepsilon_{i_1 j}\Big) - M_3 c \sum_{i \geq i_1, j \neq i} \varepsilon_{ij}.$$

We choose M_3 large enough so that $O\Big(\sum_{j\neq i_1}\varepsilon_{i_1 j}\Big)$ is absorbed by $M_3 c\sum_{i\geq i_1, j\neq i}\varepsilon_{ij}$. We deduce

$$\langle \partial J(u), Z_4(u)\rangle \leq -c\Big(\frac{1}{\lambda_{i_1}} + \sum_{i\geq i_1, i\neq j}\varepsilon_{ij}\Big). \tag{3.57}$$

Also $1/\lambda_{i_1}$ makes appear $\sum_{i\geq i_1} 1/\lambda_i$ in the upper bound of (3.57). Taking M a positive constant large enough, and let

$$\widetilde{W}_3^5(u) = MZ_4 + Z_4'.$$

Thus we derive

$$\langle \partial J(u), \widetilde{W}_3^5(u)\rangle \leq -c\Big(\sum_{i=1}^{p}\frac{|\nabla K(a_i)}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{i\neq j}\varepsilon_{ij}\Big).$$

The claim of lemma 3.14 follows. $\square$

The vector field W_3, required in proposition 3.6, will be defined by a convex combination of the vector fields $\widetilde{W}_3^1(u)$, $\widetilde{W}_3^2(u)$, $\widetilde{W}_3^3(u)$, $\widetilde{W}_3^4(u)$ and $\widetilde{W}_3^5(u)$.

Corollary 3.15. *Let $n = 4$. Assume that K satisfies the condition* (A3). *Under assumptions* (A1), (A5)–(A7)*, the critical points at infinity of J in $V(p,\varepsilon), p\geq 1$, correspond to*

$$\sum_{j=1}^{p}\frac{1}{\big(K(y_{i_j})\big)^{1/2}}P\delta_{(y_{i_j},\infty)},$$

where $(y_{i_1},\dots,y_{i_p})\in\mathcal{C}_\infty$. Moreover, such a critical point at infinity has an index equal to $5p-1-\sum_{j=1}^{p}\widetilde{i}(y_{i_j})$.

Proof. Using theorem 3.3, the only region where the λ_i's are unbounded is the one where each a_i is close to a critical point y_{j_i} where $y_{j_i}\neq y_{j_k}$, for $i\neq k$, and $(y_{j_1},\dots,y_{j_p})\in\mathcal{C}_\infty$. Let $y_{j_i}\in\mathcal{K}_2^+\cup\mathcal{K}_{>2}$ for all $1\leq i\leq s$, and $y_{j_i}\in\mathcal{K}_{<2}^+$ for all $s+1\leq i\leq p$. In this region, arguing as in [4, Appendix 2], we can find a change of variables

$$(a_1,\dots,a_p,\lambda_1,\dots,\lambda_p)\mapsto(\widetilde{a}_1,\dots,\widetilde{a}_p,\widetilde{\lambda}_1,\dots,\widetilde{\lambda}_p) =: (\widetilde{a},\widetilde{\lambda})$$

such that

$$\begin{aligned} J\Big(\sum_{i=1}^{p}\alpha_i P\delta_{a_i,\lambda_i}+\bar{v}\Big) &= \frac{S_4^{1/2}\sum_{i=1}^{p}\alpha_i^2}{\big(\sum_{i=1}^{p}\alpha_i^4 K(\widetilde{a}_i)\big)^{1/2}}\big\{1+A(\widetilde{\lambda})\big\} \\ &=: S_4^{1/2}\Psi\big(\alpha_1,\dots,\alpha_p,\widetilde{a}\big)\big\{1+A(\widetilde{\lambda})\big\}, \end{aligned} \tag{3.58}$$

where $A(\widetilde{\lambda})$ is some quantity satisfying

$$A(\widetilde{\lambda}) = o(1), \quad \text{for } \widetilde{\lambda}_i\geq A, 1\leq i\leq p,$$

with A uniform on $\widetilde{a}_i\in B(y_{j_i},\rho)$ and $S_4 := \int_{\mathbb{R}^4}\delta_{o,1}^4(x)dx$. Now, denoting by

$$h_t\Big(\sum_{i=1}^{p}\alpha_i P\delta_{\widetilde{a}_i,\widetilde{\lambda}_i}\Big) := \sum_{i=1}^{p}\alpha_i P\delta_{\widetilde{a}_i(t),\widetilde{\lambda}_i(t)}$$

the 1-parameter group generated by the pseudo gradient W in this region. Taking into account the construction of W, we derive that, for t large enough, $\widetilde{\lambda}_i(t)|\widetilde{a}_i(t)-y_{j_i}|\leq\delta$ for $i=1,\dots,p$ with $y_{j_i}\in\mathcal{K}_{<2}^+\cup\mathcal{K}_2^+$, and $\big(\widetilde{\lambda}_i(t)\big)^{\alpha_{j_i}}|\widetilde{a}_i(t)-y_{j_i}|\leq\delta$ for

$i = 1, \dots, p$ with $y_{j_i} \in \mathcal{K}_{>2}$, where $\delta > 0$ is a fixed constant small enough. Thus we obtain

$$A(\widetilde{\lambda}(t)) = c\big[-\sum_{i=s+1}^{p} \frac{\sum_{k=1}^{4} b_k(y_{j_i})}{\widetilde{\lambda}_i^{\beta_{j_i}}} +^T \Lambda M \Lambda\big], \tag{3.59}$$

where $M := M(y_{j_1}, \dots, y_{j_s})$ is the matrix defined by (1.4) and (1.5), and ${}^T\Lambda := (\frac{1}{\widetilde{\lambda}_{j_1}}, \dots, \frac{1}{\widetilde{\lambda}_{j_s}})$. This proves that we have a critical point at infinity. Now, by combining (3.58) and (3.59), we derive that the index of such critical point at infinity is equal to the index of the critical point of

$$\Psi\big(\alpha_1, \dots, \alpha_p, \widetilde{a}_1, \dots, \widetilde{a}_p\big) = \frac{\sum_{i=1}^{p} \alpha_i^2}{\big(\sum_{i=1}^{p} \alpha_i^4 K(\widetilde{a}_i)\big)^{1/2}}.$$

Observe that the function Ψ admits for the variables α_i's an absolute degenerate maximum with one dimensional nullity space. Then the index of such critical point at infinity is equal to $5p - 1 - \sum_{i=1}^{p} \widetilde{i}(y_{j_i})$. The result of corollary 3.15 follows. $\square$

4. Proof of the main result

Proof of theorem 1.2. Assume that J has no critical points in Σ^+. It follows from corollary 3.15 that the only critical points at infinity of J are

$$(\tau_p)_\infty := \sum_{j=1}^{p} \frac{1}{\big(K(y_j)\big)^{1/2}} P\delta_{(y_j,\infty)}, \quad p \geq 1,$$

where $\tau_p := (y_1, \dots, y_p) \in \mathcal{C}_\infty$. Such a critical point at infinity has an index equal to $5p - 1 - \sum_{j=1}^{p} \widetilde{i}(y_j) =: i(\tau_p)$. By using the deformation lemma of [7], we obtain

$$\Sigma^+ \simeq \cup_{\tau_p \in \mathcal{C}_\infty} W_u\big((\tau_p)_\infty\big), \tag{4.1}$$

where $W_u\big((\tau_p)_\infty\big)$ denotes the unstable manifold of the critical point at infinity $(\tau_p)_\infty$ and $\simeq$ denotes the retract by deformation. Applying now the Euler-Poincaré characteristic χ on the both sides of (4.1) and using the fact that Σ^+ is a contractible space, we obtain

$$1 = \sum_{\tau_p \in \mathcal{C}_\infty} (-1)^{i\left(\tau_p\right)}.$$

This contradicts the assumption of our theorem 1.2. This completes the proof of our existence result. $\square$

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Zakaria Boucheche

Department of mathematics, Faculty of Sciences of Sfax, 3018 Sfax, Tunisia

E-mail address: `Zakaria.Boucheche@ipeim.rnu.tn`

Hichem Chtioui

Department of mathematics, Faculty of Sciences of Sfax, 3018 Sfax, Tunisia

E-mail address: `Hichem.Chtioui@fss.rnu.tn`

Hichem Hajaiej

California State University Los Angeles, 5151 University Drive, Los Angeles, CA 90032, USA

E-mail address: `hichem.hajaiej@gmail.com`