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ENERGY DECAY IN A TIMOSHENKO-TYPE SYSTEM FOR THERMOELASTICITY OF TYPE III WITH DISTRIBUTED DELAY AND PAST HISTORY

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Abstract. In this work, we consider a one-dimensional Timoshenko system of thermoelasticity of type III with past history and distributive delay. It is known that an arbitrarily small delay may be the source of instability. We establish the well-posedness and the stability of the system for the cases of equal and nonequal speeds of wave propagation respectively. Our results show that the damping effect is strong enough to uniformly stabilize the system even in the presence of time delay under suitable conditions and improve the related results.

1. Introduction

In this article, we study the following Timoshenko-type system for thermoelasticity of type III with distributive delay and past history,

$$
\begin{aligned}
&\rho_1\varphi_{tt} - k(\varphi_x+\psi)_x + \beta\theta_{tx} = 0, \quad (x,t)\in(0,1)\times(0,\infty),\\
&\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x+\psi) + \int_0^\infty g(s)\psi_{xx}(x,t-s)ds\\
&\quad - \beta\theta_t + f(\psi) = 0, \quad (x,t)\in(0,1)\times(0,\infty),\\
&\rho_3\theta_{tt} - \delta\theta_{xx} - \ell\theta_{txx} + \gamma\varphi_{tx} + \gamma\psi_t - \int_{\tau_1}^{\tau_2}\mu(\varsigma)\theta_{txx}(x,t-\varsigma)d\varsigma\\
&\quad = 0, \quad (x,t)\in(0,1)\times(0,\infty),\\
&\varphi(x,0)=\varphi_0(x), \quad \varphi_t(x,0)=\varphi_1(x), \quad \theta(x,0)=\theta_0(x), \quad x\in(0,1),\\
&\theta_t(x,0)=\theta_1(x), \quad \psi_t(x,0)=\psi_1(x), \quad x\in(0,1),\\
&\psi(x,-t)=\psi_0(x,t), \quad (x,t)\in(0,1)\times(0,\infty),\\
&\varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=\theta(0,t)=\theta(1,t)=0, \quad t\in(0,\infty),\\
&\theta_{tx}(x,-t)=f_0(x,t), \quad (x,t)\in(0,1)\times(0,\tau_2),
\end{aligned}
\tag{1.1}
$$

where φ is the longitudinal displacement, ψ is the volume fraction, θ is the difference in temperature, the coefficients ρ_1, ρ_2, ρ_3, k, b, ℓ, β, δ, γ are positive constants,

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$\tau_1 < \tau_2$ are non-negative constants such that $\mu : [\tau_1, \tau_2] \to R$ represents distributive time delay, f is a forcing term.

In 1921, Timoshenko [18] gave, as a model for a thick beam, the following system of coupled hyperbolic equations

$$\begin{gathered} \rho u_{tt} = (K(u_x + \varphi))_x, \\ I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_x - \varphi), \end{gathered} \tag{1.2}$$

where t denotes the time variable and x is the space variable along the beam of length L, in its equilibrium configuration, u is the transverse displacement of the beam and φ is the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I and K are respectively the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.

Since then, the issue of existence and stability of Timoshenko system has attracted a great deal of attention in the last decades (e.g. [2, 3, 4, 5, 6, 7, 8, 10, 11]). Messaoudi and Said [11] considered the following Timoshenko-type system with past history

$$\begin{gathered} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + \int_0^\infty g(s)\psi_{xx}(x, t-s)ds + k(\varphi_x + \psi) = 0, \end{gathered} \tag{1.3}$$

where ρ_1, ρ_2, k, b are positive constants and g is a differentiable function satisfying, for some positive constant k_0 and $1 \le p < 3/2$, the conditions

$$g(t) > 0, \quad \widehat{b} = b - \int_0^\infty g(s)ds > 0, \quad g'(t) \le k_0 g^p(t).$$

They proved that, for the case of equal-speed propagation $\frac{\rho_1}{k} = \frac{\rho_2}{b}$, the first energy decays exponentially if $p = 1$ and polynomially if $p > 1$. When in the opposite case $\frac{\rho_1}{k} \neq \frac{\rho_2}{b}$, the decay is in the rate of $1/t^p$. Guesmia and Messaoudi [5] also considered (1.3) and established some general decay results for the equal and nonequal speed propagation cases where the relaxation function satisfies a relation of the form

$$g'(t) \le -\xi(t)g(t).$$

Guesmia and Messaoudi [6] concerned with the long-time behavior of the solution of the Timoshenko system

$$\begin{gathered} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x + \gamma h(\varphi_t) + \int_0^\infty g(s)(a(x)\varphi_x(t-s))_x ds = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) = 0. \end{gathered} \tag{1.4}$$

They showed that the dissipation given by this complementary controls guarantees the stability of the system in case of the equal-speed propagation as well as in the opposite case.

Time delays arise in many applications because most phenomena naturally depend not only on the present state but also on some past occurrences. In recent years, the stability of evolution systems with time delay effects has become an active area of research (e.g. [1, 2, 7, 8, 12, 13, 15]). Apalara [1] considered the following thermoelasic system of Timoshenko type with a linear frictional damping and an internal distributed delay acting on transverse displacement,

$$
\begin{aligned}
\rho_1\varphi_{tt} - \ell(\varphi_x + \psi)_x + \mu_1\varphi_t + \int_{\tau_1}^{\tau_2} \mu_2(s)\varphi_t(x, t-s)ds = 0,\\
\rho_2\psi_{tt} - b\psi_{xx} + \ell(\varphi_x + \psi) + \delta\theta_x = 0,\\
\rho_3\theta_t + q_x + \delta\psi_{tx} = 0,\\
\tau q_t + \beta q + \theta_x = 0.
\end{aligned} \tag{1.5}
$$

Under suitable assumptions on the weight of the delay and that of frictional damping, the author established the well-posedness result and proved that the system is exponentially stable regardless of the speeds of wave propagation.

Feng and Pelicer [2] were concerned with a Timoshenko system with time delay

$$
\begin{aligned}
\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0,\\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \mu_1\psi_t + \mu_2\psi_t(x, t-\tau) + f(\psi) = 0,
\end{aligned} \tag{1.6}
$$

where $\mu_2\psi_t(x, t-\tau)$ is time delay. They established the well-posedness of the problem with respect to weak solutions under suitable assumptions and the exponential stability of the system under the usual equal wave speed assumption.

Kafini et al. [7] considered the following Timoshenko-type system of thermoelasticity of type III with distributive delay

$$
\begin{aligned}
\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0,\\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\theta_{tx} = 0,\\
\rho_3\theta_{tt} - \delta\theta_{xx} - k\theta_{txx} - \int_{\tau_1}^{\tau_2} g(s)\theta_{txx}(x, t-s)ds + \gamma\psi_{tx} = 0,
\end{aligned} \tag{1.7}
$$

where $\tau_1 < \tau_2$ are non-negative constants such that $g : [\tau_1, \tau_2] \to R^+$ represents distributive time delay. They proved an exponential decay in the case of equal wave speeds and a polynomial decay result in the case of nonequal wave speeds with smooth initial data.

In this present work we consider (1.1), prove the well-posedness and establish the energy decay rate in case of the equal-speed propagation as well as in the opposite case.

The article is organized as follows. In Section 2, we introduce some transformations and assumptions needed in our work. In Section 3, we use the semigroup method to prove the well-posedness of problem (1.1). In Section 4, we state and prove our stability results.

2. Preliminaries

In this section, we present some materials needed in the proof of our results. Throughout this paper, c is used to denote a generic positive constant and is different in various occurrences.

Firstly, to deal with the delay term, we introduce the new variable

$$z(x, \rho, \varsigma, t) = \theta_{tx}(x, t-\varsigma\rho), \quad x \in (0,1),\ \rho \in (0,1),\ \varsigma \in (\tau_1, \tau_2),\ t > 0.$$

Then we obtain

$$\varsigma z_t(x, \rho, \varsigma, t) + z_\rho(x, \rho, \varsigma, t) = 0, \quad x \in (0,1),\ \rho \in (0,1),\ \varsigma \in (\tau_1, \tau_2),\ t > 0.$$

Then problem (1.1) is equivalent to

$$
\begin{aligned}
&\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \beta\theta_{tx} = 0, \quad (x,t) \in (0,1)\times(0,\infty),\\
&\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x+\psi) + \int_0^\infty g(s)\psi_{xx}(x,t-s)ds - \beta\theta_t + f(\psi)\\
&= 0, \quad (x,t)\in(0,1)\times(0,\infty),\\
&\rho_3\theta_{tt} - \delta\theta_{xx} - \ell\theta_{txx} + \gamma\varphi_{tx} + \gamma\psi_t - \int_{\tau_1}^{\tau_2}\mu(\varsigma)z_x(x,1,\varsigma,t)d\varsigma\\
&= 0, \quad (x,t)\in(0,1)\times(0,\infty),\\
&\varsigma z_t(x,\rho,\varsigma,t) + z_\rho(x,\rho,\varsigma,t)\\
&= 0, \quad (x,\rho,\varsigma,t)\in(0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty),\\
&\varphi(x,0)=\varphi_0(x), \quad \varphi_t(x,0)=\varphi_1(x), \quad \theta(x,0)=\theta_0(x), \quad x\in(0,1),\\
&\theta_t(x,0)=\theta_1(x), \quad \psi_t(x,0)=\psi_1(x), \quad x\in(0,1),\\
&\psi(x,-t)=\psi_0(x,t), \quad (x,t)\in(0,1)\times(0,\infty),\\
&\varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=\theta(0,t)=\theta(1,t)=0, \quad t\in(0,\infty),\\
&z(x,\rho,\varsigma,0)=f_0(x,\rho\varsigma), \quad (x,\rho,\varsigma)\in(0,1)\times(0,1)\times(\tau_1,\tau_2).
\end{aligned}
\tag{2.1}
$$

We shall use the folloing hypotheses.

(H1) $\mu : [\tau_1,\tau_2] \to R$ is a bounded function and

$$\ell - \int_{\tau_1}^{\tau_2} |\mu(\varsigma)|d\varsigma > 0. \tag{2.2}$$

(H2) $g : R_+ \to R_+$ is a C^1 function satisfying

$$g(0) > 0, \quad b - \int_0^\infty g(s)ds = l > 0, \quad \int_0^\infty g(s)ds = g_0. \tag{2.3}$$

(H3) There exists a positive nonincreasing differentiable function $\xi : R_+ \to R_+$ satisfying

$$g'(t) \le -\xi(t)g(t), \ t \ge 0. \tag{2.4}$$

(H4) $f : R \to R$ satisfies

$$|f(\psi^2) - f(\psi^1)| \le k_0(|\psi^1|^\varrho + |\psi^2|^\varrho)|\psi^1-\psi^2|, \quad \psi^1,\psi^2 \in R, \tag{2.5}$$

where $k_0 > 0$, $\varrho > 0$. In addition we assume that

$$0 \le \widehat{f}(\psi) \le f(\psi)\psi, \quad \psi \in R, \tag{2.6}$$

with $\widehat{f}(\psi) = \int_0^\psi f(s)ds$.

The first-order energy associated with (2.1) is

$$
\begin{aligned}
E(t) :=&E_1(\varphi,\psi,\theta)\\
=&\frac{\gamma}{2}\int_0^1 \Big(\rho_1\varphi_t^2 + k(\varphi_x+\psi)^2 + \rho_2\psi_t^2 + (b - \int_0^\infty g(s)ds)\psi_x^2\Big)dx\\
&+ \frac{\gamma}{2}(g\circ\psi_x) + \gamma\int_0^1 \widehat{f}(\psi)dx + \frac{\beta}{2}\int_0^1(\rho_3\theta_t^2 + \delta\theta_x^2)dx\\
&+ \frac{\beta}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varsigma|\mu(\varsigma)|z^2(x,\rho,\varsigma,t)\,d\varsigma\,d\rho\,dx,
\end{aligned}
\tag{2.7}
$$

where
$$(g \circ \nu)(t) = \int_0^1 \int_0^\infty g(s)(\nu(x,t) - \nu(x,t-s))^2 \, ds \, dx.$$

3. Well-posedness of the problem

In this section, we give a brief idea about the existence and uniqueness of solution for (2.1) using the semigroup theory [16]. Using the notation
$$\eta^t(x,s) = \psi(x,t) - \psi(x,t-s), \quad t \in R_+, \ (x,t,s) \in (0,1) \times R_+ \times R_+, \tag{3.1}$$
which was adopted in articles [11, 14] and [17] and η^t is the relative history of ψ, we have
$$\begin{gathered} \eta_t^t + \eta_s^t - \psi_t = 0, \quad (x,t,s) \in (0,1) \times R_+ \times R_+, \\ \eta^t(0,s) = \eta^t(1,s) = 0, \quad (t,s) \in R_+ \times R_+, \\ \eta^t(x,0) = 0, \quad (x,t) \in (0,1) \times R_+. \end{gathered} \tag{3.2}$$
Then the second equation of (2.1) can be formulated as
$$\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + g_0 \psi_{xx}(x,t) - \int_0^\infty g(s)\eta_{xx}^t(x,s)ds - \beta\theta_t + f(\psi) = 0.$$
Let
$$\eta_0(x,s) := \eta^0(x,s) = \psi_0(x,0) - \psi_0(x,s), \quad (x,s) \in (0,1) \times R_+.$$
Before using the semigroup theory, we introduce three new dependent variables
$$u = \varphi_t, \quad v = \psi_t, \quad \omega = \theta_t.$$
Then problem (2.1) becomes the following problem for an abstract first-order evolutionary equation,
$$\begin{gathered} \frac{d}{dt} U + \mathcal{A} U = F(U), \\ U(0) = U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, \theta_1, \eta_0, f_0)^T, \end{gathered} \tag{3.3}$$
where $U = (\varphi, u, \psi, v, \theta, \omega, \eta^t, z)$ and the linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined by
$$\mathcal{A}U = \begin{pmatrix} -u \\ -\frac{k}{\rho_1}(\varphi_x + \psi)_x + \frac{\beta}{\rho_1}\omega_x \\ -v \\ -\frac{b}{\rho_2}\psi_{xx} + \frac{k}{\rho_2}(\varphi_x + \psi) + \frac{g_0}{\rho_2}\psi_{xx} - \frac{1}{\rho_2}\int_0^\infty g(s)\eta_{xx}^t(x,t,s)ds - \frac{\beta}{\rho_2}\omega \\ -\omega \\ -\frac{\delta}{\rho_3}\theta_{xx} - \frac{\ell}{\rho_3}\omega_{xx} + \frac{\gamma}{\rho_3}u_x + \frac{\gamma}{\rho_3}v - \frac{1}{\rho_3}\int_{\tau_1}^{\tau_2} \mu(\varsigma) z_x(x,1,\varsigma,t)d\varsigma \\ \eta_s^t - v \\ \frac{1}{\varsigma} z_\rho \end{pmatrix}, \tag{3.4}$$
$$F(U) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{\rho_2} f(\psi) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.5}$$

Next, we introduce the energy space

$$\begin{aligned}\mathcal{H} &= H_0^1(0,1)\times L^2(0,1)\times H_0^1(0,1)\times L^2(0,1)\times H_0^1(0,1)\times L^2(0,1)\times L_g\\ &\quad\times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)),\end{aligned}$$

where

$$L_g=\big\{\phi:R_+\to H_0^1(0,1),\ \int_0^1\int_0^\infty g(s)\phi_x^2\,ds\,dx<\infty\big\},$$

endowed with the inner product

$$\langle\phi_1,\phi_2\rangle_{L_g}=\int_0^1\int_0^\infty g(s)\phi_{1x}(s)\phi_{2x}(s)\,ds\,dx.$$

For any $U=(\varphi,u,\psi,v,\theta,\omega,\eta^t,z)^T\in\mathcal{H}$, $\widetilde{U}=(\widetilde{\varphi},\widetilde{u},\widetilde{\psi},\widetilde{v},\widetilde{\theta},\widetilde{\omega},\widetilde{\eta}^t,\widetilde{z})^T\in\mathcal{H}$ and for $\ell-\int_{\tau_1}^{\tau_2}|\mu(\varsigma)|d\varsigma>0$ we equip $\mathcal{H}$ with the inner product defined by

$$\begin{aligned}\langle U,\widetilde{U}\rangle_{\mathcal{H}} &= \gamma\int_0^1(\rho_1u\widetilde{u}+\rho_2v\widetilde{v}+k(\varphi_x+\psi)(\widetilde{\varphi_x}+\widetilde{\psi})+b\psi_x\widetilde{\psi_x}-g_0\psi_x\widetilde{\psi_x})dx\\ &\quad+\gamma<\eta^t,\widetilde{\eta}^t>_{L_g}+\beta\int_0^1(\rho_3\omega\widetilde{\omega}+\delta\theta_x\widetilde{\theta}_x)dx\\ &\quad+\beta\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varsigma|\mu(\varsigma)|z\widetilde{z}(x,\rho,\varsigma,\cdot)\,d\varsigma\,d\rho\,dx.\end{aligned}$$

The domain of $\mathcal{A}$ is

$$\begin{aligned}D(\mathcal{A})=\Big\{&U\in\mathcal{H}:\varphi,\psi,\theta\in H^2(0,1)\cap H_0^1(0,1),u,v,\omega\in H_0^1(0,1),\\ &\eta^t\in L_g,z,z_\rho\in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2))\Big\},\end{aligned}$$

which is dense in $\mathcal{H}$.

Theorem 3.1. *Assume $U_0\in\mathcal{H}$ and* (H1)–(H4) *hold. Then, there exists a unique solution $U\in(R_+,\mathcal{H})$ of problem* (2.1)*. Moreover, if $U_0\in D(\mathcal{A})$ then*

$$U\in C(R_+,D(\mathcal{A}))\cap C^1(R_+,\mathcal{H}).$$

Proof. We use the semigroup approach. Sufficiently, we prove that $\mathcal{A}$ is a maximal monotone operator. First, we prove that $\mathcal{A}$ is monotone. For any $U\in D(\mathcal{A})$, we have

$$\begin{aligned}(\mathcal{A}U,U)_{\mathcal{H}} &= -\frac{\gamma}{2}(g'\circ\psi_x)+\ell\beta\int_0^1\omega_x^2dx-\beta\int_0^1\int_{\tau_1}^{\tau_2}\mu(\varsigma)z_x(x,1,\varsigma,\cdot)d\varsigma\omega\,dx\\ &\quad+\frac{\beta}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu(\varsigma)|z^2(x,1,\varsigma,\cdot)d\varsigma\,dx-\frac{\beta}{2}\int_{\tau_1}^{\tau_2}|\mu(\varsigma)|d\varsigma\int_0^1\omega_x^2dx.\end{aligned}$$

Using integration by parts and Young's inequality, we obtain

$$\begin{aligned}&-\beta\int_0^1\int_{\tau_1}^{\tau_2}\mu(\varsigma)z_x(x,1,\varsigma,\cdot)d\varsigma\omega\,dx\\ &=\beta\int_0^1\int_{\tau_1}^{\tau_2}\mu(\varsigma)z(x,1,\varsigma,\cdot)d\varsigma\omega_x\,dx\\ &\geq-\frac{\beta}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu(\varsigma)|z^2(x,1,\varsigma,\cdot)d\varsigma\,dx-\frac{\beta}{2}\int_{\tau_1}^{\tau_2}|\mu(\varsigma)|d\varsigma\int_0^1\omega_x^2dx.\end{aligned}$$

Consequently,

$$(\mathcal{A}U,U)_{\mathcal{H}} \ge -\frac{\gamma}{2}(g'\circ\psi_x) + \beta(\ell - \int_{\tau_1}^{\tau_2}|\mu(\varsigma)|d\varsigma)\int_0^1 \omega_x^2 dx \ge 0.$$

Thus, $\mathcal{A}$ is monotone. Next, we prove that the operator $I+\mathcal{A}$ is surjective. Given $\mathcal{F} = (k_1,k_2,k_3,k_4,k_5,k_6,k_7,k_8)^T \in \mathcal{H}$, we prove that there exists a unique $U \in D(\mathcal{A})$ such that

$$(I+\mathcal{A})U = \mathcal{F}. \tag{3.6}$$

That is,

$$\begin{gathered}
\varphi - u = k_1 \in H_0^1,\\
\rho_1 u - k(\varphi_x+\psi)_x + \beta\omega_x = \rho_1 k_2 \in L^2(0,1),\\
\psi - v = k_3 \in H_0^1,\\
\rho_2 v - b\psi_{xx} + k(\varphi_x+\psi) + \int_0^\infty g(s)ds\psi_{xx}\\
-\int_0^\infty g(s)\eta_{xx}^t(x,t,s)ds - \beta\omega = \rho_2 k_4 \in L^2(0,1),\\
\theta - \omega = k_5 \in H_0^1,\\
\rho_3\omega - \delta\theta_{xx} - \ell\omega_{xx} + \gamma u_x + \gamma v - \int_{\tau_1}^{\tau_2}\mu(\varsigma)z_x(x,1,\varsigma,\cdot)d\varsigma = \rho_3 k_6 \in L^2(0,1),\\
\eta^t + \eta_s^t - v = k_7 \in L_g,\\
\varsigma z + z_\rho = \varsigma k_8 \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)).
\end{gathered} \tag{3.7}$$

Using lines 7 and 8 in the above equation, we obtain

$$\eta^t = e^{-s}\int_0^s e^\tau(v + k_7(\tau))d\tau, \tag{3.8}$$

$$z(x,\rho,\varsigma,\cdot) = e^{-\varsigma\rho}\omega_x + \varsigma e^{-\varsigma\rho}\int_0^\rho e^{\varsigma\tau}k_8(x,\tau,\varsigma)d\tau. \tag{3.9}$$

Inserting $u = \varphi - k_1$, $v = \psi - k_3$, $\omega = \theta - k_5$, (3.8) and (3.9) in $(3.7)_2$, $(3.7)_4$ and $(3.7)_6$, we obtain

$$\begin{gathered}
\rho_1\varphi - k(\varphi_x+\psi)_x + \beta\theta_x = h_1 \in L^2(0,1),\\
\rho_2\psi - b\psi_{xx} + k(\varphi_x+\psi) + \int_0^\infty g(s)ds\psi_{xx}\\
-\int_0^\infty g(s)e^{-s}\int_0^s \psi_{xx}e^\tau d\tau ds - \beta\theta = h_2 \in L^2(0,1),\\
\rho_3\theta - \delta\theta_{xx} - \ell\theta_{xx} + \gamma\varphi_x + \gamma\psi - \int_{\tau_1}^{\tau_2}\mu(\varsigma)e^{-\varsigma\rho}\theta_{xx}d\varsigma = h_3 \in L^2(0,1),
\end{gathered} \tag{3.10}$$

where

$$\begin{gathered}
h_1 = k_2\rho_1 + k_1\rho_1 + \beta k_{5x},\\
h_2 = \rho_2 k_4 + \rho_2 k_3 - \beta k_5 + \int_0^\infty g(s)e^{-s}\int_0^s (k_7 - k_3)_{xx}e^\tau d\tau ds,
\end{gathered}$$

$$
\begin{aligned}
h_3 = \rho_3 k_5 - \ell k_{5xx} + \gamma k_{1x} + \gamma k_3 + \int_{\tau_1}^{\tau_2} \mu(\varsigma)\varsigma e^{-\varsigma\rho} \int_0^\rho e^{\varsigma\tau} k_{8x}(x,\tau,\varsigma) d\tau d\varsigma \\
+ k_6\rho_3 - \int_{\tau_1}^{\tau_2} \mu(\varsigma) e^{-\varsigma\rho} k_{5xx} d\varsigma.
\end{aligned}
$$

To solve (3.10), we consider the following variational formulation

$$
B((\varphi,\psi,\theta),\ (\varphi_1,\psi_1,\theta_1)) = G(\varphi_1,\psi_1,\theta_1), \tag{3.11}
$$

where $B : [H_0^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1)]^2 \to R$ is the bilinear form defined by

$$
\begin{aligned}
& B((\varphi,\psi,\theta),(\varphi_1,\psi_1,\theta_1)) \\
&= \gamma\rho_1 \int_0^1 \varphi\varphi_1 dx + k\gamma \int_0^1 (\varphi_x+\psi)(\varphi_{1x}+\psi_1) dx + \gamma\beta \int_0^1 \theta_x \varphi_1 dx \\
&\quad + \gamma\rho_2 \int_0^1 \psi\psi_1 dx + b\gamma \int_0^1 \psi_x \psi_{1x} dx - \gamma \int_0^\infty g(s) ds \int_0^1 \psi_x \psi_{1x} dx \\
&\quad + \gamma \int_0^1 \int_0^\infty g(s) e^{-s} \int_0^s \psi_x e^\tau d\tau ds \psi_{1x} dx - \beta\gamma \int_0^1 \theta\psi_1\, dx \\
&\quad + \rho_3\beta \int_0^1 \theta\theta_1 dx + \beta\delta \int_0^1 \theta_x\theta_{1x} dx + \ell\beta \int_0^1 \theta_x \theta_{1x} dx + \gamma\beta \int_0^1 \varphi_x \theta_1 dx \\
&\quad + \gamma\beta \int_0^1 \psi\theta_1 dx + \beta \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\varsigma\rho} \mu(\varsigma) \theta_x d\varsigma \theta_{1x} dx,
\end{aligned}
$$

and $G : [H_0^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1)]^2 \to R$ is the linear functional

$$
G[(\varphi_1,\psi_1,\theta_1)] = \gamma \int_0^1 h_1\varphi_1 dx + \gamma \int_0^1 h_2 \psi_1 dx + \beta \int_0^1 h_3\theta_1 dx.
$$

Now, for $V = H_0^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1)$ equipped with the norm

$$
\|\varphi,\psi,\theta\|_V^2 = \|\varphi_x+\psi\|_2^2 + \|\varphi\|_2^2 + \|\psi_x\|_2^2 + \|\theta\|_2^2 + \|\theta_x\|_2^2,
$$

using integration by parts we have

$$
\begin{aligned}
& B((\varphi,\psi,\theta),(\varphi,\psi,\theta)) \\
&= \gamma\rho_1 \int_0^1 \varphi^2 dx + k\gamma \int_0^1 (\varphi_x+\psi)^2 dx + \gamma\rho_2 \int_0^1 \psi^2 dx \\
&\quad + (b - \int_0^\infty g(s) ds)\gamma \int_0^1 \psi_x^2 dx + \gamma \int_0^1 \psi_x^2 dx \int_0^\infty g(s) \int_0^s e^\tau d\tau e^{-s} ds \\
&\quad + \rho_3\beta \int_0^1 \theta^2 dx + \beta\delta \int_0^1 \theta_x^2 dx + \ell\beta \int_0^1 \theta_x^2 dx + \beta \int_0^1 \theta_x^2 dx \int_{\tau_1}^{\tau_2} e^{-\varsigma\rho}\mu(\varsigma) d\varsigma, \\
&\geq \alpha_0 \|\varphi,\psi,\theta\|_V^2,
\end{aligned}
$$

for some $\alpha_0 > 0$. Thus, B is coercive.

On the other hand, using Hölder and Poincaré inequalities, we obtain

$$
|B((\varphi,\psi,\theta),(\varphi_1,\psi_1,\theta_1))| \leq c\|\varphi,\psi,\theta\|_V \|\varphi_1,\psi_1,\theta_1\|_V.
$$

Similarly

$$
|G(\varphi_1,\psi_1,\theta_1)| \leq c\|\varphi_1,\psi_1,\theta_1\|_V.
$$

Consequently, by the Lax-Milgram Lemma, system (3.10) has a unique solution

$$
(\varphi,\psi,\theta) \in H_0^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1)
$$

satisfying

$$B((\varphi,\psi,\theta),(\varphi_1,\psi_1,\theta_1)) = G(\varphi_1,\psi_1,\theta_1), \quad (\varphi_1,\psi_1,\theta_1) \in V.$$

The substitution of φ, ψ and θ into $(3.7)_1$, $(3.7)_3$ and $(3.7)_5$ yields

$$(u,v,\omega) \in H_0^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1).$$

Similarly, inserting v in (3.8) and bearing in mind $(3.7)_7$, we obtain $\eta^t \in L_g$. At the same time, inserting ω in (3.9) and using $(3.7)_8$, we obtain

$$z, z_\rho \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)).$$

Moreover, if we take $(\varphi_1,\theta_1) \equiv (0,0) \in H_0^1(0,1)\times H_0^1(0,1)$ in (3.11), we obtain

$$\begin{aligned}
&k\int_0^1 (\varphi_x+\psi)\psi_1 dx + \rho_2\int_0^1 \psi\psi_1 dx + b\int_0^1 \psi_x\psi_{1x} dx \\
&- \int_0^\infty g(s)ds\int_0^1 \psi_x\psi_{1x}dx - \beta\int_0^1 \theta\psi_1 dx + \int_0^\infty g(s)(1-e^{-s})ds\int_0^1 \psi_x\psi_{1x}dx \\
&= \int_0^1 h_2\psi_1 dx.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&b\int_0^1 \psi_x\psi_{1x}dx - \int_0^\infty g(s)ds\int_0^1 \psi_x\psi_{1x}dx + \int_0^\infty g(s)(1-e^{-s})ds\int_0^1 \psi_x\psi_{1x}dx \\
&= \int_0^1 [-k(\varphi_x+\psi) - \rho_2\psi + \beta\theta + h_2]\psi_1 dx, \quad \psi_1 \in H_0^1(0,1).
\end{aligned}$$

By noting that $-k(\varphi_x+\psi) - \rho_2\psi + \beta\theta + h_2 \in L^2(0,1)$, we obtain $\psi \in H^2(0,1)\cap H_0^1(0,1)$. Consequently using integration by parts we have

$$\begin{aligned}
&\int_0^1 [-b\psi_{xx} + \int_0^\infty g(s)ds\psi_{xx}dx - \int_0^\infty g(s)(1-e^{-s})ds\psi_{xx} \\
&+ k(\varphi_x+\psi) + \rho_2\psi - \beta\theta - h_2]\psi_1 dx = 0, \quad \psi_1 \in H_0^1(0,1).
\end{aligned}$$

Therefore,

$$-b\psi_{xx} + \int_0^\infty g(s)ds\psi_{xx}dx - \int_0^\infty g(s)(1-e^{-s})ds\psi_{xx} + k(\varphi_x+\psi) + \rho_2\psi - \beta\theta = h_2.$$

This gives $(3.10)_2$. Similarly, if we take $(\varphi_1,\psi_1) \equiv (0,0) \in H_0^1(0,1)\times H_0^1(0,1)$ in (3.11), we can show that

$$\theta \in H^2(0,1)\cap H_0^1(0,1),$$

and $(3.10)_3$ are satisfied.

If we take $(\psi_1,\theta_1) \equiv (0,0) \in H_0^1(0,1)\times H_0^1(0,1)$ in (3.11), we can show that

$$\varphi \in H^2(0,1)\cap H_0^1(0,1),$$

and $(3.10)_1$ are satisfied.

Finally, from (3.8) we can get $\eta^t \in L_g$. From (3.9), we know $z, z_\rho \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2))$. Hence, there exists a unique $U \in D(\mathcal{A})$ such that (3.6) is satisfied. Therefore, $\mathcal{A}$ is a maximal monotone operator.

Now, we prove that the operator F defined in (3.3) is locally Lipschitz in $\mathcal{H}$. Let $U = (\varphi,u,\psi,v,\theta,\omega,\eta^t,z)^T$ and $U_1 = (\varphi_1,u_1,\psi_1,v_1,\theta_1,\omega_1,\eta_1^t,z_1)^T$, then we have

$$\|F(U) - F(U_1)\|_{\mathcal{H}} \le \|f(\psi) - f(\psi_1)\|_{L^2}.$$

By using (2.5), Hölder and Poincaré inequalities, we can get

$$\|f(\psi^2) - f(\psi^1)\|_{L^2} \le c(\|\psi^1\|_{2\varrho}^{\varrho} + \|\psi^2\|_{2\varrho}^{\varrho})\|\psi^1 - \psi^2\| \le c\|\psi_x^1 - \psi_x^2\|,$$

which gives us

$$\|F(U) - F(U_1)\|_{\mathcal{H}} \le c\|U - U_1\|_{\mathcal{H}}.$$

Then the operator F is locally Lipschitz in $\mathcal{H}$. The proof complete. □

4. Proof of stability results

In this section, we state and prove our stability results for the energy of system (2.1) by using the multiplier technique. To achieve our goal, we need the following lemmas.

Lemma 4.1. *Let* $(\varphi,\ \psi,\ \theta)$ *be the solution of* (2.1)*, then we have*

$$E'(t) \le \frac{\gamma}{2}(g' \circ \psi_x)(t) - \beta\Big(\ell - \int_{\tau_1}^{\tau_2} |\mu(\varsigma)|d\varsigma\Big) \int_0^1 \theta_{tx}^2 dx. \tag{4.1}$$

Proof. Multiplying $(2.1)_1$ by $\gamma\varphi_t$, $(2.1)_2$ by $\gamma\psi_t$, $(2.1)_3$ by $\beta\theta_t$, integrating over $(0,1)$ with respect to x, multiplying equation $(2.1)_4$ by $\beta|\mu(\varsigma)|z$ and integrating over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$ with respect to ρ, x and ς, summing them up, we obtain

$$\begin{aligned} E'(t) &= \frac{\gamma}{2}(g' \circ \psi_x) - \ell\beta \int_0^1 \theta_{tx}^2 dx + \beta \int_0^1 \int_{\tau_1}^{\tau_2} \mu(\varsigma) z_x(x,1,\varsigma,t) d\varsigma \theta_t(x,t) dx \\ &\quad - \frac{\beta}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(\varsigma)| z^2(x,1,\varsigma,t) d\varsigma\, dx + \frac{\beta}{2} \int_{\tau_1}^{\tau_2} |\mu(\varsigma)| d\varsigma \int_0^1 \theta_{tx}^2(x,t) dx. \end{aligned} \tag{4.2}$$

At the same time, using integration by parts and Young's inequality, we have

$$\begin{aligned} &\beta \int_0^1 \int_{\tau_1}^{\tau_2} \mu(\varsigma) z_x(x,1,\varsigma,t) d\varsigma \theta_t(x,t) dx \\ &= -\beta \int_0^1 \int_{\tau_1}^{\tau_2} \mu(\varsigma) z(x,1,\varsigma,t) d\varsigma \theta_{tx}(x,t) dx \\ &\le \frac{\beta}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(\varsigma)| z^2(x,1,\varsigma,t) d\varsigma\, dx + \frac{\beta}{2} \int_{\tau_1}^{\tau_2} |\mu(\varsigma)| d\varsigma \int_0^1 \theta_{tx}^2(x,t) dx. \end{aligned} \tag{4.3}$$

A combination of (4.2) and (4.3) gives

$$E'(t) \le \frac{\gamma}{2}(g' \circ \psi_x) - \beta\Big(\ell - \int_{\tau_1}^{\tau_2} |\mu(\varsigma)|d\varsigma\Big) \int_0^1 \theta_{tx} dx.$$

Thus (4.1) follows. □

We note here that if $E(t) = E(t,\varphi,\psi,\theta,z) = E_1(t)$, denotes the energy defined in (2.7) then

$$E_2(t) = E(t,\varphi_t,\psi_t,\theta_t,z_t),$$

denotes the second order energy and one can easily obtain that

$$E_2'(t) \le \frac{\gamma}{2}(g' \circ \psi_{xt}) - c\int_0^1 \theta_{ttx}^2 dx,$$

in which c is some positive constant.

It is easy to obtain the following inequalities, we omit their proofs.

Lemma 4.2. *The following inequalities hold,*

$$\int_0^1 \Big(\int_0^\infty g(s)\psi_x(t-s)ds\Big)^2 dx \le 2g_0(g\circ\psi_x)(t) + 2g_0\int_0^1 \psi_x^2 dx, \tag{4.4}$$

$$\int_0^1 \Big(\int_0^\infty g(s)(\psi_x(t)-\psi_x(t-s))ds\Big)^2 dx \le g_0(g\circ\psi_x)(t), \tag{4.5}$$

$$\int_0^1 \Big(\int_0^\infty g(s)(\psi(t)-\psi(t-s))ds\Big)^2 dx \le d_1(g\circ\psi_x)(t), \tag{4.6}$$

$$\int_0^1 \Big(\int_0^\infty g'(s)(\psi(t)-\psi(t-s))ds\Big)^2 dx \le -d_2(g\circ\psi_x)(t), \tag{4.7}$$

$$\int_0^1 \Big(\int_0^\infty g'(s)(\psi_x(t)-\psi_x(t-s))ds\Big)^2 dx \le -g(0)(g'\circ\psi_x)(t), \tag{4.8}$$

in which d_1 and d_2 are positive constants.

Lemma 4.3. *Let (φ,ψ,θ) be the solution of (2.1). Then for any positive constant ε_1, the functional*

$$J_1(t) := -\rho_1\int_0^1 \varphi_t\varphi\,dx - \rho_2\int_0^1 \psi_t\psi\,dx$$

satisfies

$$\begin{aligned} J_1'(t) &\le -\rho_1\int_0^1 \varphi_t^2dx + (k+\beta\varepsilon_1)\int_0^1(\varphi_x+\psi)^2dx + \frac{\beta}{\varepsilon_1}\int_0^1\theta_t^2dx \\ &\quad - \rho_2\int_0^1\psi_t^2dx + c(1+\varepsilon_1)\int_0^1\psi_x^2dx + c(g\circ\psi_x)(t). \end{aligned} \tag{4.9}$$

Proof. By computations, using (2.1), we obtain

$$\begin{aligned} J_1'(t) &= -\rho_1\int_0^1\varphi_t^2dx + k\int_0^1(\varphi_x+\psi)^2dx - \beta\int_0^1\theta_t\varphi_x\,dx - \rho_2\int_0^1\psi_t^2dx \\ &\quad + b\int_0^1\psi_x^2dx - \beta\int_0^1\theta_t\psi\,dx - \int_0^1\psi_x\int_0^\infty g(s)\psi_x(x,t-s)\,ds\,dx \\ &\quad + \int_0^1 f(\psi)\psi\,dx. \end{aligned}$$

By using Young's inequality and Poincaré inequality, we obtain for $\varepsilon_1>0$,

$$\begin{aligned} J_1'(t) &\le -\rho_1\int_0^1\varphi_t^2dx + k\int_0^1(\varphi_x+\psi)^2dx + \frac{\beta\varepsilon_1}{2}\int_0^1\varphi_x^2dx - \rho_2\int_0^1\psi_t^2dx \\ &\quad + \frac{\beta}{\varepsilon_1}\int_0^1\theta_t^2dx + (2b+\frac{\beta\varepsilon_1}{2})\int_0^1\psi_x^2dx \\ &\quad + \frac{1}{4b}\int_0^1\Big(\int_0^\infty g(s)\psi_x(x,t-s)ds\Big)^2dx + \int_0^1 f(\psi)\psi\,dx. \end{aligned} \tag{4.10}$$

By using the Cauchy-Schwarz inequality and Poincaré inequality, we have

$$\int_0^1\varphi_x^2dx \le 2\int_0^1(\varphi_x+\psi)^2dx + 2\int_0^1\psi^2dx \le 2\int_0^1(\varphi_x+\psi)^2dx + 2\int_0^1\psi_x^2dx, \tag{4.11}$$

and

$$\int_0^1 |f(\psi)\psi|dx \le \int_0^1 |\psi|^{\varrho}|\psi\|\psi| \le \|\psi\|_{2(\varrho+1)}^{\varrho}\|\psi\|_{2(\varrho+1)}\|\psi\|dx \le c\int_0^1 \psi_x^2 dx. \quad (4.12)$$

The substitution of (4.4), (4.11) and (4.12) into (4.10) gives (4.9). □

Lemma 4.4. *Let (φ,ψ,θ) be the solution of* (2.1)*. Then for any positive constant ε_2, the functional*

$$J_2(t) := \rho_3\int_0^1 \theta_t\theta\,dx + \frac{\ell}{2}\int_0^1 \theta_x^2 dx + \gamma\int_0^1 \varphi_x\theta\,dx$$

satisfies

$$\begin{aligned} J_2'(t) \le &-\frac{\delta}{2}\int_0^1 \theta_x^2 dx + \Big(\rho_3 + \frac{\gamma^2}{2\varepsilon_2}\Big)\int_0^1 \theta_t^2 dx + \varepsilon_2\int_0^1 \psi_x^2 dx \\ &+ \frac{\gamma^2}{\delta}\int_0^1 \psi_t^2 dx + \varepsilon_2\int_0^1 (\varphi_x+\psi)^2 dx \\ &+ \frac{c}{\delta}\int_0^1\int_{\tau_1}^{\tau_2} z^2(x,1,\varsigma,t)d\varsigma\,dx. \end{aligned} \quad (4.13)$$

Proof. By differentiating $J_2(t)$ and using (2.1), we obtain

$$\begin{aligned} J_2'(t) = &\rho_3\int_0^1 \theta_t^2 dx + \gamma\int_0^1 \varphi_x\theta_t dx - \delta\int_0^1 \theta_x^2 dx - \gamma\int_0^1 \psi_t\theta\,dx \\ &- \int_0^1\int_{\tau_1}^{\tau_2} \mu(\varsigma)z(x,1,\varsigma,t)d\varsigma\theta_x\,dx. \end{aligned}$$

By using Young's and Poincaré inequalities, we obtain for any $\varepsilon_2 > 0$

$$\gamma\int_0^1 \varphi_x\theta_t dx \le \varepsilon_2\int_0^1 (\varphi_x+\psi)^2 dx + \varepsilon_2\int_0^1 \psi_x^2 dx + \frac{\gamma^2}{2\varepsilon_2}\int_0^1 \theta_t^2 dx, \quad (4.14)$$

$$\gamma\int_0^1 \psi_t\theta\,dx \le \frac{\delta}{4}\int_0^1 \theta_x^2 dx + \frac{\gamma^2}{\delta}\int_0^1 \psi_t^2 dx. \quad (4.15)$$

By using (H1), we have

$$\begin{aligned} &\int_0^1\int_{\tau_1}^{\tau_2} \mu(\varsigma)z(x,1,\varsigma,t)d\varsigma\theta_x\,dx \\ &\le \frac{\delta}{4}\int_0^1 \theta_x^2 dx + \frac{1}{\delta}\int_0^1 \Big(\int_{\tau_1}^{\tau_2} \mu(\varsigma)z(x,1,\varsigma,t)d\varsigma\Big)^2 dx \\ &\le \frac{\delta}{4}\int_0^1 \theta_x^2 dx + \frac{c}{\delta}\int_0^1\int_{\tau_1}^{\tau_2} z^2(x,1,\varsigma,t)d\varsigma\,dx. \end{aligned} \quad (4.16)$$

Thus, (4.13) is established. □

Lemma 4.5. *Let $(\varphi,\ \psi,\ \theta)$ be the solution of* (2.1)*. Then for any positive constant ε_3 the functional*

$$J_3(t) := -\rho_2\int_0^1 \psi_t\int_0^\infty g(s)(\psi(t)-\psi(t-s))\,ds\,dx$$

satisfies

$$
\begin{aligned}
J_3'(t) &\le \varepsilon_3 c\int_0^1 \psi_x^2 dx - (\rho_2 g_0 - \varepsilon_3\rho_2)\int_0^1 \psi_t^2 dx + c(\varepsilon_3 + \frac{1}{\varepsilon_3})(g\circ\psi_x)(t) \\
&\quad + \varepsilon_3 k\int_0^1 (\varphi_x+\psi)^2 dx + \varepsilon_3\beta^2\int_0^1 \theta_t^2 dx - \frac{c}{4\varepsilon_3}(g'\circ\psi_x)(t).
\end{aligned}
\tag{4.17}
$$

Proof. First, we note that

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Big(\int_0^\infty g(s)(\psi(t)-\psi(t-s))ds\Big) \\
&= \frac{\partial}{\partial t}\Big(\int_{-\infty}^t g(t-s)(\psi(t)-\psi(s))ds\Big) \\
&= \int_{-\infty}^t g'(t-s)(\psi(t)-\psi(s))ds + \int_{-\infty}^t g(t-s)\psi_t(t)ds \\
&= g_0\psi_t(t) + \int_0^\infty g'(s)(\psi(t)-\psi(t-s))ds.
\end{aligned}
$$

Then, by differentiating $J_3(t)$ and using (2.1), we find

$$
\begin{aligned}
J_3'(t) &= b\int_0^1 \psi_x \int_0^\infty g(s)(\psi_x(t)-\psi_x(t-s))\,ds\,dx - g_0\rho_2\int_0^1 \psi_t^2 dx \\
&\quad - \rho_2\int_0^1 \psi_t \int_0^\infty g'(s)(\psi(t)-\psi(t-s))\,ds\,dx \\
&\quad + k\int_0^1 (\varphi_x+\psi)\int_0^\infty g(s)(\psi(t)-\psi(t-s))\,ds\,dx \\
&\quad - \beta\int_0^1 \theta_t \int_0^\infty g(s)(\psi(t)-\psi(t-s))\,ds\,dx \\
&\quad + \int_0^1 f(\psi)\int_0^\infty g(s)(\psi(t)-\psi(t-s))\,ds\,dx \\
&\quad - \int_0^1 \int_0^\infty g(s)\psi_x(t-s)ds \int_0^\infty g(s)(\psi_x(t)-\psi_x(t-s))\,ds\,dx.
\end{aligned}
\tag{4.18}
$$

By using Young's and Poincaré inequalities,

$$
\begin{aligned}
&b\int_0^1 \psi_x \int_0^\infty g(s)(\psi_x(t)-\psi_x(t-s))\,ds\,dx \\
&\le \varepsilon_3 b\int_0^1 \psi_x^2 dx + \frac{b}{4\varepsilon_3}\int_0^1 \Big(\int_0^\infty g(s)(\psi_x(t)-\psi_x(t-s))ds\Big)^2 dx \\
&\le \varepsilon_3 b\int_0^1 \psi_x^2 dx + \frac{bg_0}{4\varepsilon_3}(g\circ\psi_x)(t),
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
&-\rho_2\int_0^1 \psi_t \int_0^\infty g'(s)(\psi(t)-\psi(t-s))\,ds\,dx \\
&\le \varepsilon_3\int_0^1 \rho_2\psi_t^2 dx + \frac{\rho_2}{4\varepsilon_3}\int_0^1 \Big(\int_0^\infty g'(s)(\psi(t)-\psi(t-s))ds\Big)^2 dx \\
&\le \varepsilon_3\int_0^1 \rho_2\psi_t^2 dx - \frac{\rho_2 d_2}{4\varepsilon_3}(g'\circ\psi_x)(t),
\end{aligned}
\tag{4.20}
$$

$$
\begin{aligned}
&k\int_0^1(\varphi_x+\psi)\int_0^\infty g(s)(\psi(t)-\psi(t-s))\,ds\,dx\\
&\le \varepsilon_3 k\int_0^1(\varphi_x+\psi)^2dx+\frac{k}{4\varepsilon_3}\int_0^1\Big(\int_0^\infty g(s)(\psi(t)-\psi(t-s))ds\Big)^2dx \qquad (4.21)\\
&\le \varepsilon_3 k\int_0^1(\varphi_x+\psi)^2dx+\frac{kd_1}{4\varepsilon_3}(g\circ\psi_x)(t),
\end{aligned}
$$

$$
\begin{aligned}
&\beta\int_0^1\theta_t\int_0^\infty g(s)(\psi(t)-\psi(t-s))\,ds\,dx\\
&\le \varepsilon_3\beta^2\int_0^1\theta_t^2dx+\frac{1}{4\varepsilon_3}\int_0^1\Big(\int_0^\infty g(s)(\psi(t)-\psi(t-s))ds\Big)^2dx \qquad (4.22)\\
&\le \varepsilon_3\beta^2\int_0^1\theta_t^2dx+\frac{d_1}{4\varepsilon_3}(g\circ\psi_x)(t),
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 f(\psi)\int_0^\infty g(s)(\psi(t)-\psi(t-s))\,ds\,dx\\
&\le k_0\int_0^1|\psi|^\varrho|\psi|\int_0^\infty g(s)(\psi(t)-\psi(t-s))\,ds\,dx\\
&\le k_0\|\psi\|^\varrho_{2(\varrho+1)}\|\psi\|_{2(\varrho+1)}\Big(\int_0^1(\int_0^\infty g(s)(\psi(t)-\psi(t-s))ds)^2dx\Big)^{1/2} \qquad (4.23)\\
&\le \varepsilon_3 c\int_0^1\psi_x^2dx+\frac{d_1}{4\varepsilon_3}(g\circ\psi_x)(t),
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1\int_0^\infty g(s)\psi_x(t-s)ds\int_0^\infty g(s)(\psi_x(t)-\psi_x(t-s))\,ds\,dx\\
&\le \varepsilon_3\int_0^1\Big(\int_0^\infty g(s)\psi_x(t-s)ds\Big)^2dx\\
&\quad+\frac{1}{4\varepsilon_3}\int_0^1\Big(\int_0^\infty g(s)(\psi_x(t)-\psi_x(t-s))ds\Big)^2dx \qquad (4.24)\\
&\le \Big(2\varepsilon_3+\frac{1}{4\varepsilon_3}\Big)g_0(g\circ\psi_x)+2\varepsilon_2 g_0\int_0^1\psi_x^2dx.
\end{aligned}
$$

By substituting (4.19)-(4.24) into (4.18), we obtain (4.17). □

As in [9], we introduce the multiplier ω which is the solution of

$$
-w_{xx}=\psi_x,\quad w(0)=w(1)=0. \qquad (4.25)
$$

Lemma 4.6. *The solution of* (4.25) *satisfies*

$$
\begin{gathered}
\int_0^1 w_x^2dx\le\int_0^1\psi^2dx\le\int_0^1\psi_x^2dx,\\
\int_0^1 w_t^2dx\le\int_0^1 w_{xt}^2dx\le\int_0^1\psi_t^2dx.
\end{gathered}
$$

Lemma 4.7. *Let* (φ,ψ,θ) *be the solution of* (2.1). *Then for any positive constant* ε_4 *the functional*

$$
J_4(t):=\int_0^1(\rho_1\varphi_t w+\rho_2\psi_t\psi)dx,
$$

satisfies

$$\begin{aligned} J_4'(t) \le & -\frac{l}{2}\int_0^1 \psi_x^2(t)dx + \frac{3\beta^2}{l}\int_0^1 \theta_t^2(t)dx + \varepsilon_4 \int_0^1 \varphi_t^2 dx \\ & + (\rho_2 + \frac{\rho_1}{4\varepsilon_4})\int_0^1 \psi_t^2 dx + \frac{3(b-l)}{2l}(g\circ\psi_x)(t) - \int_0^1 \widehat{f}(\psi)dx. \end{aligned} \tag{4.26}$$

Proof. A simple differentiation of $J_4(t)$ and together with (2.1) give

$$\begin{aligned} J_4'(t) = & \beta\int_0^1 \theta_t w_x\,dx + k\int_0^1 w_x^2 dx + \rho_1\int_0^1 \varphi_t w_t dx + \beta\int_0^1 \theta_t\psi\,dx \\ & - b\int_0^1 \psi_x^2 dx - k\int_0^1 \psi^2 dx + \rho_2\int_0^1 \psi_t^2 dx \\ & + \int_0^1\int_0^\infty g(s)\psi_x(x,t-s)ds\psi_x(t)dx - \int_0^1 f(\psi)\psi\,dx, \end{aligned} \tag{4.27}$$

where we have used integration by parts, (4.25) and the boundary conditions in (2.1). By using Young's, Poincaré inequalities, Lemma 4.2 and Lemma 4.6, we have

$$\beta\int_0^1 \theta_t\omega_x\,dx \le \frac{l}{6}\int_0^1 \omega_x^2\,dx + \frac{3\beta^2}{2l}\int_0^1 \theta_t^2 dx \le \frac{l}{6}\int_0^1 \psi_x^2\,dx + \frac{3\beta^2}{2l}\int_0^1 \theta_t^2 dx,$$
$$\rho_1\int_0^1 \varphi_t\omega_t dx \le \varepsilon_4\int_0^1 \varphi_t^2 dx + \frac{\rho_1^2}{4\varepsilon_4}\int_0^1 \omega_t^2 dx \le \varepsilon_4\int_0^1 \varphi_t^2 dx + \frac{\rho_1^2}{4\varepsilon_4}\int_0^1 \psi_t^2 dx,$$
$$\beta\int_0^1 \theta_t\psi\,dx \le \frac{l}{6}\int_0^1 \psi_x^2\,dx + \frac{3\beta^2}{2l}\int_0^1 \theta_t^2 dx,$$

$$\begin{aligned} & \int_0^1\int_0^\infty g(s)\psi_x(x,t-s)ds\psi_x(t)dx \\ & = \int_0^1\int_0^\infty g(s)(\psi_x(x,t-s) - \psi_x(t) + \psi_x(t))ds\psi_x(t)dx \\ & = \int_0^1\int_0^\infty g(s)(\psi_x(x,t-s) - \psi_x(t))ds\psi_x(t)dx + \int_0^\infty g(s)ds\int_0^1 \psi_x^2(t)dx \\ & \le \frac{l}{6}\int_0^1 \psi_x^2(t)dx + \frac{3}{2l}\int_0^1 \Big(\int_0^\infty g(s)(\psi_x(x,t-s) - \psi_x(t))ds\Big)^2 dx \\ & \quad + \int_0^\infty g(s)ds\int_0^1 \psi_x^2(t)dx \\ & \le \frac{l}{6}\int_0^1 \psi_x^2(t)dx + \frac{3(b-l)}{2l}(g\circ\psi_x)(t) + \int_0^\infty g(s)ds\int_0^1 \psi_x^2(t)dx. \end{aligned}$$

Thus, (4.26) is established. □

Lemma 4.8. *Let* (φ,ψ,θ) *be the solution of* (2.1)*. Then for any positive constant* ε_5 *the functional*

$$J_5(t) := \rho_2\int_0^1 \psi_t(\varphi_x + \psi)dx + \frac{b\rho_1}{k}\int_0^1 \varphi_t\psi_x\,dx - \frac{\rho_1}{k}\int_0^1 \varphi_t\int_0^\infty g(s)\psi_x(t-s)\,ds\,dx$$

satisfies

$$
\begin{aligned}
J_5'(t) \le & \Big[\varphi_x(b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds)\Big]_{x=0}^{x=1} + \rho_2 \int_0^1 \psi_t^2 dx \\
& - \frac{k}{2}\int_0^1 (\varphi_x+\psi)^2 dx + c(1+\frac{1}{\varepsilon_5})\int_0^1 \theta_{tx}^2 dx \\
& + c\big(1+\frac{1}{\varepsilon_5}\big)\int_0^1 \psi_x^2 dx + c\varepsilon_5\big(g\circ\psi_x\big)(t) - \frac{c}{\varepsilon_5}(g'\circ\psi_x)(t) \\
& + \varepsilon_5\int_0^1 \varphi_t^2 dx - \int_0^1 \widehat{f}(\psi)dx + \big(\frac{b\rho_1}{k}-\rho_2\big)\int_0^1 \varphi_t\psi_{xt}dx.
\end{aligned}
\tag{4.28}
$$

Proof. First, we note that

$$
\begin{aligned}
& \frac{d}{dt}\Big(\int_0^\infty g(s)\psi_x(t-s)ds\Big) \\
& = \frac{d}{dt}\Big(\int_{-\infty}^t g(t-s)\psi_x(s)ds\Big) \\
& = g(0)\psi_x(t) + \int_{-\infty}^t g'(t-s)\psi_x(s)ds \\
& = g(0)\psi_x(t) + \int_0^\infty g'(s)\big(\psi_x(t-s)-\psi_x(t)+\psi_x(t)\big)ds \\
& = g(0)\psi_x(t) + \int_0^\infty g'(s)\big(\psi_x(t-s)-\psi_x(t)\big)ds + \int_0^\infty g'(s)ds\psi_x(t) \\
& = \int_0^\infty g'(s)\left(\psi_x(t-s)-\psi_x(t)\right)ds.
\end{aligned}
$$

By using equation (2.1) and integration by parts, we obtain

$$
\begin{aligned}
J_5'(t) = & \Big[\varphi_x(b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds)\Big]_{x=0}^{x=1} + \rho_2\int_0^1 \psi_t^2 dx - k\int_0^1(\varphi_x+\psi)^2 dx \\
& + \beta\int_0^1 \theta_t(\varphi_x+\psi)dx + \frac{\beta}{k}\int_0^1 \theta_{tx}\int_0^\infty g(s)\left(\psi_x(t-s)-\psi_x(t)\right)ds\,dx \\
& + \frac{\rho_1}{k}\int_0^1 \varphi_t\int_0^\infty g'(s)\left(\psi_x(t)-\psi_x(t-s)\right)ds\,dx + \big(\frac{b\rho_1}{k}-\rho_2\big)\int_0^1\varphi_t\psi_{xt}dx \\
& + \frac{\beta}{k}\int_0^\infty g(s)ds\int_0^1 \theta_{xt}\psi_x\,dx - \frac{b\beta}{k}\int_0^1 \theta_{xt}\psi_x\,dx \\
& - \int_0^1 f(\psi)\varphi_x\,dx - \int_0^1 \psi f(\psi)\,dx.
\end{aligned}
$$

By using Young's, Poincaré inequalities, Lemma 4.2, we know that for any $\varepsilon_5 > 0$,

$$
\beta\int_0^1 \theta_t(\varphi_x+\psi)dx \le \frac{k}{4}\int_0^1(\varphi_x+\psi)^2 dx + \frac{\beta^2}{k}\int_0^1 \theta_{tx}^2 dx, \tag{4.29}
$$

$$
\begin{aligned}
& \frac{\beta}{k}\int_0^1 \theta_{tx}\int_0^\infty g(s)(\psi_x(t-s)-\psi_x(t)ds)^2 dx \\
& \le c\varepsilon_5(g\circ\psi_x)(t) + \frac{1}{4\varepsilon_5}\int_0^1 \theta_{tx}^2 dx,
\end{aligned}
\tag{4.30}
$$

$$
\begin{aligned}
&\frac{\rho_1}{k}\int_0^1 \varphi_t \int_0^\infty g'(s)\left(\psi_x(t)-\psi_x(t-s)\right)\,ds\,dx \\
&\le -\frac{c}{\varepsilon_5}(g'\circ\psi_x)(t)+\varepsilon_5\int_0^1 \varphi_t^2 dx,
\end{aligned} \tag{4.31}
$$

$$
\frac{\beta}{k}\int_0^\infty g(s)ds\int_0^1 \theta_{xt}\psi_x\,dx-\frac{b\beta}{k}\int_0^1 \theta_{xt}\psi_x\,dx \le \varepsilon_5\int_0^1 \theta_{xt}^2 dx+\frac{c}{\varepsilon_5}\int_0^1 \psi_x^2\,dx, \tag{4.32}
$$

$$
\begin{aligned}
\int_0^1 |\varphi_x f(\psi)|dx &\le \|\varphi_x\|\|\psi\|_{2(\varrho+1)}^{\varrho}\|\psi\|_{2(\varrho+1)} \\
&\le \frac{k}{8}\int_0^1 \varphi_x^2 dx+\frac{2}{k}\int_0^1 \psi_x^2 dx \\
&\le \frac{k}{4}\int_0^1 (\varphi_x+\psi)^2 dx+(\frac{2}{k}+\frac{k}{4})\int_0^1 \psi_x^2 dx.
\end{aligned} \tag{4.33}
$$

Substituting (4.29)-(4.33) into $J_5'(t)$, gives (4.28). □

Considering the boundary terms that appears in (4.28), we define the function

$$
q(x)=2-4x,\quad x\in[0,1].
$$

Lemma 4.9. *Let (φ,ψ,θ) be the solution of* (2.1). *Then we have that for a positive constant ε_6,*

$$
\begin{aligned}
&\Big[\varphi_x\Big(b\psi_x-\int_0^\infty g(s)\psi_x(t-s)ds\Big)\Big]_{x=0}^{x=1} \\
&\le -\frac{\varepsilon_6}{k}\frac{d}{dt}\int_0^1 \rho_1 q(x)\varphi_t\varphi_x\,dx+c(\varepsilon_6+\frac{1}{\varepsilon_6}+\frac{1}{\varepsilon_6^3})\int_0^1 \psi_x^2 dx+c\varepsilon_6\int_0^1 (\varphi_x+\psi)^2 dx \\
&\quad -\frac{\rho_2}{4\varepsilon_6}\frac{d}{dt}\int_0^1 q(x)\psi_t\Big(b\psi_x-\int_0^\infty g(s)\psi_x(t-s)ds\Big)dx+c(\frac{1}{\varepsilon_6}+\frac{1}{\varepsilon_6^3})(g\circ\psi_x)(t) \\
&\quad +\frac{c}{\varepsilon_6}\int_0^1 \psi_t^2 dx-\frac{c}{\varepsilon_6}(g'\circ\psi_x)(t)+c(\varepsilon_6+\frac{1}{\varepsilon_6})\int_0^1 \theta_{tx}^2 dx+\frac{2\rho_1\varepsilon_6}{k}\int_0^1 \varphi_t^2 dx.
\end{aligned} \tag{4.34}
$$

Proof. By using Young's and Poincaré inequalities, we obtain for any $\varepsilon_6>0$,

$$
\begin{aligned}
&\Big[\varphi_x\Big(b\psi_x-\int_0^\infty g(s)\psi_x(t-s)ds\Big)\Big]_{x=0}^{x=1} \\
&=\varphi_x(1)\Big(b\psi_x(1)-\int_0^\infty g(s)\psi_x(1,t-s)ds \\
&\quad -\varphi_x(0)\Big(b\psi_x(0)-\int_0^\infty g(s)\psi_x(0,t-s)ds\Big)\Big) \\
&\le \frac{1}{4\varepsilon_6}\Big[(b\psi_x(1)-\int_0^\infty g(s)\psi_x(1,t-s)ds)^2 \\
&\quad +\Big(b\psi_x(0)-\int_0^\infty g(s)\psi_x(0,t-s)ds\Big)^2\Big]+\varepsilon_6[\varphi_x(1)^2+\varphi_x(0)^2].
\end{aligned}
$$

By computation we have

$$
\frac{d}{dt}\int_0^1 \rho_2 q(x)\psi_t\Big(b\psi_x-\int_0^\infty g(s)\psi_x(t-s)ds\Big)dx
$$

$$
\begin{aligned}
&= -\Big[(b\psi_x(1) - \int_0^\infty g(s)\psi_x(1,t-s)ds)^2 + \Big(b\psi_x(0) - \int_0^\infty g(s)\psi_x(0,t-s)ds\Big)^2\Big] \\
&\quad + 2\int_0^1 (b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds)^2 dx \\
&\quad + \beta\int_0^1 q(x)\theta_t\Big(b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds\Big)dx \\
&\quad - k\int_0^1 q(x)(\varphi_x+\psi)\Big(b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds\Big)dx + 2\rho_2 b\int_0^1 \psi_t^2 dx \\
&\quad - \int_0^1 q(x)f(\psi)(b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds)dx \\
&\quad + \rho_2\int_0^1 q(x)\psi_t\int_0^\infty g'(s)(\psi_x(t)-\psi_x(t-s))\,ds\,dx,
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\frac{1}{4\varepsilon_6}\Big[(b\psi_x(1) - \int_0^\infty g(s)\psi_x(1,t-s)ds)^2 + (b\psi_x(0) - \int_0^\infty g(s)\psi_x(0,t-s)ds)^2\Big] \\
&= -\frac{1}{4\varepsilon_6}\frac{d}{dt}\int_0^1 \rho_2 q(x)\psi_t\Big(b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds\Big)dx \\
&\quad + \frac{1}{2\varepsilon_6}\int_0^1 (b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds)^2 dx \\
&\quad + \frac{\beta}{4\varepsilon_6}\int_0^1 q(x)\theta_t\Big(b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds\Big)dx \\
&\quad - \frac{k}{4\varepsilon_6}\int_0^1 q(x)(\varphi_x+\psi)(b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds)dx + \frac{\rho_2 b}{2\varepsilon_6}\int_0^1 \psi_t^2 dx \\
&\quad - \frac{1}{4\varepsilon_6}\int_0^1 q(x)f(\psi)\Big(b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds\Big)dx \\
&\quad + \frac{\rho_2}{4\varepsilon_6}\int_0^1 q(x)\psi_t\int_0^\infty g'(s)(\psi_x(t)-\psi_x(t-s))\,ds\,dx.
\end{aligned}
$$

In what follows, we use Young's and Poincaré inequalities, Lemma 4.2, (H2), and the fact $0 \le q(x) \le 4$, $x \in [0,1]$, and obtain

$$
\begin{aligned}
&\frac{1}{2\varepsilon_6}\int_0^1 (b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds)^2 dx \\
&\le \frac{1}{\varepsilon_6}\int_0^1 b^2\psi_x^2\,dx + \frac{1}{\varepsilon_6}\int_0^1 \Big(\int_0^\infty g(s)\psi_x(t-s)ds\Big)^2 dx \\
&\le \frac{c}{\varepsilon_6}\int_0^1 \psi_x^2\,dx + \frac{c}{\varepsilon_6}(g\circ\psi_x)(t),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\beta}{4\varepsilon_6}\int_0^1 q(x)\theta_t(b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds)dx \\
&\le \frac{\varepsilon_6^2}{8\varepsilon_6}\int_0^1 q^2(x)\theta_t^2 dx + \frac{\beta^2}{8\varepsilon_6^3}\int_0^1 (b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds)^2 dx \\
&\le c\varepsilon_6\int_0^1 \theta_{tx}^2 dx + \frac{c}{\varepsilon_6^3}(g\circ\psi_x)(t) + \frac{c}{\varepsilon_6^3}\int_0^1 \psi_x^2\,dx,
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{k}{4\varepsilon_6}\int_0^1 q(x)(\varphi_x+\psi)(b\psi_x-\int_0^\infty g(s)\psi_x(t-s)ds)dx\\
&\le \frac{\varepsilon_6^2}{8\varepsilon_6}\int_0^1 q^2(x)(\varphi_x+\psi)^2dx+\frac{k^2}{8\varepsilon_6^3}\int_0^1 (b\psi_x-\int_0^\infty g(s)\psi_x(t-s)ds)^2dx\\
&\le c\varepsilon_6\int_0^1(\varphi_x+\psi)^2dx+\frac{c}{\varepsilon_6^3}\int_0^1\psi_x^2\,dx+\frac{c}{\varepsilon_6^3}(g\circ\psi_x)(t),
\end{aligned}
$$

$$
-\frac{1}{4\varepsilon_6}\int_0^1 q(x)f(\psi)\Big(b\psi_x-\int_0^\infty g(s)\psi_x(t-s)ds\Big)dx\le\frac{c}{\varepsilon_6}\int_0^1\psi_x^2\,dx+\frac{c}{\varepsilon_6}(g\circ\psi_x)(t),
$$

$$
\frac{\rho_2}{4\varepsilon_6}\int_0^1 q(x)\psi_t\int_0^\infty g'(s)(\psi_x(t)-\psi_x(t-s))\,ds\,dx\le\frac{c}{\varepsilon_6}\int_0^1\psi_t^2dx-\frac{c}{\varepsilon_6}(g'\circ\psi_x)(t).
$$

At the same time we can get

$$
\begin{aligned}
\frac{d}{dt}\int_0^1\frac{\rho_1}{k}q(x)\varphi_t\varphi_x\,dx&=-[\varphi_x^2(1)+\varphi_x^2(0)]+2\int_0^1\varphi_x^2dx+\int_0^1 q(x)\varphi_x\psi_x\,dx\\
&\quad-\frac{\beta}{k}\int_0^1 q(x)\theta_{tx}\varphi_x\,dx+\frac{2\rho_1}{k}\int_0^1\varphi_t^2dx,
\end{aligned}
$$

which gives

$$
\begin{aligned}
\varepsilon_6[\varphi_x^2(1)+\varphi_x^2(0)]&=-\frac{d}{dt}\frac{\varepsilon_6}{k}\int_0^1\rho_1 q(x)\varphi_t\varphi_x\,dx+2\varepsilon_6\int_0^1\varphi_x^2dx+\varepsilon_6\int_0^1 q(x)\varphi_x\psi_x\,dx\\
&\quad-\frac{\beta\varepsilon_6}{k}\int_0^1 q(x)\theta_{tx}\varphi_x\,dx+\frac{2\rho_1\varepsilon_6}{k}\int_0^1\varphi_t^2dx.
\end{aligned}
$$

By using Young's and Poincaré inequalities we have

$$
2\varepsilon_6\int_0^1\varphi_x^2dx\le4\varepsilon_6\int_0^1(\varphi_x+\psi)^2dx+4\varepsilon_6\int_0^1\psi_x^2dx,
$$

$$
\varepsilon_6\int_0^1 q(x)\varphi_x\psi_x\,dx\le c\varepsilon_6\int_0^1(\varphi_x+\psi)^2dx+c\varepsilon_6\int_0^1\psi_x^2dx.
$$

$$
-\frac{\beta}{k}\int_0^1 q(x)\theta_{tx}\varphi_x\,dx\le c\varepsilon_6\int_0^1(\varphi_x+\psi)^2dx+c\varepsilon_6\int_0^1\psi_x^2dx+c\varepsilon_6\int_0^1\theta_{tx}^2dx.
$$

Thus, we obtain (4.34). □

Lemma 4.10. *Let (φ,ψ,θ) be the solution of (2.1). Then for some positive constant α_1 the functional*

$$
J_6(t):=\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varsigma e^{-\varsigma\rho}|\mu(\varsigma)|z^2(x,\rho,\varsigma,t)\,d\varsigma\,d\rho\,dx
$$

satisfies

$$
\begin{aligned}
J_6'(t)&\le-\alpha_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varsigma|\mu(\varsigma)|z^2(x,\rho,\varsigma,t)\,d\varsigma\,d\rho\,dx+\ell\int_0^1\theta_{tx}^2dx\\
&\quad-\alpha_1\int_0^1\int_{\tau_1}^{\tau_2}|\mu(\varsigma)|z^2(x,1,\varsigma,t)d\varsigma\,dx.
\end{aligned}\tag{4.35}
$$

Proof. Differentiating $J_6(t)$ and using (2.1), we obtain

$$
J_6'(t)=2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varsigma e^{-\varsigma\rho}|\mu(\varsigma)|z(x,\rho,\varsigma,t)z_t(x,\rho,\varsigma,t)\,d\varsigma\,d\rho\,dx
$$

$$
\begin{aligned}
&= -2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-\varsigma\rho}|\mu(\varsigma)|z(x,\rho,\varsigma,t)z_\rho(x,\rho,\varsigma,t)\,d\varsigma\,d\rho\,dx\\
&= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \frac{\partial}{\partial\rho}\left(e^{-\varsigma\rho}|\mu(\varsigma)|z^2(x,\rho,\varsigma,t)\right)\,d\varsigma\,d\rho\,dx\\
&\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varsigma e^{-\varsigma\rho}|\mu(\varsigma)|z^2(x,\rho,\varsigma,t)\,d\varsigma\,d\rho\,dx\\
&= -\int_0^1\int_{\tau_1}^{\tau_2}|\mu(\varsigma)|\left(e^{-\varsigma}z^2(x,1,\varsigma,t)-z^2(x,0,\varsigma,t)\right)d\varsigma\,dx\\
&\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varsigma e^{-\varsigma\rho}|\mu(\varsigma)|z^2(x,\rho,\varsigma,t)\,d\varsigma\,d\rho\,dx.
\end{aligned}
$$

Using the fact that $z(x,0,\varsigma,t)=\theta_{tx}(x,t)$ and $e^{-\varsigma}\le e^{-\varsigma\rho}\le 1$, for all $\rho\in[0,1]$, we obtain

$$
\begin{aligned}
J_6'(t) \le& -\int_0^1\int_{\tau_1}^{\tau_2}|\mu(\varsigma)|e^{-\varsigma}z^2(x,1,\varsigma,t)d\varsigma\,dx+\int_{\tau_1}^{\tau_2}|\mu(\varsigma)|d\varsigma\int_0^1\theta_{tx}^2(x,t)dx\\
&-\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varsigma e^{-\varsigma}|\mu(\varsigma)|z^2(x,\rho,\varsigma,t)\,d\varsigma\,d\rho\,dx.
\end{aligned}
$$

Because $-e^{-\varsigma}$ is an increasing function, we have $-e^{-\varsigma}\le -e^{-\tau_2}$ for all $\varsigma\in[\tau_1,\tau_2]$. Finally, setting $\alpha_1=e^{-\tau_2}$ and recall (2.2), we obtain (4.35). □

Now we define the Lyapunov functional $L(t)$ by

$$
\begin{aligned}
L(t) =& NE(t)+\frac{1}{8}J_1(t)+J_2(t)+N_1J_3(t)+N_2J_4(t)+J_5(t)+N_3J_6(t)\\
&+\frac{\varepsilon_6\rho_1}{k}\int_0^1 q(x)\varphi_t\varphi_x\,dx+\frac{\rho_2}{4\varepsilon_6}\int_0^1 q(x)\psi_t(b\psi_x-\int_0^\infty g(s)ds)dx.
\end{aligned}
$$

where N, N_1, N_2, N_3, ε_6 are positive constants to be chosen properly later.

Lemma 4.11. *Let $(\varphi,\ \psi,\ \theta)$ be the solution of (2.1). For N large enough, there exist two positive constants α_2 and α_3 satisfies*

$$
\alpha_2E(t)\le L(t)\le\alpha_3E(t).
$$

Proof. By the same arguments as in [2], using $\int_0^1\varphi_x(t)\le 2\int_0^1(\varphi_x+\psi)^2dx+2\int_0^1\psi_x^2(t)dx$ and Lemma 2, we can deduce

$$
\begin{aligned}
|L(t)-NE(t)| \le& \gamma_1\int_0^1\varphi_t^2dt+\gamma_2\int_0^1\psi_t^2\,dt+\gamma_3\int_0^1(\varphi_x+\psi)^2dx+\gamma_4\int_0^1\psi_x^2\,dx\\
&+\gamma_5\int_0^1\theta_t^2dx+\gamma_6\int_0^1\theta_x^2\,dx+\gamma_7(g\circ\psi_x)+\gamma_8\int_0^1\widehat{f}(\psi)dx\\
&+\gamma_9\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varsigma|\mu(\varsigma)|z^2(x,\rho,\varsigma,t)\,d\varsigma\,d\rho\,dx\\
\le& cE(t)
\end{aligned}
$$

in which γ_i $(i=1,\dots,9)$ are positive constants as in [2]. □

Lemma 4.12. *Assume that* (H1)–(H4) *hold, then, there exist two positive constants* β_1 *and* β_2 *such that for* $t > 0$,

$$\begin{aligned} &\xi(t)L'(t) + \beta_1 E'(t) \\ &\le -\alpha_4 \xi(t) E(t) + \beta_2 \xi(t) \int_t^\infty g(s)ds + \xi(t)\big(\frac{\rho_1 b}{k} - \rho_2\big) \int_0^1 \varphi_t(x,t)\psi_{xt}(x,t)dx, \end{aligned} \tag{4.36}$$

with $\beta_1 = \frac{2\alpha_5}{\gamma}$ *and* $\beta_2 = \alpha_5 \left(\frac{8E(0)}{\gamma(b-g_0)} + 2c_0 \right)$.

Proof. By differentiating $L(t)$, using Lemma 4.3-Lemma 4.10 and letting $\varepsilon_3 = \frac{1}{4N_1}$, we obtain

$$\begin{aligned} L'(t) &\le [\frac{N\gamma}{2} - c(N_1^2 + \frac{1}{\varepsilon_5} + \frac{1}{\varepsilon_6})](g' \circ \psi_x)(t) - [\frac{\rho_1}{8} - \varepsilon_4 N_2 - \varepsilon_5 - \frac{2\rho_1\varepsilon_6}{k}] \int_0^1 \varphi_t^2 dx \\ &\quad - [\frac{k}{8} - c(\varepsilon_1 + \varepsilon_2 + \varepsilon_6)] \int_0^1 (\varphi_x + \psi)^2 dx - (N_2 + 1) \int_0^1 \widehat{f}(\psi) dx - \frac{\delta}{2} \int_0^1 \theta_x^2 dx \\ &\quad - [N_1 \rho_2 \int_0^\infty g(s)ds - \frac{9}{8}\rho_2 - c(1 + \frac{1}{\varepsilon_6} + (1 + \frac{1}{\varepsilon_4})N_2)] \int_0^1 \psi_t^2 dx \\ &\quad - [\frac{N_2 l}{2} - c(1 + \varepsilon_1 + \varepsilon_2 + \frac{1}{\varepsilon_5} + \varepsilon_6 + \frac{1}{\varepsilon_6} + \frac{1}{\varepsilon_6^3})] \int_0^1 \psi_x^2 dx \\ &\quad + (\frac{b\rho_1}{k} - \rho_2) \int_0^1 \varphi_t \psi_{xt} dx - \alpha_1 N_3 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varsigma |\mu(\varsigma)| z^2(x,\rho,\varsigma,t)\, d\varsigma\, d\rho\, dx \\ &\quad - (\alpha_1 N_3 - \frac{c}{\delta}) \int_0^1 \int_{\tau_1}^{\tau_2} \varsigma |\mu(\varsigma)| z^2(x,1,\varsigma,t) d\varsigma\, dx \\ &\quad + c(1 + N_1^2 + N_2 + \varepsilon_5 + \frac{1}{\varepsilon_6} + \frac{1}{\varepsilon_6^3})(g \circ \psi_x)(t) \\ &\quad - [Nc - c(1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + N_2 + \frac{1}{\varepsilon_5} + \frac{1}{\varepsilon_6} + N_3)] \int_0^1 \theta_{tx}^2 dx. \end{aligned}$$

We choose $\varepsilon_1, \varepsilon_2, \varepsilon_5, \varepsilon_6$ small enough such that

$$\mu_1 = \frac{\rho_1}{8} - \varepsilon_5 - \frac{2\rho_1\varepsilon_6}{k} > 0, \quad \frac{k}{8} - c(\varepsilon_1 + \varepsilon_2 + \varepsilon_6) > 0,$$

and then we choose N_2 large enough such that

$$\frac{N_2 l}{2} - c(1 + \varepsilon_1 + \varepsilon_2 + \frac{1}{\varepsilon_5} + \varepsilon_6 + \frac{1}{\varepsilon_6} + \frac{1}{\varepsilon_6^3}) > 0.$$

We then select ε_4 so small that

$$\mu_1 - \varepsilon_4 N_2 > 0.$$

Next, we pick N_1, N_3 large enough such that

$$N_1 \rho_2 \int_0^\infty g(s)ds - \frac{9}{8}\rho_2 - c(1 + \frac{1}{\varepsilon_6} + (1 + \frac{1}{\varepsilon_4})N_2) > 0, \quad \alpha_1 N_3 - \frac{c}{\delta} > 0.$$

Finally, we choose N large enough such Lemma 4.11 remains valid and

$$\frac{N\gamma}{2} - c(N_1^2 + \frac{1}{\varepsilon_5} + \frac{1}{\varepsilon_6}), \quad Nc - c(1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + N_2 + \frac{1}{\varepsilon_5} + \frac{1}{\varepsilon_6} + N_3) > 0.$$

Consequently, by using Poincaré inequality and (2.7), we obtain

$$\begin{aligned} L'(t) \le -\alpha_4 E(t) + \alpha_5 \int_0^1 \int_0^\infty g(s)(\psi_x(x,t) - \psi_x(x,t-s))^2\,ds\,dx \\ + \big(\frac{\rho_1 b}{k} - \rho_2\big) \int_0^1 \varphi_t(x,t)\psi_{xt}(x,t)dx, \end{aligned} \tag{4.37}$$

where α_4 and α_5 are positive constants.

Using (H3) and (4.1), we obtain that for all $t \in R_+$,

$$\begin{aligned} &\xi(t) \int_0^1 \int_0^t g(s)(\psi_x(x,t) - \psi_x(x,t-s))^2\,ds\,dx \\ &\le \int_0^1 \int_0^t \xi(s) g(s)(\psi_x(x,t) - \psi_x(x,t-s))^2\,ds\,dx \\ &\le -\int_0^1 \int_0^t g'(s)(\psi_x(x,t) - \psi_x(x,t-s))^2\,ds\,dx \\ &\le -\int_0^1 \int_0^\infty g'(s)(\psi_x(x,t) - \psi_x(x,t-s))^2\,ds\,dx \\ &\le -\frac{2}{\gamma} E'(t). \end{aligned}$$

On the other hand, using the definition of $E(t)$ and the fact that $E(t)$ is nonincreasing, we ask for $t, s \in R_+$,

$$\begin{aligned} \int_0^1 (\psi_x(x,t) - \psi_x(x,t-s))^2 dx &\le 2\int_0^1 \psi_x^2(x,t)dx + 2\int_0^1 \psi_x^2(x,t-s)dx \\ &\le 4\sup_{s>0} \int_0^1 \psi_x^2(x,s)dx + 2\sup_{\tau<0} \int_0^1 \psi_x^2(x,\tau)d\tau \\ &\le 4\sup_{s>0} \int_0^1 \psi_x^2(x,s)dx + 2\sup_{\tau>0} \int_0^1 \psi_{0x}^2(x,\tau)d\tau \\ &\le \frac{8E(0)}{\gamma(b-g_0)} + 2c_0. \end{aligned}$$

Then, we obtain

$$\begin{aligned} &\xi(t) \int_0^1 \int_t^\infty g(s)(\psi_x(x,t) - \psi_x(x,t-s))^2\,ds\,dx \\ &\le \Big(\frac{8E(0)}{\gamma(b-g_0)} + 2c_0\Big) \xi(t) \int_t^\infty g(s)ds. \end{aligned}$$

Then, we deduce that, for all $t \in R_+$,

$$\begin{aligned} &\xi(t) \int_0^1 \int_0^\infty g(s)(\psi_x(x,t) - \psi_x(x,t-s))^2\,ds\,dx \\ &\le -\frac{2}{\gamma} E'(t) + \Big(\frac{8E(0)}{\gamma(b-g_0)} + 2c_0\Big) \xi(t) \int_t^\infty g(s)ds. \end{aligned}$$

The proof is complete. □

Theorem 4.13. *Assume that* (H1)–(H4) *hold.*

1. If $\frac{k}{\rho_1}=\frac{b}{\rho_2}$ holds, then, for any $((\varphi_0,\varphi_1),(\psi_0,\psi_1),(\theta_0,\theta_1)) \in (H_0^1(0,1)\times L^2(0,1))^3$ satisfying, for some $c_0\ge 0$,

$$\int_0^1 \psi_{0x}^2(x,s)dx \le c_0,\ \forall s>0,$$

there exist constants ϵ_1, $\epsilon_2>0$ such that, for all $t\in R_+$ and for all $\epsilon_0\in[0,\epsilon_1]$,

$$E(t)\le \epsilon_2\Big(1+\int_0^t (g(s))^{1-\epsilon_0}ds\Big)e^{-\epsilon_0\int_0^t \xi(s)ds}+\epsilon_2\int_t^\infty g(s)ds. \tag{4.38}$$

2. If $\frac{k}{\rho_1}\neq\frac{b}{\rho_2}$, then, for any $((\varphi_0,\varphi_1),(\psi_0,\psi_1),(\theta_0,\theta_1)) \in (H_0^1(0,1)\times L^2(0,1))^3$ satisfying, for some $c_0\ge 0$

$$\max\Big\{\int_0^1 \psi_{0x}^2(x,s)dx,\int_0^1 \psi_{0xs}^2(x,s)dx\Big\}\le c_0,\quad s>0,$$

there exists a constant $\epsilon_2>0$ such that, for all $t\in R_+$,

$$E(t)\le \frac{\epsilon_2(1+\int_0^t\xi(s)\int_s^\infty g(\tau)d\tau ds)}{\int_0^t\xi(s)ds}. \tag{4.39}$$

Proof. First, we define

$$L_1(t)=\xi(t)L(t)+\beta_1E(t),\quad r(t)=\xi(t)\int_t^\infty g(s)ds.$$

Clearly, $L_1(t)$ and $E(t)$ are equivalent, that is, exist positive constants α_6 and α_7, such that

$$\alpha_6E(t)\le L_1(t)\le\alpha_7E(t)$$

Then using Lemma 4.12 we have

$$L_1'(t)\le -\epsilon_1\xi(t)L_1(t)+\beta_2r(t)+\Big(\frac{\rho_1b}{k}-\rho_2\Big)\xi(t)\int_0^1\varphi_t(x,t)\psi_{xt}(x,t)dx, \tag{4.40}$$

with $\epsilon_1=\frac{\alpha_4}{\alpha_7}$. This inequality still holds, for any $\epsilon_0\in[0,\epsilon_1]$, that is

$$L_1'(t)\le -\epsilon_0\xi(t)L_1(t)+\beta_2r(t)+\Big(\frac{\rho_1b}{k}-\rho_2\Big)\xi(t)\int_0^1\varphi_t(x,t)\psi_{xt}(x,t)dx. \tag{4.41}$$

Now, we distinguish two cases.

Case 1: If $\frac{k}{\rho_1}=\frac{b}{\rho_2}$ holds. Because the last term in (4.41) vanishes, then (4.41) implies that, for all $t\in R_+$,

$$\frac{d}{dt}\Big(e^{\epsilon_0\int_0^t\xi(s)ds}L_1(t)\Big)\le\beta_2e^{\epsilon_0\int_0^t\xi(s)ds}r(t).$$

Therefore, by integrating over $[0,T]$ with $T\ge 0$, we have

$$L_1(T)\le e^{-\epsilon_0\int_0^T\xi(s)ds}\Big(L_1(0)+\beta_2\int_0^T e^{\epsilon_0\int_0^t\xi(s)ds}r(t)dt\Big),$$

which implies

$$E(T)\le\frac{1}{\alpha_6}e^{-\epsilon_0\int_0^T\xi(s)ds}\Big(L_1(0)+\beta_2\int_0^T e^{\epsilon_0\int_0^t\xi(s)ds}r(t)dt\Big). \tag{4.42}$$

Because

$$e^{\epsilon_0\int_0^t\xi(s)ds}r(t)dt=\frac{1}{\epsilon_0}\frac{d}{dt}\Big(e^{\epsilon_0\int_0^t\xi(s)ds}\Big)\int_t^\infty g(s)ds$$

$$= \frac{1}{\epsilon_0} \frac{d}{dt}\Big(e^{\epsilon_0 \int_0^t \xi(s)ds} \int_t^\infty g(s)ds \Big) + g(t) e^{\epsilon_0 \int_0^t \xi(s)ds},$$

by integration we obtain

$$\begin{aligned}
&\int_0^T e^{\epsilon_0 \int_0^t \xi(s)ds} r(t)dt \\
&= \frac{1}{\epsilon_0}\Big(e^{\epsilon_0 \int_0^T \xi(s)ds} \int_T^\infty g(s)ds - \int_0^\infty g(s)ds + \int_0^T e^{\epsilon_0 \int_0^t \xi(s)ds} g(t)dt \Big).
\end{aligned}$$

Consequently

$$\begin{aligned}
E(T) \le \frac{1}{\alpha_6}\Big(L_1(0) e^{-\epsilon_0 \int_0^T \xi(s)ds} + \frac{\beta_2}{\epsilon_0}\int_T^\infty g(s)ds \Big) \\
+ \frac{\beta_2}{\alpha_6 \epsilon_0} e^{-\epsilon_0 \int_0^T \xi(s)ds} \int_0^T e^{\epsilon_0 \int_0^t \xi(s)ds} g(t)dt.
\end{aligned} \tag{4.43}$$

On the other hand, for all $t \in R_+$,

$$\begin{aligned}
&\frac{d}{dt}\Big(e^{\epsilon_0 \int_0^t \xi(s)ds} (g(t))^{\epsilon_0} \Big) \\
&= \epsilon_0 e^{\epsilon_0 \int_0^t \xi(s)ds} \xi(t) (g(t))^{\epsilon_0} - \epsilon_0 e^{\epsilon_0 \int_0^t \xi(s)ds} g^{\epsilon_0 - 1}(t) g'(t) \\
&= \epsilon_0 e^{\epsilon_0 \int_0^t \xi(s)ds} (g(t))^{\epsilon_0} (\xi(t) + g^{-1}(t) g'(t)) \\
&\le 0,
\end{aligned}$$

and then $e^{\epsilon_0 \int_0^t \xi(s)ds} (g(t))^{\delta_0} \le (g(0))^{\epsilon_0}$. Therefore

$$\int_0^T e^{\epsilon_0 \int_0^t \xi(s)ds} g(t)dt \le (g(0))^{\epsilon_0} \int_0^T (g(t))^{1-\epsilon_0} dt. \tag{4.44}$$

Finally, (4.43) and (4.44) give (4.38) for any classical solution of (2.1) with

$$\epsilon_1 = \frac{1}{\alpha_6} \max\Big\{ L_1(0), \frac{\beta_2}{\epsilon_0}, \frac{\beta_2}{\epsilon_0} (g(0))^{\epsilon_0} \Big\}.$$

By denseness arguments, (4.38) remains valid for any weak solution of (2.1).

Case 2: $\frac{k}{\rho_1} \ne \frac{b}{\rho_2}$. We estimate the last term of (4.41) as follows: for any $\varepsilon > 0$, there exists a positive constant c_ε (depending on ε) such that

$$\begin{aligned}
&\Big(\frac{\rho_1 b}{k} - \rho_2 \Big) \int_0^1 \varphi_t(x,t) \psi_{xt}(x,t) dx \\
&= \frac{\rho_1 b - \rho_2 k}{g_0 k} \int_0^1 \varphi_t(x,t) \int_0^\infty g(s)(\psi_{xt}(x,t) - \psi_{xt}(x,t-s))\, ds\, dx \\
&\quad + \frac{\rho_1 b - \rho_2 k}{g_0 k} \int_0^1 \varphi_t(x,t) \int_0^\infty g(s) \psi_{xt}(x,t-s)\, ds\, dx.
\end{aligned}$$

By using Young's inequality we obtain

$$\begin{aligned}
&\frac{\rho_1 b - \rho_2 k}{g_0 k} \int_0^1 \varphi_t(x,t) \int_0^\infty g(s)(\psi_{xt}(x,t) - \psi_{xt}(x,t-s))\, ds\, dx \\
&\le c \int_0^1 |\varphi_t(x,t)| \int_0^\infty g(s) |\psi_{xt}(x,t) - \psi_{xt}(x,t-s)|\, ds\, dx \\
&\le \frac{\varepsilon}{2} E(t) + c_\varepsilon \int_0^1 \int_0^\infty g(s)(\psi_{xt}(x,t) - \psi_{xt}(x,t-s))^2\, ds\, dx.
\end{aligned}$$

At the same time we have

$$\int_0^\infty g(s)\psi_{xt}(x,t-s)ds=\int_0^\infty g'(s)\left(\psi_{xt}(t-s)-\psi_x(t)\right)ds,$$

$$\begin{aligned}&\frac{\rho_1 b-\rho_2 k}{g_0 k}\int_0^1\varphi_t(x,t)\int_0^\infty g(s)\psi_{xt}(x,t-s)\,ds\,dx\\&=\frac{\rho_1 b-\rho_2 k}{g_0 k}\int_0^1\varphi_t(x,t)\int_0^\infty g'(s)\left(\psi_{xt}(t-s)-\psi_x(t)\right)\,ds\,dx\\&\le\frac{\varepsilon}{2}E(t)-c_\varepsilon(g'\circ\psi_{xt})(t)\\&\le\frac{\varepsilon}{2}E(t)-\frac{2c_\varepsilon}{\gamma}E'(t),\end{aligned}$$

$$\begin{aligned}&\Big(\frac{\rho_1 b}{k}-\rho_2\Big)\int_0^1\varphi_t(x,t)\psi_{xt}(x,t)dx\\&\le\varepsilon E(t)+c_\varepsilon\int_0^1\int_0^\infty g(s)(\psi_{xt}(x,t)-\psi_{xt}(x,t-s))^2\,ds\,dx-\frac{2c_\varepsilon}{\gamma}E'(t).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\xi(t)\Big(\frac{\rho_1 b}{k}-\rho_2\Big)\int_0^1\varphi_t(x,t)\psi_{xt}(x,t)dx\\&\le\varepsilon\xi(t)E(t)+c_\varepsilon\xi(t)\int_0^1\int_0^\infty g(s)(\psi_{xt}(x,t)-\psi_{xt}(x,t-s))^2\,ds\,dx-\frac{2c_\varepsilon}{\gamma}\xi(t)E'(t).\end{aligned}$$

Consequently,

$$\begin{aligned}L_1'(t)&\le-\alpha_8\xi(t)E(t)+\beta_2 r(t)-\frac{2c_\epsilon}{\gamma}\xi(t)E'(t)\\&\quad+c_\epsilon\xi(t)\int_0^1\int_0^\infty g(s)(\psi_{xt}(x,t)-\psi_{xt}(x,t-s))^2\,ds\,dx,\end{aligned}\tag{4.45}$$

where $\alpha_8=\epsilon_0\alpha_6-\varepsilon$. By using the definition of $E_2(t)$ and $E_2'(t)$, we have

$$\xi(t)\int_0^1\int_0^t g(s)(\psi_{xt}(x,t)-\psi_{xt}(x,t-s))^2\,ds\,dx\le-\frac{2}{\gamma}E_2'(t),\tag{4.46}$$

$$\begin{aligned}&\xi(t)\int_0^1\int_t^\infty g(s)(\psi_{xt}(x,t)-\psi_{xt}(x,t-s))^2\,ds\,dx\\&\le\Big(\frac{8E_2(0)}{\gamma(b-g_0)}+2c_0\Big)r(t).\end{aligned}\tag{4.47}$$

Hence, combining (4.45), (4.46) and (4.47) we obtain

$$\begin{aligned}&\frac{d}{dt}\Big(L_1(t)+\frac{2c_\varepsilon}{\gamma}E_2(t)+\frac{2c_\varepsilon}{\gamma}\xi(t)E(t)\Big)\\&\le-\alpha_8\xi(t)E(t)+\beta_3 r(t)+\frac{2c_\varepsilon}{\gamma}\xi'(t)E(t),\end{aligned}\tag{4.48}$$

where $\beta_3=\beta_2+(\frac{8E_2(0)}{\gamma(b-g_0)}+2c_0)c_\varepsilon$. Because ξ is nonincreasing, the last term of (4.48) is nonpositive, therefore, by integration on [0,T] and using the fact $E(t)$ is nonincreasing, we obtain

$$\alpha_8 E(T)\int_0^T \xi(t)dt \le L_1(0)+\frac{2c_\varepsilon}{\gamma}E_2(0)+\frac{2c_\varepsilon}{\xi(0)}E(0)+\beta_3\int_0^T r(t)dt,$$

which gives (4.39) with

$$\epsilon_2=\frac{1}{\alpha_8}\max\{L_1(0)+\frac{2c_\varepsilon}{\gamma}E_2(0)+\frac{2c_\varepsilon}{\xi}(0)E(0),\beta_3\}.$$

This completes the proof. □

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