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SOLUTIONS TO THE MAXIMAL SPACELIKE HYPERSURFACE EQUATION IN GENERALIZED ROBERTSON-WALKER SPACETIMES

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Abstract. We apply some generalized maximum principles for establishing uniqueness and nonexistence results concerning maximal spacelike hypersurfaces immersed in a generalized Robertson-Walker (GRW) spacetime, which is supposed to obey the so-called timelike convergence condition (TCC). As application, we study the uniqueness and nonexistence of entire solutions of a suitable maximal spacelike hypersurface equation in GRW spacetimes obeying the TCC.

1. Introduction

In the previous decades, the study of spacelike hypersurfaces immersed in a Lorentz manifold has been of substantial interest from both physical and mathematical points of view. For instance, it was pointed out by Marsden and Tipler [24] and Stumbles [37] that spacelike hypersurfaces with constant mean curvature in a spacetime play an important role in General Relativity, since they can be used as initial hypersurfaces where the constraint equations can be split into a linear system and a nonlinear elliptic equation.

From a mathematical point of view, spacelike hypersurfaces are also interesting because of their Bernstein-type properties. One can truly say that the first remarkable results in this branch were the rigidity theorems of Calabi [13] and Cheng and Yau [15], who showed (the former for $n \leq 4$, and the latter for general n) that the only maximal (that is, with zero mean curvature) complete spacelike hypersurfaces of the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ are the spacelike hyperplanes. However, in the case that the mean curvature is a positive constant, Treibergs [38] astonishingly showed that there are many entire solutions of the corresponding constant mean curvature equation in $\mathbb{L}^{n+1}$, which he was able to classify by their projective boundary values at infinity.

Later on, Ishihara [22] showed that the only complete maximal spacelike hypersurfaces immersed in a Lorentz manifold with nonnegative constant curvature are the totally geodesic ones. For the case of ambient spacetimes with negative constant

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curvature, he obtained a sharp estimate for the norm of the second fundamental form of a maximal spacelike hypersurface. In [14], the first author jointly with Camargo have obtained rigidity results for complete maximal spacelike hypersurfaces in the anti-de Sitter space, imposing suitable conditions on both the norm of the second fundamental form and a certain height function naturally attached to these hypersurfaces.

In this article, we are interested in the study of complete maximal spacelike hypersurfaces immersed in generalized Robertson-Walker (GRW) spacetimes. By GRW spacetimes, we mean Lorentzian warped products $-I\times_f M^n$ with Riemannian fibre M^n and warping function f. In particular, when the Riemannian fibre M^n has constant sectional curvature then $-I\times_f M^n$ is classically called a Robertson-Walker (RW) spacetime (for the details, see Section 2).

Many authors have approached problems in this subject. We may cite the works [3, 10, 11, 12, 31, 32, 33], where Romero et al. obtained rigidity and uniqueness results for the spacelike slices and complete maximal surfaces immersed in a GRW spacetime obeying either the *timelike convergence condition* or the *null convergence condition*. Let us recall that a spacetime obeys the timelike (null) convergence condition if its Ricci curvature is nonnegative on timelike (null or lightlike) directions.

Related to the compact case, Alías, Romero and Sánchez [8] proved that in a GRW spacetime satisfying the timelike convergence condition, every compact spacelike hypersurface of constant mean curvature must be totally umbilical. In this setting, they also showed how their result solve a certain Bernstein-type problem. Later on, Alías and Colares [5] studied the problem of uniqueness for compact spacelike hypersurfaces immersed with constant higher order mean curvature in GRW spacetimes. In order to establish one of their main results (cf. [5, Theorem 9.2]), they supposed that the ambient spacetime obeys a new notion of convergence condition, the so-called *strong null convergence condition* which corresponds to a suitable restriction on the sectional curvature of the Riemannian fibre of the GRW spacetime.

Here, we deal with complete noncompact maximal spacelike hypersurfaces immersed in a GRW spacetime. In this setting, by assuming that the ambient spacetime obeys the timelike convergence condition (TCC), we apply some generalized maximum principles in order to establish uniqueness and nonexistence results concerning these hypersurfaces (see Theorems 3.2, 3.7 and 3.9, and Corollaries 3.4 and 3.5). As application, we study the uniqueness and nonexistence of entire solutions of a suitable maximal spacelike hypersurface equation in GRW spacetimes obeying the TCC (see Theorems 4.1, 4.2 and 4.3). We point out that our uniqueness and nonexistence results can be regarded as extensions of several others appearing in the current literature, for instance, those ones in [4, 11, 14, 16, 32, 33, 34].

2. Preliminaries

In this section, we introduce some basic notation and facts which will appear along the paper.

2.1. **GRW spacetimes and spacelike hypersurfaces.**

Let M^n be a connected, n-dimensional ($n\geq 2$) oriented Riemannian manifold, $I\subseteq\mathbb{R}$ a 1-dimensional manifold (either a circle or an open interval of $\mathbb{R}$), and $f:I\to\mathbb{R}$ a positive smooth function. In the product differentiable manifold $\overline{M}^{n+1} = I \times M^n$, let π_I and π_M denote the projections onto the factors I and M^n, respectively.

A particular class of Lorentzian manifolds is the one obtained by furnishing $\overline{M}^{n+1}$ with the metric

$$\langle v, w\rangle_p = -\langle (\pi_I)_* v, (\pi_I)_* w\rangle + (f \circ \pi_I)(p)^2 \langle (\pi_M)_* v, (\pi_M)_* w\rangle,$$

for all $p \in \overline{M}^{n+1}$ and all $v, w \in T_p\overline{M}$. Following the terminology introduced in [8], such a space is called a *generalized Robertson-Walker* (GRW) spacetime, f is known as the warping function and we shall write $\overline{M}^{n+1} = -I \times_f M^n$ to denote it. In particular, when the Riemannian fibre M^n has constant sectional curvature, then $-I \times_f M^n$ is classically called a *Robertson-Walker* (RW) spacetime, and it is a spatially homogeneous spacetime (cf. [27]).

As it was observed in [7], we note that spatial homogeneity, which is reasonable as a first approximation of the large scale structure of the universe, may not be realistic when one considers a more accurate scale. For that reason, GRW spacetimes could be suitable spacetimes to model universes with inhomogeneous spacelike geometry. Besides, small deformations of the metric on the fiber of RW spacetimes fit into the class of GRW spacetimes (see, for instance, [21] and [30]).

We recall that a smooth immersion $\psi : \Sigma^n \to -I \times_f M^n$ of an n-dimensional connected manifold Σ^n is said to be a *spacelike hypersurface* if the induced metric via ψ is a Riemannian metric on Σ^n, which, as usual, is also denoted for $\langle\cdot,\cdot\rangle$. In that case, since

$$\partial_t = (\partial/\partial_t)_{(t,x)}\,, \quad , (t, x) \in -I \times_f M^n,$$

is a unitary timelike vector field globally defined on the ambient spacetime, then there exists a unique timelike unitary normal vector field N globally defined on the spacelike hypersurface Σ^n which is in the same time-orientation as ∂_t. By using the Cauchy-Schwarz inequality, we obtain

$$\langle N, \partial_t\rangle \le -1 < 0 \quad \text{on } \Sigma^n. \tag{2.1}$$

We will refer to that normal vector field N as the future-pointing Gauss map of the spacelike hypersurface Σ^n.

For $t_0 \in I$, we orient the (spacelike) *slice* $M^n_{t_0} = \{t_0\} \times M^n$ by using its unit normal vector field ∂_t. According to [8], M_{t_0} has constant mean curvature $H = \frac{f'}{f}(t_0)$ with respect to ∂_t.

Let $\overline{\nabla}$ and ∇ denote the Levi-Civita connections in $-I \times_f M^n$ and Σ^n, respectively. Then the Gauss and Weingarten formulas for the spacelike hypersurface $\psi : \Sigma^n \to -I \times_f M^n$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y - \langle AX, Y\rangle N, \tag{2.2}$$

$$AX = -\overline{\nabla}_X N, \tag{2.3}$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$, where $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ stands for the shape operator (or Weingarten endomorphism) of Σ^n with respect to its future-pointing Gauss map N.

As in [27], the curvature tensor R of the spacelike hypersurface Σ^n is given by

$$R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where $[\cdot,\cdot]$ denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(\Sigma)$.

A well-known fact is that the curvature tensor R of the spacelike hypersurface Σ^n can be described in terms of the shape operator A and the curvature tensor $\overline{R}$ of the ambient spacetime $\overline{M}^{n+1}$ by the so-called Gauss equation given by

$$R(X,Y)Z = (\overline{R}(X,Y)Z)^{\top} - \langle AX, Z\rangle AY + \langle AY, Z\rangle AX, \tag{2.4}$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$, where $(\cdot)^{\top}$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M})$ along Σ^n.

2.2. **Height and support functions and the normal hyperbolic angle.** We consider two particular functions naturally attached to a spacelike hypersurface Σ^n immersed into a GRW spacetime $\overline{M}^{n+1} = -I \times_f M^n$, namely, the (vertical) *height function* $h = (\pi_I)|_{\Sigma}$ and the *support function* $\langle N, \partial_t\rangle$, where we recall that N denotes the future-pointing Gauss map of Σ^n.

A simple computation shows that

$$\overline{\nabla}\pi_I = -\langle \overline{\nabla}\pi_I, \partial_t\rangle\partial_t = -\partial_t,$$

so that

$$\nabla h = (\overline{\nabla}\pi_I)^{\top} = -\partial_t^{\top} = -\partial_t - \langle N, \partial_t\rangle N. \tag{2.5}$$

Therefore,

$$|\nabla h|^2 = \langle N, \partial_t\rangle^2 - 1, \tag{2.6}$$

where $|\cdot|$ stands for the norm of a vector field on Σ^n.

We define the *hyperbolic angle* θ of Σ^n as being the smooth function $\theta : \Sigma^n \to [0, +\infty)$ given by

$$\cosh\theta = -\langle N, \partial_t\rangle \geq 1. \tag{2.7}$$

Therefore, from (2.6) and (2.7) we obtain

$$\sinh^2\theta = |\nabla h|^2. \tag{2.8}$$

2.3. **Energy curvature conditions.** We recall that a GRW spacetime $\overline{M}^{n+1} = -I \times_f M^n$ obeys the *null convergence condition* (NCC) when

$$\overline{\mathrm{Ric}}(Z,Z) \geq 0, \tag{2.9}$$

for all null vector field $Z \in \mathfrak{X}(\overline{M})$.

From [27, Corollary 7.43] we have that

$$\begin{aligned}\overline{\mathrm{Ric}}(Z,W) &= \mathrm{Ric}_M(Z^*, W*) + (n((\log f)')^2 + (\log f)'')\langle Z, W\rangle \\ &\quad - (n-1)(\log f)''\langle Z, \partial_t\rangle\langle W, \partial_t\rangle,\end{aligned} \tag{2.10}$$

where Ric_M denotes the Ricci tensor of M and $Z^* = Z + \langle Z, \partial_t\rangle\partial_t$ stands for the projection of the vector field Z onto M^n. Consequently, from (2.10) we have that the NCC holds in $\overline{M}^{n+1}$ if, and only if,

$$\mathrm{Ric}_M \geq (n-1)\left(f^2(\log f)''\right)\langle\cdot,\cdot\rangle_M. \tag{2.11}$$

A more restrictive energy condition is the *timelike converge condition*, that is

$$\overline{\mathrm{Ric}}(Z,Z) \geq 0, \tag{2.12}$$

for all timelike vector field $Z \in \mathfrak{X}(\overline{M})$. Note that, by a continuity argument, It turns out that the TCC implies NCC. Moreover, it is not difficult check that $\overline{M}^{n+1}$ satisfies the TCC if, and only if, (2.11) holds and $f'' \leq 0$.

3. Uniqueness and nonexistence results in GRW spacetimes

This section is devoted to present our main results which are concerning the uniqueness and nonexistence of spacelike hypersurfaces immersed in a GRW spacetime obeying the TCC. For this, we start quoting an extension of Hopf's theorem on a complete noncompact Riemannian manifold due to Yau [39]. In what follows, $\mathcal{L}^1(\Sigma)$ denotes the space of Lebesgue integrable functions on Σ^n.

Lemma 3.1. *Let Σ^n be an n-dimensional, complete Riemannian manifold and let $g : \Sigma^n \to \mathbb{R}$ be a smooth function. If g is a subharmonic (or superharmonic) function with $|\nabla g| \in \mathcal{L}^1(\Sigma)$, then g must actually be harmonic.*

In what follows, a slab

$$[t_1, t_2] \times M^n = \{(t, q) \in -I \times_f M^n : t_1 \leq t \leq t_2\}$$

is called a *timelike bounded region*.

Our first result is a sort of improvement to [16, Theorem 4.6].

Theorem 3.2. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime obeying the TCC.*

(i) *The only complete maximal spacelike hypersurfaces Σ^n contained in a timelike bounded region of $\overline{M}^{n+1}$, whose hyperbolic angle and second fundamental form are bounded, $f''(h) < 0$ and with $|\nabla h| \in \mathcal{L}^1(\Sigma)$, are the totally geodesic slices of $\overline{M}^{n+1}$.*

(ii) *there are not exist complete maximal spacelike hypersurfaces Σ^n contained in a timelike bounded region of $\overline{M}^{n+1}$ having bounded hyperbolic angle and second fundamental form, $f'(h) \neq 0$ and with $|\nabla h| \in \mathcal{L}^1(\Sigma)$.*

Proof. From [23, Proposition 3.1] we have

$$\begin{aligned}
&\frac{1}{2}\Delta \sinh^2\theta \\
&\geq n\frac{f'(h)^2}{f(h)^2} + \langle A^2\nabla h, \nabla h\rangle - 2\frac{f'(h)}{f(h)}\operatorname{Hess}(h)(\nabla h, \nabla h) \\
&\quad + \cosh^2\theta \operatorname{Ric}_M(N^*, N^*) + 2\frac{f'(h)}{f(h)}\cosh\theta\langle A\nabla h, \nabla h\rangle \\
&\quad + (2n+1)\frac{f'(h)^2}{f(h)^2}\sinh^2\theta - n\frac{f''(h)}{f(h)}\sinh^2\theta + (n+1)\frac{f'(h)^2}{f(h)^2}\sinh^4\theta \\
&\quad - n\frac{f''(h)}{f(h)}\sinh^4\theta.
\end{aligned} \tag{3.1}$$

On the other hand, it is not difficult to verify that

$$\nabla\cosh\theta = A(\nabla h) - \frac{f'(h)}{f(h)}\langle N, \partial_t\rangle\nabla h, \tag{3.2}$$

$$\sinh^2\theta = f(h)^2\langle N^*, N^*\rangle_M. \tag{3.3}$$

Using inequality (2.11) and equation (3.3), from (3.1) we have

$$\begin{aligned}\frac{1}{2}\Delta\sinh^2\theta \geq 2\frac{f'(h)}{f(h)}&(\cosh\theta\langle A\nabla h,\nabla h\rangle - \operatorname{Hess}(h)(\nabla h,\nabla h))\\ &+(n-1)\cosh^2\theta\sinh^2\theta(\log f)''(h)+(2n+1)\frac{f'(h)^2}{f(h)^2}\sinh^2\theta\\ &-n\frac{f''(h)}{f(h)}\sinh^2\theta+(n+1)\frac{f'(h)^2}{f(h)^2}\sinh^4\theta-n\frac{f''(h)}{f(h)}\sinh^4\theta.\end{aligned}\tag{3.4}$$

Also from equation (3.2) we obtain

$$\begin{aligned}\cosh\theta\langle A(\nabla h),\nabla h\rangle-\operatorname{Hess}(h)(\nabla h,\nabla h)&=\cosh\theta\frac{f'(h)}{f(h)}\langle N,\partial_t\rangle|\nabla h|^2\\ &=-\cosh^2\theta\sinh^2\theta\frac{f'(h)}{f(h)}.\end{aligned}\tag{3.5}$$

Hence, inserting (3.5) into (3.4), with a straightforward computation we obtain

$$\frac{1}{2}\Delta\sinh^2\theta\geq n\Big(\frac{f'(h)}{f(h)}\Big)^2\sinh^2\theta-n\frac{f''(h)}{f(h)}\sinh^4\theta.\tag{3.6}$$

So, let us assume the situation of item (i). From inequality (3.6) we obtain

$$\frac{1}{2}\Delta\sinh^2\theta\geq -n\frac{f''(h)}{f(h)}\sinh^4\theta.\tag{3.7}$$

In particular, since we are supposing that $f''(h)<0$, from (3.7) we conclude that $\sinh^2\theta$ is a subharmonic function on Σ^n.

On the other hand, since we are supposing that A and θ are bounded and that Σ^n is contained in a timelike bounded region of $\overline{M}^{n+1}$, from (3.2) we have

$$|\nabla\sinh^2\theta|=2\cosh\theta\Big|\Big(A+\frac{f'(h)}{f(h)}\cosh\theta I\Big)\nabla h\Big|\leq C|\nabla h|,\tag{3.8}$$

for some positive constant C. Thus, since we are also assuming that $|\nabla h|\in\mathcal{L}^1(\Sigma)$, from (3.8) we obtain that $|\nabla\sinh^2\theta|\in\mathcal{L}^1(\Sigma)$.

Consequently, we can apply Lemma 3.1 to obtain that $\sinh^2\theta$ is, in fact, harmonic on Σ^n. Therefore, returning to (3.7) and using once more the hypothesis $f''(h)<0$, we conclude that θ vanishes identically on Σ^n, that is, Σ^n must be a totally geodesic slice of $\overline{M}^{n+1}$.

Now, let us prove item (ii). For this, suppose by contradiction that there exists such a spacelike hypersurface Σ^n. From inequality (3.6), we also have

$$\frac{1}{2}\Delta\sinh^2\theta\geq n\Big(\frac{f'(h)}{f(h)}\Big)^2\sinh^2\theta\geq 0.\tag{3.9}$$

Thus, we can apply again Lemma 3.1 to conclude that $\sinh^2\theta$ is a harmonic function. So, returning to (3.6) we must be $\sinh^2\theta\equiv 0$. Hence, using the identity (2.8), we have that $\cosh^2\theta=1$ on Σ^n. Therefore, there exists $t_0\in I$ such that $\Sigma^n\subset M^n_{t_0}$ and, for completeness, Σ^n is a totally geodesic slice with $f'(t_0)=0$ and we arrive to a contradiction. □

Remark 3.3. We recall that a spacetime $\overline{M}^{n+1}$ obeys the *ubiquitous energy condition* if its Ricci curvature satisfies $\overline{\operatorname{Ric}}(Z,Z)>0$, for all timelike vector field $Z\in\mathfrak{X}(\overline{M})$. This last energy condition is stronger than the TCC and roughly means a real presence of matter at any point of the spacetime. It is not difficult to verify that if $\overline{M}^{n+1} = -I \times_f M^n$ is a GRW spacetime obeying the ubiquitous energy condition then $f^{''} < 0$. We observe that the open subset of the anti-de Sitter space $\mathbb{H}_1^{n+1}$ which is modeled by the GRW spacetime $-(-\pi/2, \pi/2) \times_{\cos t} \mathbb{H}^n$ (cf. Example 3 in Section 4 of [25]), the so-called Einstein-de Sitter cosmological model $-(0,\infty) \times_{t^{2/3}} \mathbb{R}^3$ and certain big bang cosmological models (see, for instance, [27, Chapter 12], Chapter 5 of [9] or [21, Chapter 5]) are examples of GRW spacetimes obeying the ubiquitous energy condition. So, in this case, the hypothesis $f''(h) < 0$ in Theorem 3.2 is automatically satisfied.

It is worth to make a discussion on the meaning of our assumption in Theorem 3.2 concerning the integrability of $|\nabla h|$ on the spacelike hypersurface Σ^n both from geometric and physical viewpoints. From the first viewpoint, it is a natural extension to the case in which the spacelike hypersurface is compact. On the other hand, some physical interpretation is now in order.

According to [25],

$$V = V(t,p) = f(t)\partial_t \tag{3.10}$$

is a closed conformal vector field globally defined on a GRW spacetime $\overline{M}^{n+1} = -I \times_f M^n$. So, following the concepts of [35] (see also [17, 23]), given a spacelike hypersurface Σ^n immersed in $\overline{M}^{n+1}$ with future-pointing Gauss map N, we can write $V_q = e(q)N_q + V_q^\top$, for each $q \in \Sigma^n$, where $e(q) = -\langle V_q, N_q\rangle > 0$ and $V_q^\top$ are, respectively, the energy and the n-momentum that the instantaneous observer N_q measures for V_q. Moreover, the quantity $\frac{1}{e(q)}V_q^\top$ is the relative velocity (and, hence, $\frac{1}{e(q)}|V_q^\top|$ is the relative speed) of V_q with respect to N_q. Note that

$$|V_q^\top| = \sqrt{-\langle V_q, V_q\rangle}\sinh\theta(q), \tag{3.11}$$

where $\theta(q)$ is the hyperbolic angle between V_q and N_q. Thus, from (3.11) we obtain

$$|V_q^\top| = e(q)\tanh\theta(q) \le e(q). \tag{3.12}$$

Furthermore, from (2.5) and (3.10) we also have that

$$|V_q^\top| = f(h(q))|\nabla h(p)|. \tag{3.13}$$

Consequently, assuming that Σ^n is contained in a timelike bounded region of $\overline{M}^{n+1}$, from (3.12) and (3.13) we see that the integrability of $|\nabla h|$ can be regarded as been the n-momentum of N having integrable norm on Σ^n and, in particular, such condition is satisfied when Σ^n has *finite total energy*, that is,

$$\int_\Sigma e(q)d\Sigma < +\infty.$$

So, from Theorem 3.2 we obtain the following result.

Corollary 3.4. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime obeying the TCC.*

(i) *The only complete maximal spacelike hypersurfaces Σ^n contained in a timelike bounded region of $\overline{M}^{n+1}$, whose hyperbolic angle and second fundamental form are bounded, $f''(h) < 0$ and with finite total energy, are the totally geodesic slices of $\overline{M}^{n+1}$.*

(ii) *there are not exist complete maximal spacelike hypersurfaces Σ^n contained in a timelike bounded region of $\overline{M}^{n+1}$ having bounded hyperbolic angle and second fundamental form, $f'(h) \neq 0$ and with finite total energy.*

In [22], Ishihara proved that a n-dimensional complete maximal spacelike hypersurface immersed in the anti-de Sitter space $\mathbb{H}_1^{n+1}$ must have the squared norm of the second fundamental form bounded from above by n. Taking into account Ishihara's result, Theorem 3.2 allows us to obtain the following refinement of [14, Theorem 1.2].

Corollary 3.5. *The only complete maximal spacelike hypersurface Σ^n contained in a timelike bounded region of $-(-\pi/2, \pi/2) \times_{\cos t} \mathbb{H}^n \subset \mathbb{H}_1^{n+1}$, whose hyperbolic angle is bounded and with $|\nabla h| \in \mathcal{L}^1(\Sigma)$, is the totally geodesic slice $\{0\} \times \mathbb{H}^n$.*

A Riemannian manifold Σ^n is said to be *stochastically complete* if, for some (and, hence, for any) $(x,t) \in \Sigma \times (0, +\infty)$, the heat kernel $p(x,y,t)$ of the Laplace-Beltrami operator Δ satisfies the conservation property

$$\int_{\Sigma} p(x,y,t)d\mu(y) = 1. \tag{3.14}$$

From the probabilistic viewpoint, stochastically completeness is the property of a stochastic process to have infinite life time. For the Brownian motion on a manifold, the conservation property (3.14) means that the total probability of the particle to be found in the state space is constantly equal to one (cf. [18, 19, 20, 36]).

On the other hand, Pigola, Rigoli and Setti showed that stochastic completeness turns out to be equivalent to the validity of a weak form of the Omori-Yau maximum principle (see [28, Theorem 1.1] or [29, Theorem 3.1]), as is expressed below.

Lemma 3.6. *A Riemannian manifold Σ^n is stochastically complete if, and only if, for every $g \in \mathcal{C}^2(\Sigma)$ satisfying $\sup_\Sigma g < +\infty$, there exists a sequence of points $\{p_k\} \subset \Sigma^n$ such that*

$$\lim_{k\to\infty} g(p_k) = \sup_{\Sigma} g \quad \text{and} \quad \limsup_{k\to\infty} \Delta g(p_k) \leq 0.$$

Our next result is an extension of those in [11, 14, 16, 32, 33, 34] for the case that the maximal spacelike hypersurface is supposed to be stochastically complete. For this, we observe that the slices of a GRW spacetime which satisfies (2.11) have Ricci curvature bounded from below and, consequently, they are stochastically complete.

Theorem 3.7. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime obeying the TCC.*

(i) *The only stochastically complete maximal spacelike hypersurfaces contained in timelike bounded region $\mathcal{U} \subset \overline{M}^{n+1}$, whose hyperbolic angle is bounded and such that $f'' < 0$ in $\mathcal{U}$, are the totally geodesic slices of $\overline{M}^{n+1}$.*

(ii) *There are not exist stochastically complete maximal spacelike hypersurfaces contained in a timelike bounded region $\mathcal{U} \subset \overline{M}^{n+1}$, with bounded hyperbolic angle and such that $f' \neq 0$ in $\mathcal{U}$.*

Proof. Let us assume the situation of item (i). From (3.7) we have that

$$\frac{1}{2}\Delta \sinh^2\theta \geq -n\frac{f''(h)}{f(h)}\sinh^4\theta.$$

Consequently, since Σ^n is contained in a timelike bounded region $\mathcal{U} \subset \overline{M}^{n+1}$ with $f'' < 0$ in $\mathcal{U}$, there exists a positive constant C such that

$$\frac{1}{2}\Delta \sinh^2 \theta \geq C \sinh^4 \theta. \tag{3.15}$$

On the other hand, since we are supposing that θ is bounded, we can apply Lemma 3.6 in order to obtain a sequence of points $\{p_k\}_{k\in\mathbb{N}} \subset \Sigma^n$ such that

$$0 \leq \sup_{\Sigma} \sinh^2 \theta = \lim_{k\to\infty} \sinh^2 \theta(p_k) \quad \text{and} \quad \limsup_{k\to\infty} \Delta \sinh^2 \theta(p_k) \leq 0 \tag{3.16}$$

Considering (3.16) into inequality (3.15), we obtain

$$0 \geq \limsup_{k\to\infty} \Delta \sinh^2 \theta(p_k) \geq C \sup_{M} \sinh^4 \theta \geq 0. \tag{3.17}$$

Therefore, from (3.17) we conclude that $\theta = 0$ on Σ^n and, hence, Σ^n must be a totally geodesic slice of $\overline{M}^{n+1}$.

Now, we consider the case of item (ii). Suppose, for contradiction, that there exists such a stochastically complete maximal hypersurface Σ^n. From (3.7) we also have that

$$\frac{1}{2}\Delta \sinh^2 \theta \geq n\frac{f'(h)^2}{f(h)^2} \sinh^2 \theta.$$

Thus, as in the previous item, there exists a positive constant C such that

$$\frac{1}{2}\Delta \sinh^2 \theta \geq C \sinh^2 \theta. \tag{3.18}$$

On the other hand, as we are supposing that θ is bounded, from (2.8), we can apply Lemma 3.6 to obtain the sequence of points $\{p_k\}_{k\in\mathbb{N}} \subset \Sigma^n$ such that

$$0 \leq \sup_{\Sigma} \sinh^2 \theta = \lim_{k\to\infty} \sinh^2 \theta(p_k) \quad \text{and} \quad \limsup_{k\to\infty} \Delta \sinh^2 \theta(p_k) \leq 0. \tag{3.19}$$

Now, applying (3.19) into inequality (3.18) we obtain

$$0 \geq \limsup_{k\to\infty} \Delta \sinh^2 \theta(p_k) \geq 2C \sup_{M} \sinh^2 \theta \geq 0.$$

So, we conclude that $\sinh^2 \theta \equiv 0$. Using equation (2.8), we have that $\cosh^2 \theta = 1$ on Σ^n. Therefore, there exists $t_0 \in I$ such that with $f'(t_0) = 0$ and $\Sigma^n \subset M^n_{t_0}$ and, hence, we arrive to a contradiction. □

According to the terminology due to Alías and Colares [5], a GRW spacetime is said to obey the *strong null convergence condition* (SNCC) when the sectional curvature K_M of its fiber M^n satisfies the inequality

$$K_M \geq \sup_{I}(f^2(\log f)''), \tag{3.20}$$

It is not difficult to see that the SNCC implies in the NCC.

Paraphrasing the definition of the SNCC, we say that a GRW spacetime obeys the *strong timelike convergence condition* (STCC) when (3.20) is satisfied and $f'' \leq 0$. Clearly all GRW spacetime which satisfies the STCC also satisfies the TCC. Consequently, taking into account the discussion made in Section 4.3 of [21] concerning the physical interpretation of the TCC, we conclude that the assumption of the ambient GRW spacetime to obey the STCC can be regarded as a mathematical way to express that gravity, on average, attracts.

To establish our next result, we quote the following consequence of the generalized maximum principle of Omori-Yau [26, 39] which was obtained by Akutagawa [1].

Lemma 3.8. *Let Σ^n denote an n-dimensional complete Riemannian manifold having Ricci curvature bounded from below. If $g \in \mathcal{C}^2(\Sigma)$ is nonnegative and satisfies $\Delta g \geq C g^{\beta}$, for some real numbers $C > 0$ and $\beta > 1$, then $g \equiv 0$.*

We will apply the previous lemma to obtain an extension of several results in [11, 14, 16, 32, 33, 34] for the context of complete maximal spacelike hypersurfaces immersed in a GRW spacetime obeying the STCC.

Theorem 3.9. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime obeying the STCC. The only complete maximal spacelike hypersurfaces contained in a timelike bounded region $\mathcal{U} \subset \overline{M}^{n+1}$ with $f'' < 0$ in $\mathcal{U}$ are the spacelike totally geodesic slices of $\overline{M}^{n+1}$.*

Proof. Firstly, to apply Lemma 3.8, we claim that the Ricci curvature of Σ^n is bounded from below. Indeed, set $X \in \mathfrak{X}(\Sigma)$ and a local orthonormal frame $\{E_1, \cdots, E_n\}$ of $\mathfrak{X}(\Sigma)$. Then, since Σ^n is maximal, it follows from (2.4) that the Ricci curvature Ric of Σ^n is given by

$$\operatorname{Ric}(X,X) = \sum_i \langle \overline{R}(X,E_i)X, E_i\rangle + |AX|^2 \geq \sum_i \langle \overline{R}(X,E_i)X, E_i\rangle. \tag{3.21}$$

Consequently, from (3.21) we obtain that $\operatorname{Ric}(X,X)$ is bounded from below if, and only if, $\sum_i \langle \overline{R}(X,E_i)X, E_i\rangle$ is bounded from below.

On the other hand, by using [5, equation (33)] (see also [27, Proposition 7.42]) and taking into account equation (2.5), we obtain

$$\begin{aligned}\sum_i \langle \overline{R}(X,E_i)X, E_i\rangle &= \sum_i \langle R_M(X^*,E_i^*)X^*, E_i^*\rangle + (n-1)((\log f)'(h))^2|X|^2\\ &\quad - (n-2)(\log f)''(h)\langle X, \nabla h\rangle^2 - (\log f)''(h)|\nabla h|^2|X|^2.\end{aligned} \tag{3.22}$$

where R_M is the curvature tensor of M^n, $E_i^* = (\pi_M)_*(E_i)$ and $X^* = (\pi_M)_*(X)$.

By computing the first parcel of the right side of (3.22), we have

$$\begin{aligned}\sum_i \langle R_M(X^*,E_i^*)X^*, E_i^*\rangle &\geq \frac{1}{f^2(h)}((n-1)|X|^2 + |\nabla h|^2|X|^2\\ &\quad + (n-2)\langle X,\nabla h\rangle^2) \min_i K_M(X^*,E_i^*).\end{aligned} \tag{3.23}$$

Thus, considering (3.20) into (3.23), we obtain

$$\begin{aligned}\sum_i \langle R_M(X^*,E_i^*)X^*, E_i^*\rangle &\geq ((n-1)|X|^2 + |\nabla h|^2|X|^2\\ &\quad + (n-2)\langle X,\nabla h\rangle^2)(\log f)''(h).\end{aligned} \tag{3.24}$$

Substituting (3.24) in (3.22), we have

$$\sum_i \langle \overline{R}(X,E_i)X, E_i\rangle \geq (n-1)\frac{f''(h)}{f(h)}|X|^2. \tag{3.25}$$

Hence, since Σ^n is supposed to be contained into a timelike bounded region of $\overline{M}^{n+1}$, from (3.25) we obtain that the Ricci curvature of Σ^n is bounded from below.

Moreover, in a similar way of that in the proof of Theorem 3.7, we see that inequality (3.15) still holds. Therefore, we can apply Lemma 3.8 to conclude that θ vanishes identically on Σ^n and, hence, Σ^n must be a totally geodesic slice of $\overline{M}^{n+1}$. $\square$

4. Maximal spacelike hypersurface equation in GRW spacetimes

The goal of this section is to apply our previous uniqueness and nonexistence results on maximal hypersurfaces in order to study entire solutions of a suitable maximal hypersurface equation in GRW spacetimes obeying the TCC. For this, we will first recall some basic facts concerning entire graphs in GRW spacetimes.

Let $\Omega \subseteq M^n$ be a connected domain of M^n. For every $u \in \mathcal{C}^\infty(\Omega)$ such that $|Du|_M < f(u)$ where $|Du|_M$ stands for the length of the gradient Du of u, we will consider the vertical graph over Ω is determined by a smooth function $u \in \mathcal{C}^\infty(\Omega)$ and it is given by

$$\Sigma(u) = \{(u(x), x) : x \in \Omega\} \subset -I \times_f M^n. \tag{4.1}$$

The metric induced on Ω from the Lorentzian metric on the ambient space via $\Sigma(u)$ is

$$\langle \cdot, \cdot \rangle = -du^2 + f^2(u)\langle \cdot, \cdot \rangle_{M^n}. \tag{4.2}$$

The graph is said to be entire if $\Omega = M^n$. It can be easily seen that a graph $\Sigma(u)$ is a spacelike hypersurface if, and only if, $|Du|_M < f(u)$.

Observe that by [8, Lemma 3.1], in the case where M^n is a simply connected manifold, every complete spacelike hypersurface Σ^n in $-I \times_f M^n$ such that the warping function f is bounded on Σ^n is an entire spacelike graph in such space. In particular, this happens for complete spacelike hypersurfaces bounded away from the infinity of $-I \times_f M^n$. However, in contrast to the case of graphs into a Riemannian space, an entire spacelike graph in a Lorentzian spacetime is not necessarily complete, in the sense that the induced Riemannian metric (4.2) is not necessarily complete on M^n. For instance, Albujer [2] have obtained explicit examples of non-complete entire maximal graphs in $-\mathbb{R} \times \mathbb{H}^2$.

It is not difficult to see that the future-pointing Gauss map of $\Sigma(u)$ is given by

$$N = \frac{f(u)}{\sqrt{f^2(u) - |Du|_M^2}}\Big(\partial_t + \frac{1}{f^2(u)} Du\Big). \tag{4.3}$$

Moreover, the shape operator A of $\Sigma(u)$ with respect to its orientation (4.3) is given by

$$\begin{aligned} AX = &-\frac{1}{f(u)\sqrt{f^2(u) - |Du|_M^2}} D_X Du - \frac{f'(u)}{\sqrt{f^2(u) - |Du|_M^2}} X \\ &+ \Big(\frac{-\langle D_X Du, Du\rangle_M}{f(u)\big(f^2(u) - |Du|_M^2\big)^{3/2}} + \frac{f'(u)\langle Du, X\rangle}{\big(f^2(u) - |Du|_M^2\big)^{3/2}}\Big) Du, \end{aligned} \tag{4.4}$$

for any tangent vector field X. Consequently, denoting by div the divergence operator on $\Sigma(u)$, the mean curvature function $H(u)$ associated to A is given by

$$H(u) = -\operatorname{div}\Big(\frac{Du}{nf(u)\sqrt{f(u)^2 - |Du|_M^2}}\Big) - \frac{f'(u)}{n\sqrt{f(u)^2 - |Du|_M^2}}\Big(n + \frac{|Du|_M^2}{f(u)^2}\Big).$$

The differential equation $H(u) = 0$ with the constraints $|Du|_M < f(u)$ is called the *maximal spacelike hypersurface equation* in $\overline{M}$, and its solutions provide maximal spacelike graphs in $\overline{M}$.

Motivated by this previous digression, we will consider the following maximal spacelike hypersurface equation

$$\begin{gathered}\operatorname{div}\Big(\frac{Du}{f(u)\sqrt{f(u)^2-|Du|_M^2}}\Big) = -\frac{f'(u)}{\sqrt{f(u)^2-|Du|_M^2}}\Big(n+\frac{|Du|_M^2}{f(u)^2}\Big)\\ |Du|_M \le \alpha f(u),\end{gathered} \tag{4.5}$$

where $0 < \alpha < 1$ is constant. We observe that (4.5) is uniformly elliptic and that the constraint on $|Du|_M$ assures the boundedness of the hyperbolic angle θ of $\Sigma(u)$. Indeed, from (4.3) we obtain that

$$|\nabla h|^2 = \frac{|Du|_M^2}{f^2(u) - |Du|_M^2}. \tag{4.6}$$

Hence, using (2.8) and (4.6) we see that $|Du|_M \le \alpha f(u)$ implies $\cosh\theta \le \frac{1}{\sqrt{1-\alpha^2}}$.

To study equation (4.5), we also recall that

$$|u|_{\mathcal{C}^2(M)} = \max_{|\gamma|\le 2} |D^\gamma u|_{L^\infty(M)}.$$

Our next result corresponds to a nonparametric version of Theorem 3.2.

Theorem 4.1. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime obeying the TCC.*

- (i) *The only entire solutions of (4.5) such that $|u|_{\mathcal{C}^2(M)} < +\infty$, $f''(u) < 0$ and $|Du|_M \in \mathcal{L}^1(M)$ are the constant functions $u = c$, with $f'(c) = 0$.*
- (ii) *There are not exist entire solutions u of (4.5) such that $|u|_{\mathcal{C}^2(M)} < +\infty$, $f'(u) \ne 0$ and $|Du|_M \in \mathcal{L}^1(M)$.*

Proof. Since we are assuming that $|u|_{\mathcal{C}^2(M)} < +\infty$ and $|Du|_M \le \alpha f(u)$ for some constant $0 < \alpha < 1$, from (4.4) we obtain that $|A|$ is bounded on $\Sigma(u)$. Therefore, reasoning as in the proof of [6, Corollary 5.1], we can apply Theorem 3.2 to get the result. $\square$

From Theorem 3.7 we obtain the following result.

Theorem 4.2. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime obeying the TCC.*

- (i) *The only entire solutions of (4.5) which are stochastically complete and such that $f''(u) < 0$ are the constant functions $u = c$, with $f'(c) = 0$.*
- (ii) *There are not exist entire solutions u of (4.5) which are stochastically complete and such that $f'(u) \ne 0$.*

To close our paper, we quote the nonparametric version of Theorem 3.9.

Theorem 4.3. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime obeying the STCC and let $\mathcal{U}$ be a timelike bounded region of $\overline{M}^{n+1}$ such that $f'' < 0$ in $\mathcal{U}$. The only entire solutions of (4.5) contained into $\mathcal{U}$ are the constant functions $u = c$, with $f'(c) = 0$.*

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References

[1] K. Akutagawa; *On spacelike hypersurfaces with constant mean curvature in the de Sitter space*, Math. Z., **196** (1987), 13–19.

[2] A. L. Albujer; *New examples of entire maximal graphs in $\mathbb{H}^2 \times \mathbb{R}_1$*, Diff. Geom. Appl., **26** (2008), 456–462.

[3] J. A. Aledo, A. Romero, R. M. Rubio; *Constant mean curvature spacelike hypersurfaces in Lorentzian warped products and Calabi-Bernstein type problems*, Nonl. Anal., **106** (2014), 57–69.

[4] J. A. Aledo, R. M. Rubio, J. J. Salamanca; *Complete spacelike hypersurfaces in generalized Robertson-Walker and the null convergence condition: Calabi-Bernstein problems*, RACSAM **111** (2017), 115–128.

[5] L. J. Alías, A. G. Colares; *Uniqueness of spacelike hypersurfaces with constant higher order mean curvature in Generalized Robertson-Walker spacetimes*, Math. Proc. Cambridge Philos. Soc., **143** (2007), 703–729.

[6] L. J. Alías, A. G. Colares, H. F. de Lima; *On the rigidity of complete spacelike hypersurfaces immersed in a generalized Robertson-Walker spacetime*, Bull. Brazilian Math. Soc., **44** (2013), 195–217.

[7] L. J. Alías, S. Montiel; *Uniqueness of spacelike hypersurfaces with constant mean curvature in generalized Robertson-Walker spacetimes*, Differential Geometry, Valencia, (2001), 59–69, World Sci. Publ., River Edge, NJ, 2002.

[8] L. J. Alías, A. Romero, M. Sánchez; *Uniqueness of complete spacelike hypersurfaces of constant mean curvature in Generalized Robertson-Walker spacetimes*, Gen. Relat. Grav., **27** (1995), 71–84.

[9] J. K. Beem, P. E. Ehrlich, K. L. Easley; *Global Lorentzian Geometry*, Second Edition, CRC Press, New York, 1996.

[10] M. Caballero, A. Romero, R. M. Rubio; *Constant mean curvature spacelike surfaces in three-dimensional generalized Robertson-Walker spacetimes*, Lett. Math. Phys. **93** (2010), 85–105.

[11] M. Caballero, A. Romero, R. M. Rubio; *Uniqueness of maximal surfaces in generalized Robertson-Walker spacetimes and Calabi-Bernstein type problems*, J. Geom. Phys. **60** (2010), 394–402.

[12] M. Caballero, A. Romero, R. M. Rubio; *Complete CMC spacelike surfaces with bounded hyperbolic angle in generalized Robertson-Walker spacetimes*, Int. J. Geom. Meth. Mod. Phys. **7** (2010), 961–978.

[13] E. Calabi; *Examples of Bernstein problems for some nonlinear equations*, Proc. Sympos. Pure Math., **15** (1970), 223–230.

[14] F. Camargo, H. F. de Lima; *New characterizations of totally geodesic hypersurfaces in the anti-de Sitter space $\mathbb{H}_1^{n+1}$*, J. Geom. Phys., **60** (2010), 1326–1332.

[15] S. Y. Cheng, S. T. Yau; *Maximal Spacelike Hypersurfaces in the Lorentz-Minkowski Space*, Ann. of Math., **104** (1976), 407–419.

[16] H.F. de Lima, U.L. Parente; *On the geometry of maximal spacelike hypersurfaces immersed in a generalized Robertson-Walker spacetime*, Ann. Mat. Pura Appl. **192** (2013), 649–663.

[17] H. F. de Lima, M. A. L. Velásquez; *Uniqueness of complete spacelike hypersurfaces via their higher order mean curvatures in a conformally stationary spacetime*, Math. Nachr., **287** (2014), 1223–1240.

[18] M. Émery; *Stochastic Calculus on Manifolds*, Springer-Verlag, Berlin, 1989.

[19] A. A. Grigor'yan; *Stochastically complete manifolds and summable harmonic functions*, Izv. Akad. Nauk SSSR Ser. Mat. **52** (1988), 1102–1108; translation in Math. USSR-Izv. **33** (1989), 425–432.

[20] A. A. Grigor'yan; *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*, Bull. American Math. Soc., **36** (1999), 135–249.

[21] S. W. Hawking, G. F. R. Ellis; *The Large Scale Structure of Spacetime*, Cambridge Univ. Press, Cambridge, 1973.

[22] T. Ishihara; *Maximal spacelike submanifolds of a pseudo-Riemannian space of constant curvature*, Mich. Math. J. **35** (1988), 345–352.

[23] J. M. Latorre, A. Romero; *Uniqueness of Noncompact Spacelike Hypersurfaces of Constant Mean Curvature in Generalized Robertson-Walker Spacetimes*, Geom. Ded. **93** (2002), 1–10.

[24] J. Marsden, F. Tipler; *Maximal hypersurfaces and foliations of constant mean curvature in general relativity*, Bull. American Phys. Soc. **23**, (1978) 84.
[25] S. Montiel; *Uniqueness of spacelike hypersurface of constant mean curvature in foliated spacetimes*, Math. Ann. **314** (1999), 529–553.
[26] H. Omori; *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan **19** (1967), 205–214.
[27] B. O'Neill; *Semi-Riemannian Geometry, with Applications to Relativity*, Academic Press, New York, 1983.
[28] S. Pigola, M. Rigoli, A. G. Setti; *A remark on the maximum principle and stochastic completeness*, Proc. American Math. Soc., **131** (2003), 1283–1288.
[29] S. Pigola, M. Rigoli, A. G. Setti; *Maximum principles on Riemannian manifolds and applications*, Mem. Amererican Math. Soc., **174** (2005), Number 822.
[30] M. Rainer, H.-J. Schmidt; *Inhomogeneous cosmological models with homogeneous inner hypersurface geometry*, Gen. Relat. Grav., **27** (1995), 1265–1293.
[31] A. Romero, R. M. Rubio; *On the mean curvature of spacelike surfaces in certain three-dimensional Robertson-Walker spacetimes and Calabi-Bernstein's type problems*, Ann. Glob. Anal. Geom., **37** (2010), 21–31.
[32] A. Romero, R. M. Rubio, J. J. Salamanca; *Uniqueness of complete maximal hypersurfaces in spatially parabolic generalized Robertson-Walker spacetimes*, Class. Quantum Grav., **30** (2013), 115007.
[33] A. Romero, R. M. Rubio, J. J. Salamanca; *A new approach for uniqueness of complete maximal hypersurfaces in spatially parabolic GRW spacetimes*, J. Math. Anal. Appl., **419** (2014), 355–372.
[34] R. M. Rubio; *Complete constant mean curvature spacelike hypersurfaces in the Einstein-De Sitter spacetime*, Rep. Math. Phys. **74** (2014), 127–133.
[35] R. K. Sachs, H. Wu; *General Relativity for Mathematicians*, Graduate Texts in Mathematics, Vol. 48, Springer-Verlag, New York, 1977.
[36] D. Stroock; *An Introduction to the Analysis of Paths on a Riemannian Manifold*, Math. Surveys and Monographs, volume 4, American Math. Soc, 2000.
[37] S. Stumbles; *Hypersurfaces of constant mean extrinsic curvature*, Ann. Phys., **133**, (1980) 28–56.
[38] A. E. Treibergs; *Entire Spacelike Hypersurfaces of Constant Mean Curvature in Minkowski Space*, Invent. Math., **66** (1982), 39–56.
[39] S. T. Yau; *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana Univ. Math. J., **25** (1976), 659–670.

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