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LIOUVILLE-TYPE THEOREMS FOR STABLE SOLUTIONS OF SINGULAR QUASILINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^N$

CAISHENG CHEN, HONGXUE SONG, HONGWEI YANG

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Abstract. We prove a Liouville-type theorem for stable solution of the singular quasilinear elliptic equations

$$-\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = f(x)|u|^{q-1}u, \quad \text{in } \mathbb{R}^N,$$
$$-\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = f(x)e^u, \quad \text{in } \mathbb{R}^N$$

where $2 \le p < N$, $-\infty < a < (N-p)/p$ and the function $f(x)$ is continuous and nonnegative in $\mathbb{R}^N \setminus \{0\}$ such that $f(x) \ge c_0|x|^b$ as $|x| \ge R_0$, with $b > -p(1+a)$ and $c_0 > 0$. The results hold for $1 \le p-1 < q = q_c(p,N,a,b)$ in the first equation, and for $2 \le N < q_0(p,a,b)$ in the second equation. Here q_0 and q_c are exponents, which are always larger than the classical critical ones and depend on the parameters a, b.

1. Introduction and main results

Recently, Ghergu and Rădulescu [19] studied the singular elliptic problem

$$-\operatorname{div}(|x|^{-2a}\nabla u) = K(x)|x|^{-bq}|u|^{q-2}u + \lambda g(x), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $-\infty < a < (N-2)/2$, $a \le b < a+1$, $q = 2N/(N-2(1+a-b))$ and $N \ge 2$. Under some natural assumptions on the positive potential $K(x)$, the authors established the existence of some $\lambda_0 > 0$ such that the problem (1.1) has at least two distinct solutions provided that $\lambda \in (0,\lambda_0)$. For $\lambda = 0$, there exists an interesting question: does (1.1) admit a nontrivial solution?

D'Ambrosio and Mitidieri [7] considered the existence and nonexistence of nontrivial weak solution to the following quasilinear elliptic equation with singular weights and critical exponent

$$\operatorname{div}(A(x,u,\nabla u)) + V(x)|u|^{p-2}u = a(x)|u|^{q-1}u, \quad x \in \mathbb{R}^N \tag{1.2}$$

Here A contains the $p-$Laplacian operator $A(x,t,\xi) = |\xi|^{p-2}\xi$ and the mean curvature operator $A(x,t,\xi) = \xi/\sqrt{1+|\xi|^2}$ for $\xi \in \mathbb{R}^N$, $p > 1$, $V \ge 0$ is a singular potential function, $a(x) : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative measurable function and $q > p-1$. Similar consideration can be found [1, 2, 16, 17, 18, 20] and the references therein.

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In this article, motivated by Chen [5], Dancer et al. [8] and the references mentioned above, we study the nonexistence of stable solutions to the singular quasilinear elliptic equation

$$-\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = g(x,u), \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

In particular, we are interested in the Liouville-type theorems for stable solutions of the singular quasilinear elliptic equations

$$-\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = f(x)|u|^{q-1}u, \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

$$-\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = f(x)e^u, \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where $2 \le p < N$, $-\infty < a < (N-p)/p$, $f(x) \in C(\mathbb{R}^N \setminus \{0\})$ is nonnegative. The exact assumption on $f(x)$ will be given in (H1) below.

For $a \neq 0$ and $p \neq 2$, to our knowledge, there is very little information on the nonexistence of stable solutions for problems (1.4) and (1.5).

In this article, we are concerned about stable solutions of (1.4) and (1.5) in the following sense.

Definition 1.1 ([7, 27])**.** Let $g(x,\cdot) : \mathbb{R} \to \mathbb{R}$ be a C^1 function for almost every $x \in \mathbb{R}^N$. We say that u is a weak solution of (1.3) if $u \in C^{1,\omega}_{\rm loc}(\mathbb{R}^N)(0 < \omega < 1)$ satisfies $g(x,u) \in L^1_{\rm loc}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^{p-2}\nabla u \nabla\zeta\, dx = \int_{\mathbb{R}^N} g(x,u)\zeta\, dx, \quad \forall \zeta \in C^1_0(\mathbb{R}^N). \tag{1.6}$$

Let u be a weak solution of (1.3). We say that u is stable if $g_u(x,u) \in L^1_{\rm loc}(\mathbb{R}^N)$ and

$$\begin{aligned} Q_u(\zeta) := &\int_{\mathbb{R}^N} |x|^{-ap}\Big(|\nabla u|^{p-2}|\nabla\zeta|^2 + (p-2)|\nabla u|^{p-4}(\nabla u\cdot\nabla\zeta)^2\Big)dx \\ &- \int_{\mathbb{R}^N} g_u(x,u)\zeta^2 dx \ge 0, \end{aligned} \tag{1.7}$$

for every $\zeta \in C^1_0(\mathbb{R}^N)$.

The Morse index of a solution u, $i(u)$ is defined as the maximal dimension of all subspace X of $C^1_0(\mathbb{R}^N)$ such that $Q_u(\zeta) < 0$ for any $\zeta \in X \setminus \{0\}$. Clearly, u is stable if and only if $i(u) = 0$.

We note that the $C^{1,\omega}$ regularity assumption is natural to the solution of (1.3) due to the results in [1, 8, 10, 29].

Remark 1.2. If u is a stable weak solution of (1.5), then from (1.7) it follows that

$$\int_{\mathbb{R}^N} f(x)e^u\zeta^2 dx \le (p-1)\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^{p-2}|\nabla\zeta|^2 dx, \quad \forall \zeta \in C^1_0(\mathbb{R}^N). \tag{1.8}$$

Similarly, if u is a stable nonnegative solution of (1.4), we have from (1.7) that

$$q\int_{\mathbb{R}^N} f(x)u^{q-1}\zeta^2 dx \le (p-1)\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^{p-2}|\nabla\zeta|^2 dx, \quad \forall \zeta \in C^1_0(\mathbb{R}^N). \tag{1.9}$$

We recall that Liouville-type theorem is the nonexistence of nontrivial solution in the entire space $\mathbb{R}^N$. The classical Liouville theorem stated that a bounded harmonic (or holomorphic) function defined in entire space $\mathbb{R}^N$ must be constant. This theorem, known as Liouville Theorem, was first announced in 1844 by Liouville [24] for the special case of a doubly-periodic function. Later in the same year, Cauchy [4] published the first proof of the above stated theorem. In 1981, Gidas and Spruck established in pioneering article [21] the optimal Liouville type result for nonnegative solutions to the singular equation (1.4) with $p = 2, a = 0$ and $f(x) = |x|^b$:

$$-\Delta u = |x|^b |u|^{q-1} u, \quad x \in \mathbb{R}^N. \tag{1.10}$$

They proved that (1.10) with $b = 0$ has no positive solution if and only if $1 < q < q_s = \frac{N+2}{N-2}$ if $N > 2$ and $q_s = \infty$ if $N = 2$.

The case $b \neq 0$ is less completely understood. Let us first recall that if $b \leq -2$, then (1.10) has no positive solution in any domain Ω containing the origin, see [3, 21]. We therefore restrict ourselves to the case $b > -2$ in the rest of this article. Let us introduce the Hardy-Sobolev exponent

$$q_s(b) = \frac{N+2+2b}{N-2} (= \infty), \quad \text{if } N = 2. \tag{1.11}$$

In the class of radial solutions, the Liouville property was completely solved [3, 26].

Proposition 1.3. *Let $N \geq 2, b > -2$ and $q > 1$.*

(i) *If $q < q_s(b)$, then (1.10) has no positive radial solution in $\mathbb{R}^N$.*
(ii) *If $q \geq q_s(b)$, then (1.10) possesses a bounded, positive radial solution in $\mathbb{R}^N$.*

So far, for the radial solutions, the results have been clean and neat. On the other hand, using Farina's approach in [13], Fazly [15] established Liouville type theorem of the weighted Lane-Emden equation

$$-\Delta u = (1 + |x|^2)^{b/2} |u|^{q-1} u, \quad \text{in } \mathbb{R}^N. \tag{1.12}$$

Proposition 1.4 ([15, Theorems 2.3])**.** *Let u be a nonnegative entire semi-stable solution of (1.12) with $b > -2, q \geq 2$. Then u is the trivial solution if the space dimension N satisfies*

$$2 \leq N < 2 + \frac{2(2+b)}{q-1}\Big(q + \sqrt{q^2 - q}\Big). \tag{1.13}$$

Remark 1.5. Obviously, If $b > -2$ and $N > 10 + 4b$, then (1.13) implies that

$$\begin{aligned} 1 &< q < q_c \\ &:= \frac{(N-2)(N-6-2b) - 2(2+b)^2 + 2(2+b)\sqrt{(2+b)(2N-2+b)}}{(N-2)(N-10-4b)}. \end{aligned} \tag{1.14}$$

Similar works can be founded in [9, 11, 20, 23, 28] and the references therein.

To our knowledge, there are only few works on exponential case (1.5) as compared with (1.10) and (1.12). Farina in [14] proved that $\Delta u + e^u = 0$ has no stable classical solution in $\mathbb{R}^N$ for $2 \leq N \leq 9$. Dancer and farina in [8] proved that (1.5) with $p = 2$ and $f(x) = 1$ admits classical entire solutions which are stable outside a compact set if and only if $N \geq 10$. Recently, Wang and Ye in [28] proved

Theorem 1.6. *Let $p = 2, a = 0$ and $f(x) = |x|^b$ with $b > -2$. For $2 \leq N < 10+4b$, there is no weak stable solution of (1.5).*

In this paper, the first aim is to show the nonexistence of stable solutions to (1.5) with the weighted functions $f(x)$ and $2 \leq p < N$. Since $p > 2$, the test functions in the above references does not work. For the estimation of solution, we need to choose some special test functions to investigate our problem.

Throughout this paper, we make the following assumption on $f(x)$.

(H1) $f(x) \in C(\mathbb{R}^N \setminus \{0\})$ is nonnegative in $\mathbb{R}^N$. In addition, there exist $b > -p(1+a), c_0 > 0$ and $R_0 > 0$ such that $f(x) \geq c_0|x|^b$ for all $|x| \geq R_0$.

Denote

$$q_0(p,a,b) = \frac{p(p+3)(1+a)+4b}{p-1}.$$

Our main results in this paper are as follows.

Theorem 1.7. *Suppose that the function $f(x)$ satisfies* (H1) *and $2 \leq p < N < q_0(p,a,b)$. Then there is no weak stable solution of* (1.5).

Open problem. When $N > q_0(p,a,b)$ or $1 < p < 2$, does equation (1.5) admit a stable solution?

Remark 1.8. If $p = 2$, $a = b = 0$, then $q_0(2,0,0) = 10$. The result in Theorem 1.7 coincides with that in [14]. If $p = 2$, $a = 0$ and $b > -2$, $q_0(2,0,b) = 10 + 4b$. It is the critical exponent $q_c = 10 + 4b$ in [28].

Theorem 1.9. *Suppose that the function $f(x)$ satisfies* (H1) *and $p \geq 2, b > -p(1+a)$. Let $u \in C^{1\delta}_{\rm loc}(\mathbb{R}^N)$ be a stable solution of problem* (1.4)*. Assume that*

$$\begin{cases} p-1 < q < \infty, & \text{if } N \leq q_0(p,a,b), \\ p-1 < q < q_c = q_c(p,a,b), & \text{if } N > q_0(p,a,b) \end{cases} \tag{1.15}$$

with the critical exponent

$$\begin{aligned} &q_c(p,N,a,b) \\ &= \Big((p-1)[N^2(p-1) - p(1+a)(N(p+2) - p(1+a)) \\ &\quad + b(N(p-4) - p^2(1+a)) - 2b^2]\Big) \\ &\quad \div \Big((N - p(1+a))[N(p-1) - p(p+3)(1+a) - 4b]\Big) \\ &\quad + \frac{2(p(1+a)+b)\sqrt{(p-1)(p(1+a)+b)[p(N-1-a)+b(p-1)]}}{(N-p(1+a))[N(p-1)-p(p+3)(1+a)-4b]}. \end{aligned} \tag{1.16}$$

Then $u \equiv 0$ in $\mathbb{R}^N$.

Remark 1.10. If $a = 0$, then

$$\begin{aligned} &q_c(p,N,0,b) \\ &= \frac{(p-1)[N^2(p-1) - p(N(p+2)-p) + b(N(p-4)-p^2) - 2b^2]}{(N-p)[(N(p-1)-p(p+3)-4b]} \\ &\quad + \frac{2(p+b)\sqrt{(p+b)(p-1)[p(N-1)+b(p-1)]}}{(N-p)[(N(p-1)-p(p+3)-4b]}. \end{aligned} \tag{1.17}$$

It is the critical exponent q_c in [5]. Furthermore, if $a = b = 0$, then

$$\begin{aligned} &q_c(p,N,0,0) \\ &= \frac{(p-1)[N^2(p-1) - p(N(p+2)-p)] + 2p^2\sqrt{(p-1)(N-1)}}{(N-p)[(N(p-1)-p(p+3)]}. \end{aligned} \tag{1.18}$$

It equals the critical exponent p_c in [6]. Also, we observe that the critical exponent $q_c(p,N,0,0)$ is always greater than the classic critical exponent $\frac{N(p-1)+p}{N-p}$. If $a =$

$b = 0$ and $p = 2$, we find

$$q_c(2, N, 0, 0) = \frac{N^2 - 8N + 4 + 8\sqrt{N-1}}{(N-2)(N-10)}. \tag{1.19}$$

It is the critical exponent p_c in [13] and the exponent $q_c(p, N)$ in [22] and $p(N, \alpha)$ in [12], and coincides with that in [25].

Remark 1.11. Clearly, problems (1.4) and (1.5) are an extension of problems in [14, 22, 25, 28] respectively. Our conclusions in Theorems 1.7 and 1.9 extend results in the above references.

Remark 1.12. For (1.1), we let $\lambda = 0$ and $N(1 + \frac{8b}{N-2(1+a-b)}) < 10(1 + a)$. Then, an application of Theorem 1.9 shows that there are no stable solutions.

The rest of the paper is devoted to the proof of Theorems 1.7 and 1.9. In the following, we denote by C_j $(j = 1, 2, \dots)$ positive constants, which may vary from line to line.

2. Proof of Theorem 1.7

To prove the nonexistence of solutions to (1.5), we use the test function method, which has been used in [6, 14] and references therein. Since $2 \le p < N$, some modification in choosing functions is necessary. The proof is based on argument by contradiction which involves a priori estimate for a solution of (1.5) by carefully choosing the special test function and scaling argument.

For any nonnegative function $\varphi \in C_0^1(\mathbb{R}^N)$ and $\alpha > 0$, we denote $\zeta = e^{p\alpha u}\varphi^p$. Then, it follows from (1.5) and (1.6) that

$$\begin{aligned} &\int_{\mathbb{R}^N} |x|^{-ap} e^{p\alpha u} |\nabla u|^p \varphi^p dx \\ &= \frac{1}{p\alpha} \int_{\mathbb{R}^N} f(x) e^{(p\alpha+1)u} \varphi^p dx - \frac{1}{\alpha} \int_{\mathbb{R}^N} |x|^{-ap} e^{p\alpha u} \varphi^{p-1} |\nabla u|^{p-2} \nabla u \cdot \nabla\varphi dx. \end{aligned} \tag{2.1}$$

By Young inequality with any $\varepsilon > 0$, one sees that

$$\begin{aligned} &\frac{1}{\alpha} \int_{\mathbb{R}^N} |x|^{-ap} e^{p\alpha u} \varphi^{p-1} |\nabla u|^{p-1} |\nabla\varphi| dx \\ &\le \varepsilon \int_{\mathbb{R}^N} |x|^{-ap} e^{p\alpha u} |\nabla u|^p \varphi^p dx + C_\varepsilon \int_{\mathbb{R}^N} |x|^{-ap} e^{p\alpha u} |\nabla\varphi|^p dx \end{aligned} \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned} &(1 - \varepsilon) \int_{\mathbb{R}^N} |x|^{-ap} e^{p\alpha u} \varphi^{p-1} |\nabla u|^{p-1} |\nabla\varphi| dx \\ &\le \frac{1}{p\alpha} \int_{\mathbb{R}^N} f(x) e^{(p\alpha+1)u} \varphi^p dx + C_\varepsilon \int_{\mathbb{R}^N} |x|^{-ap} e^{p\alpha u} |\nabla\varphi|^p dx \end{aligned} \tag{2.3}$$

On the other hand, we take $\zeta = (e^{p\alpha u}\varphi^p)^{1/2}$ in (1.8) and obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N} f(x)e^{(p\alpha+1)u}\varphi^p dx \\
&\le (p-1)\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^{p-2}|\nabla(e^{p\alpha u/2}\varphi^{p/2})|^2 dx \\
&= \frac{p^2(p-1)}{4}\int_{\mathbb{R}^N} |x|^{-ap}e^{p\alpha u}\Big(\alpha^2|\nabla u|^p\varphi^p + 2\alpha\varphi^{p-1}|\nabla u|^{p-2}\nabla u\nabla\varphi \\
&\quad + |\nabla u|^{p-2}|\nabla\varphi|^2\varphi^{p-2}\Big)dx
\end{aligned}
\tag{2.4}
$$

Similarly, by Young inequality, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |x|^{-ap}e^{p\alpha u}\varphi^{p-2}|\nabla u|^{p-2}|\nabla\varphi|^2 dx \\
&\le \varepsilon\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p e^{p\alpha u}\varphi^p dx + C_\varepsilon\int_{\mathbb{R}^N} |x|^{-ap}e^{p\alpha u}|\nabla\varphi|^p dx
\end{aligned}
\tag{2.5}
$$

Then an application of (2.3)-(2.5) gives

$$
\begin{aligned}
&\int_{\mathbb{R}^N} f(x)e^{(p\alpha+1)u}\varphi^p dx \\
&\le \frac{p^2(p-1)(\alpha^2+\varepsilon+2\alpha\varepsilon)}{4}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p e^{p\alpha u}\varphi^p dx \\
&\quad + C_\varepsilon\int_{\mathbb{R}^N} |x|^{-ap}e^{p\alpha u}|\nabla\varphi|^p dx \\
&\le \frac{p^2(p-1)(\alpha^2+\varepsilon+2\alpha\varepsilon)}{4p\alpha(1-\varepsilon)}\int_{\mathbb{R}^N} f(x)e^{(p\alpha+1)u}\varphi^p dx \\
&\quad + C_\varepsilon\int_{\mathbb{R}^N} |x|^{-ap}e^{p\alpha u}|\nabla\varphi|^p dx.
\end{aligned}
\tag{2.6}
$$

Then, one sees that

$$
\lambda_1\int_{\mathbb{R}^N} f(x)e^{(p\alpha+1)u}\varphi^p dx \le C_\varepsilon\int_{\mathbb{R}^N} |x|^{-ap}e^{p\alpha u}|\nabla\varphi|^p dx,
\tag{2.7}
$$

where

$$
\lambda_1 = \lambda_0 - \frac{\lambda_2\varepsilon}{1-\varepsilon},\quad \lambda_0 = 1 - \frac{p(p-1)\alpha}{4},\quad \lambda_2 = \frac{p(p-1)(1+\alpha)^2}{4\alpha}.
\tag{2.8}
$$

in which we choose $0 < \alpha < \frac{4}{p(p-1)}$ such that $\lambda_0 > 0$. Furthermore, let $\varepsilon > 0$ be so small that $\lambda_1 > 0$.

By Hölder inequality, from (2.7) we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^N} fe^{(p\alpha+1)u}\varphi^p dx &\le C_\varepsilon\Big(\int_{\mathbb{R}^N} fe^{(p\alpha+1)u}\varphi^p dx\Big)^{\frac{p\alpha}{p\alpha+1}} \\
&\quad \times\Big(\int_{\mathbb{R}^N} f^{-p\alpha}|x|^{-pa(p\alpha+1)}\varphi^{-p^2\alpha}|\nabla\varphi|^{p(p\alpha+1)}dx\Big)^{\frac{1}{p\alpha+1}},
\end{aligned}
\tag{2.9}
$$

where $f = f(x)$. This implies

$$
\int_{\mathbb{R}^N} f(x)e^{(p\alpha+1)u}\varphi^p dx \le C_\varepsilon\int_{\mathbb{R}^N} f^{-p\alpha}|x|^{-pa(p\alpha+1)}\varphi^{-p^2\alpha}|\nabla\varphi|^{p(p\alpha+1)}dx
\tag{2.10}
$$

We now choose $\varphi_0(s) \in C_0^1[0,\infty)$ defined by

$$\varphi_0(s) = \begin{cases} 1, & 0 \le s \le 1, \\ 2(2-s)^k - (2-s)^{2k}, & 1 < s < 2, \\ 0, & s > 2, \end{cases} \tag{2.11}$$

where $k = p\alpha + 1 > 1$. It is not difficult to verify that $0 \le \varphi_0(s) \le 1$ and $|\varphi_0'(s)| \le \beta_0 \varphi_0^{1-1/k}(s)$ with $\beta_0 = 2^{1/k}k$.

We let $\varphi = \varphi(x) = \varphi_0(\frac{|x|}{R})$ with $R > R_0$, where R_0 is given in (H1). Then, setting $x = R\xi$, we get

$$\begin{aligned} \int_{\mathbb{R}^N} f^{-p\alpha}\varphi^{-p^2\alpha}|\nabla\varphi|^{p(p\alpha+1)}dx &\le CR^\theta \int_{1\le|\xi|\le 2} \Big(\frac{|\varphi_0'(|\xi|)}{\varphi_0(|\xi|)^{1-\frac{1}{p\alpha+1}}}\Big)^{p(p\alpha+1)} d\xi \\ &\le CR^\theta \beta_0^{p(p\alpha+1)} \end{aligned} \tag{2.12}$$

where C is a positive constant independent of R and $\theta = N - p(1+a) - p\alpha(b + p(1+a))$. Noticing that $0 < p(p-1)\alpha < 4$ and $N < q_0(p,a,b) = \frac{p(p+3)(1+a)+4b}{p-1}$, we can so choose that α such that $N < p(1+a) + p\alpha(b + p(1+a))$. Then, letting $R \to +\infty$, we obtain from (2.12) that

$$\int_{\mathbb{R}^N} f(x)e^{(p\alpha+1)u}dx = 0 \tag{2.13}$$

This is impossible. This completes the proof.

3. Proof of Theorem 1.9

We first establish some estimation for solutions of (1.4).

Lemma 3.1. *Let $u \in C^{1,\omega}_{\rm loc}(\mathbb{R}^N)$ be a stable weak solution of (1.4) with $q > p-1 \ge 1$. Then for every $k \in (1,\ k_0(q))$, where*

$$k_0(t) = \frac{2t - p + 1 + 2\sqrt{t(t-p+1)}}{p-1}, \quad t > p-1, \tag{3.1}$$

and for any integer n with $n \ge \max\{2, \frac{q+k}{q-p+1}\}$, there exists a constant $C = C(q,p,n,k)$ such that

$$\begin{aligned} &\int_{\mathbb{R}^N} f(x)|u|^{q+k}\varphi^{pn}dx + \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p|u|^{k-1}\varphi^{pn}dx \\ &\le C\int_{\mathbb{R}^N} |x|^{-ap}|\nabla\varphi|^{\frac{p(q+k)}{q-p+1}}|f(x)|^{-\frac{k+p-1}{q-p+1}}dx, \end{aligned} \tag{3.2}$$

where $\varphi \in C_0^1(\mathbb{R}^N)$ is the nonnegative cut-off function, in which $\varphi(x) = \varphi_0(\frac{|x|}{R})$ for any $R > 0$ with $\varphi_0(s) \in C_0^1(\mathbb{R}^+), 0 \le \varphi_0(s) \le 1$ and

$$\varphi_0(s) = \begin{cases} 1, & 0 \le s \le 1, \\ 0, & s \ge 2. \end{cases} \tag{3.3}$$

Proof. By the definition of $\varphi(x)$, we know that there exists $C > 0$ such that $|\nabla\varphi(x)| \le CR^{-1}$ in $x \in \overline{B}_{2R} \setminus \overline{B}_R$ and $|\nabla\varphi(x)| = 0$ if $x \in B_R \cup B_{2R}^c$, where $B_r = \{x \in \mathbb{R}^N : |x| < r\}$.

Let $u \in C^{1,\omega}_{\rm loc}(\mathbb{R}^N)$ be a stable solution of (1.4) and $k > 1$. Multiplying (1.4) by $|u|^{k-1}u\varphi^p$ and integrating by parts, we find

$$\begin{aligned} &k\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p|u|^{k-1}\varphi^p dx\\ &\le p\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^{p-1}|\nabla\varphi||u|^k\varphi^{p-1}dx+\int_{\mathbb{R}^N}f(x)|u|^{q+k}\varphi^p dx. \end{aligned}\tag{3.4}$$

Then applying Young's inequality with parameter $\epsilon \in (0,1)$, we have

$$\begin{aligned} &p\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^{p-1}|\nabla\varphi||u|^k\varphi^{p-1}dx\\ &\le \epsilon\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p|u|^{k-1}\varphi^p dx+C_1\int_{\mathbb{R}^N}|x|^{-ap}|\nabla\varphi|^p|u|^{k+p-1}dx. \end{aligned}\tag{3.5}$$

Then from (3.4) and (3.5) it follows that

$$\begin{aligned} &(k-\epsilon)\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p|u|^{k-1}\varphi^p dx\\ &\le \int_{\mathbb{R}^N}f(x)|u|^{q+k}\varphi^p dx+C_1\int_{\mathbb{R}^N}|x|^{-ap}|\nabla\varphi|^p|u|^{k+p-1}dx. \end{aligned}\tag{3.6}$$

On the other hand, taking $\zeta = |u|^{\frac{k-1}{2}}u\varphi^{\frac{p}{2}}$ in (1.9), one sees that

$$\begin{aligned} &\frac{q}{p-1}\int_{\mathbb{R}^N}f(x)|u|^{q+k}\varphi^p dx\\ &\le \frac{1}{4}(1+k)^2\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p|u|^{k-1}\varphi^p dx\\ &\quad+\frac{p^2}{4}\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^{p-2}|\nabla\varphi|^2|u|^{k+1}\varphi^{p-2}dx\\ &\quad+\frac{1}{2}p(1+k)\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^{p-1}|\nabla\varphi||u|^k\varphi^{p-1}dx. \end{aligned}\tag{3.7}$$

By Young's inequality with $\epsilon > 0$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^{p-2}|\nabla\varphi|^2|u|^{k+1}\varphi^{p-2}dx\\ &\le \epsilon\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p|u|^{k-1}\varphi^p dx+C_2\int_{\mathbb{R}^N}|x|^{-ap}|\nabla\varphi|^p|u|^{k+p-1}dx, \end{aligned}\tag{3.8}$$

$$\begin{aligned} &\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^{p-1}|\nabla\varphi||u|^k\varphi^{p-1}dx\\ &\le \epsilon\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p|u|^{k-1}\varphi^p dx+C_3\int_{\mathbb{R}^N}|x|^{-ap}|\nabla\varphi|^p|u|^{k+p-1}dx. \end{aligned}\tag{3.9}$$

Then it follows from (3.7)-(3.9) that

$$\begin{aligned} \frac{q}{p-1}\int_{\mathbb{R}^N}f(x)|u|^{q+k}\varphi^p dx &\le \beta_\epsilon\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p|u|^{k-1}\varphi^p dx\\ &\quad+C_4\int_{\mathbb{R}^N}|x|^{-ap}|\nabla\varphi|^p|u|^{k+p-1}dx \end{aligned}\tag{3.10}$$

with $\beta_\epsilon = \frac{1}{4}[(1+k)^2 + p^2\epsilon + 2p(1+k)\epsilon] > 0$. Furthermore, we obtain from (3.6) and (3.10) that

$$\alpha_\epsilon \int_{\mathbb{R}^N} f(x)|u|^{q+k}\varphi^p dx \le C_5 \int_{\mathbb{R}^N} |x|^{-ap}|\nabla\varphi|^p |u|^{k+p-1} dx \tag{3.11}$$

with some constant $C_5 > 0$ and

$$\alpha_\epsilon = \frac{q}{p-1} - \frac{\beta_\epsilon}{k-\epsilon}, \quad \lim_{\epsilon\to 0^+} \alpha_\epsilon = \alpha_0 := \frac{q}{p-1} - \frac{(1+k)^2}{4k}. \tag{3.12}$$

The fact $\alpha_0 > 0$ implies that $k \in (1, k_0(q))$. Now, the application of (3.6) and (3.11) yields

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p |u|^{k-1}\varphi^p dx \le C_5 \int_{\mathbb{R}^N} |x|^{-ap}|\nabla\varphi|^p |u|^{k+p-1} dx. \tag{3.13}$$

We claim that the estimate (3.2) holds. Choose the integer $n \ge \max\{2, \frac{q+k}{q-p+1}\}$. Then one sees that

$$\varphi^{\frac{p(n-1)(q+k)}{k+p-1}}(x) \le \varphi^{pn}(x), \quad x \in \mathbb{R}^N. \tag{3.14}$$

Then, replacing φ by φ^n in (3.11), we find

$$\begin{aligned}
&\int_{\mathbb{R}^N} f(x)|u|^{q+k}\varphi^{pn} dx \\
&\le \Big(\int_{\mathbb{R}^N} f(x)|u|^{q+k}\varphi^{p(n-1)\lambda} dx\Big)^{\frac{1}{\lambda}} \Big(\int_{\mathbb{R}^N} f^{-\frac{\lambda'}{\lambda}} |x|^{-pa\lambda'}|\nabla\varphi|^{p\lambda'} dx\Big)^{\frac{1}{\lambda'}} \\
&\le \Big(\int_{\mathbb{R}^N} f(x)|u|^{q+k}\varphi^{pn} dx\Big)^{\frac{1}{\lambda}} \Big(\int_{\mathbb{R}^N} f^{-\frac{\lambda'}{\lambda}} |x|^{-pa\lambda'}|\nabla\varphi|^{p\lambda'} dx\Big)^{\frac{1}{\lambda'}},
\end{aligned} \tag{3.15}$$

where $\lambda = (q+k)/(k+p-1) > 1, \lambda' = (q+k)/(q-p+1) > 1$. So, it derives by (3.15) that

$$\int_{\mathbb{R}^N} f(x)|u|^{q+k}\varphi^{pn} dx \le C \int_{\mathbb{R}^N} |x|^{-ap}|\nabla\varphi|^{\frac{p(q+k)}{q-p+1}} f^{-\frac{k+p-1}{q-p+1}} dx, \tag{3.16}$$

Similarly, replacing φ by φ^n in (3.13), from (3.15) and (3.16) we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} |x|^{-pa}|\nabla u|^p |u|^{k-1}\varphi^{pn} dx &\le C \int_{\mathbb{R}^N} |x|^{-pa}\varphi^{p(n-1)}|u|^{k+p-1}|\nabla\varphi|^p dx \\
&\le C \int_{\mathbb{R}^N} |x|^{-ap}|\nabla\varphi|^{\frac{p(q+k)}{q-p+1}} f^{-\frac{k+p-1}{q-p+1}} dx.
\end{aligned}$$

So, we obtain (3.2) and the proof of Lemma 3.1 is complete. $\square$

Proof of Theorem 1.9. From estimate (3.2) and the definition of the function $\varphi(x)$, it follows that

$$\int_{\mathbb{R}^N} f(x)|u|^{q+k}\varphi^{pn} dx + \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p |u|^{k-1}\varphi^{pn} dx \le CR^\tau, \tag{3.17}$$

where assumption (H1) has been used and

$$\tau = N - p(1+a) - \frac{(p(1+a)+b)(k+p-1)}{t-p+1}. \tag{3.18}$$

Clearly, if $\tau < 0$, the desired result follows by letting $R \to \infty$ in (3.17). In the following, we consider the case in which $\tau < 0$. Define the function

$$g(t) = \frac{k_0(t) + p - 1}{t - p + 1}, \quad t > p - 1, \tag{3.19}$$

where $k_0(t)$ is given in (3.1). Obviously,

$$\lim_{t \to (p-1)^+} g(t) = +\infty, \quad \lim_{t \to +\infty} g(t) = g_\infty := \frac{4}{p-1}. \tag{3.20}$$

Since

$$g'(t) = \frac{-1}{(t-p+1)^2}\Big[1 + \frac{t-p+1}{\sqrt{t(t-p+1)}}\Big] < 0, \quad \text{for } t > p - 1, \tag{3.21}$$

the function $g(t)$ is decreasing in $t > p - 1$. So, we have $g_\infty < g(t) < +\infty$ for $t > p - 1$.

Therefore, if $N - p(1+a) \leq (p(1+a)+b)g_\infty$, then $N - p(1+a) < (p(1+a)+b)g(t)$ for any $t > p - 1$. Hence if we fix $k \in [1, k_0(t))$ suitably near $k_0(t)$, we obtain

$$N - p(1+a) < \frac{(p(1+a)+b)(k+p-1)}{t-p+1}. \tag{3.22}$$

For this reason, the desired result follows by letting $R \to \infty$ in (3.17).

Assume now $N - p(1+a) > (p(1+a)+b)g_\infty$. Since g is decreasing, we get in this case a critical value $q_c(p, N, a, b)$ such that $N - p(1+a) < (p(1+a)+b)g(q)$ for $p-1 < q < q_c(p, N, a)$. From this, the desired result follows again by letting $R \to \infty$ in (3.17). Clearly, $q_c(p, N, a, b)$ may be deduced from the equation $N - p(1+a) = (p(1+a)+b)g(q)$, which is given the value in (1.16). Then we complete the proof. $\square$

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Caisheng Chen (corresponding author)
College of Science, Hohai University, Nanjing 210098, China
E-mail address: `cshengchen@hhu.edu.cn`

Hongxue Song
College of Science, Hohai University, Nanjing 210098, China.
College of Science, Nanjing University of Posts and Telecommunications, Nanjing 210023, China
E-mail address: `songhx@njupt.edu.cn`

Hongwei Yang
College of Mathematics and System Science, Shandong University of Science and Technology, Qingdao 266590, China
E-mail address: `hwyang1979@163.com`