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INVERSE PROBLEM FOR A TWO-DIMENSIONAL STRONGLY DEGENERATE HEAT EQUATION

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Abstract. This article concerns the existence and uniqueness of solutions in the problem of identifying the leading coefficient in a two-dimensional heat equation. We suppose that unknown coefficient depends on the time variable and the equation is strongly degenerate. Applying Schauder fixed-point theorem, we find conditions for existence of a classical solution.

1. Introduction

We consider an inverse problem for a two-dimensional degenerate heat equation in a rectangular domain. Direct problems of this type are mathematical models of various processes such as seawater desalination, movement of liquid in porous medium, financial market behavior, etc. [1, 2, 4]. Such problems are comparatively well studied (see, for instance, [3]).

Inverse problems arise when certain parameters of these processes are unknown. Various types of inverse problems for non-degenerate equations are well investigated and some results may be found in monographs and references therein [7, 13, 15]. The beginning of the research of inverse problems for one-dimensional degenerate parabolic equations was made in [8, 9, 10, 12, 16]. There the conditions of existence and uniqueness of solution for these problems were established in the cases of week and strong power degeneration. Later some results were obtained for parabolic equations with arbitrary types of degeneration [17, 18] and for free-boundary domains [5, 6]. Inverse problem for a two-dimensional weakly degenerate heat equation in a rectangular domain was considered in [11].

In this paper we establish conditions for existence and uniqueness of solution to an inverse problem for a two-dimensional strongly degenerate heat equation.

2. Statement of the problem and main result

In the domain $Q_T := \{(x, y, t) : 0 < x < h, 0 < y < l, 0 < t < T\}$ we consider a two-dimensional heat equation with unknown leading coefficient depending on the time variable. We suppose that the equation degenerates at the initial moment

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as a power with a given exponent $\beta \geq 1$. We choose the case of mixed Dirichlet-Neumann boundary conditions. The additional condition (so-called *overdetermination condition*) is taken accordingly to the physical sense and it presents the value of the heat flux on the part of the boundary of domain. So, the problem consists of finding a pair of functions $(a(t), u(x, y, t)), a(t) > 0, t \in [0, T]$ that satisfy the degenerate heat equation

$$u_t = t^\beta a(t)\Delta u + f(x, y, t), \quad (x, y, t) \in Q_T, \tag{2.1}$$

the initial condition

$$u(x, y, 0) = \varphi(x, y, 0), \quad (x, y) \in \overline{D} := [0, h] \times [0, l], \tag{2.2}$$

the boundary conditions

$$u(0, y, t) = \mu_1(y, t),\ u(h, y, t) = \mu_2(y, t), \quad (y, t) \in [0, l] \times [0, T], \tag{2.3}$$

$$u_y(x, 0, t) = \nu_1(x, t),\ u_y(x, l, t) = \nu_2(x, t), \quad (x, t) \in [0, h] \times [0, T] \tag{2.4}$$

and the overdetermination condition

$$a(t)u_x(0, y_0, t) = \kappa(t), \quad t \in (0, T], \tag{2.5}$$

where $y_0 \in (0, l)$ is some arbitrary fixed point. Our goal is to determine conditions of the existence and uniqueness of solution to problem (2.1)–(2.5) in the spaces of continuously differentiable functions. The corresponding result is presented in a theorem, for which we use the following assumptions:

- (A1) $\beta \geq 1,\ \varphi \in C^2(\overline{D}),\ \mu_i \in C^{2,1}([0, l] \times (0, T]) \cap C^{1,1}([0, l] \times [0, T]),\ \nu_i \in C^{1,0}([0, h] \times (0, T]) \cap C([0, h] \times [0, T])$ for $i = 1, 2,\ f \in C^{2,2,0}(\overline{Q}_T),\ \kappa(t) = \kappa_0(t)t^{\frac{1-\beta}{2}}$ with $\kappa_0 \in C[0, T]$;
- (A2) $\varphi_x(x, y) \geq 0$ for $(x, y) \in \overline{D}$; $\mu_{1_t}(y, t) - f(0, y, t) < 0,\ \mu_{2_t}(y, t) - f(h, y, t) \geq 0,\ \mu_{1_{yy}}(y, t) \geq 0,\ \mu_{2_{yy}}(y, t) \leq 0$ for $(y, t) \in [0, l] \times (0, T]$; $\nu_{1_x}(x, t) \leq 0,\ \nu_{2_x}(x, t) \geq 0$ for $(x, t) \in [0, h] \times [0, T]$; $f_x(x, y, t) \geq 0$ for $(x, y, t) \in \overline{Q}_T$; $\kappa_0(t) > 0$ for $t \in [0, T]$;
- (A3) $\mu_1(y, 0) = \varphi(0, y),\ \mu_2(y, 0) = \varphi(h, y)$ for $y \in [0, l]$; $\nu_1(x, 0) = \varphi_y(x, 0),\ \nu_2(x, 0) = \varphi_y(x, l)$ for $x \in [0, h]$; $\mu_{1y}(0, t) = \nu_1(0, t),\ \mu_{1y}(l, t) = \nu_2(0, t),\ \mu_{2y}(0, t) = \nu_1(h, t),\ \mu_{2y}(l, t) = \nu_2(h, t)$ for $t \in [0, T]$.

Theorem 2.1. *Under assumptions* (A1)–(A3)*, there exists unique solution* (a, u) *to* (2.1)–(2.5)*, which belongs to the space* $C[0, T] \times (C^{2,2,1}(Q_T) \cap C^{0,1,1}(\overline{Q}_T))$*, where* $a(t) > 0$ *and* $t \in [0, T]$.

Here we have the usual notation [14] for the Banach spaces $C^{k,m,n}(Q_T)$ consisting of functions that are continuous in Q_T, with their derivatives of k-th order with respect to x, m-th order with respect to y, and n-th order with respect to t. The spaces $C^{k,m}([0, h] \times (0, T])$ and others are defined similarly.

3. Existence of solutions

There are many approaches to study existence of solutions to inverse problems. Among them, we choose the method of reducing the problem (2.1)–(2.5) to an equation with respect to the coefficient $a(t)$ and applying the Schauder fixed-point theorem.

Assume temporarily that $a = a(t) > 0$ from class $C[0,T]$ is known and find the solution to the direct problem (2.1)–(2.4) [7]:

$$
\begin{aligned}
u(x,y,t) = &\int_0^l \int_0^h G_{12}(x,y,t,\xi,\eta,0)\varphi(\xi,\eta)d\xi d\eta \\
&+ \int_0^t \int_0^l G_{12_\xi}(x,y,t,0,\eta,\tau)\tau^\beta a(\tau)\mu_1(\eta,\tau)\, d\eta\, d\tau \\
&- \int_0^t \int_0^l G_{12_\xi}(x,y,t,h,\eta,\tau)\tau^\beta a(\tau)\mu_2(\eta,\tau)\, d\eta\, d\tau \\
&- \int_0^t \int_0^h G_{12}(x,y,t,\xi,0,\tau)\tau^\beta a(\tau)\nu_1(\xi,\tau)d\xi d\tau \\
&+ \int_0^t \int_0^h G_{12}(x,y,t,\xi,l,\tau)\tau^\beta a(\tau)\nu_2(\xi,\tau)d\xi d\tau \\
&+ \int_0^t \int_0^l \int_0^h G_{12}(x,y,t,\xi,\eta,\tau)f(\xi,\eta,\tau)\, d\xi\, d\eta\, d\tau.
\end{aligned} \tag{3.1}
$$

Here we denote by $G_{ij}(x,y,t,\xi,\eta,\tau)$, $i \in \{1,2\}$ the Green functions for the heat equation (2.1) with boundary conditions of i-th kind with respect to x and j-th kind with respect to y. They are defined the equality [7]

$$
\begin{aligned}
&G_{ij}(x,y,t,\xi,\eta,\tau) \\
&= \frac{1}{4\pi(\theta(t)-\theta(\tau))} \sum_{m,n=-\infty}^{\infty} \Big(\exp\Big(-\frac{(x-\xi+2nh)^2}{4(\theta(t)-\theta(\tau))}\Big) \\
&\quad + (-1)^i \exp\Big(-\frac{(x+\xi+2nh)^2}{4(\theta(t)-\theta(\tau))}\Big)\Big)\Big(\exp\Big(-\frac{(y-\eta+2ml)^2}{4(\theta(t)-\theta(\tau))}\Big) \\
&\quad + (-1)^j \exp\Big(-\frac{(y+\eta+2ml)^2}{4(\theta(t)-\theta(\tau))}\Big)\Big), \\
&\qquad \theta(t) = \int_0^t \tau^\beta a(\tau)d\tau, \quad i,j \in \{1,2\}.
\end{aligned} \tag{3.2}
$$

Note that G_{ij} may be expressed as a product of two Green functions for one-dimensional heat equations: $G_{ij}(x,y,t,\xi,\eta,\tau) = G_i(x,t,\xi,\tau)G_j(y,t,\eta,\tau)$.

Using Green function's properties and integrating by parts, find the derivative $u_x(x,y,t)$:

$$
\begin{aligned}
&u_x(x,y,t) \\
&= \int_0^l \int_0^h G_{22}(x,y,t,\xi,\eta,0)\varphi_\xi(\xi,\eta)d\xi d\eta \\
&\quad - \int_0^t \int_0^l G_{22}(x,y,t,0,\eta,\tau)(\mu_{1_\tau}(\eta,\tau) - \tau^\beta a(\tau)\mu_{1_{\eta\eta}}(\eta,\tau) - f(0,\eta,\tau))\, d\eta\, d\tau \\
&\quad + \int_0^t \int_0^l G_{22}(x,y,t,h,\eta,\tau)(\mu_{2_\tau}(\eta,\tau) - \tau^\beta a(\tau)\mu_{2_{\eta\eta}}(\eta,\tau) - f(h,\eta,\tau))\, d\eta\, d\tau \\
&\quad - \int_0^t \int_0^h G_{22}(x,y,t,\xi,0,\tau)\tau^\beta a(\tau)\nu_{1_\xi}(\xi,\tau)d\xi d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t\int_0^h G_{22}(x,y,t,\xi,l,\tau)\tau^\beta a(\tau)\nu_{2_\xi}(\xi,\tau)d\xi d\tau \\
&+\int_0^t\int_0^l\int_0^h G_{22}(x,y,t,\xi,\eta,\tau)f_\xi(\xi,\eta,\tau)\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

Substituting expression in (2.5) we obtain the equation

$$
\begin{aligned}
a(t)=\kappa(t)\Big(&\int_0^l\int_0^h G_{22}(0,y_0,t,\xi,\eta,0)\varphi_\xi(\xi,\eta)d\xi d\eta \\
&-\int_0^t\int_0^l G_{22}(0,y_0,t,0,\eta,\tau)(\mu_{1_\tau}(\eta,\tau)-\tau^\beta a(\tau)\mu_{1_{\eta\eta}}(\eta,\tau) \\
&-f(0,\eta,\tau))\,d\eta\,d\tau \\
&+\int_0^t\int_0^l G_{22}(0,y_0,t,h,\eta,\tau)(\mu_{2_\tau}(\eta,\tau)-\tau^\beta a(\tau)\mu_{2_{\eta\eta}}(\eta,\tau) \\
&-f(h,\eta,\tau))\,d\eta\,d\tau \\
&-\int_0^t\int_0^h G_{22}(0,y_0,t,\xi,0,\tau)\tau^\beta a(\tau)\nu_{1_\xi}(\xi,\tau)d\xi d\tau \\
&+\int_0^t\int_0^h G_{22}(0,y_0,t,\xi,l,\tau)\tau^\beta a(\tau)\nu_{2_\xi}(\xi,\tau)d\xi d\tau \\
&+\int_0^t\int_0^l\int_0^h G_{22}(0,y_0,t,\xi,\eta,\tau)f_\xi(\xi,\eta,\tau)\,d\xi\,d\eta\,d\tau\Big)^{-1}, \quad t\in(0,T].
\end{aligned}
\tag{3.3}
$$

It is easy to see this equation and problem (2.1)–(2.5) are equivalent. Indeed, if $(a(t),u(x,y,t))$ is a solution to (2.1)–(2.5), then the function $a(t)$ satisfies (3.3) as it is shown above. On the other hand, if $a(t)$ is a solution to (3.3), then we substitute it into (2.1) and find the solution to problem (2.1)–(2.4) under the form (3.1). Condition (2.5) follows from (3.3).

To establish estimates for solutions of (3.3),we denote $a_{\max}:=\max_{[0,T]}a(t)$ and consider the following integral from (3.3),

$$
\begin{aligned}
&\int_0^t\int_0^l G_{22}(0,y_0,t,0,\eta,\tau)(f(0,\eta,\tau)-\mu_{1\tau}(\eta,\tau))\,d\eta\,d\tau \\
&\ge \min_{[0,l]\times[0,T]}(f(0,y,t)-\mu_{1_t}(y,t))\int_0^t\int_0^l G_{22}(0,y_0,t,0,\eta,\tau)\,d\eta\,d\tau \\
&\ge C_1\int_0^t\frac{d\tau}{\sqrt{\theta(t)-\theta(\tau)}} \\
&\ge \frac{C_2}{\sqrt{a_{\max}}}\int_0^t\frac{d\tau}{\sqrt{t^{\beta+1}-\tau^{\beta+1}}}\ge\frac{C_3}{t^{\frac{\beta-1}{2}}\sqrt{a_{\max}}}.
\end{aligned}
\tag{3.4}
$$

Here we used the equality

$$
\int_0^l G_2(y,t,\eta,\tau)d\eta=1, \tag{3.5}
$$

that can be verified by direct computation. The constants C_i, $i\in\{1,2,3\}$ depend on given data. It follows from (A2) that all other summands in denominator of

(3.3) are non-negative, therefore

$$a(t) \le \frac{\kappa(t) t^{\frac{\beta-1}{2}} \sqrt{a_{\max}}}{C_3}.$$

Taking into account the assumptions $\kappa(t) = \kappa_0(t) t^{\frac{1-\beta}{2}}$, $\kappa_0(t) > 0$, $t \in [0,T]$, we obtain

$$a(t) \le \frac{\kappa_0(t)\sqrt{a_{\max}}}{C_3}, \quad t \in [0,T].$$

Hence, we get the estimate

$$a(t) \le A_1 < \infty, \quad t \in [0,T], \tag{3.6}$$

where A_1 is a known constant depending on the problem data.

Now we estimate $a(t)$ from below. Denote integrals in the denominator of (3.3) as I_i, $i = \overline{1,6}$. Taking into account (3.5), we find

$$I_1 \le \max_{\overline{D}} \varphi_x(x,y) := C_4.$$

Denote $a_{\min} := \min_{[0,T]} a(t)$. Next we have the estimate

$$\begin{aligned} I_2 &= \int_0^t \int_0^l G_{22}(0, y_0, t, 0, \eta, \tau)(f(0,\eta,\tau) - \mu_{1\tau}(\eta,\tau) + \tau^\beta a(\tau)\mu_{1_{\eta\eta}}(\eta,\tau))\, d\eta\, d\tau \\ &\le \max_{[0,l]\times[0,T]} (f(0,y,t) - \mu_{1_t}(y,t)) \int_0^t \int_0^l G_{22}(0, y_0, t, 0, \eta, \tau)\, d\eta\, d\tau \\ &\quad + \max_{[0,l]\times[0,T]} t^\beta \mu_{1_{yy}}(y,t) \int_0^t \int_0^l G_{22}(0, y_0, t, 0, \eta, \tau) a(\tau)\, d\eta\, d\tau. \end{aligned}$$

Using (3.5), (3.6) and the known estimates of the Green function [7],

$$G_2(0,t,0,\tau) \le \frac{C_5}{\sqrt{\theta(t) - \theta(\tau)}} + C_6, \tag{3.7}$$

we obtain

$$I_3 \le \frac{C_7}{t^{\frac{\beta-1}{2}} \sqrt{a_{\min}}} + C_8.$$

Taking into account estimates [7]

$$G_2(0,t,h,\tau) \le C_7$$

and (3.5), we have $I_2 \le C_9$.

To estimate I_4 we use equality (3.5) and estimates (3.6), (3.7):

$$\begin{aligned} I_4 &\le C_{10} \int_0^t G_2(y_0, t, 0, \tau) \tau^\beta a(\tau) d\tau \\ &= \frac{C_{10}}{2\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{\theta(t) - \theta(\tau)}} \sum_{m=-\infty}^{\infty} \Big(\exp\Big(-\frac{(y_0 - \eta + 2ml)^2}{4(\theta(t) - \theta(\tau))} \Big) \\ &\quad + \exp\Big(-\frac{(y_0 + \eta + 2ml)^2}{4(\theta(t) - \theta(\tau))} \Big) \Big) \tau^\beta a(\tau) d\tau \\ &= \frac{C_{10}}{2\sqrt{\pi}} \int_0^{\theta(t)} \frac{1}{\sqrt{z}} \sum_{m=-\infty}^{\infty} \Big(\exp\Big(-\frac{(y_0 - \eta + 2ml)^2}{4z} \Big) \\ &\quad + \exp\Big(-\frac{(y_0 + \eta + 2ml)^2}{4z} \Big) \Big) dz \end{aligned}$$

$$\le C_{11}\sqrt{\theta(t)} \le C_{12}.$$

The integral I_5 is estimated similarly. From (3.5) we obtain $I_6 \le C_{13}$.

The above estimates lead us to the inequality

$$a(t) \ge \frac{\kappa(t)}{\frac{C_{14}}{t^{\frac{\beta-1}{2}}\sqrt{a_{\min}}} + C_{15}}.$$

Taking into account assumptions on $\kappa(t)$, we transform it to

$$a(t) \ge \frac{\kappa_0(t)\sqrt{a_{\min}}}{C_{14} + C_{15}t^{\frac{\beta-1}{2}}\sqrt{a_{\min}}}, \quad t \in [0,T]$$

or

$$\sqrt{a_{\min}} \ge \frac{C_{16}}{C_{14} + C_{17}\sqrt{a_{\min}}}.$$

Solving this inequality we obtain an estimate for $a(t)$ from below:

$$a(t) \ge A_0 > 0, \quad t \in [0,T], \tag{3.8}$$

where A_0 is determined by given data. Therefore, the a priori estimates of solutions to the equation (3.3) are established.

Now we put equation (3.3) in the form

$$a(t) = Pa(t), \quad t \in [0,T]. \tag{3.9}$$

It follows from (3.6), (3.8) that the operator P maps the set $\mathcal{N} := \{a \in C[0,T] : A_0 \le a(t) \le A_1\}$ into itself. By the Arzela-Ascoli theorem and the above estimates, it is clear that the compactness of P on $\mathcal{N}$ will be established if we prove that $\forall \varepsilon > 0$ exists such a $\delta > 0$, that inequality $|Pa(t_2) - Pa(t_1)| < \varepsilon$ holds if $|t_1 - t_2| < \delta$ for all $t_1, t_2 \in [0,T]$ and for any $a \in \mathcal{N}$.

Firstly, let show that there exists $\lim_{t\to 0} a(t)$. It follows from the above estimates and assumptions (A2) that the limit of denominator in (3.3) is equal to $\lim_{t\to 0} I_2(t)$. Taking into account the equalities

$$\begin{gathered}\lim_{\tau\to t}\int_0^l G_2(y_0,t,\eta,\tau)(f(0,\eta,\tau) - \mu_{1_\tau}(\eta,\tau))d\eta = f(0,y_0,t) - \mu_{1_t}(y_0,t),\\ \int_0^t \frac{d\tau}{\sqrt{t^{\beta+1} - \tau^{\beta+1}}} = t^{\frac{1-\beta}{2}}\int_0^1 \frac{dz}{\sqrt{1 - z^{\beta+1}}}\end{gathered} \tag{3.10}$$

and using the mean-value theorem, from (3.3) we have

$$\lim_{t\to 0} a(t) = \sqrt{\frac{\pi}{\beta+1}}\frac{\kappa_0(0)\sqrt{\lim_{t\to 0} a(t^*)}}{(f(0,y_0,0) - \mu_{1_t}(y_0,0))\int_0^1 \frac{dz}{\sqrt{1-z^{\beta+1}}}},$$

where $t^* \in [0,t]$. From this we obtain

$$\lim_{t\to 0} a(t) = \frac{\pi}{\beta+1}\Big(\frac{\kappa_0(0)}{(f(0,y_0,0) - \mu_{1_t}(y_0,0))\int_0^1 \frac{dz}{\sqrt{1-z^{\beta+1}}}}\Big)^2.$$

Consider the difference

$$|Pa(t_1) - Pa(t_2)| = \Big|\frac{\kappa_0(t_1)}{t_1^{\frac{\beta-1}{2}}u_x(0,y_0,t_1)} - \frac{\kappa_0(t_2)}{t_2^{\frac{\beta-1}{2}}u_x(0,y_0,t_2)}\Big|$$

$$= \frac{|\kappa_0(t_1) t_2^{\frac{\beta-1}{2}} u_x(0,y_0,t_2) - \kappa_0(t_2) t_1^{\frac{\beta-1}{2}} u_x(0,y_0,t_1)|}{t_1^{\frac{\beta-1}{2}} u_x(0,y_0,t_1) t_2^{\frac{\beta-1}{2}} u_x(0,y_0,t_2)}.$$

It follows from (3.4) that

$$t_1^{\frac{\beta-1}{2}} u_x(0,y_0,t_1) t_2^{\frac{\beta-1}{2}} u_x(0,y_0,t_2) \ge C_{18} > 0.$$

Thus, it is sufficient to estimate the difference $u_x(0,y_0,t_1) t_2^{\frac{\beta-1}{2}} - u_x(0,y_0,t_2) t_1^{\frac{\beta-1}{2}}$. Consider for example the expression

$$\begin{aligned}\Delta :=& \Big| t_2^{\frac{\beta-1}{2}} \int_0^{t_2} \int_0^l G_{22}(0,y_0,t_2,0,\eta,\tau)\mu(\eta,\tau)\,d\eta\,d\tau \\ &- t_1^{\frac{\beta-1}{2}} \int_0^{t_1} \int_0^l G_{22}(0,y_0,t_1,0,\eta,\tau)\mu(\eta,\tau)\,d\eta\,d\tau \Big|,\end{aligned}$$

where $\mu(y,t)$ is some continuous on $[0,l]\times[0,T]$ function. To find the limit

$$\lim_{t\to 0} t^{\frac{\beta-1}{2}} \int_0^t \int_0^l G_{22}(0,y_0,t,0,\eta,\tau)\mu(\eta,\tau)\,d\eta\,d\tau,$$

we decompose the Green function into two summands

$$G_{22}(0,y_0,t,0,\eta,\tau) = \frac{1}{\pi(\theta(t)-\theta(\tau))} \exp\Big(-\frac{(y_0-\eta)^2}{4(\theta(t)-\theta(\tau))}\Big) + G_{22}^0(0,y_0,t,0,\eta,\tau).$$

It follows from the Green function estimates (3.7) that

$$\lim_{t\to 0} t^{\frac{\beta-1}{2}} \int_0^t \int_0^l G_{22}^0(0,y_0,t,0,\eta,\tau)\mu(\eta,\tau)\,d\eta\,d\tau = 0.$$

On the other hand, using (3.10) we find

$$\begin{aligned}&\lim_{t\to 0} t^{\frac{\beta-1}{2}} \int_0^t \int_0^l \frac{1}{\pi(\theta(t)-\theta(\tau))} \Big(\exp\Big(-\frac{(y_0-\eta)^2}{4(\theta(t)-\theta(\tau))}\Big) \\ &+ \exp\Big(-\frac{(y_0+\eta)^2}{4(\theta(t)-\theta(\tau))}\Big)\Big)\mu(\eta,\tau)\,d\eta\,d\tau \\ &= \sqrt{\frac{\beta+1}{\pi}} \frac{\mu(y_0,0)}{\sqrt{a(0)}} \int_0^1 \frac{dz}{\sqrt{1-z^{\beta+1}}}.\end{aligned}$$

It means that $\forall\varepsilon>0$ there exists such $t_0\in(0,T]$ that

$$\Big| t^{\frac{\beta-1}{2}} \int_0^t \int_0^l G_{22}(0,y_0,t,0,\eta,\tau)\mu(\eta,\tau)\,d\eta\,d\tau - \sqrt{\frac{\beta+1}{\pi}} \frac{\mu(y_0,0)}{\sqrt{a(0)}} \int_0^1 \frac{dz}{\sqrt{1-z^{\beta+1}}} \Big| < \varepsilon,$$

when $0<t<t_0$. Therefore, if $t_i<t_0, i\in\{1,2\}$, then $\Delta<2\varepsilon$.

Consider the case when $t_i \ge t_0, i\in\{1,2\}$ and suppose for the definiteness that $t_1<t_2$. Present Δ as follows:

$$\begin{aligned}\Delta =& \Big| t_2^{\frac{\beta-1}{2}} \int_{t_1}^{t_2} \int_0^l G_{22}(0,y_0,t_2,0,\eta,\tau)\mu(\eta,\tau)\,d\eta\,d\tau \\ &+ \int_0^{t_1} \int_0^l \big(t_2^{\frac{\beta-1}{2}} G_{22}(0,y_0,t_2,0,\eta,\tau) - t_1^{\frac{\beta-1}{2}} G_{22}(0,y_0,t_1,0,\eta,\tau)\big)\mu(\eta,\tau)\,d\eta\,d\tau \Big|.\end{aligned} \tag{3.11}$$

Using (3.7), we obtain

$$
\begin{aligned}
&\big|t_2^{\frac{\beta-1}{2}}\int_{t_1}^{t_2}\int_0^l G_{22}(0,y_0,t_2,0,\eta,\tau)\mu(\eta,\tau)\,d\eta\,d\tau\big| \\
&\le t_2^{\frac{\beta-1}{2}}\int_{t_1}^{t_2}\int_0^l\Big(\frac{C_{19}}{\sqrt{\theta(t_2)-\theta(\tau)}}+C_{20}\Big)d\tau \\
&\le C_{21}|t_2-t_1|+C_{22}t_2^{\frac{\beta-1}{2}}\int_{t_1}^{t_2}\frac{d\tau}{\sqrt{t_2^{\beta+1}-\tau^{\beta+1}}} \\
&= C_{21}|t_2-t_1|+C_{22}\int_{\frac{t_1}{t_2}}^{1}\frac{dz}{\sqrt{1-z^{\beta+1}}} \\
&\le C_{21}|t_2-t_1|+C_{22}\int_{\frac{t_1}{t_2}}^{1}\frac{dz}{\sqrt{1-z}} \\
&\le C_{21}|t_2-t_1|+C_{23}\sqrt{t_2-t_1}.
\end{aligned}
$$

Consider the second summand from (3.11):

$$
\begin{aligned}
&\big|\int_0^{t_1}\int_0^l\big(t_2^{\frac{\beta-1}{2}}G_{22}(0,y_0,t_2,0,\eta,\tau)-t_1^{\frac{\beta-1}{2}}G_{22}(0,y_0,t_1,0,\eta,\tau)\big)\mu(\eta,\tau)\,d\eta\,d\tau\big| \\
&\le \big|t_2^{\frac{\beta-1}{2}}-t_1^{\frac{\beta-1}{2}}\big|\,\big|\int_0^{t_1}\int_0^l G_{22}(0,y_0,t_2,0,\eta,\tau)\mu(\eta,\tau)\,d\eta\,d\tau\big| \\
&\quad + t_1^{\frac{\beta-1}{2}}\int_0^{t_1}\int_0^l |G_{22}(0,y_0,t_2,0,\eta,\tau)-G_{22}(0,y_0,t_1,0,\eta,\tau)||\mu(\eta,\tau)|\,d\eta\,d\tau \\
&:= \Delta_1+\Delta_2.
\end{aligned}
$$

Substituting the product of two one-dimensional Green functions instead of the Green function $G_{22}(0,y_0,t_2,0,\eta,\tau)$ and taking into account (3.5), (3.7), we find

$$
\begin{aligned}
\Delta_1 &\le C_{24}\big|t_2^{\frac{\beta-1}{2}}-t_1^{\frac{\beta-1}{2}}\big|\int_0^{t_1}\int_0^l G_2(0,t_2,0,\tau)G_2(y_0,t_2,\eta,\tau)\,d\eta\,d\tau \\
&\le C_{24}\big|t_2^{\frac{\beta-1}{2}}-t_1^{\frac{\beta-1}{2}}\big|\int_0^{t_1}G_2(0,t_2,0,\tau)d\tau \\
&\le C_{25}\big|t_2^{\frac{\beta-1}{2}}-t_1^{\frac{\beta-1}{2}}\big|t_1^{\frac{1-\beta}{2}} = C_{25}\big|\big(\frac{t_2}{t_1}\big)^{\frac{\beta-1}{2}}-1\big|<\epsilon,
\end{aligned}
$$

when $|t_2-t_1|<\delta_1$. By the same reasoning we obtain

$$
\begin{aligned}
\Delta_2 &\le C_{24}t_1^{\frac{\beta-1}{2}}\Big(\int_0^{t_1}\int_0^l|G_2(0,t_2,0,\tau)-G_2(0,t_1,0,\tau)|G_2(y_0,t_2,\eta,\tau)\,d\eta\,d\tau \\
&\quad+\int_0^{t_1}\int_0^l G_2(0,t_1,0,\tau)|G_2(y_0,t_2,\eta,\tau)-G_2(y_0,t_1,\eta,\tau)|\,d\eta\,d\tau\Big) \\
&\le C_{24}t_1^{\frac{\beta-1}{2}}\Big(\int_0^{t_1}|G_2(0,t_2,0,\tau)-G_2(0,t_1,0,\tau)|d\tau \\
&\quad+\int_0^{t_1}\int_0^l G_2(0,t_1,0,\tau)|G_2(y_0,t_2,\eta,\tau)-G_2(y_0,t_1,\eta,\tau)|\,d\eta\,d\tau\Big) \\
&:= \Delta_{2,1}+\Delta_{2,2}\,.
\end{aligned}
$$

Now we put the Green function $G_2(0,t,0,\tau)$ in the form

$$G_2(0,t,0,\tau)=\frac{1}{\sqrt{\pi(\theta(t)-\theta(\tau))}}+\frac{2}{\sqrt{\pi(\theta(t)-\theta(\tau))}}\sum_{n=1}^{\infty}\exp\Big(-\frac{n^2h^2}{\theta(t)-\theta(\tau)}\Big).$$

Then

$$\begin{aligned}\Delta_{2,1}\le C_{24}t_1^{\frac{\beta-1}{2}}\Big(&\frac{1}{\sqrt{\pi}}\int_0^{t_1}\Big(\frac{1}{\sqrt{\theta(t_1)-\theta(\tau)}}-\frac{1}{\sqrt{\theta(t_2)-\theta(\tau)}}\Big)d\tau\\ &+\int_0^{t_1}\Big|\sum_{n=1}^{\infty}\Big(\frac{2}{\sqrt{\pi(\theta(t_1)-\theta(\tau))}}\exp\Big(-\frac{n^2h^2}{\theta(t_1)-\theta(\tau)}\Big)\\ &-\frac{2}{\sqrt{\pi(\theta(t_2)-\theta(\tau))}}\exp\Big(-\frac{n^2h^2}{\theta(t_2)-\theta(\tau)}\Big)\Big)\Big|d\tau\Big).\end{aligned}\tag{3.12}$$

Now we transform and estimate the first summand:

$$\begin{aligned}&C_{26}t_1^{\frac{\beta-1}{2}}\int_0^{t_1}\Big(\frac{1}{\sqrt{\theta(t_1)-\theta(\tau)}}-\frac{1}{\sqrt{\theta(t_2)-\theta(\tau)}}\Big)d\tau\\ &=C_{26}t_1^{\frac{\beta-1}{2}}\int_0^{t_1}\frac{\theta(t_2)-\theta(t_1)}{\sqrt{(\theta(t_1)-\theta(\tau))}}\frac{d\tau}{(\theta(t_2)-\theta(\tau))(\sqrt{\theta(t_1)-\theta(\tau)}+\sqrt{\theta(t_2)-\theta(\tau)}}\\ &\le C_{27}t_1^{\frac{\beta-1}{2}}\int_0^{t_1}\frac{(t_2^{\beta+1}-t_1^{\beta+1})d\tau}{\big(\sqrt{t_1^{\beta+1}-\tau^{\beta+1}}+\sqrt{t_2^{\beta+1}-\tau^{\beta+1}}\big)\sqrt{\big(t_1^{\beta+1}-\tau^{\beta+1}\big)\big(t_2^{\beta+1}-\tau^{\beta+1}\big)}}\\ &=C_{27}t_1^{\frac{\beta-1}{2}}\int_0^{t_1}\Big(\frac{1}{\sqrt{t_1^{\beta+1}-\tau^{\beta+1}}}-\frac{1}{\sqrt{t_2^{\beta+1}-\tau^{\beta+1}}}\Big)d\tau.\end{aligned}$$

After the change of variable $z=\tau/t_1$ we obtain

$$\begin{aligned}&C_{26}t_1^{\frac{\beta-1}{2}}\int_0^{t_1}\Big(\frac{1}{\sqrt{\theta(t_1)-\theta(\tau)}}-\frac{1}{\sqrt{\theta(t_2)-\theta(\tau)}}\Big)d\tau\\ &\le C_{27}\int_0^1\Big(\frac{1}{\sqrt{1-z^{\beta+1}}}\\ &-\frac{1}{\sqrt{(\frac{t_2}{t_1})^{\beta+1}-z^{\beta+1}}}\Big)dz<\epsilon,\quad\text{when }\Big|\frac{t_2}{t_1}-1\Big|<\delta_2,\end{aligned}\tag{3.13}$$

as the function

$$I(\sigma):=\int_0^1\frac{dz}{\sqrt{\sigma-z^{\beta+1}}}$$

is continuous for $\sigma\ge 1$.

Now we estimate the second summand from (3.12):

$$\begin{aligned}&C_{26}t_1^{\frac{\beta-1}{2}}\int_0^{t_1}\Big|\sum_{n=1}^{\infty}\Big(\frac{2}{\sqrt{\pi(\theta(t_1)-\theta(\tau))}}\exp\Big(-\frac{n^2h^2}{\theta(t_1)-\theta(\tau)}\Big)\\ &-\frac{2}{\sqrt{\pi(\theta(t_2)-\theta(\tau))}}\exp\Big(-\frac{n^2h^2}{\theta(t_2)-\theta(\tau)}\Big)\Big)\Big|d\tau\\ &=\frac{2C_{26}}{\sqrt{\pi}}t_1^{\frac{\beta-1}{2}}\int_0^{t_1}\Big|\int_{\theta(t_2)-\theta(\tau)}^{\theta(t_1)-\theta(\tau)}\frac{\partial}{\partial z}\Big(\frac{1}{\sqrt{z}}\sum_{n=1}^{\infty}\exp\Big(-\frac{n^2h^2}{z}\Big)dz\Big|d\tau\end{aligned}$$

$$\le C_{28}(\theta(t_2)-\theta(t_1)) \le C_{29}(t_2^{\beta+1}-t_1^{\beta+1}) < \epsilon, \quad \text{when } |t_2-t_1|<\delta_3.$$

To estimate $\Delta_{2,2}$, we extract the main term from the Green function,

$$\begin{aligned}
\Delta_{2,2} \le C_{24} t_1^{\frac{\beta-1}{2}} \Big(\int_0^{t_1} G_2(0,t_1,0,\tau)d\tau \\
&\times \int_0^l \Big| \frac{1}{\sqrt{\theta(t_2)-\theta(\tau)}} \exp\Big(-\frac{(\eta-y_0)^2}{4(\theta(t_2)-\theta(\tau))}\Big) \\
&- \frac{1}{\sqrt{\theta(t_1)-\theta(\tau)}} \exp\Big(-\frac{(\eta-y_0)^2}{4(\theta(t_1)-\theta(\tau))}\Big)\Big| d\eta \\
&+ \int_0^{t_1} G_2(0,t_1,0,\tau)d\tau \int_0^l |G_2^0(y_0,t_2,\eta,\tau)-G_2^0(y_0,t_1,\eta,\tau)|d\eta\Big).
\end{aligned} \tag{3.14}$$

Taking into account that $G_2^0(y_0,t,\eta,\tau)$ has no singularity, we find

$$\begin{aligned}
\int_0^l |G_2^0(y_0,t_2,\eta,\tau)-G_2^0(y_0,t_1,\eta,\tau)|d\eta &\le \int_0^l \Big|\int_{t_1}^{t_2} G_{2_t}^0(y_0,t,\eta,\tau)dt\Big| d\eta \\
&\le C_{30}|t_1-t_2|.
\end{aligned}$$

We consider the expression

$$\begin{aligned}
&\int_0^l \Big| \frac{1}{\sqrt{\theta(t_2)-\theta(\tau)}} \exp\Big(-\frac{(\eta-y_0)^2}{4(\theta(t_2)-\theta(\tau))}\Big) \\
&- \frac{1}{\sqrt{\theta(t_1)-\theta(\tau)}} \exp\Big(-\frac{(\eta-y_0)^2}{4(\theta(t_1)-\theta(\tau))}\Big)\Big| d\eta \\
&\le \int_0^l \frac{1}{\sqrt{\theta(t_2)-\theta(\tau)}} \Big(\exp\Big(-\frac{(\eta-y_0)^2}{4(\theta(t_2)-\theta(\tau))}\Big) \\
&\quad - \exp\Big(-\frac{(\eta-y_0)^2}{4(\theta(t_1)-\theta(\tau))}\Big)\Big) d\eta \\
&\quad + \int_0^l \exp\Big(-\frac{(\eta-y_0)^2}{4(\theta(t_1)-\theta(\tau))}\Big)\Big(\frac{1}{\sqrt{\theta(t_1)-\theta(\tau)}} - \frac{1}{\sqrt{\theta(t_2)-\theta(\tau)}}\Big) d\eta\,.
\end{aligned} \tag{3.15}$$

After the change of variable $z=\frac{\eta-y_0}{2\sqrt{\theta(t_2)-\theta(\tau)}}$ in the first summand we have

$$\begin{aligned}
&\int_0^l \Big| \frac{1}{\sqrt{\theta(t_2)-\theta(\tau)}} \exp\Big(-\frac{(\eta-y_0)^2}{4(\theta(t_2)-\theta(\tau))}\Big) \\
&- \frac{1}{\sqrt{\theta(t_1)-\theta(\tau)}} \exp\Big(-\frac{(\eta-y_0)^2}{4(\theta(t_1)-\theta(\tau))}\Big)\Big| d\eta \\
&= 2\int_{\frac{-y_0}{2\sqrt{\theta(t_2)-\theta(\tau)}}}^{\frac{l-y_0}{2\sqrt{\theta(t_2)-\theta(\tau)}}} \Big(\exp(-z^2) - \exp\Big(-z^2\frac{\theta(t_2)-\theta(\tau)}{\theta(t_1)-\theta(\tau)}\Big)\Big) dz \\
&\le 2\int_{-\infty}^{\infty} \Big(\exp(-z^2) - \exp\Big(-z^2\frac{\theta(t_2)-\theta(\tau)}{\theta(t_1)-\theta(\tau)}\Big)\Big) dz \\
&= 2\sqrt{\pi}\Big(1-\frac{\sqrt{\theta(t_1)-\theta(\tau)}}{\sqrt{\theta(t_2)-\theta(\tau)}}\Big).
\end{aligned}$$

The change of variable $z = \frac{\eta - y_0}{2\sqrt{\theta(t_1)-\theta(\tau)}}$ in the second summand gives

$$\begin{aligned}
&\int_0^l \exp\Big(-\frac{(\eta-y_0)^2}{4(\theta(t_1)-\theta(\tau))}\Big)\Big(\frac{1}{\sqrt{\theta(t_1)-\theta(\tau)}} - \frac{1}{\sqrt{\theta(t_2)-\theta(\tau)}}\Big)d\eta \\
&= 2\Big(1 - \frac{\sqrt{\theta(t_1)-\theta(\tau)}}{\sqrt{\theta(t_2)-\theta(\tau)}}\Big)\int_{\frac{-y_0}{2\sqrt{\theta(t_2)-\theta(\tau)}}}^{\frac{l-y_0}{2\sqrt{\theta(t_2)-\theta(\tau)}}} e^{-z^2}dz \\
&\le 2\sqrt{\pi}\Big(1 - \frac{\sqrt{\theta(t_1)-\theta(\tau)}}{\sqrt{\theta(t_2)-\theta(\tau)}}\Big).
\end{aligned}$$

Now we return to the estimate of the first summand in (3.14), using (3.7) and (3.13),

$$\begin{aligned}
&C_{24}t_1^{\frac{\beta-1}{2}}\int_0^{t_1} G_2(0,t_1,0,\tau)d\tau \int_0^l \Big|\frac{1}{\sqrt{\theta(t_2)-\theta(\tau)}}\exp\Big(-\frac{(\eta-y_0)^2}{4(\theta(t_2)-\theta(\tau))}\Big) \\
&- \frac{1}{\sqrt{\theta(t_1)-\theta(\tau)}}\exp\Big(-\frac{(\eta-y_0)^2}{4(\theta(t_1)-\theta(\tau))}\Big)\Big|d\eta \\
&\le C_{31}t_1^{\frac{\beta-1}{2}}\int_0^{t_1}\Big(\frac{1}{\sqrt{\theta(t_1)-\theta(\tau)}} - \frac{1}{\sqrt{\theta(t_2)-\theta(\tau)}}\Big)d\tau \\
&\quad + C_{32}t_1^{\frac{\beta-1}{2}}\int_0^{t_1}\Big(1 - \frac{\sqrt{\theta(t_1)-\theta(\tau)}}{\sqrt{\theta(t_2)-\theta(\tau)}}\Big)d\tau < \epsilon,
\end{aligned}$$

when $|\frac{t_2}{t_1} - 1| < \delta_4$.

So, we obtained the estimate of Δ. Other expressions from $Pa(t)$ are estimated by a similar way. Therefore, the conditions of Schauder fixed-point theorem are satisfied for (3.9), and the existence of the solution to the problem (2.1)–(2.4) is proved.

4. Uniqueness of the solution

Consider the function

$$H(t) := \frac{\sqrt{\pi}\kappa(t)}{\sqrt{\beta+1}\int_0^t \frac{\mu_0(\tau)d\tau}{\sqrt{t^{\beta+1}-\tau^{\beta+1}}}},$$

where

$$\mu_0(t) := \int_0^l G_2(y_0,t,\eta,\tau)(f(0,\eta,\tau) - \mu_{1_\tau}(\eta,\tau))d\eta. \tag{4.1}$$

It is easy to see that there exists the limit

$$\lim_{t\to+0} H(t) = \frac{\sqrt{\pi}\kappa_0(0)}{\sqrt{\beta+1}\mu_0(0)\int_0^t \frac{d\sigma}{\sqrt{1-\sigma^{\beta+1}}}} > 0.$$

It means that $H(t)$ is continuous function on $[0,T]$.

Using the function $H(t)$, we will establish the estimates of solutions for the equation (3.3). Taking into account assumption (A2), from (3.3) we have

$$a(t) \le \frac{\kappa(t)}{\int_0^t\int_0^l G_{22}(0,y_0,t,0,\eta,\tau)(f(0,\eta,\tau) - \mu_{1_\tau}(\eta,\tau))\,d\eta\,d\tau}$$

$$
\begin{aligned}
&= \frac{\kappa(t)}{\int_0^t G_2(0,t,0,\tau)\mu_0(\tau)d\tau} \le \frac{\kappa(t)\sqrt{\pi\tilde a_{\max}(t)}}{\sqrt{\beta+1}\int_0^t \frac{\mu_0(\tau)d\tau}{\sqrt{t^{\beta+1}-\tau^{\beta+1}}}} \\
&\le \sqrt{\tilde a_{\max}(t)}H_{\max}(t),
\end{aligned}
$$

where $\tilde a_{\max}(t) := \max_{[0,t]} a(\tau)$, $H_{\max}(t) := \max_{[0,t]} H(\tau)$. From this follows that

$$
a(t) \le H^2_{\max}(t), \quad t\in[0,T]. \tag{4.2}
$$

To estimate $a(t)$ from below, we put the denominator from (3.3) as a sum $\frac{1}{\sqrt{\pi}}\int_0^t \frac{\mu_0(\tau)d\tau}{\sqrt{\theta(t)-\theta(\tau)}} + S(t)$. Using the notation $\tilde a_{\min}(t) := \min_{[0,t]} a(\tau)$, $H_{\min}(t) := \min_{[0,t]} H(\tau)$ we obtain

$$
\frac{1}{\sqrt{\pi}}\int_0^t \frac{\mu_0(\tau)d\tau}{\sqrt{\theta(t)-\theta(\tau)}} \le \frac{\sqrt{\beta+1}}{\sqrt{\pi\tilde a_{\min}(t)}}\int_0^t \frac{\mu_0(\tau)d\tau}{\sqrt{t^{\beta+1}-\tau^{\beta+1}}} = \frac{\kappa(t)}{H(t)\sqrt{\tilde a_{\min}(t)}}.
$$

Repeating the reasons of the estimation of the denominator of (3.3) from above, we establish that $S(t)\le C_{33}$. Then

$$
a(t) \ge \frac{\kappa(t)}{\frac{\kappa(t)}{H(t)\sqrt{\tilde a_{\min}(t)}} + C_{33}} = \frac{H(t)\sqrt{\tilde a_{\min}(t)}}{1+\frac{C_{33}H(t)\sqrt{\tilde a_{\min}(t)}}{\kappa(t)}}.
$$

It follows from the assumptions (A2) and the estimate (4.2) that

$$
\frac{C_{33}H(t)\sqrt{\tilde a_{\min}(t)}}{\kappa(t)} \le C_{34}t^{\frac{\beta-1}{2}}.
$$

Therefore,

$$
a(t) \ge \frac{H_{\min}(t)\sqrt{\tilde a_{\min}(t)}}{1+C_{34}t^{\frac{\beta-1}{2}}}, \quad t\in[0,T],
$$

or

$$
a(t) \ge \frac{H^2_{\min}(t)}{(1+C_{34}t^{\frac{\beta-1}{2}})^2}, \quad t\in[0,T]. \tag{4.3}
$$

Having the estimates (4.2) and (4.3) of solutions of equation (3.3), we can pass to the proof of the uniqueness of solution. Put the equation (3.3) in the form

$$
a(t) = \frac{\kappa(t)}{\frac{1}{\sqrt{\pi}}\int_0^t \frac{\mu_0(\tau)d\tau}{\sqrt{\theta(t)-\theta(\tau)}} + S(t,a(t))}, \quad t\in[0,T]. \tag{4.4}
$$

Suppose that this equation has two solutions $a_1(t)$ and $a_2(t)$. Denote $b(t) := a_1(t) - a_2(t)$. The function $b(t)$ satisfies the equation

$$
\begin{aligned}
b(t) &= \kappa(t)\Big(\frac{1}{\sqrt{\pi}}\int_0^t\Big(\frac{1}{\sqrt{\theta_2(t)-\theta_2(\tau)}} - \frac{1}{\sqrt{\theta_1(t)-\theta_1(\tau)}}\Big)\mu_0(\tau)d\tau + S(t,a_2(t)) \\
&\quad - S(t,a_1(t))\Big)\Big(\frac{1}{\sqrt{\pi}}\int_0^t \frac{\mu_0(\tau)d\tau}{\sqrt{\theta_1(t)-\theta_1(\tau)}} + S(t,a_1(t))\Big)^{-1} \\
&\quad \times\Big(\frac{1}{\sqrt{\pi}}\int_0^t \frac{\mu_0(\tau)d\tau}{\sqrt{\theta_2(t)-\theta_2(\tau)}} + S(t,a_2(t))\Big)^{-1}.
\end{aligned} \tag{4.5}
$$

It is clear from above that $S(t,a_i(t))\geq 0, i\in\{1,2\}$. Then it follows from (4.5) that

$$\begin{aligned}
&|b(t)|\\
&\leq \kappa(t)\Big(\frac{1}{\sqrt{\pi}}\int_0^t \Big|\frac{1}{\sqrt{\theta_2(t)-\theta_2(\tau)}}-\frac{1}{\sqrt{\theta_1(t)-\theta_1(\tau)}}\Big|\mu_0(\tau)d\tau+|S(t,a_2(t))\\
&\quad -S(t,a_1(t))|\Big)\Big(\frac{1}{\pi}\int_0^t\frac{\mu_0(\tau)d\tau}{\sqrt{\theta_1(t)-\theta_1(\tau)}}\int_0^t\frac{\mu_0(\tau)d\tau}{\sqrt{\theta_2(t)-\theta_2(\tau)}}\Big)^{-1}.
\end{aligned}\tag{4.6}$$

Transform the expression

$$\begin{aligned}
&\frac{1}{\sqrt{\theta_2(t)-\theta_2(\tau)}}-\frac{1}{\sqrt{\theta_1(t)-\theta_1(\tau)}}\\
&=\frac{\theta_2(t)-\theta_2(\tau)-(\theta_1(t)-\theta_1(\tau))}{\sqrt{(\theta_1(t)-\theta_1(\tau))(\theta_2(t)-\theta_2(\tau))}}\frac{1}{\sqrt{\theta_1(t)-\theta_1(\tau)}+\sqrt{\theta_2(t)-\theta_2(\tau)}}.
\end{aligned}$$

Using estimate (4.3), we find

$$\Big|\frac{1}{\sqrt{\theta_2(t)-\theta_2(\tau)}}-\frac{1}{\sqrt{\theta_1(t)-\theta_1(\tau)}}\Big|\leq\frac{\sqrt{\beta+1}(1+C_{34}t^{\frac{\beta-1}{2}})^3}{2H^3_{\min}(t)\sqrt{t^{\beta+1}-\tau^{\beta+1}}}b_{\max}(t),$$

where $b_{\max}(t):=\max_{[0,t]}|b(\tau)|$. Therefore, we obtain

$$\begin{aligned}
&\frac{\kappa(t)}{\sqrt{\pi}}\int_0^t\Big|\frac{1}{\sqrt{\theta_2(t)-\theta_2(\tau)}}-\frac{1}{\sqrt{\theta_1(t)-\theta_1(\tau)}}\Big|\mu_0(\tau)d\tau\\
&\leq\frac{\kappa^2(t)(1+C_{34}t^{\frac{\beta-1}{2}})^3}{2H^4_{\min}(t)}b_{\max}(t).
\end{aligned}\tag{4.7}$$

Now we estimate the difference $S(t,a_2(t))-S(t,a_1(t))$. Consider

$$\begin{aligned}
\Delta_1R:=&\frac{1}{2\pi}\Big|\int_0^h\int_0^l\Big(\frac{1}{\theta_2(t)}\sum_{m,n=-\infty}^{\infty}\exp\Big(-\frac{(\xi+2nh)^2}{4\theta_2(t)}\Big)\\
&\times\Big(\exp\Big(-\frac{(y_0-\eta+2ml)^2}{4\theta_2(t)}\Big)+\exp\Big(-\frac{(y_0+\eta+2ml)^2}{4\theta_2(t)}\Big)\Big)\\
&-\frac{1}{\theta_1(t)}\sum_{m,n=-\infty}^{\infty}\exp\Big(-\frac{(\xi+2nh)^2}{4\theta_1(t)}\Big)\Big(\exp\Big(-\frac{(y_0-\eta+2ml)^2}{4\theta_1(t)}\Big)\\
&+\exp\Big(-\frac{(y_0+\eta+2ml)^2}{4\theta_1(t)}\Big)\Big)\Big)\varphi_\xi(\xi,\eta)d\eta d\xi\Big|\\
\leq& C_{35}\int_0^h\int_0^l\Big|\int_{\theta_1(t)}^{\theta_2(t)}\frac{\partial}{\partial z}\Big(\frac{1}{z}\sum_{m,n=-\infty}^{\infty}\exp\Big(-\frac{(\xi+2nh)^2}{4z}\Big)\\
&\times\Big(\exp\Big(-\frac{(y_0-\eta+2ml)^2}{4z}\Big)+\exp\Big(-\frac{(y_0+\eta+2ml)^2}{4z}\Big)\Big)\Big)dz\Big|d\eta d\xi.
\end{aligned}$$

After differentiating and using (3.5) we obtain

$$\begin{aligned}
\Delta_1R\leq C_{36}\Big|&\int_{\theta_2(t)}^{\theta_1(t)}dz\int_0^h\int_0^l\Big(\frac{1}{z^2}\sum_{m,n=-\infty}^{\infty}\exp\Big(-\frac{(\xi+2nh)^2}{4z}\Big)\\
&\times\Big(\exp\Big(-\frac{(y_0-\eta+2ml)^2}{4z}\Big)+\exp\Big(-\frac{(y_0+\eta+2ml)^2}{4z}\Big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{z^3}\sum_{m,n=-\infty}^{\infty}(\xi+2nh)^2\exp\Big(-\frac{(\xi+2nh)^2}{4z}\Big)\\
&\times\Big((y_0-\eta+2ml)^2\exp\Big(-\frac{(y_0-\eta+2ml)^2}{4z}\Big)+(y_0+\eta+2ml)^2\\
&\times\exp\Big(-\frac{(y_0+\eta+2ml)^2}{4z}\Big)\Big)\Big)d\eta d\xi\Big|\\
\le{}& C_{37}\Big|\int_{\theta_2(t)}^{\theta_1(t)}\frac{dz}{z}\Big|\le C_{37}\frac{|\theta_1(t)-\theta_2(t)|}{\min\{\theta_1(t),\theta_2(t)\}}.
\end{aligned}
$$

Applying (4.3), we arrive at the estimate

$$
\Delta_1 R\le C_{38}\Big(\frac{(1+C_{34}t^{\frac{\beta-1}{2}})^2}{t^{\beta-1}H^2_{\rm min}(t)}+1\Big)t^{\beta-1}b_{\rm max}(t)\le C_{39}b_{\rm max}(t).
$$

Now we estimate the expression

$$
\begin{aligned}
\Delta_2 R:={}&\frac{1}{\sqrt{\pi}}\Big|\int_0^t\sum_{n=1}^{\infty}\Big(\frac{1}{\sqrt{\theta_2(t)-\theta_2(\tau)}}\exp\Big(-\frac{n^2h^2}{\theta_2(t)-\theta_2(\tau)}\Big)-\frac{1}{\sqrt{\theta_1(t)-\theta_1(\tau)}}\\
&\times\exp\Big(-\frac{n^2h^2}{\theta_1(t)-\theta_1(\tau)}\Big)\Big)\mu_0(\tau)d\tau\Big|\\
&\le C_{40}\int_0^t d\tau\Big|\int_{\theta_1(t)-\theta_1(\tau)}^{\theta_2(t)-\theta_2(\tau)}\Big|\frac{\partial}{\partial z}\Big(\frac{1}{z}\sum_{n=1}^{\infty}\exp\Big(-\frac{n^2h^2}{z}\Big)\Big|dz\Big|\\
&\le C_{41}|\theta_1(t)-\theta_2(t)|\le C_{43}b_{\rm max}(t).
\end{aligned}
$$

Other summands from $S(t,a_2(t))-S(t,a_1(t))$ are estimated in a similar way. Hence,

$$
|S(t,a_1(t))-S(t,a_2(t))|\le C_{42}b_{\rm max}(t). \tag{4.8}
$$

To estimate the denominator in (4.6), note that

$$
\frac{\kappa(t)}{\frac{1}{\sqrt{\pi}}\int_0^t\frac{\mu_0(\tau)d\tau}{\sqrt{\theta_i(t)-\theta_i(\tau)}}}\le\frac{\kappa(t)\sqrt{a_{\rm max}(t)}}{\frac{\sqrt{\beta+1}}{\sqrt{\pi}}\int_0^t\frac{\mu_0(\tau)d\tau}{\sqrt{t^{\beta+1}-\tau^{\beta+1}}}}\le H^2_{\rm max}(t), \tag{4.9}
$$

for $i\in\{1,2\}$.

By applying estimates (4.7)–(4.9) to (4.6), we obtain

$$
\begin{aligned}
|b(t)|&\le\Big(\frac{(1+C_{34}t^{\frac{\beta-1}{2}})^3}{2H^4_{\rm min}(t)}+C_{43}t^{\frac{\beta-1}{2}}\Big)H^4_{\rm max}(t)b_{\rm max}(t)\\
&\le\Big(\frac{(1+C_{34}t^{\frac{\beta-1}{2}})^3H^4_{\rm max}(t)}{2H^4_{\rm min}(t)}+C_{42}t^{\frac{\beta-1}{2}}H^4_{\rm max}(t)\Big)b_{\rm max}(t).
\end{aligned} \tag{4.10}
$$

As $\lim_{t\to 0}H_{\rm min}(t)=\lim_{t\to 0}H_{\rm max}(t)$, there exists such a number $t_0\in(0,T]$ that the following inequality holds:

$$
\Big(\frac{(1+C_{34}t^{\frac{\beta-1}{2}})^3H^4_{\rm max}(t)}{2H^4_{\rm min}(t)}+C_{42}t^{\frac{\beta-1}{2}}H^4_{\rm max}(t)\Big)<1,\quad t\in[0,t_0]. \tag{4.11}
$$

Then we conclude from (4.10) that $b_{\rm max}(t)\le 0$, $t\in[0,t_0]$, that is impossible. Therefore, $b(t)\equiv 0, t\in[0,t_0]$. It follows from the uniqueness of solution to problem (2.1)–(2.5) that the function $u(x,y,t)$ is uniquely defined in $\overline{Q}_{t_0}$.

Now let us show that the solution to (2.1)–(2.5) is unique in the whole domain $\overline{Q}_T$. Suppose that there exist two solutions $(a_i(t), u_i(x,y,t))$, $i \in \{1,2\}$ to the problem. For their difference $a := a_1 - a_2, u := u_1 - u_2$ we obtain the problem

$$u_t = t^\beta a_1(t)\Delta u + t^\beta a(t)\Delta u_2(x,y,t), \quad (x,y,t) \in Q_T, \tag{4.12}$$

$$u(x,y,0) = 0, \quad (x,y) \in \overline{D}, \tag{4.13}$$

$$u(0,y,t) = u(h,y,t) = 0, \quad (y,t) \in [0,l] \times [0,T], \tag{4.14}$$

$$u_y(x,0,t) = u_y(x,l,t) = 0, \quad (x,t) \in [0,h] \times [0,T], \tag{4.15}$$

$$a_1(t)u_x(0,y_0,t) = -a(t)u_{2x}(0,y_0,t), \quad t \in [0,T]. \tag{4.16}$$

Using the Green function $\tilde{G}_{12}(x,y,t,\xi,\eta,\tau)$ of the problem (4.12)–(4.15), we find that

$$u(x,y,t) = \int_0^t \int_D \tilde{G}_{12}(x,y,t,\xi,\eta,\tau)\tau^\beta a(\tau)\Delta u_2(\xi,\eta,\tau)\, d\xi\, d\eta\, d\tau. \tag{4.17}$$

Substituting it into the overdetermination condition (4.16):

$$a(t) = -\frac{a_1(t)}{u_{2x}(0,y_0,t)} \int_0^t \int_0^l \int_0^h \tilde{G}_{12_x}(0,y_0,t,\xi,\eta,\tau)\tau^\beta a(\tau)\Delta u_2(\xi,\eta,\tau)\, d\xi\, d\eta\, d\tau, \tag{4.18}$$

for $t \in [0,T]$. Thus, we obtain a homogeneous Volterra integral equation of the second kind for $a(t)$. Put it as

$$a(t) = \int_0^t K(t,\tau)a(\tau)d\tau, \quad t \in [0,T]. \tag{4.19}$$

To study the behavior of the kernel $K(t,\tau)$, we find

$$\begin{aligned} u_2(x,y,t) &= \int_0^l \int_0^h \hat{G}_{12}(x,y,t,\xi,\eta,0)\varphi(\xi,\eta)d\xi d\eta \\ &\quad + \int_0^t \int_0^l \hat{G}_{12_\xi}(x,y,t,0,\eta,\tau)\tau^\beta a(\tau)\mu_1(\eta,\tau)\, d\eta\, d\tau \\ &\quad - \int_0^t \int_0^l \hat{G}_{12_\xi}(x,y,t,h,\eta,\tau)\tau^\beta a(\tau)\mu_2(\eta,\tau)\, d\eta\, d\tau \\ &\quad - \int_0^t \int_0^h \hat{G}_{12}(x,y,t,\xi,0,\tau)\tau^\beta a(\tau)\nu_1(\xi,\tau)d\xi d\tau \\ &\quad + \int_0^t \int_0^h \hat{G}_{12}(x,y,t,\xi,l,\tau)\tau^\beta a(\tau)\nu_2(\xi,\tau)d\xi d\tau \\ &\quad + \int_0^t \int_0^l \int_0^h \hat{G}_{12}(x,y,t,\xi,\eta,\tau)f(\xi,\eta,\tau)\, d\xi\, d\eta\, d\tau, \end{aligned} \tag{4.20}$$

where $\hat{G}_{12}(x,y,t,\xi,\eta,\tau)$ is the Green function of problem (2.2)–(2.4) for the equation

$$u_t = t^\beta a_2(t)\Delta u + f(x,y,t).$$

By differentiating (4.20) twice by x, y and integrating by parts, we obtain

$u_{2_{xx}}(x,y,t)$

$$
\begin{aligned}
&= \int_0^l \int_0^h \hat{G}_{12}(x,y,t,\xi,\eta,0)\varphi_{\xi\xi}(\xi,\eta)d\xi d\eta \\
&\quad + \int_0^t \int_0^l \hat{G}_{12_\xi}(x,y,t,0,\eta,\tau)(\mu_{1_\tau}(\eta,\tau) - \tau^\beta a_2(\tau)\mu_{1_{\eta\eta}}(\eta,\tau) - f(0,\eta,\tau))\,d\eta\,d\tau \\
&\quad - \int_0^t \int_0^l \hat{G}_{12_\xi}(x,y,t,h,\eta,\tau)(\mu_{2_\tau}(\eta,\tau) - \tau^\beta a_2(\tau)\mu_{2_{\eta\eta}}(\eta,\tau) - f(h,\eta,\tau))\,d\eta\,d\tau \\
&\quad - \int_0^t \int_0^h \hat{G}_{12}(x,y,t,\xi,0,\tau)t^\beta a_2(\tau)\nu_{1_{\xi\xi}}(\xi,\tau)d\xi d\tau \\
&\quad + \int_0^t \int_0^h \hat{G}_{12}(x,y,t,\xi,l,\tau)\tau^\beta a_2(\tau)\nu_{2_{\xi\xi}}(\xi,\tau)d\xi d\tau \\
&\quad + \int_0^t \int_0^l \int_0^h \hat{G}_{12}(x,y,t,\xi,\eta,\tau)f_{\xi\xi}(\xi,\eta,)\tau)\,d\xi\,d\eta\,d\tau,
\end{aligned}
$$

$$
\begin{aligned}
&u_{2yy}(x,y,t) \\
&= \int_0^l \int_0^h \hat{G}_{12}(x,y,t,\xi,\eta,0)\varphi_{\eta\eta}(\xi,\eta)d\xi d\eta \\
&\quad + \int_0^t \int_0^l \hat{G}_{12_\xi}(x,y,t,0,\eta,\tau)\tau^\beta a_2(\tau)\mu_{1_{\eta\eta}}(\eta,\tau)\,d\eta\,d\tau \\
&\quad - \int_0^t \int_0^l \hat{G}_{12_\xi}(x,y,t,h,\eta,\tau)\tau^\beta a_2(\tau)\mu_{2_{\eta\eta}}(\eta,\tau)\,d\eta\,d\tau \\
&\quad - \int_0^t \int_0^h \hat{G}_{12}(x,y,t,\xi,0,\tau)(\nu_{1_\tau}(\eta,\tau) - t^\beta a_2(\tau)\nu_{1_{\xi\xi}}(\xi,\tau) - f(\xi,0,\tau))d\xi d\tau \\
&\quad + \int_0^t \int_0^h \hat{G}_{12}(x,y,t,\xi,0,\tau)(\nu_{2_\tau}(\eta,\tau) - t^\beta a_2(\tau)\nu_{2_{\xi\xi}}(\xi,\tau) - f(\xi,l,\tau))d\xi d\tau \\
&\quad + \int_0^t \int_0^l \int_0^h \hat{G}_{12}(x,y,t,\xi,\eta,\tau)f_{\eta\eta}(\xi,\eta,)\tau)\,d\xi\,d\eta\,d\tau.
\end{aligned}
$$

From the above expression, we establish the following estimate for the kernel of (4.19)):

$$
|K(t,\tau)| \le \frac{C_{44}}{\sqrt{t(t-\tau)}}.
$$

This means that

$$
K(t,\tau) \equiv \frac{1}{\sqrt{t(t-\tau)}}K_1(t,\tau),
$$

and the equation (4.19)) can be presented as

$$
a(t) = \int_0^t \frac{1}{\sqrt{t(t-\tau)}}K_1(t,\tau)a(\tau)d\tau, \quad t \in [0,T], \tag{4.21}
$$

where $|K_1(t,\tau)| \le C_{44}$.

It was proved that there exists such an interval $[0,t_0]$ where $a(t) \equiv 0$. Then the equation (4.21)) can be transformed to the form

$$
a(t) = \int_{t_0}^t \frac{1}{\sqrt{t(t-\tau)}}K_1(t,\tau)a(\tau)d\tau, \quad t \in [t_0,T], \tag{4.22}
$$

and the following estimate holds:

$$\Big|\frac{1}{\sqrt{t(t-\tau)}}K_1(t,\tau)\Big| \le \frac{C_{45}}{\sqrt{t-\tau}}, \quad t \in [t_0, T].$$

Then the properties of the integral Volterra equations of the second kind imply that equation (4.22)) has only the trivial solution. Hence, $a(t) \equiv 0$, $t \in [0,T]$. Using it in the equation (4.12)), we obtain $u(x,y,t) \equiv 0, (x,y,t) \in \overline{Q}_T$ because of the the uniqueness of solution of the problem (4.12)–(4.14) [14]. The proof of Theorem 2.1 is complete.

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