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FRACTIONAL-LIKE DERIVATIVE OF LYAPUNOV-TYPE FUNCTIONS AND APPLICATIONS TO STABILITY ANALYSIS OF MOTION

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Abstract. This article discusses the application of a fractional-like derivative of Lyapunov-type functions in the stability analysis of solutions of perturbed motion equations with a fractional-like derivative of the state vector. The main theorems of the direct Lyapunov method for this class of motion equations are established.

1. Introduction

It is known that the Lyapunov function method (or the direct method of Lyapunov) is extended to many classes of equations of perturbed motion, including systems with distributed parameters and sets of equations in metric spaces. See, for example [9] and the references therein. Recall that the stability of motion theory in the sense of Lyapunov was created by him as a result of his work in 1889–1892 [8]. The key element of the direct Layapunov method is the opportunity to calculate the total derivative of a composition of functions (chain rule) corresponding to an auxiliary function under consideration and the perturbed motion equations.

The great interest in equations with fractional derivatives over the last two decades (see [4, 7, 11, 12, 14] and the bibliography therein) has prompted many researchers to generalize the direct Lyapunov method to this class of equations. However, the lack of a simple formula for calculating the fractional derivative of a composition of functions does not allow us to get results similar to those obtained for many types equations for which the total derivative of the Lyapunov function is calculated as in the classic analysis. Along with the most common definitions of Riemann-Liouville, Hadamard, Grünwald-Letnikov, in 1969 Caputo (see [5]) proposed his definition of a fractional derivative.

In contrast to the classical definitions of fractional derivatives, the Caputo definition allows ones to choose the initial values of the solutions of fractional differential equations in the same way as for a system of ordinary differential equations. This result made it possible to simplify somewhat the analysis of the equations of motion with a Caputo's fractional derivative. But, as for classical definitions the problem for the evaluation of the Caputo-type fractional derivative of a composition of functions remains open.

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It should be noted that some estimates of the Caputo derivative for simple functions of Lyapunov (see [3, 10, 11] and the bibliography there) have expanded the possibilities of the direct Lyapunov method when analyzing the equations of perturbed motion with Caputo derivatives of the state vector. The monograph [7] contains the results for equations of this type, obtained up to 2009. Many of these results are generalized later for functional fractional differential equations and impulsive fractional differential equations [14].

Recently, in [1], a definition of a fractional derivative named "conformable fractional derivative" has been proposed by the authors. In the opinion of the authors, it is natural to named the new derivative as a "fractional-like derivative" (FLD). In this article the same expression is used, since it reflects the essence of the new definition of a fractional derivative.

This article is organized as follows. Section 2 provides definitions of a fractional-like derivative and some rules for computing it for simple functions. In Section 3 our concept of a fractional-like derivative of a Lyapunov-type function is introduced, and a Yoshizawa-type relation is established for such derivatives. In addition, it is shown here that for some simple Lyapunov functions of the type of quadratic forms, the fractional-like derivative is an upper bound to the Caputo derivative of these functions. In Section 4 sufficient conditions for stability, asymptotic stability and instability of the trivial solution of equations of perturbed motion with a fractional-like derivative of the state vector are presented. In Section 5 we prove the main theorems of the comparison principle on the basis of the Lyapunov scalar and vector functions for fractional-like equations. In Section 6 sufficient conditions for the stability of motion on a finite interval are given. Finally, in Section 7 concluding remarks are presented.

2. Fractional-like derivatives

Let $q \in (0,1]$, $\mathbb{R}_+ = [0,\infty)$, $t_0 \in \mathbb{R}_+$ and given a continuous function $x(t) : [t_0,\infty) \to \mathbb{R}$.

Definition 2.1 ([1, 6])**.** For any $q \in (0,1]$ the fractional-like derivative $\mathcal{D}^q_{t_0}(x(t))$ of the function $x(t)$ of order $0 < q \le 1$ is defined by

$$\mathcal{D}^q_{t_0}(x(t)) = \lim \Big\{ \frac{x(t+\theta(t-t_0)^{1-q}) - x(t)}{\theta}, \theta \to 0 \Big\}.$$

If $t_0 = 0$, then $\mathcal{D}^q_{t_0}(x(t))$ has the form [6]

$$\mathcal{D}^q_0(x(t)) = \lim \Big\{ \frac{x(t+\theta t^{1-q}) - x(t)}{\theta}, \theta \to 0 \Big\}.$$

In the case $t_0 = 0$, we will denote $\mathcal{D}^q_0(x(t)) = \mathcal{D}^q(x(t))$.

If $\mathcal{D}^q(x(t))$ exists on an open interval of the type $(0,b)$, then

$$\mathcal{D}^q(x(0)) = \lim_{t\to 0^+} \mathcal{D}^q(x(t)).$$

If the fractional-like derivative of $x(t)$ of order q exists on (t_0,∞), then the function $x(t)$ is said to be q-differentiable on the interval (t_0,∞).

Proposition 2.2 ([6])**.** *Let $q \in (0,1]$ and $x(t), y(t)$ be q-differentiable at a point $t > 0$. Then:*

(a) $\mathcal{D}^q_{t_0}(ax(t) + by(t)) = a\mathcal{D}^q_{t_0}(x(t)) + b\mathcal{D}^q_{t_0}(y(t))$ *for all* $a, b \in \mathbb{R}$;
(b) $\mathcal{D}^q_{t_0}(t^p) = pt^{p-q}$ *for any* $p \in \mathbb{R}$;

(c) $\mathcal{D}^q_{t_0}(x(t)y(t)) = x(t)\mathcal{D}^q_{t_0}(y(t)) + y(t)\mathcal{D}^q_{t_0}(x(t))$;

(d)
$$\mathcal{D}^q_{t_0}\Big(\frac{x(t)}{y(t)}\Big) = \frac{y(t)\mathcal{D}^q_{t_0}(x(t)) - x(t)\mathcal{D}^q_{t_0}(y(t))}{y^2(t)};$$

(e) $\mathcal{D}^q_{t_0}(x(t)) = 0$ *for any* $x(t) = \lambda$, *where* λ *is an arbitrary constant.*

Proposition 2.3 ([1, 13])**.** *Let* $h(y(t)) : (t_0, \infty) \to \mathbb{R}$. *If* $h(\cdot)$ *is differentiable with respect to* $y(t)$ *and* $y(t)$ *is* q*-differentiable, where* $0 < q \le 1$, *then for any* $t \in \mathbb{R}_+$, $t \ne t_0$ *and* $y(t) \ne 0$
$$\mathcal{D}^q_{t_0}h(y(t)) = h'(y(t))\mathcal{D}^q_{t_0}(y(t)),$$
where $h'(t)$ *is a partial derivative of* h.

The fractional-like integral of order $0 < q \le 1$ with a lower limit t_0 is defined by (see [6])
$$I^q_{t_0}x(t) = \int_{t_0}^t (s - t_0)^{q-1}x(s)ds.$$

Proposition 2.4 ([6])**.** *Let the function* $x(t) : (t_0, \infty) \to \mathbb{R}$ *be* q*-differentiable for* $0 < q \le 1$. *Then for all* $t > t_0$,
$$I^q_{t_0}(\mathcal{D}^q_{t_0}x(t)) = x(t) - x(t_0).$$

3. Fractional-like derivatives of Lyapunov-type functions

Consider a system of differential equations with fractional-like derivative of the state vector
$$\mathcal{D}^q_{t_0}x(t) = f(t, x(t)), \tag{3.1}$$
$$x(t_0) = x_0, \tag{3.2}$$
where $x \in \mathbb{R}^n$, $f \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, $t_0 \ge 0$. It is further assumed that for $(t_0, x_0) \in int(\mathbb{R}_+ \times \mathbb{R}^n)$ the initial value problem (IVP) (3.1)–(3.2) has a solution $x(t, t_0, x_0) \in C^q([t_0, \infty), \mathbb{R}^n)$ for all $t \ge t_0$. In addition, it is assumed that $f(t, 0) = 0$ for all $t \ge t_0$.

Let for equation (3.1) a Lyapunov-type function $V(t, x) \in C^q(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$ be constructed in some way such that $V(t, 0) = 0$ for all $t \in \mathbb{R}^n$. Introduce the notation $B_r = \{x \in \mathbb{R}^n : \|x\| < r\}$, $r > 0$.

Definition 3.1. Let V be a continuous and q-differentiable function (scalar or vector), $V : \mathbb{R}_+ \times B_r \to \mathbb{R}^s$ ($s = 1$ or $s = m$, respectively), and $x(t, t_0, x_0)$ be the solution of the IVP (3.1)–(3.2), which exists and is defined on $\mathbb{R}_+ \times B_r$. Then for $(t, x) \in \mathbb{R}_+ \times B_r$ the expression:

(1)
$$\begin{aligned} &{}^+\mathcal{D}^q_{t_0}V(t, x) \\ &= \limsup\Big\{\frac{V(t + \theta(t - t_0)^{1-q}, x(t + \theta(t - t_0)^{1-q}, t, x)) - V(t, x)}{\theta}, \theta \to 0^+\Big\}, \end{aligned} \tag{3.3}$$
is the upper right fractional-like derivative of the Lyapunov function,

(2)
$$\begin{aligned} &{}_+\mathcal{D}^q_{t_0}V(t, x) \\ &= \liminf\Big\{\frac{V(t + \theta(t - t_0)^{1-q}, x(t + \theta(t - t_0)^{1-q}, t, x)) - V(t, x)}{\theta}, \theta \to 0^+\Big\}, \end{aligned}$$

is the lower right fractional-like derivative of the Lyapunov function,

(3)

$$
\begin{aligned}
&{}^{-}\mathcal{D}^q_{t_0}V(t,x)\\
&= \limsup \Big\{\frac{V(t+\theta(t-t_0)^{1-q}, x(t+\theta(t-t_0)^{1-q}, t, x)) - V(t,x)}{\theta},\ \theta \to 0^-\Big\},
\end{aligned}
$$

is the upper left fractional-like derivative of the Lyapunov function,

(4)

$$
\begin{aligned}
&{}_{-}\mathcal{D}^q_{t_0}V(t,x)\\
&= \liminf \Big\{\frac{V(t+\theta(t-t_0)^{1-q}, x(t+\theta(t-t_0)^{1-q}, t, x)) - V(t,x)}{\theta}, \theta \to 0^-\Big\},
\end{aligned}
$$

is the lower left fractional-like derivative of the Lyapunov function.

An efficient application of the upper right fractional-like derivatives of Lyapunov functions in the construction of his direct method is based on the following result (cf. [15]).

Lemma 3.2. *Let $V(t,x)$ be continuous, q-differentiable and locally Lipschitz with respect to its second variable x on $\mathbb{R}_+ \times B_r$. Then the fractional-like derivative of the function $V(t,x)$ with respect to the solution $x(t,t_0,x_0)$ is defined by*

$$
\begin{aligned}
&{}^{+}\mathcal{D}^q_{t_0}V(t,x)\\
&= \limsup \Big\{\frac{V(t+\theta(t-t_0)^{1-q}, x+\theta(t-t_0)^{1-q}f(t,x)) - V(t,x)}{\theta}, \theta \to 0^+\Big\},
\end{aligned}
\tag{3.4}
$$

where $(t,x) \in \mathbb{R}_+ \times B_r$.

If $V(t,x(t)) = V(x(t))$, $0 < q \le 1$, the function V is differentiable on x, and the function $x(t)$ is q-differentiable on t for $t > t_0$, then

$$
{}^{+}\mathcal{D}^q_{t_0}\ V(t,x) = V'(x(t))\mathcal{D}^q_{t_0}x(t),
$$

where V' is a partial derivative of the function V.

Taking relations (3.3) and (3.4) into account, we obtain the result by Yoshizawa [15] for a fractional-like derivative of the function $V(t,x)$ in the form

$$
{}^{+}\mathcal{D}^q_{t_0}V(t,x(t,t_0,x_0)) = {}^{+}\mathcal{D}^q_{t_0}\ V(t,x)\big|_{(3.1)}.
$$

Definition 3.3. If the function $V(t,x)$ together with one of its fractional-like derivatives resolves the problem of stability (instability) of the solutions of (3.1), we will call $V(t,x)$ a Lyapunov function for the fractional-like system (3.1).

Example 3.4. Let $t > t_0$, $V(t,x) = V_1(x) = x^2(t), x \in \mathbb{R}$. Then, according to (c) in Proposition 2.2, we have

$$
\begin{aligned}
{}^{+}\mathcal{D}^q_{t_0}\ V(x(t)) &=^{+} \mathcal{D}^q_{t_0}(x(t)x(t)) = x(t)^{+}\mathcal{D}^q_{t_0}(x(t)),\\
&+^{+}\mathcal{D}^q_{t_0}(x(t))x(t) = 2x(t)^{+}\mathcal{D}^q_{t_0}(x(t))
\end{aligned}
\tag{3.5}
$$

for all $t \ge t_0$. Consider the following scalar fractional-like equation for $0 < q \le 1$,

$$
\mathcal{D}^q_{t_0}x(t) = f(t,x(t)), \quad t \ge t_0, \tag{3.6}
$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, f(t,0) = 0$ for $t \ge t_0$. For the function $V(x) = \frac{1}{2}x^2(t)$, considering (3.5), we obtain

$$
{}^{+}\mathcal{D}^q_{t_0}V_1(x(t))\big|_{(3.6)} = x(t)^{+}\mathcal{D}^q_{t_0}x(t) = x(t)f(t,x(t)) \tag{3.7}
$$

in the domain of the function $f(t,x)$.

Let $V(t,x) = V_2(x) = x^T x, x \in \mathbb{R}^n$. Then, according to (c) in Proposition 2.2, we have

$$
{}^+\mathcal{D}^q_{t_0}(V_2(x(t)) ={}^+ \mathcal{D}^q_{t_0}(x^T(t)x(t)) = 2x^T(t)\,{}^+\mathcal{D}^q_{t_0}x(t)). \tag{3.8}
$$

Example 3.5. Let $x_1, x_2 : [t_0,\infty) \to \mathbb{R}$ and x_1, x_2 be q-differentiable. Consider the equations of perturbed motion with fractional-like derivatives in the form

$$
\begin{aligned}
\mathcal{D}^q_{t_0}x_1(t) &= -\mu(t)x_2 - \nu(t)x_1,\\
\mathcal{D}^q_{t_0}x_2(t) &= \mu(t)x_1 - \nu(t)x_2.
\end{aligned} \tag{3.9}
$$

where $\mu(t)$ and $\nu(t)$ are continuous single-valued functions defined on $t \ge t_0$.

For the function $V_2(x_1,x_2) = \frac{1}{2}(x_1^2 + x_2^2)$ according to (c) in Proposition 2.2, we obtain

$$
\begin{aligned}
&{}^+\mathcal{D}^q_{t_0}\Big(\frac{1}{2}(x_1^2 + x_2^2)\Big) = x_1(t)\,{}^+\mathcal{D}^q_{t_0}x_1(t) + x_2(t)\,{}^+\mathcal{D}^q_{t_0}x_2(t)\\
&= -2\nu(t)(x_1^2(t) + x_2^2(t)).
\end{aligned} \tag{3.10}
$$

Remark 3.6. In [2] the authors obtain the following estimate for a fractional derivative in the Caputo sense of the Lyapunov function $V(t,x_1,x_2) = (x_1^2 + x_2^2)$ with respect to the system (3.9)

$$
{}^c_{t_0}D^q_t(V(t,x_1,x_2)) \le -2(x_1^2(t) + x_2^2(t))
$$

for $x \in \mathbb{R}^2$. Comparing this estimate with the estimate (3.10), we see that when estimating a fractional Caputo derivative we "lose" the effect of the function $\nu(t)$ on the properties of the zero solution of the system of equations (3.9).

Lemma 3.7. *Let $x \in \mathbb{R}$, $y \in \mathbb{R}^n$ and P is an $n \times n$ constant matrix. Then for the functions $V_1 = x^2(t)$, $V_2 = y^T(t)y(t)$, and $V_3 = y^T(t)Py(t)$ the following estimates hold:*

(a) ${}^c_{t_0}D^q_t(x^2(t)) \le{}^+ \mathcal{D}^q_{t_0}(x^2(t))$ *for* $x \in \mathbb{R}$*;*
(b) ${}^c_{t_0}D^q_t(y^T(t)y(t)) \le{}^+ \mathcal{D}^q_{t_0}(y^T(t)y(t))$ *for* $y \in \mathbb{R}^n$*;*
(c) ${}^c_{t_0}D^q_t(y^T(t)Py(t)) \le{}^+ \mathcal{D}^q_{t_0}(y^T(t)Py(t))$*, for* $y \in \mathbb{R}^n$*.*

Proof. We apply [3, Lemma 1] to the Caputo fractional derivative of the function V_1 and obtain

$$
{}^c_{t_0}D^q_t\ (x^2(t)) \le 2x(t)\ {}^c_{t_0}D^q_t\ (x(t)).
$$

Similar estimates we can obtain for the functions V_2 and V_3. Taking this into account the equalities (3.5) and (3.8), we obtain assertions (a)–(c) of Lemma 3.7. □

From Lemma 3.7 it follows that the fractional-like derivative of a Lyapunov-type function is an upper bound of the Caputo fractional derivatives of this Lyapunov function.

4. Direct Lyapunov's method and main results

The Lyapunov-type stability definitions for a fractional-like system (3.1) remain the same as for ordinary differential equations and differential equations with Caputo's fractional derivatives. See, for example, [7, 14] and the references therein.

In our main theorems we will use the Hahn class of functions $K = \{a \in C[\mathbb{R}_+,\mathbb{R}_+] : a(u)$ is strictly increasing and $a(0) = 0\}$.

Theorem 4.1. *Assume that for the fractional-like system* (3.1) *there exist a q-differentiable function $V(t,x)$, $V(t,0)=0$ for $t \geq t_0$ and functions $a, b \in K$ such that*

(i) $V(t,x) \geq a(\|x\|)$, $(t,x) \in \mathbb{R}_+ \times B_r$,
(ii) $V(t,x) \leq b(\|x\|)$, $(t,x) \in \mathbb{R}_+ \times B_r$,
(iii)
$$^{+}\mathcal{D}^q_{t_0}(V(t,x(t))) \leq 0 \quad \textit{for } (t,x) \in \mathbb{R}_+ \times B_r. \tag{4.1}$$

Then the state $x=0$ of (3.1) *is uniformly stable.*

Proof. Let $x(t) = x(t,t_0,x_0)$ be the solution of (3.1) for $(t_0,x_0) \in (\mathbb{R}_+ \times B_r)$ defined for all $t \geq t_0$. Let $t_0 \in \mathbb{R}_+$ and $0 < \varepsilon < r$ be given. By conditions (i), (ii) of Theorem 4.1 we can choose $\delta = \delta(\varepsilon) > 0$ so that
$$b(\delta) < a(\varepsilon). \tag{4.2}$$

We will prove that $\|x_0\| < \delta$ implies $\|x(t)\| < \varepsilon$ for all $t \geq t_0$. If this is not true there exists a solution $x(t,t_0,x_0) = x(t)$ of (3.1) such that for $\|x_0\| < \delta$ there is $t_1 > t_0$ for which
$$\|x(t_1)\| = \varepsilon, \quad \|x(t)\| < \varepsilon \quad \text{for all } t \in [t_0,t_1).$$
By Proposition 2.4 and condition (4.1), the Lyapunov relation
$$V(t,x(t)) - V(t_0,x_0) = I^q_{t_0}(^{+}\mathcal{D}^q_{t_0}(V(t,x(t))))$$
becomes
$$V(t,x(t)) - V(t_0,x_0) \leq 0. \tag{4.3}$$
For $t = t_1$ we have from (4.3),
$$a(\varepsilon) \leq V(t_1,x(t_1)) \leq V(t_0,x_0) \leq b(\|x_0\|) < a(\varepsilon). \tag{4.4}$$
This inequality contradicts condition (4.2). This completes the proof. □

Example 3.4 continued. From (3.7) and Theorem 4.1 it follows that the state $x=0$ of the fractional-like equation (3.6) is uniformly stable if
$$x(t)f(t,x(t)) \leq 0$$
for $(t,x) \in \mathbb{R}_+ \times B_r$.

Theorem 4.2. *Let the condition of Theorem 4.1 be satisfied and instead of* (4.1) *the following estimate hold*
$$^{+}\mathcal{D}^q_{t_0}(V(t,x(t))) \leq -d(\|x\|) \tag{4.5}$$
for $(t,x) \in \mathbb{R}_+ \times B_r$, where $d \in K$. Then the state $x=0$ of system (3.1) *is uniformly asymptotically stable.*

Proof. Since all conditions of Theorem 4.1 are satisfied the state $x=0$ is uniformly stable. We will prove that it is uniformly asymptotically stable.

Let $0 < \varepsilon < r$ and $\delta = \delta(\varepsilon) > 0$ be the same as in in Theorem 4.1. For $\varepsilon_0 \leq r$ we choose $\delta_0 = \delta_0(\varepsilon_0) > 0$ and consider the solution $x(t,t_0,x_0)$ with initial data $t_0 \in \mathbb{R}_+$ and $\|x_0\| < \delta_0$. Let for $t_0 < t \leq t_0 + T(\varepsilon)$, where $T(\varepsilon) \geq \big(qb(\delta_0)/d(\delta(\varepsilon))\big)^{1/q}$

for $x(t)$ we have $\|x(t)\| \geq \delta(\varepsilon)$. We will show that this is not possible under the conditions of Theorem 4.2. From the Lyapunov relation we obtain

$$\begin{aligned} V(t,x(t)) - V(t_0,x_0) &= I^q_{t_0}({}^+\mathcal{D}^q_{t_0}(V(t,x(t)) \\ &\leq -I^q_{t_0}(d(\|x(t)\|)) = -\int_{t_0}^t (s-t_0)^{q-1} d(\|x(s)\|) ds. \end{aligned} \tag{4.6}$$

From (4.6) we obtain

$$\begin{aligned} V(t,x(t)) &\leq V(t_0,x_0) - \int_{t_0}^t (s-t_0)^{q-1} d(\|x(s)\|) ds \\ &\leq b(\delta_0) - d(\delta(\varepsilon)) \frac{(t-t_0)^q}{q}. \end{aligned} \tag{4.7}$$

For $t = t_0 + T(\varepsilon)$ by (4.7) we have

$$\begin{aligned} 0 < a(\delta(\varepsilon)) &\leq V(t_0 + T(\varepsilon), x(t_0 + T(\varepsilon)) \\ &\leq b(\delta_0) - d(\delta(\varepsilon)) \frac{T(\varepsilon)^q}{q} \leq 0, \end{aligned}$$

which is a contradiction.

The above contradiction shows that there exists $t_1 \in [t_0, t_0 + T(\varepsilon)]$ such that $\|x(t_1)\| < \delta(\varepsilon)$. Hence $\|x(t)\| < \varepsilon$ for all $t \geq t_0 + T(\varepsilon)$ as far as $\|x_0\| < \delta_0$ and $\lim \|x(t)\| = 0$ as $t \to \infty$ uniformly on $t_0 \in \mathbb{R}_+$. This completes the proof. □

Example 3.5 continued. It follows from (3.5) and conditions of Theorem 4.2 that the state $x_1 = x_2 = 0$ of (3.9) will be uniformly asymptotically stable if the function $\nu(t)$ satisfies the condition $\nu(t) \geq \nu_0 > 0$, since in this case we have

$${}^+\mathcal{D}^q_{t_0}(V(x_1(t), x_2(t))) \leq -2\nu_0(x_1^2(t) + x_2^2(t)) < 0$$

for all $t \geq t_0$ and $0 < q \leq 1$.

In the next theorem, we will establish conditions for the instability of the state $x = 0$ of system (3.1).

Theorem 4.3. *Let for the system* (3.1) *there exists a q-differentiable function* $V(t,x) : \mathbb{R}_+ \times B_\varepsilon \to \mathbb{R}$, *such that on* $[t_0, \infty) \times G(h)$, *where* $G(h) \subset B_\varepsilon$, $t_0 \geq 0$, *the following conditions are satisfied:*

1. $0 < V(t,x) \leq c < \infty$ *for some constant c;*
2. ${}^+\mathcal{D}^q_{t_0}\ V(t,x)\big|_{(3.1)} \geq a(V(t,x))$, *where* $a \in K$, $0 < q \leq 1$;
3. *the state* $x = 0$ *belongs to* $\partial G(h)$;
4. $V(t,x)$=0 *for* $[t_0, \infty) \times (\partial G(h) \cap B_\varepsilon)$.

Then the state $x = 0$ *of system* (3.1) *is unstable.*

Proof. It follows from condition (3) of Theorem 4.3 that for any $\delta > 0$ there exists a $x_0 \in G(h) \cap B_\delta$ such that $V(t_0, x_0) > 0$. For the solution $x(t) = x(t, t_0, x_0)$ while $x(t) \in G(h)$ from conditions (3.1), (2) we have

$$\begin{aligned} c \geq V(t,x(t)) - V(t_0,x_0) &\geq I^q_{t_0} a(V(t,x(t)) \\ &\geq V(t_0,x_0) + a(V(t_0,x_0)) \frac{(t-t_0)^q}{q}. \end{aligned} \tag{4.8}$$

From this inequality it follows that the solution $x(t)$ must leave the domain $G(h)$ at some moment $t_1 > t_0$.

Since condition (4) of Theorem 4.3 is satisfied, then $x(t)$ can not leave the domain $G(h)$ across the boundary $\partial G(h)$, because $G(h) \subset B_\varepsilon$. Therefore $x(t)$ will leave B_ε, i.e. $\|x(t_1)\| \geq \varepsilon$. This completes the proof. □

From Theorem 4.3, we have the following corollary.

Corollary 4.4. *Suppose that all conditions of Theorem 4.3 hold and conditions (1) and (2) are replaced by the following conditions, respectively:*

(1*) $0 < V(t,x) \leq b(\|x\|)$,

(2*) ${}^+\mathcal{D}^q_{t_0} V(t,x) \geq a(\|x\|)$, *where* $a, b \in K$.

Then the state $x = 0$ *of system* (3.1) *is unstable.*

Corollary 4.5. *Suppose that all conditions of Theorem 4.3 hold and condition (2) is replaced by*

$$ {}^+\mathcal{D}^q_{t_0} V(t,x) = \lambda V(t,x) + W(x(t)), t \in [t_0, \infty), \quad x \in G(h),\ \lambda > 0, \tag{4.9} $$

where the function W *is continuous and* $W(x) \geq 0$. *Then the state* $x = 0$ *of system* (3.1) *is unstable.*

Proof. Relation (4.9) can be represented in the integral form

$$ \begin{aligned} V(t, x(t)) &= V(t_0, x(t_0)) \exp\Big(\lambda \frac{(s-t_0)^q}{q}\Big) + \int_{t_0}^t \exp\Big(\lambda \frac{(s-t_0)^q}{q}\Big) \\ &\quad \times \exp\Big(-\lambda \frac{(s-t_0)^q}{q}\Big)(s-t_0)^{q-1} W(x(s)) ds. \end{aligned} $$

From the above relation, since the second term is positive by the conditions of Corollary 4.5, for any $0 < q \leq 1$ we have

$$ V(t, x(t)) \geq V(t_0, x(t_0)) \exp\Big(\lambda \frac{(t-t_0)^q}{q}\Big), \quad t \geq t_0, \tag{4.10} $$

Let the initial state of the solution $x(t) = x(t, t_0, x_0)$ be $x_0 \in U$, where U is a neighborhood of the origin $x = 0$. Since for any $t \geq t_0$ the estimate (4.10) is satisfied with respect to the solution $x(t)$, then for $t \to \infty$ the function $V(t, x(t))$ increases while, by the conditions of Theorem 4.3 it is bounded. Hence for $x(t)$ there exists t^* such that $x(t^*)$ will leave B_r. This proves the instability of the state $x = 0$ of system (3.1). □

Example 4.6. Consider the fractional-like system for $0 < q \leq 1$,

$$ \begin{aligned} \mathcal{D}^q_{t_0} x(t) &= n(t) y - x g(t,x,y), \quad x(t_0) = x_0; \\ \mathcal{D}^q_{t_0} y(t) &= -n(t) x - y g(t,x,y), \quad y(t_0) = y_0, \end{aligned} \tag{4.11} $$

where $n(t)$ is a continuous function for all $t \geq t_0$, $g(t,x,y)$ is a sum of a convergent power series, $g(t,0,0) = 0$ for $t \geq t_0$. Applying the function $2V(x,y) = x^2 + y^2$ to system (4.11) we have

$$ {}^+\mathcal{D}^q_{t_0}\ V(x(t)), y(t)) = -(x^2 + y^2) g(t,x,y). \tag{4.12} $$

Performing a q-integration of (4.12), we obtain the Lyapunov relation

$$ V(x(t), y(t)) - V(x_0, y_0) \leq -r^2 \int_{t_0}^t \frac{g(s, x(s), y(s))}{(s-t_0)^{1-q}} ds \tag{4.13} $$

on the domain $x^2 + y^2 \leq r^2$ of the equilibrium state $x = y = 0$. From the relation (4.12) and inequality (4.13) it follows that:

(a) By Theorem 4.1 the state $x = y = 0$ of (4.11) is uniformly stable provided the function is such that $g(t, x, y) \geq 0$ for $t \geq t_0$;
(b) By Theorem 4.2 the state $x = y = 0$ of system (4.11) is uniformly asymptotically stable, if $g(t, x, y) > 0$ on the domain $x^2 + y^2 \leq r^2$ for $t \geq t_0$;
(c) By Theorem 4.3 the state $x = y = 0$ of system (4.11) is unstable if $g(t, x, y) < 0$ for $t \geq t_0$ on a sufficiently small neighborhood.

5. Comparison principle

We continue our consideration of system (3.1) together with the q-differentiable function $V(t, x) \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$. Consider the total fractional-like derivative of the function $V(t, x)$ of the type (3.4).

As in the general theory of stability of motion, the application of the comparison principle allows us to indicate in a general form the structure of the stability conditions for fractional-like equations of perturbed motion. We will show that the following comparison theorem holds.

Theorem 5.1. *Assume that:*

(1) *For the system* (3.1) *there exists a q-differentiable function $V(t, x)$ with fractional-like derivative of the type* (3.4)*;*
(2) *There exists a function $g(t, u) \in C(\mathbb{R}_+^2, \mathbb{R})$ such that*
$$ {}^{+}\mathcal{D}_{t_0}^{q}\, V(t, x) \leq g(t, V(t, x)), \tag{5.1}$$
for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ and $0 < q \leq 1$;
(3) *There exists a maximal solution $r(t) = r(t, t_0, r_0) \in C^q([t_0, \infty), \mathbb{R})$ of the comparison scalar fractional-like equation*
$$\mathcal{D}_{t_0}^{q}\, u(t) = g(t, u),\ u(t_0) = u_0 \geq 0 \tag{5.2}$$
for all $t \geq t_0$.

Then along the solutions of system (3.1) *the estimate*
$$V(t, x(t)) \leq r(t), \tag{5.3}$$
is valid for all $t \geq t_0$ whenever $V(t_0, x_0) \leq u_0$.

Proof. Let the solution $x(t) = x(t, t_0, x_0)$ of the IVP (3.1)–(3.2) exists on $t \in [t_0, \infty)$ and $V(t_0, x_0) \leq u_0$. Denote by $m(t) = V(t, x(t))$ and evaluate the fractional-like derivative of the function $m(t)$ by the formula (3.4). From condition (2) of Theorem 5.1 we obtain
$$\mathcal{D}_{t_0}^{q}\, m(t) \leq g(t, V(t, x)) = g(t, m(t)). \tag{5.4}$$
Similar to [7, Theorem 2.8.3] we have
$$\mathcal{D}_{t_0}^{q}\, u(t) = g(t, u) + \varepsilon, u(t_0) = u_0 + \varepsilon \geq 0, \quad \varepsilon > 0. \tag{5.5}$$
From this equality it follows that
$$\mathcal{D}_{t_0}^{q} u(t, \varepsilon) = g(t, u(t, \varepsilon)) + \varepsilon > g(t, u(t, \varepsilon)),$$
so $m(t) < u(t, \varepsilon)$ and, hence $\lim u(t, \varepsilon) = r(t)$ as $\varepsilon \to 0$, uniformly on t for $t_0 \leq t < T < +\infty$. This completes the proof. □

Further, we will represent the fractional-like system (3.1) in the form

$$\mathcal{D}^q_{t_0}\ x_i(t) = f_i(t, x_i) + R_i(t, x_1(t), \dots, x_m(t)), \tag{5.6}$$

where $x_i \in \mathbb{R}^{n_i}$, $\sum_{i=1}^m n_i = n$, $f_i \in C(\mathbb{R}_+ \times \mathbb{R}^{n_i}, \mathbb{R}^{n_i})$, $R_i \in C(\mathbb{R}_+ \times \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m}, \mathbb{R}^{n_i})$, and $f_i(t, 0) = 0$ for $t \ge t_0$.

Suppose that for the independent subsystems

$$\mathcal{D}^q_{t_0}\ x_i(t) = f_i(t, x_i), \quad i = 1, 2, \dots, m \tag{5.7}$$

Lyapunov-type functions $V_i(t, x_i)$ exist such that

$${}^+\mathcal{D}^q_{t_0}\ (V_i(t, x_i(t)))\big|_{(27)} \le -d_i(\|x_i\|) + w_i(t, x_i, x). \tag{5.8}$$

Here $d_i(\cdot)$ are functions of the Hahn class of functions K, $w_i(t, \cdot, \cdot)$ are continuous with respect to t functions, and $w_i(t, 0, 0) = 0$ for $t \ge t_0$.

If for the right-hand side of (5.8) there is a majorizing function $H(t; u)$ which is quasi-monotonic (see [7, 14, 15]) non-decreasing with respect to u and such that

$${}^+\mathcal{D}^q_{t_0} V(t, x(t)) \le H(t, V(t, x(t))), \tag{5.9}$$

where $V(t, x) = (V_1(t, x_1), \dots, V_m(t, x_m))$, then the following theorem holds.

Theorem 5.2. *Assume that $V \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^m_+)$ is q-differentiable,*

$${}^+\mathcal{D}^q_{t_0}(V(t, x(t))) \le H(t, V(t, x(t))),$$

where $H \in C(\mathbb{R}_+ \times \mathbb{R}^m_+, \mathbb{R}^m)$ and for all $t \ge t_0$ there exists the maximal solution $u(t)$ of the fractional-like equation

$$\mathcal{D}^q_{t_0}\ u(t) = H(t, u), \quad u(t_0) = u_0,$$

for values $0 < q \le 1$. Then $V(t_0, x_0) \le u_0$ implies

$$V(t, x(t)) \le u(t), \quad t \ge t_0. \tag{5.10}$$

Proof. The proof of Theorem 5.2 is similar to the proof of [7, Theorem 4.2.1], taking into account that the total fractional-like derivative of the Lyapunov is evaluated according to Proposition 2.3. □

Estimates (5.3) and (5.10) allow us to establish stability criteria for the state $x = 0$ of system (3.1) in the same way as it is done the monograph [7].

Corollary 5.3. *If in the estimate* (5.1) *the majorizing function*

$$g(t, V(t, x)) \le kV(t, x), \quad k = const > 0,$$

then

$$V(t, x(t)) \le V(t_0, x_0) \exp\Big(k\frac{(t - t_0)^q}{q}\Big) \tag{5.11}$$

for all $t \in [t_0, t_0 + T]$ and any values of $0 < q \le 1$.

Corollary 5.4. *If in estimate* (4.13) *the majorizing function*

$$g(t, V(t, x)) \le k(t)V(t, x),$$

where $k(t)$ is a q-differentiable function, then

$$V(t, x(t)) \le V(t_0, x_0) \exp\Big(\int_{t_0}^t k(s)(s - t_0)^{q-1} ds\Big) \tag{5.12}$$

for all $t \in [t_0, t_0 + T]$ and any values of $0 < q \le 1$.

6. Stability on a finite interval

For the fractional-like system (3.1) we give the following definition of the stability on a finite interval.

Definition 6.1. System (3.1) is stable on a finite interval, if for given $0 < c_1 < c_2$, t_0 and $T > 0$ the solution $x(t)$ satisfies the estimate

$$V(t, x(t)) \le c_2 \quad \text{for all } t \in [t_0, t_0 + T]$$

whenever $V(t_0, x_0) < c_1$.

The following theorem follows directly from the estimates (5.11) and (5.12).

Theorem 6.2. *If for system* (3.1) *there exists a q-differentiable function* $V(t, x) \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$*, such that the conditions of corollaries 3 or 4 are satisfied, then system* (3.1) *is stable on a finite interval if one of the conditions:*

(1) $\exp\big(k\frac{(t-t_0)^q}{q}\big) \le \frac{c_2}{c_1}$ *for all* $t \in [t_0, t_0 + T]$ *and any values of* $0 < q \le 1$*;*
(2) $\exp\big(\int_{t_0}^t k(s)(s - t_0)^{q-1} ds\big) \le \frac{c_2}{c_1}$ *for all* $t \in [t_0, t_0 + T]$ *and any values of* $0 < q \le 1$ *is satisfied, respectively.*

Concluding remarks. For systems of equations with Caputo fractional derivatives of the state vector there exist several definitions of fractional derivatives of a Lyapunov-type function (see, [2, 7]). The actual calculation of the Caputo fractional derivative for a Lyapunov-type function is difficult due to the absence of a chain rule for this derivative, as for other fractional derivatives (Riemann-Liouville, Grünwald-Letnikov, etc.). For this reason, when considering particular examples, it is necessary to estimate the fractional derivative of the Lyapunov function [2, 3, 10, 11].

In this paper, the direct Lyapunov method is extended to systems of equations of perturbed motion with fractional-like derivatives. Theorems of the direct Lyapunov method and the comparison principle are established for the scalar and vector Lyapunov functions, taking into account that for a fractional-like derivative, a chain rule takes place. The relationship between a fractional-like derivative and a Caputo fractional derivative (Lemma 3.7) indicates that the fractional-like derivative of Lyapunov functions under consideration is a majorant for the Caputo fractional derivative of these functions. This circumstance should be taking into account when considering specific problems of the stability of motions.

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