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FRACTIONAL MINIMIZATION PROBLEM ON THE NEHARI MANIFOLD

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Abstract. In the framework of fractional Sobolev space, using Nehari manifold and concentration compactness principle, we study a minimization problem in the whole space involving the fractional Laplacian. Firstly, we give a Lions type lemma in fractional Sobolev space, which is crucial in the proof of our main result. Then, by showing a relative compactness of minimizing sequence, we obtain the existence of minimizer for the above-mentioned fractional minimization problem. Furthermore, we also point out that the minimizer is actually a ground state solution for the associated fractional Schrödinger equation

1. Introduction and statement of main results

In this article, we are interested in the minimizing problem

$$I_0 := \inf \big\{ I(u) : u \in H^\alpha(\mathbb{R}^N) \setminus \{0\},\ \langle I'(u), u\rangle = 0 \big\}. \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual duality between the fractional Sobolev space $H^\alpha(\mathbb{R}^N)$ (see Section 2 for more details) and its dual space. The energy functional $I : H^\alpha(\mathbb{R}^N) \to \mathbb{R}^N$ is defined by

$$I(u) = \frac{1}{2} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N} |u|^2\,dx - \frac{1}{q}\int_{\mathbb{R}^N} |u|^q\,dx,$$

where $0 < \alpha < 1$, $2 < q < 2^*_\alpha = 2N/(N-2\alpha)$. It is standard to verify that $I \in C^1(H^\alpha(\mathbb{R}^N), \mathbb{R})$. Then, a necessary condition for $u \in H^\alpha(\mathbb{R}^N)$ to be a critical point of I is that $\langle I'(u), u\rangle = 0$, which defines the Nehari manifold

$$\mathcal{N} := \{u \in H^\alpha(\mathbb{R}^N) \setminus \{0\} : \langle I'(u), u\rangle = 0\}.$$

The minimizing problem (1.1) is often referred to as the minimizing problem with artificial constraint.

In recent years, fractional spaces and the corresponding fractional problems arise in many different applications, such as phase transitions [1, 31], optimization [16], conservation laws [4], minimal surfaces [7], materials science [3], water waves [9, 10] and so on. Some interesting topics concerning the fractional Laplacian, such as, the nonlinear fractional Schrödinger equation (see [20, 21]), the nonlinear fractional

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Kirchhoff equation (see [19, 29, 34, 35]), the fractional porous medium equation (see [8, 33]) and so on, have attracted recently much research interest. Indeed, the literature on fractional operators and their applications to partially differential equations is quite large, here we would like to mention a few, see for instance [2, 6, 11, 13, 27, 30, 36] and the references therein.

Based on the theory of fractional Sobolev space, we will study the existence of minimizer for minimizing problem (1.1) by showing a relative compactness of minimizing sequences of problem (1.1). A main difficulty is due to the loss of the compactness for the embedding $H^\alpha(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$. To overcome this difficulty, we use the concentration compactness principle established by Lions in [22, 23]. With the help of this principle, a noncompact minimizing sequence can be changed into a new sequence possessing some compact. Now, we give the main result as follows:

Theorem 1.1. *Problem* (1.1) *has a minimizer, i.e. there exists* $v \in \mathcal{N}$ *such that* $I(v) = \inf_{u\in\mathcal{N}} I(u)$.

In fact, it is easy to see that v is exactly a ground state solution for the fractional Schrödinger equation

$$(-\Delta)^\alpha u + V(x)u = |u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where the potential $V(x) = 1$. More precisely, set $G(u) = \langle I'(u), u\rangle$. Then it follows from Lagrange multiplier rule that, for some $\lambda \in \mathbb{R}$,

$$I'(v) = \lambda G'(v).$$

Note that $\langle I'(v), v\rangle = 0$ implies that

$$\langle G'(v), v\rangle = (2-q)\int_{\mathbb{R}^N} |v|^q\,dx < 0,$$

which means that $\lambda = 0$. Thus, $I'(v) = 0$.

It is worthy pointing out that problem (1.1) was studied by several researchers recently. For example, Byeon *et al.* in [5] investigated a Pohozaev type minimization problem for a nonlinear fractional Schrödinger equation involving the Berestycki-Lions type C^0 nonlinearity. In this article, we will take a quite different approach from [5] to study problem (1.1).

Finally, let us sketch the main advances related to our study and the key techniques used in this article. In [17], by using the mountain pass lemma, the authors obtained the existence of positive solutions to problem (1.2). Especially, they used a comparison argument to overcome the difficulty that the Palais-Smale sequences may lose compactness in the whole space $\mathbb{R}^N$. In [15], the existence of a radially symmetric solution for problem (1.2) has been obtained by applying symmetric decreasing rearrangement. The existence of bound state solutions for problem (1.2) with unbounded potential have been derived by Lagrange multiplier method and Nehari manifold in [12]. In [18], the author also used the concentration compactness principle in fractional Sobolev space to get the existence of a positive ground state solution for problem (1.2) with some positive Lagrange multiplier $V(x) = V_0$. For example, see also [32, 37, 38] for some recent results for problem (1.2) involving the critical exponents exploited by the fractional version of concentration compactness principle. In this paper, we first give a Lions type lemma in fractional Sobolev space. Then, using a different version of concentration compactness principle, we obtain the main result. To the best of our knowledge, there are few papers to use the above-mentioned approach to study the existence of ground state solutions for problem (1.2) without critical nonlinearities.

2. Preliminaries

For the convenience of reader, in this part we recall some definitions and basic properties of fractional Sobolev spaces. For a deeper treatment on these spaces and their applications to fractional Laplacian problems of elliptic type, we refer to [14, 24] and the references therein.

Let $0 < \alpha < 1 < p < \infty$ be real numbers with $p\alpha < N$, and let p_α^* be the fractional Sobolev critical exponent defined by $p_\alpha^* = Np/(N - p\alpha)$. The fractional Sobolev space $W^{\alpha,p}(\mathbb{R}^N)$ is defined by

$$W^{\alpha,p}(\mathbb{R}^N) = \Big\{u \in L^p(\mathbb{R}^N) : [u]_{\alpha,p} < \infty\Big\},$$

where $[u]_{\alpha,p}$ denotes the Gagliardo norm, that is

$$[u]_{\alpha,p} = \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+p\alpha}} dx\, dy\Big)^{1/p},$$

and $W^{\alpha,p}(\mathbb{R}^N)$ is equipped with the norm

$$\|u\|_{W^{\alpha,p}(\mathbb{R}^N)} = \Big(\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{\alpha,p}\Big)^{1/p}.$$

As it is well known, $W^{\alpha,p}(\mathbb{R}^N) = \big(W^{\alpha,p}(\mathbb{R}^N), \|\cdot\|_{W^{\alpha,p}(\mathbb{R}^N)}\big)$ is a uniformly convex Banach space (see also [28]), which implies that it is reflexive.

In the case $p = 2$, $W^{\alpha,2}(\mathbb{R}^N) := H^\alpha(\mathbb{R}^N)$ turns out to be a Hilbert space with scalar product

$$\langle u,\ v\rangle_{H^\alpha(\mathbb{R}^N)} = \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2\alpha}}\, dx\, dy + \int_{\mathbb{R}^N} u(x)v(x)\, dx.$$

Theorem 2.1 ([14, Theorem 6.7])**.** *Let $\Omega \subseteq \mathbb{R}^N$ be an extension domain for the space $W^{\alpha,p}(\Omega)$. Then there exists a positive constant $C = C(N, p, \alpha, \Omega)$ such that for any $u \in W^{\alpha,p}(\Omega)$,*

$$\|u\|_{L^\nu(\Omega)} \le C\|u\|_{W^{\alpha,p}(\Omega)},$$

for any $\nu \in [p, p_\alpha^*]$, i.e. the embedding $W^{\alpha,p}(\Omega) \hookrightarrow L^\nu(\Omega)$ is continuous for any $\nu \in [p, p_\alpha^*]$.

Remark 2.2. By Theorem 2.1, the embedding $W^{\alpha,p}(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N)$ is continuous for any $\nu \in [p, p_\alpha^*]$, that is, there exists a positive constant $C_1 = C(N, p, \alpha)$ such that for any $u \in W^{\alpha,p}(\mathbb{R}^N)$,

$$\|u\|_{L^\nu(\mathbb{R}^N)} \le C_1\|u\|_{W^{\alpha,p}(\mathbb{R}^N)}. \tag{2.1}$$

We could also obtain that the embedding $W^{\alpha,p}(B(y, R)) \hookrightarrow L^\nu(B(y, R))$ is continuous for any $\nu \in [p, p_\alpha^*]$, where $B(y, R) \subset \mathbb{R}^N$ is an open ball with center y and radius R. Then, there exists a positive constant $C_2 = C(N, p, \alpha, B(y, R))$ such that for any $u \in W^{\alpha,p}(B(y, R))$,

$$\|u\|_{L^\nu(B(y,R))} \le C_2\|u\|_{W^{\alpha,p}(B(y,R))}. \tag{2.2}$$

In fact, we could modify the proof of [14, (5.3) in Proposition 2.2] which is important in the discussion of Theorem 2.1 and obtain that the constant C_2 depends only on N, p, α and the radius R.

Theorem 2.3 ([28, Lemma 2.1])**.** *If $1 \le \nu < p_s^*$, the embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^\nu_{\rm loc}(\mathbb{R}^N)$ is compact.*

To prove the main result, we give the following lemma of Lions type in fractional Sobolev space.

Lemma 2.4. *If $\{u_n\}_n$ is bounded in $W^{\alpha,p}(\mathbb{R}^N)$ and*

$$\sup_{y\in\mathbb{R}^N}\int_{B(y,R)} |u_n|^\nu\,dx \to 0, \quad \text{as } n\to\infty,$$

for some $R>0$ and some ν satisfying $p\le \nu < p_\alpha^*$, then $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for any s with $p<s<p_\alpha^*$. Furthermore, if $\nu = p_\alpha^*$, then $u_n\to 0$ in $L^{p_\alpha^*}(\mathbb{R}^N)$.

Proof. Choose $\{y_i\}_i \subset \mathbb{R}^N$ such that $\mathbb{R}^N \subset \cup_{i=1}^\infty B(y_i,R)$ and each $y\in\mathbb{R}^N$ is covered by at most $N+1$ of such balls. Note that

$$\mathbb{R}^{2N} \subset \big(\cup_{i=1}^\infty B(y_i,R)\big)\times\big(\cup_{i=1}^\infty B(y_i,R)\big) \subset \cup_{i,j=1}^\infty\big(B(y_i,R)\times B(y_j,R)\big).$$

Then $z\in\mathbb{R}^{2N}$ is contained in at most $(N+1)^2$ of $\{B(y_i,R)\times B(y_j,R)\}_{i,j}$.

We first consider the case $p\le\nu<p_\alpha^*$. Let $m = p + (1-p/p_\alpha^*)\nu$ and $\kappa = p/m$. Then, $\nu<m<p_\alpha^*$, $0<\kappa<1$ and they satisfy the following equation

$$\frac{1}{m} = \frac{1-\kappa}{\nu} + \frac{\kappa}{p_\alpha^*}.$$

By the Hölder inequality and (2.2), we have

$$\begin{aligned}
\int_{B(y_i,R)} \big|u_n(x)\big|^m dx &\le \|u_n\|^{m(1-\kappa)}_{L^\nu(B(y_i,R))}\|u_n\|^{m\kappa}_{L^{p_\alpha^*}(B(y_i,R))}\\
&\le C_2^p\Big(\sup_{y\in\mathbb{R}^N}\int_{B(y,R)}\big|u_n(x)\big|^\nu dx\Big)^{m(1-\kappa)\nu}\|u_n\|^p_{W^{\alpha,p}(B(y_i,R))}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\sum_{i=1}^\infty \|u_n\|^p_{W^{\alpha,p}(B(y_i,R))}\\
&\le (N+1)\int_{\mathbb{R}^N}|u_n(x)|^p\,dx + \sum_{i,j=1}^\infty\int_{B(y_i,R)}\int_{B(y_j,R)}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+p\alpha}}\,dx\,dy\\
&\le (N+1)\int_{\mathbb{R}^N}|u_n(x)|^p\,dx + (N+1)^2\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+p\alpha}}\,dx\,dy,
\end{aligned}$$

which implies

$$\begin{aligned}
\int_{\mathbb{R}^N}\big|u_n(x)\big|^m dx &\le \sum_{i=1}^\infty\int_{B(y_i,R)}\big|u_n(x)\big|^m dx\\
&\le C_2^p\Big(\sup_{y\in\mathbb{R}^N}\int_{B(y,R)}\big|u_n(x)\big|^\nu dx\Big)^{\frac{m(1-\kappa)}{\nu}}\sum_{i=1}^\infty\|u_n\|^p_{W^{\alpha,p}(B(y_i,R))}\\
&\le C\Big(\sup_{y\in\mathbb{R}^N}\int_{B(y,R)}\big|u_n(x)\big|^\nu dx\Big)^{\frac{m(1-\kappa)}{\nu}} \to 0, \quad \text{as } n\to\infty.
\end{aligned}$$

(i) If $m < s < p_s^*$, take $\lambda = \frac{s-m}{p_\alpha^*-m} \cdot \frac{p_\alpha^*}{s}$, then $\frac{1}{s} = \frac{1-\lambda}{m} + \frac{\lambda}{p_\alpha^*}$. By the Hölder inequality and (2.1), we have

$$\begin{aligned}\|u_n\|_{L^s(\mathbb{R}^N)} \le \|u_n\|_{L^m(\mathbb{R}^N)}^{1-\lambda}\|u_n\|_{L^{p_\alpha^*}(\mathbb{R}^N)}^{\lambda} &\le C\|u_n\|_{L^m(\mathbb{R}^N)}^{1-\lambda}\|u_n\|_{W^{\alpha,p}(\mathbb{R}^N)}^{\lambda}\\ &\le C\|u_n\|_{L^m(\mathbb{R}^N)}^{1-\lambda},\end{aligned}$$

then $\|u_n\|_{L^s(\mathbb{R}^N)} \to 0$, as $n \to \infty$.

(ii) If $p < s < m$, take $\lambda = \frac{s-p}{m-p} \cdot \frac{m}{s}$, then $\frac{1}{s} = \frac{1-\lambda}{p} + \frac{\lambda}{m}$, which implies

$$\|u_n\|_{L^s(\mathbb{R}^N)} \le \|u_n\|_{L^p(\mathbb{R}^N)}^{1-\lambda}\|u_n\|_{L^m(\mathbb{R}^N)}^{\lambda} \le C\|u_n\|_{L^m(\mathbb{R}^N)}^{1-\lambda}.$$

Thus $\|u_n\|_{L^s(\mathbb{R}^N)} \to 0$, as $n \to \infty$.

Finally, in the case $\nu = p_\alpha^*$, we have

$$\int_{B(y_i,R)} |u_n(x)|^{p_\alpha^*} dx \le \Big(\sup_{y\in\mathbb{R}^N}\int_{B(y,R)} |u_n(x)|^{p_\alpha^*} dx\Big)^{\frac{p_\alpha^*-p}{p_\alpha^*}}\Big(\int_{B(y_i,R)} |u_n(x)|^{p_\alpha^*} dx\Big)^{\frac{p}{p_\alpha^*}}.$$

Then we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |u_n(x)|^{p_\alpha^*} dx &\le \sum_{i=1}^{\infty}\int_{B(y_i,R)} |u_n(x)|^{p_\alpha^*} dx\\ &\le C\Big(\sup_{y\in\mathbb{R}^N}\int_{B(y,R)} |u_n(x)|^{p_\alpha^*} dx\Big)^{\frac{p_\alpha^*-p}{p_\alpha^*}} \to 0,\end{aligned}$$

as $n \to \infty$. The lemma is thus proved. □

3. Proof of Theorem 1.1

In this section, by using concentration compactness principle, we will show that the minimizing sequence of problem (1.1) is relative compactness.

First of all, several technical results will be established. For any $a \ge 0$, define

$$I_{-a} = \inf\big\{I(u) + \frac{a}{2} : u \in H^\alpha(\mathbb{R}^N)\setminus\{0\},\ \langle I'(u), u\rangle = -a\big\}.$$

Theorem 3.1. *For any $a \ge 0$, I_{-a} satisfies the following properties:*

1. *$I_0 > 0$;*
2. *for any $b > a \ge 0$, $I_{-b} > I_{-a}$;*
3. *I_{-a}, as a function of a, is continuous in $a \in (0,\infty)$.*

Proof. (1) In fact, for any $u \in N$, we have

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \int_{\mathbb{R}^N} |u|^2\,dx = \int_{\mathbb{R}^N} |u|^q\,dx,$$

which implies

$$I(u) = \big(\frac{1}{2} - \frac{1}{q}\big)\Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \int_{\mathbb{R}^N} |u|^2\,dx\Big) \ge 0. \tag{3.1}$$

If $I_0 = 0$, there exists $\{u_n\}_n \subset \mathcal{N}$ such that $I(u_n) \to 0$, as $n \to \infty$. Then

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \int_{\mathbb{R}^N} |u_n|^2\,dx \to 0,$$

which implies that $u_n \to 0$ in $H^\alpha(\mathbb{R}^N)$, as $n \to \infty$. By (2.1), we derive

$$
\begin{aligned}
0 = \langle I'(u_n), u_n\rangle &= \|u_n\|^2_{H^\alpha(\mathbb{R}^N)} - \|u_n\|^q_{L^q(\mathbb{R}^N)} \\
&\geq \|u_n\|^2_{H^\alpha(\mathbb{R}^N)} - C_1^q \|u_n\|^q_{H^\alpha(\mathbb{R}^N)} \\
&\geq \frac{1}{2}\|u_n\|^2_{H^\alpha(\mathbb{R}^N)},
\end{aligned} \tag{3.2}
$$

as n is sufficiently large. That is a contradiction. So, we obtain $I_0 > 0$.

(2) For any $b > a \geq 0$, we have that $I_{-b} \geq I_{-a}$. In fact, for any $t > 0$,

$$
\langle I'(tu), tu\rangle = t^2 \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + t^2\int_{\mathbb{R}^N} |u|^2\,dx - t^q \int_{\mathbb{R}^N} |u|^q\,dx,
$$

which implies that for any $u \in H^\alpha(\mathbb{R}^N)\setminus\{0\}$, there exists $0 < t(u) < 1$ such that

$$
\langle I'(t(u)u), t(u)u\rangle > 0.
$$

Then, for any $u \in H^\alpha(\mathbb{R}^N)$ such that $\langle I'(u), u\rangle = -b$, there exists $t_u \in (t(u), 1)$ such that

$$
\langle I'(t_u u), t_u u\rangle = -a.
$$

Then we have

$$
\begin{aligned}
&I(u) + \frac{b}{2} - I(t_u u) - \frac{a}{2} \\
&= \frac{1-t_u^2}{2}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \frac{1-t_u^2}{2}\int_{\mathbb{R}^N}|u|^2\,dx \\
&\quad - \frac{1-t_u^q}{q}\int_{\mathbb{R}^N}|u|^q\,dx + \frac{b}{2} - \frac{a}{2}.
\end{aligned}
$$

Denote

$$
g(t) = \frac{1-t^2}{2}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \frac{1-t^2}{2}\int_{\mathbb{R}^N}|u|^2\,dx - \frac{1-t^q}{q}\int_{\mathbb{R}^N}|u|^q\,dx.
$$

For any $0 < t < 1$,

$$
\begin{aligned}
g'(t) &= -t\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy - t\int_{\mathbb{R}^N}|u|^2\,dx + t^{q-1}\int_{\mathbb{R}^N}|u|^q\,dx \\
&= -t\Big(\int_{\mathbb{R}^N}|u|^q\,dx - b\Big) + t^{q-1}\int_{\mathbb{R}^N}|u|^q\,dx \\
&< bt,
\end{aligned}
$$

which implies that $g(t) - \frac{b}{2}t^2$ is decreasing on $(0,1)$. Then

$$
g(t_u) - \frac{b}{2}t_u^2 > g(1) - \frac{b}{2} = -\frac{b}{2},
$$

which yields

$$
I(u) + \frac{b}{2} - I(t_u u) - \frac{a}{2} > \frac{b}{2}t_u^2 - \frac{a}{2}.
$$

By the definition of I_{-a}, we obtain

$$
I(u) + \frac{b}{2} > I(t_u u) + \frac{a}{2} + \frac{b}{2}t_u^2 - \frac{a}{2} \geq I_{-a} + \frac{b}{2}t_u^2 - \frac{a}{2}. \tag{3.3}
$$

In addition, we have

$$
\begin{aligned}
&\langle I'(u), u\rangle - \langle I'(t_u u), t_u u\rangle \\
&= (1-t_u^2)\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + (1-t_u^2)\int_{\mathbb{R}^N} |u|^2\,dx \\
&\quad - (1-t_u^q)\int_{\mathbb{R}^N} |u|^q\,dx \\
&= (1-t_u^2)\Big(\int_{\mathbb{R}^N} |u|^q\,dx\Big) + (t_u^q - 1)\int_{\mathbb{R}^N} |u|^q\,dx \\
&= (t_u^q - t_u^2)\int_{\mathbb{R}^N} |u|^q\,dx - b(1-t_u^2) \\
&< -b(1-t_u^2),
\end{aligned} \tag{3.4}
$$

which implies that $-b+a < -b(1-t_u^2)$. Thus

$$
bt_u^2 > a. \tag{3.5}
$$

Let $\{u_n\}_n \subset H^\alpha(\mathbb{R}^N)$ such that $\langle I'(u_n), u_n\rangle = -b$ and $I(u_n) + \frac{b}{2} \to I_{-b}$, as $n \to \infty$. We have

$$
\begin{aligned}
&I(u_n) + \frac{b}{2} \\
&\geq \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N} |u_n|^2\,dx - \frac{1}{q}\int_{\mathbb{R}^N} |u_n|^q\,dx + \frac{b}{2} \\
&= \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N} |u_n|^2\,dx \\
&\quad - \frac{1}{q}\Big(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \int_{\mathbb{R}^N} |u_n|^2\,dx + b\Big) + \frac{b}{2} \\
&\geq \big(\frac{1}{2} - \frac{1}{q}\big)\Big(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \int_{\mathbb{R}^N} |u_n|^2\,dx\Big),
\end{aligned}
$$

which means that $\{u_n\}_n$ is bounded in $H^\alpha(\mathbb{R}^N)$. Hence, Theorem 2.1 yields that $\{u_n\}_n$ is also bounded in $L^q(\mathbb{R}^N)$.

In the following, we will verify that there exists $t_0 \in (0,1)$ such that $t_{u_n} \to t_0$, as $n \to \infty$. Indeed, if the conclusion is not satisfied, we may assume that there exists a subsequence, still denoted by $\{t_{u_n}\}_n$, such that $t_{u_n} \to 0$. Thus, $t_{u_n} u_n \to 0$ in $H^\alpha(\mathbb{R}^N)$. Similar to the discussion of (3.2), we obtain a contradiction. By (3.4), we obtain

$$
\langle I'(u_n), u_n\rangle - \langle I'(t_{u_n} u_n), t_{u_n} u_n\rangle = (t_{u_n}^q - t_{u_n}^2)\int_{\mathbb{R}^N} |u_n|^q\,dx - b(1-t_{u_n}^2), \tag{3.6}
$$

which gives

$$
a - b \geq C(t_{u_n}^q - t_{u_n}^2) - b(1-t_{u_n}^2).
$$

If $t_0 = 1$, let $n \to \infty$, we obtain that $a - b \geq 0$, this is absurd.

By (3.6), we have

$$
a - bt_0^2 = (t_0^q - t_0^2)\lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^q\,dx.
$$

Given that $\int_{\mathbb{R}^N} |u_n|^q\,dx \to 0$ as $n\to\infty$, we can deduce

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \int_{\mathbb{R}^N} |u_n|^2\,dx \to -b < 0,$$

which is a contradiction. Therefore, we obtain that $bt_0^2 > a$.

By (3.3) we obtain

$$I(u_n) + \frac{b}{2} > I_{-a} + \frac{b}{2}t_{u_n}^2 - \frac{a}{2}.$$

Letting $n\to\infty$, we have

$$I_{-b} \ge I_{-a} + \frac{b}{2}t_0^2 - \frac{a}{2} > I_{-a}.$$

(3) For any $a>0$, it suffices to verify that

$$I_{-a} = \lim_{n\to\infty} I_{-(a+\frac{1}{n})} = \lim_{n\to\infty} I_{-(a-\frac{1}{n})}.$$

For any $n\in\mathbb{N}$, using the result in (2), we obtain

$$I_{-(a+\frac{1}{n})} > I_{-a} > I_{-(a-\frac{1}{n})}. \tag{3.7}$$

Next, we will verify that

$$I_{-a} = \lim_{n\to\infty} I_{-(a+\frac{1}{n})}.$$

Take $\{u_n\}_n \subset H^\alpha(\mathbb{R}^N)$ such that $\langle I'(u_n),u_n\rangle = -a$ and $I(u_n)+a/2 \to I_{-a}$, as $n\to\infty$. Similar to the discussion in (2), $\{u_n\}_n$ is bounded in $H^\alpha(\mathbb{R}^N)$ and there exists $\{t_n\}_n \subset (1,+\infty)$ such that $\langle I'(t_nu_n),t_nu_n\rangle = -(a+1/n)$ for any $n\in\mathbb{N}$.

Denote $v_n = t_nu_n$ and $s_n = 1/t_n$. We obtain that $s_n\in(0,1)$, $\langle I'(s_nv_n),s_nv_n\rangle = -a$, $\langle I'(v_n),v_n\rangle = -(a+1/n)$ and $I(s_nv_n)+a/2 \to I_{-a}$, as $n\to\infty$. Similar to (3.5), we obtain that $(a+1/n)s_n^2 > a$, which implies that $s_n\to 1$. Then $t_n\to 1$ and $\{v_n\}_n$ is bounded in $H^\alpha(\mathbb{R}^N)$. Furthermore, we obtain that $I(v_n)+a/2\to I_{-a}$. As $I(v_n)+(a+1/n)/2 \ge I_{-(a+1/n)}$, we obtain that $I_{-a} \ge \limsup_{n\to\infty} I_{-(a+1/n)}$. It follows from (3.7) that $I_{-a} = \lim_{n\to\infty} I_{-(a+1/n)}$.

Next, we will verify that

$$I_{-a} = \lim_{n\to\infty} I_{-(a-\frac{1}{n})}.$$

For any $\varepsilon>0$, there exists $u_n\in H^\alpha(\mathbb{R}^N)$ such that

$$\langle I'(u_n),u_n\rangle = -\big(a-\frac{1}{n}\big),$$
$$I_{-(a-\frac{1}{n})} + \varepsilon > I(u_n) + \frac{a-\frac{1}{n}}{2}.$$

Then, the sequence $\{u_n\}_n$ is bounded in $H^\alpha(\mathbb{R}^N)$ and there exists $t_n>1$ such that $\langle I'(t_nu_n),t_nu_n\rangle = -a$.

Denote $v_n = t_nu_n$ and $s_n = 1/t_n$. Thus $s_n\in(0,1)$, $\langle I'(s_nv_n),s_nv_n\rangle = -(a-\frac{1}{n})$ and $\langle I'(v_n),v_n\rangle = -a$. We obtain that $as_n^2 > a-\frac{1}{n}$, which implies that $s_n\to 1$. Then $t_n\to 1$ and $\{v_n\}_n$ is bounded in $H^\alpha(\mathbb{R}^N)$. Note that

$$\begin{aligned}
I_{-(a-\frac{1}{n})} + \varepsilon &> I(s_nv_n) + \frac{a}{2} - \frac{1}{2n}\\
&= I(s_nv_n) - \frac{1}{2n} - I(v_n) + I(v_n) + \frac{a}{2}\\
&\ge I(s_nv_n) - \frac{1}{2n} - I(v_n) + I_{-a},
\end{aligned}$$

we obtain that $I_{-(a-\frac{1}{n})}+\varepsilon > I_{-a}-\varepsilon$, as n is sufficiently large. It follows from (3.7) that $\lim_{n\to\infty} I_{-(a-\frac{1}{n})} = I_{-a}$. Combining with the above discussions, we obtain the desired result. $\square$

Based on the following concentration compactness lemma [22, 23], we will prove the main result.

Lemma 3.2. *Let $\{\rho_n\}_n$ be a sequence in $L^1(\mathbb{R}^N)$ satisfying $\rho_n \geq 0$ and*

$$\int_{\mathbb{R}^N} \rho_n(x)\,dx \to \lambda > 0,$$

as $n \to \infty$. Then there exists a subsequence, still denoted by $\{\rho_n\}_n$, satisfying one of the following three possibilities:

(1) *(Compactness) There exists a sequence $\{x_n\}_n$ in $\mathbb{R}^N$ such that $\{\rho_n\}_n$ is tight, that is, for any $\varepsilon > 0$, there exists $R > 0$ such that*

$$\int_{B(x_n,R)} \rho_n(x)\,dx \geq \lambda - \varepsilon.$$

(2) *(Vanishing) For any $R > 0$,*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} \rho_n(x)\,dx = 0.$$

(3) *(Dichotomy) There exists $\beta \in (0, \lambda)$ such that for any $\varepsilon > 0$, there exist $R > 0$, $\{y_n\}_n \subset \mathbb{R}^N$ and $R_n \to \infty$ satisfying: for n sufficiently large,*

$$\Big|\int_{B(y_n,R)} \rho_n(x)\,dx - \beta\Big| < \varepsilon,$$

$$\Big|\int_{\mathbb{R}^N\setminus B(y_n,R_n)} \rho_n(x)\,dx - (\lambda-\beta)\Big| < \varepsilon.$$

Proof of Theorem 1.1. Take $\{u_n\}_n \subset H^\alpha(\mathbb{R}^N)\setminus\{0\}$ be a minimizing sequence of (1.1), i.e.

$$\langle I'(u_n), u_n\rangle = 0 \quad\text{and}\quad I(u_n) \to I_0 > 0, \tag{3.8}$$

as $n \to \infty$. It follows from (3.1) that $\{u_n\}_n$ is bounded in $H^\alpha(\mathbb{R}^N)$. We assume that

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \int_{\mathbb{R}^N} |u_n|^2\,dx \to \lambda, \tag{3.9}$$

as $n \to \infty$. Then $\lambda > 0$.

In the following, we will apply the concentration compactness principle to the case

$$\rho_n(y) = \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx + |u_n(y)|^2$$

and show that for such $\{\rho_n\}_n$, the cases "vanishing" and "dichotomy" do not hold.

(1) If the case "vanishing" takes place, by Lemma 3.2 we obtain

$$\lim_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B(z,R)} \Big(\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx + |u_n(y)|^2\Big)\,dy = 0,$$

for any $R > 0$. Thus

$$\lim_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B(z,R)} |u_n|^2\,dy = 0.$$

As $2 < q < 2^*_\alpha$, it follows from Lemma 2.4 that $u_n \to 0$ in $L^q(\mathbb{R}^N)$, as $n \to \infty$. Note that

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \int_{\mathbb{R}^N} |u_n|^2\,dx = \int_{\mathbb{R}^N} |u_n|^q\,dx,$$

we obtain $u_n \to 0$ in $H^\alpha(\mathbb{R}^N)$. That is a contradiction.

(2) In the case of dichotomy, by Lemma 3.2, there exists $\beta \in (0,\lambda)$ such that for any $\varepsilon > 0$, there exist $R > \varepsilon^{-1}$, $\{z_n\}_n \subset \mathbb{R}^N$, $R_n \to \infty$ and $n_0 \in \mathbb{N}$, then

$$\Big|\int_{B(z_n,R)} \rho_n(y)\,dy - \beta\Big| < \varepsilon,$$
$$\Big|\int_{\mathbb{R}^N\setminus B(z_n,R_n)} \rho_n(y)\,dy - (\lambda-\beta)\Big| < \varepsilon,$$

for $n \ge n_0$. Since $R_n \to \infty$, we assume that $R_n > 6R$, for $n \ge n_0$. Note that

$$\begin{aligned}
&\int_{B(z_n,R_n)\setminus B(z_n,R)} \rho_n(y)\,dy \\
&= \int_{\mathbb{R}^N} \rho_n(y)\,dy - \int_{\mathbb{R}^N\setminus B(z_n,R_n)} \rho_n(y)\,dy - \int_{B(z_n,R)} \rho_n(y)\,dy;
\end{aligned}$$

thus

$$\int_{B(z_n,R_n)\setminus B(z_n,R)} \Big(\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx + |u_n(y)|^2\Big)\,dy < 3\varepsilon. \tag{3.10}$$

Take $\varphi, \psi \in C^\infty(\mathbb{R}^N, \mathbb{R})$ such that $0 \le \varphi \le 1$; $\varphi(x) = 1$ in $B(0,1)$, $\varphi(x) = 0$ in $\mathbb{R}^N \setminus B(0,2)$ and $0 \le \psi \le 1$; $\psi(x) = 1$ in $\mathbb{R}^N \setminus B(0,1)$, $\psi(x) = 0$ in $B(0,1/2)$. Define

$$u_n^{(1)}(x) = u_n(x)\varphi\big(\frac{|x-z_n|}{R}\big) := u_n(x)\varphi_R(x),$$
$$u_n^{(2)}(x) = u_n(x)\psi\big(\frac{|x-z_n|}{R_n}\big) := u_n(x)\psi_{R_n}(x).$$

We have

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|u_n^{(1)}(x) - u_n^{(1)}(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&= \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)\varphi_R(x) - u_n(y)\varphi_R(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&= \iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))^2\varphi_R^2(y)}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&\quad + \iint_{\mathbb{R}^{2N}} \frac{u_n^2(x)(\varphi_R(x)-\varphi_R(y))^2}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&\quad + \iint_{\mathbb{R}^{2N}} \frac{2u_n(x)\varphi_R(y)(u_n(x)-u_n(y))(\varphi_R(x)-\varphi_R(y))}{|x-y|^{N+2\alpha}}\,dx\,dy.
\end{aligned}$$

Since $\{u_n\}_n$ is bounded in $H^\alpha(\mathbb{R}^N)$, by using the Hölder inequality we obtain

$$\begin{aligned}
&\Big|\iint_{\mathbb{R}^{2N}} \frac{u_n(x)\varphi_R(y)(u_n(x)-u_n(y))(\varphi_R(x)-\varphi_R(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\Big| \\
&\le \Big(\iint_{\mathbb{R}^{2N}} \frac{\varphi_R^2(y)(u_n(x)-u_n(y))^2}{|x-y|^{N+2\alpha}}\,dx\,dy\Big)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\times \Big(\iint_{\mathbb{R}^{2N}} \frac{u_n^2(x)(\varphi_R(x)-\varphi_R(y))^2}{|x-y|^{N+2\alpha}}\,dx\,dy\Big)^{1/2}\\
&\le C\Big(\iint_{\mathbb{R}^{2N}} \frac{u_n^2(x)(\varphi_R(x)-\varphi_R(y))^2}{|x-y|^{N+2\alpha}}\,dx\,dy\Big)^{1/2}.
\end{aligned}
$$

Next we show that

$$
\lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{u_n^2(x)(\varphi_R(x)-\varphi_R(y))^2}{|x-y|^{N+2\alpha}}\,dx\,dy = 0.
$$

Note that

$$
\begin{aligned}
\mathbb{R}^N\times\mathbb{R}^N &= \big((\mathbb{R}^N\setminus B(z_n,2R))\cup B(z_n,2R)\big)\times\big((\mathbb{R}^N\setminus B(z_n,2R))\cup B(z_n,2R)\big)\\
&= \big((\mathbb{R}^N\setminus B(z_n,2R))\times(\mathbb{R}^N\setminus B(z_n,2R))\big)\cup\big(B(z_n,2R)\times\mathbb{R}^N\big)\\
&\quad\cup\big((\mathbb{R}^N\setminus B(z_n,2R))\times B(z_n,2R)\big)\,.
\end{aligned}
$$

(i) If $(x,y)\in\big(\mathbb{R}^N\setminus B(z_n,2R)\big)\times\big(\mathbb{R}^N\setminus B(z_n,2R)\big)$, then $\varphi_R(x)=\varphi_R(y)=0$.
(ii) $(x,y)\in B(z_n,2R)\times\mathbb{R}^N$. If $|x-y|\le R$, then

$$
|y-z_n|\le|x-y|+|x-z_n|\le 3R,
$$

which implies

$$
\begin{aligned}
&\int_{B(z_n,2R)}dx\int_{\{y\in\mathbb{R}^N:|x-y|\le R\}}\frac{u_n^2(x)(\varphi_R(x)-\varphi_R(y))^2}{|x-y|^{N+2\alpha}}dy\\
&=\int_{B(z_n,2R)}dx\int_{\{y\in\mathbb{R}^N:|x-y|\le R\}}\frac{u_n^2(x)|\nabla\varphi(\xi)|^2|\frac{x-y}{R}|^2}{|x-y|^{N+2\alpha}}dy\\
&\le CR^{-2}\int_{B(z_n,2R)}dx\int_{\{y\in\mathbb{R}^N:|x-y|\le R\}}\frac{u_n^2(x)}{|x-y|^{N+2\alpha-2}}dy\\
&=CR^{-2\alpha}\int_{B(z_n,2R)}u_n^2(x)dx,
\end{aligned}
$$

where $\xi=\frac{y-z_n}{R}+\tau\frac{x-z_n}{R}$ and $\tau\in(0,1)$. If $|x-y|>R$, then we have

$$
\begin{aligned}
&\int_{B(z_n,2R)}dx\int_{\{y\in\mathbb{R}^N:|x-y|>R\}}\frac{u_n^2(x)(\varphi_R(x)-\varphi_R(y))^2}{|x-y|^{N+2\alpha}}dy\\
&\le C\int_{B(z_n,2R)}dx\int_{\{y\in\mathbb{R}^N:|x-y|>R\}}\frac{u_n^2(x)}{|x-y|^{N+2\alpha}}dy\\
&=CR^{-2\alpha}\int_{B(z_n,2R)}u_n^2(x)dx.
\end{aligned}
$$

(iii) $(x,y)\in\big(\mathbb{R}^N\setminus B(z_n,2R)\big)\times B(z_n,2R)$. If $|x-y|\le R$, then $|x-z_n|\le |x-y|+|y-z_n|\le 3R$. Furthermore,

$$
\begin{aligned}
&\int_{\mathbb{R}^N\setminus B(z_n,2R)}dx\int_{\{y\in B(z_n,2R):|x-y|\le R\}}\frac{u_n^2(x)(\varphi_R(x)-\varphi_R(y))^2}{|x-y|^{N+2\alpha}}dy\\
&\le CR^{-2}\int_{B(z_n,3R)}dx\int_{\{y\in B(z_n,2R):|x-y|\le R\}}\frac{u_n^2(x)}{|x-y|^{N+2\alpha-2}}dy\\
&\le CR^{-2\alpha}\int_{B(z_n,3R)}u_n^2(x)dx.
\end{aligned}
$$

Notice that there exists $k > 4$ such that

$$\begin{aligned}&\big(\mathbb{R}^N \setminus B(z_n, 2R)\big) \times B(z_n, 2R)\\ &\subset (B(z_n, kR) \times B(z_n, 2R)) \cup \big((\mathbb{R}^N \setminus B(z_n, kR)) \times B(z_n, 2R)\big).\end{aligned}$$

If $|x - y| > R$, we obtain

$$\begin{aligned}&\int_{B(z_n,kR)} dx \int_{\{y\in B(z_n,2R):|x-y|>R\}} \frac{u_n^2(x)(\varphi_R(x) - \varphi_R(y))^2}{|x-y|^{N+2\alpha}} dy\\ &\le C \int_{B(z_n,kR)} dx \int_{\{y\in B(z_n,2R):|x-y|>R\}} \frac{u_n^2(x)}{|x-y|^{N+2\alpha}} dy\\ &\le CR^{-2\alpha} \int_{B(z_n,kR)} u_n^2(x) dx.\end{aligned}$$

If $(x, y) \in \big(\mathbb{R}^N \setminus B(z_n, kR)\big) \times B(z_n, 2R)$, we obtain

$$|x - y| \ge |x - z_n| - |y - z_n| \ge \frac{|x - z_n|}{2} + \frac{k}{2} R - 2R > \frac{|x - z_n|}{2},$$

which implies

$$\begin{aligned}&\int_{\mathbb{R}^N\setminus B(z_n,kR)} dx \int_{\{y\in B(z_n,2R):|x-y|>R\}} \frac{u_n^2(x)(\varphi_R(x) - \varphi_R(y))^2}{|x-y|^{N+2\alpha}} dy\\ &\le C \int_{\mathbb{R}^N\setminus B(z_n,kR)} dx \int_{\{y\in B(z_n,2R):|x-y|>R\}} \frac{u_n^2(x)}{|x-z_n|^{N+2\alpha}} dy\\ &\le CR^N \int_{\mathbb{R}^N\setminus B(z_n,kR)} \frac{u_n^2(x)}{|x-z_n|^{N+2\alpha}} dx\\ &\le Ck^{-N} \Big(\int_{\mathbb{R}^N\setminus B(z_n,kR)} |u_n(x)|^{2^*_\alpha}\, dx \Big)^{2/2^*_\alpha}.\end{aligned}$$

Since $\{u_n\}_n$ is bounded in $H^\alpha(\mathbb{R}^N)$, from (i), (ii) and (iii) we have

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}} \frac{u_n^2(x)(\varphi_R(x) - \varphi_R(y))^2}{|x-y|^{N+2\alpha}}\, dx\, dy\\ &= \Big(\iint_{B(z_n,2R)\times\mathbb{R}^N} + \iint_{(\mathbb{R}^N\setminus B(z_n,2R))\times B(z_n,2R)} \Big) \frac{u_n^2(x)(\varphi_R(x) - \varphi_R(y))^2}{|x-y|^{N+2\alpha}}\, dx\, dy\\ &\le CR^{-2\alpha} \int_{B(z_n,2R)} u_n^2(x)\, dx + CR^{-2\alpha} \int_{B(z_n,3R)} u_n^2(x)\, dx\\ &\quad + CR^{-2\alpha} \int_{B(z_n,kR)} u_n^2(x)\, dx + Ck^{-N} \Big(\int_{\mathbb{R}^N\setminus B(z_n,kR)} |u_n(x)|^{2^*_\alpha}\, dx \Big)^{2/2^*_\alpha}\\ &\le CR^{-2\alpha} + Ck^{-N}\\ &\le C\varepsilon^{2\alpha} + Ck^{-N},\end{aligned}$$

which yields

$$\begin{aligned}&\lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{u_n^2(x)(\varphi_R(x) - \varphi_R(y))^2}{|x-y|^{N+2\alpha}}\, dx\, dy\\ &= \lim_{k\to\infty} \lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{u_n^2(x)(\varphi_R(x) - \varphi_R(y))^2}{|x-y|^{N+2\alpha}}\, dx\, dy = 0.\end{aligned} \tag{3.11}$$

Thus

$$
\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)\varphi_R(x)-u_n(y)\varphi_R(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&=\iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))^2\varphi_R^2(y)}{|x-y|^{N+2\alpha}}\,dx\,dy+o(1).
\end{aligned}
\tag{3.12}
$$

Hence we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{|u_n^{(1)}(x)-u_n^{(1)}(y)|^2}{|x-y|^{N+2\alpha}}\,dx+|u_n^{(1)}(y)|^2\Big)\,dy\\
&=\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^2\varphi_R^2(y)}{|x-y|^{N+2\alpha}}\,dx\,dy+\int_{\mathbb{R}^N}\varphi_R^2(y)|u_n(y)|^2\,dy+o(1)\\
&=\int_{B(z_n,2R)\setminus B(z_n,R)}\varphi_R^2(y)\Big(\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx+|u_n(y)|^2\Big)\,dy\\
&\quad+\int_{B(z_n,R)}\Big(\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx+|u_n(y)|^2\Big)\,dy+o(1).
\end{aligned}
$$

Notice that $B(z_n,2R)\setminus B(z_n,R)\subset B(z_n,R_n)\setminus B(z_n,R)$ and $\varphi_R^2(y)\le 1$, it follows from (3.10) that

$$
\int_{B(z_n,2R)\setminus B(z_n,R)}\varphi_R^2(y)\Big(\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx+|u_n(y)|^2\Big)\,dy\to 0,
$$

as $n\to\infty$. Then, as $n\to\infty$, we have

$$
\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{|u_n^{(1)}(x)-u_n^{(1)}(y)|^2}{|x-y|^{N+2\alpha}}\,dx+|u_n^{(1)}(y)|^2\Big)\,dy\to\beta. \tag{3.13}
$$

Similarly, as $n\to\infty$, we obtain

$$
\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{|u_n^{(2)}(x)-u_n^{(2)}(y)|^2}{|x-y|^{N+2\alpha}}\,dx+|u_n^{(2)}(y)|^2\Big)\,dy\to\lambda-\beta. \tag{3.14}
$$

Note that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}|u_n|^2\,dx-\int_{\mathbb{R}^N}|u_n^{(1)}|^2\,dx-\int_{\mathbb{R}^N}|u_n^{(2)}|^2\,dx\\
&=\int_{B(z_n,2R)\setminus B(z_n,R)}|u_n|^2\,dx+\int_{B(z_n,\frac12 R_n)\setminus B(z_n,2R)}|u_n|^2\,dx\\
&\quad+\int_{B(z_n,R_n)\setminus B(z_n,\frac12 R_n)}|u_n|^2\,dx-\int_{B(z_n,2R)\setminus B(z_n,R)}|u_n\varphi_R|^2\,dx\\
&\quad-\int_{B(z_n,R_n)\setminus B(z_n,\frac12 R_n)}|u_n\psi_{R_n}|^2\,dx.
\end{aligned}
$$

Then from (3.10) we deduce that

$$
\int_{\mathbb{R}^N}|u_n|^2\,dx-\int_{\mathbb{R}^N}|u_n^{(1)}|^2\,dx-\int_{\mathbb{R}^N}|u_n^{(2)}|^2\,dx\to 0, \tag{3.15}
$$

as $n \to \infty$. Combining (3.9), (3.13) with (3.14) we obtain that as $n \to \infty$,

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy - \iint_{\mathbb{R}^{2N}} \frac{|u_n^{(1)}(x)-u_n^{(1)}(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy \\ &- \iint_{\mathbb{R}^{2N}} \frac{|u_n^{(2)}(x)-u_n^{(2)}(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy \to 0. \end{aligned} \tag{3.16}$$

Denoting $v_n = u_n - u_n^{(1)} - u_n^{(2)}$, we obtain

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy \\ &= \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n^{(1)}(x)-u_n^{(2)}(x)-u_n(y)+u_n^{(1)}(y)+u_n^{(2)}(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy \\ &= \int_{B(z_n,R)} dy \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n^{(1)}(x)-u_n^{(2)}(x)|^2}{|x-y|^{N+2\alpha}}\,dx \\ &\quad + \int_{B(z_n,R_n)\setminus B(z_n,R)} dy \\ &\quad \times \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n^{(1)}(x)-u_n^{(2)}(x)-u_n(y)+u_n^{(1)}(y)+u_n^{(2)}(y)|^2}{|x-y|^{N+2\alpha}}\,dx \\ &\quad + \int_{\mathbb{R}^N\setminus B(z_n,R_n)} dy \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n^{(1)}(x)-u_n^{(2)}(x)|^2}{|x-y|^{N+2\alpha}}\,dx. \end{aligned} \tag{3.17}$$

Note that

$$\begin{aligned} &\int_{B(z_n,R)} dy \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n^{(1)}(x)-u_n^{(2)}(x)|^2}{|x-y|^{N+2\alpha}}\,dx \\ &= \int_{B(z_n,R)} dy \int_{B(z_n,R)} \frac{|u_n(x)-u_n^{(1)}(x)-u_n^{(2)}(x)|^2}{|x-y|^{N+2\alpha}}\,dx \\ &\quad + \int_{B(z_n,R)} dy \int_{B(z_n,R_n)\setminus B(z_n,R)} \frac{|u_n(x)-u_n^{(1)}(x)-u_n^{(2)}(x)|^2}{|x-y|^{N+2\alpha}}\,dx \\ &\quad + \int_{B(z_n,R)} dy \int_{\mathbb{R}^N\setminus B(z_n,R_n)} \frac{|u_n(x)-u_n^{(1)}(x)-u_n^{(2)}(x)|^2}{|x-y|^{N+2\alpha}}\,dx, \end{aligned}$$

then

$$\begin{aligned} &\int_{B(z_n,R)} dy \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n^{(1)}(x)-u_n^{(2)}(x)|^2}{|x-y|^{N+2\alpha}}\,dx \\ &= \int_{B(z_n,R)} dy \int_{B(z_n,R_n)\setminus B(z_n,R)} \frac{|u_n(x)-u_n^{(1)}(x)-u_n^{(2)}(x)|^2}{|x-y|^{N+2\alpha}}\,dx \\ &= \int_{B(z_n,R)} dy \int_{B(z_n,R_n)\setminus B(z_n,R)} \frac{|u_n(x)-\varphi_R(x)u_n(x)-\psi_{R_n}(x)u_n(x)|^2}{|x-y|^{N+2\alpha}}\,dx \\ &\le 2\int_{B(z_n,R)} dy \int_{B(z_n,R_n)\setminus B(z_n,R)} \frac{(\varphi_R(x)-1)^2|u_n(x)|^2}{|x-y|^{N+2\alpha}}\,dx \\ &\quad + 2\int_{B(z_n,R)} dy \int_{B(z_n,R_n)\setminus B(z_n,R)} \frac{\psi_{R_n}^2(x)|u_n(x)|^2}{|x-y|^{N+2\alpha}}\,dx \end{aligned}$$

$$
\begin{aligned}
&= 2\int_{B(z_n,R)} dy \int_{B(z_n,R_n)\setminus B(z_n,R)} \frac{(\varphi_R(x)-\varphi_R(y))^2|u_n(x)|^2}{|x-y|^{N+2\alpha}}\,dx \\
&\quad + 2\int_{B(z_n,R)} dy \int_{B(z_n,R_n)\setminus B(z_n,R)} \frac{(1-\psi_{R_n}(x)-(1-\psi_{R_n}(y)))^2|u_n(x)|^2}{|x-y|^{N+2\alpha}}\,dx.
\end{aligned}
$$

From (3.11), we have

$$
\int_{B(z_n,R)} dy \int_{B(z_n,R_n)\setminus B(z_n,R)} \frac{(\varphi_R(x)-\varphi_R(y))^2|u_n(x)|^2}{|x-y|^{N+2\alpha}}\,dx \to 0.
$$

Similarly, we obtain as $n\to\infty$,

$$
\int_{B(z_n,R)} dy \int_{B(z_n,R_n)\setminus B(z_n,R)} \frac{(1-\psi_{R_n}(x)-(1-\psi_{R_n}(y)))^2|u_n(x)|^2}{|x-y|^{N+2\alpha}}\,dx \to 0.
$$

Then

$$
\int_{B(z_n,R)} dy \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n^{(1)}(x)-u_n^{(2)}(x)|^2}{|x-y|^{N+2\alpha}}\,dx \to 0, \tag{3.18}
$$

as $n\to\infty$. Similarly, we obtain as $n\to\infty$,

$$
\int_{\mathbb{R}^N\setminus B(z_n,R_n)} dy \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n^{(1)}(x)-u_n^{(2)}(x)|^2}{|x-y|^{N+2\alpha}}\,dx \to 0. \tag{3.19}
$$

We have

$$
\begin{aligned}
&\int_{B(z_n,R_n)\setminus B(z_n,R)} dy \\
&\times \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n^{(1)}(x)-u_n^{(2)}(x)-u_n(y)+u_n^{(1)}(y)+u_n^{(2)}(y)|^2}{|x-y|^{N+2\alpha}}\,dx \\
&\le 3^2\int_{B(z_n,R_n)\setminus B(z_n,R)} dy \\
&\times \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^2+|u_n^{(1)}(x)-u_n^{(1)}(y)|^2+|u_n^{(2)}(x)-u_n^{(2)}(y)|^2}{|x-y|^{N+2\alpha}}\,dx
\end{aligned}
$$

Similar to the discussion of (3.12), we obtain

$$
\begin{aligned}
&\int_{B(z_n,R_n)\setminus B(z_n,R)} dy \int_{\mathbb{R}^N} \frac{|u_n^{(1)}(x)-u_n^{(1)}(y)|^2}{|x-y|^{N+2\alpha}}\,dx \\
&= \int_{B(z_n,R_n)\setminus B(z_n,R)} dy \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p\varphi_R^p(y)}{|x-y|^{N+2\alpha}}\,dx + o(1).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\int_{B(z_n,R_n)\setminus B(z_n,R)} dy \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^2\varphi_R^2(y)}{|x-y|^{N+2\alpha}}\,dx \\
&\le \int_{B(z_n,R_n)\setminus B(z_n,R)} dy \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx,
\end{aligned}
$$

it follows from (3.10) that

$$
\int_{B(z_n,R_n)\setminus B(z_n,R)} dy \int_{\mathbb{R}^N} \frac{|u_n^{(1)}(x)-u_n^{(1)}(y)|^2}{|x-y|^{N+2\alpha}}\,dx \to 0,
$$

as $n \to \infty$. Similarly, we have

$$\int_{B(z_n,R_n)\setminus B(z_n,R)} dy \int_{\mathbb{R}^N} \frac{|u_n^{(2)}(x) - u_n^{(2)}(y)|^2}{|x-y|^{N+2\alpha}}\,dx \to 0,$$

which implies as $n \to \infty$,

$$\begin{aligned}
&\int_{B(z_n,R_n)\setminus B(z_n,R)} dy \\
&\times \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n^{(1)}(x) - u_n^{(2)}(x) - u_n(y) + u_n^{(1)}(y) + u_n^{(2)}(y)|^2}{|x-y|^{N+2\alpha}}\,dx \to 0.
\end{aligned}$$

Combining (3.17), (3.18) and (3.19), we obtain

$$\iint_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy \to 0,$$

as $n \to \infty$. It follows from (3.10) that

$$\begin{aligned}
\int_{\mathbb{R}^N} |v_n|^2\,dy &= \int_{B(z_n,R_n)\setminus B(z_n,R)} |u_n - u_n^{(1)} - u_n^{(2)}|^2\,dy \\
&\le \int_{B(z_n,R_n)\setminus B(z_n,R)} 3^2(|u_n|^2 + |u_n^{(1)}|^2 + |u_n^{(2)}|^2)\,dy \to 0,
\end{aligned}$$

as $n \to \infty$. Then, $v_n \to 0$ in $H^\alpha(\mathbb{R}^N)$. Using (2.1), we obtain $v_n \to 0$ in $L^q(\mathbb{R}^N)$, as $n \to \infty$. We have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u_n|^q\,dx - \int_{\mathbb{R}^N} |u_n^{(1)}|^q\,dx - \int_{\mathbb{R}^N} |u_n^{(2)}|^q\,dx \\
&= \int_{\mathbb{R}^N} |u_n|^q\,dx - \int_{B(z_n,R)} |u_n^{(1)}|^q\,dx - \int_{B(z_n,2R)\setminus B(z_n,R)} |u_n^{(1)}|^q\,dx \\
&\quad - \int_{\mathbb{R}^N\setminus B(z_n,R_n)} |u_n^{(2)}|^q\,dx - \int_{B(z_n,R_n)\setminus B(z_n,\frac12 R_n)} |u_n^{(2)}|^q\,dx \\
&= \int_{B(z_n,R_n)\setminus B(z_n,R)} |u_n|^q\,dx - \int_{B(z_n,2R)\setminus B(z_n,R)} |u_n^{(1)}|^q\,dx \\
&\quad - \int_{B(z_n,R_n)\setminus B(z_n,\frac12 R_n)} |u_n^{(2)}|^q\,dx \\
&= \int_{B(z_n,2R)\setminus B(z_n,R)} |u_n^{(1)} + v_n|^q\,dx + \int_{B(z_n,\frac12 R_n)\setminus B(z_n,2R)} |v_n|^q\,dx \\
&\quad + \int_{B(z_n,R_n)\setminus B(z_n,\frac12 R_n)} |u_n^{(2)} + v_n|^q\,dx - \int_{B(z_n,2R)\setminus B(z_n,R)} |u_n^{(1)}|^q\,dx \\
&\quad - \int_{B(z_n,R_n)\setminus B(z_n,\frac12 R_n)} |u_n^{(2)}|^q\,dx.
\end{aligned}$$

Note that

$$\begin{aligned}
&\Big|\int_{B(z_n,2R)\setminus B(z_n,R)} |u_n^{(1)} + v_n|^q\,dx - \int_{B(z_n,2R)\setminus B(z_n,R)} |u_n^{(1)}|^q\,dx\Big| \\
&\le \int_{B(z_n,2R)\setminus B(z_n,R)} \big||u_n^{(1)} + v_n|^q - |u_n^{(1)}|^q\big|\,dx
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_{B(z_n,2R)\setminus B(z_n,R)}\Big(|u_n^{(1)}|^{q-1}|v_n|+|v_n|^q\Big)\,dx\\
&\le C\Big(\int_{B(z_n,2R)\setminus B(z_n,R)}|u_n^{(1)}|^q\,dx\Big)^{\frac{q-1}{q}}\Big(\int_{B(z_n,2R)\setminus B(z_n,R)}|v_n|^q\,dx\Big)^{1/q}\\
&\quad + C\int_{B(z_n,2R)\setminus B(z_n,R)}|v_n|^q\,dx,
\end{aligned}
$$

which implies that, as $n\to\infty$,

$$
\int_{B(z_n,2R)\setminus B(z_n,R)}|u_n^{(1)}+v_n|^q\,dx-\int_{B(z_n,2R)\setminus B(z_n,R)}|u_n^{(1)}|^q\,dx\to 0.
$$

Similarly, we deduce that, as $n\to\infty$,

$$
\int_{B(z_n,R_n)\setminus B(z_n,\frac12 R_n)}|u_n^{(2)}+v_n|^q\,dx-\int_{B(z_n,R_n)\setminus B(z_n,\frac12 R_n)}|u_n^{(2)}|^q\,dx\to 0.
$$

Then

$$
\int_{\mathbb{R}^N}|u_n|^q\,dx-\int_{\mathbb{R}^N}|u_n^{(1)}|^q\,dx-\int_{\mathbb{R}^N}|u_n^{(2)}|^q\,dx\to 0,
$$

as $n\to\infty$. Combining (3.15) and (3.16), we obtain

$$
I(u_n)-I(u_n^{(1)})-I(u_n^{(2)})\to 0, \tag{3.20}
$$

$$
\langle I'(u_n),u_n\rangle-\langle I'(u_n^{(1)}),u_n^{(1)}\rangle-\langle I'(u_n^{(2)}),u_n^{(2)}\rangle\to 0, \tag{3.21}
$$

as $n\to\infty$. If $\langle I'(u_n^{(1)}),u_n^{(1)}\rangle=0$, then $I(u_n^{(1)})\ge I_0$.

We obtain $\limsup_{n\to\infty} I(u_n^{(2)})\le 0$ and $\langle I'(u_n^{(2)}),u_n^{(2)}\rangle\to 0$, as $n\to\infty$. From (3.14), we have

$$
\begin{aligned}
I(u_n^{(2)}) &= \big(\frac12-\frac1q\big)\Big(\iint_{\mathbb{R}^{2N}}\frac{|u_n^{(2)}(x)-u_n^{(2)}(y)|^2}{|x-y|^{N+2\alpha}}\,dx+|u_n^{(2)}(y)|^2\,dy\Big)\\
&\quad+\frac1q\langle I'(u_n^{(2)}),u_n^{(2)}\rangle\\
&\to\big(\frac12-\frac1q\big)(\lambda-\beta)\ne 0,
\end{aligned}
$$

which is a contradiction.

In the following, we assume that $\langle I'(u_n^{(1)}),u_n^{(1)}\rangle=-a_n<0$. Then $I(u_n^{(1)})+a_n/2\ge I_{-a_n}>I_0$. If $a_n\to 0$, then $\langle I'(u_n^{(1)}),u_n^{(1)}\rangle\to 0$. Similar to the above discussions, we obtain a contradiction. Note that $\{u_n^{(1)}\}_n$ is bounded in $H^\alpha(\mathbb{R}^N)$, we assume that $a_n\to a>0$.

Denote $\langle I'(u_n^{(2)}),u_n^{(2)}\rangle=b_n$ and define

$$
I_{b_n}=\inf\big\{I(u)-\frac{b_n}{2}:u\in H^\alpha(\mathbb{R}^N)\setminus\{0\},\langle I'(u),u\rangle=b_n\big\}.
$$

It is easy to verify that $I_{b_n}\ge 0$. From (3.21), we have that $b_n\to a$. We assume that $b_n>0$ as n is sufficiently large. Then, we obtain

$$
I(u_n^{(1)})+\frac{a_n}{2}+I(u_n^{(2)})-\frac{b_n}{2}\ge I_{-a_n}+I_{b_n}\ge I_{-a_n},
$$

which implies $I_0 \geq \lim_{n\to\infty} I_{-a_n} = I_{-a} > I_0$. This is a contradiction. Therefore, by Lemma 3.2, *the case "compactness" holds*. There exists $\{z_n\}_n \subset \mathbb{R}^N$ such that for any $\varepsilon > 0$, there exists $R > 0$, then

$$\int_{B(z_n,R)} \Big(\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2\alpha}}\, dx + |u_n(y)|^2 \Big)\, dy \geq \lambda - \varepsilon.$$

Denote $v_n(x) = u_n(x + z_n)$, using (3.8) and (3.9), we obtain as $n \to \infty$,

$$\begin{gathered} \iint_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^2}{|x-y|^{N+2\alpha}}\, dx\, dy + \int_{\mathbb{R}^N} |v_n(y)|^2\, dy \to \lambda, \\ I(v_n) \to I_0 \text{ and } \langle I'(v_n), v_n\rangle = 0. \end{gathered} \tag{3.22}$$

Thus, $\{v_n\}_n$ is bounded in $H^\alpha(\mathbb{R}^N)$. Up to a subsequence, still denoted by $\{v_n\}_n$, we assume that $v_n \to v$ weakly in $H^\alpha(\mathbb{R}^N)$. Moreover, using Theorem 2.3 we obtain that $v_n \to v$ in $L^q(B_r(0))$, for any $r > 0$. Passing to a subsequence, still denoted by $\{v_n\}_n$, a diagonal process enables us to assume that $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$, as $n \to \infty$. Note that

$$\begin{aligned} &\int_{B(0,R)} \Big(\int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^2}{|x-y|^{N+2\alpha}}\, dx + |v_n(y)|^2 \Big)\, dy \\ &= \int_{B(0,R)} \Big(\int_{\mathbb{R}^N} \frac{|u_n(x+z_n) - v_n(y+z_n)|^2}{|x-y|^{N+2\alpha}}\, dx + |u_n(y+z_n)|^2 \Big)\, dy \\ &= \int_{B(z_n,R)} \Big(\int_{\mathbb{R}^N} \frac{|u_n(x) - v_n(y)|^2}{|x-y|^{N+2\alpha}}\, dx + |u_n(y)|^2 \Big)\, dy \geq \lambda - \varepsilon, \end{aligned}$$

it follows from (3.22) that

$$\int_{B(0,R)} \Big(\int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^2}{|x-y|^{N+2\alpha}}\, dx + |v_n(y)|^2 \Big)\, dy \to \lambda.$$

Then,

$$\int_{\mathbb{R}^N \setminus B(0,R)} \Big(\int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^2}{|x-y|^{N+2\alpha}}\, dx + |v_n(y)|^2 \Big)\, dy \to 0.$$

Using Theorem 2.1, we obtain

$$\int_{\mathbb{R}^N \setminus B(0,R)} |v_n|^q\, dx \to 0,$$

as $n \to \infty$. Thus, by the Fatou Lemma we have

$$\int_{\mathbb{R}^N \setminus B(0,R)} |v|^q\, dx = 0,$$

which implies that $v = 0$ a.e. in $\mathbb{R}^N \setminus B(0,R)$. As $v_n \to v$ in $L^q(B_R(0))$, it follows that

$$\int_{\mathbb{R}^N} |v_n - v|^q\, dx \to 0, \quad \text{as } n \to \infty,$$

i.e. $v_n \to v$ in $L^q(\mathbb{R}^N)$. Then

$$\begin{aligned} I_0 &= \lim_{n\to\infty} I(v_n) \\ &= \lim_{n\to\infty} \Big(\frac{1}{2} \iint_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^2}{|x-y|^{N+2\alpha}}\, dx\, dy + \frac{1}{2} \int_{\mathbb{R}^N} |v_n(y)|^2\, dy - \frac{1}{q} \int_{\mathbb{R}^N} |v_n(y)|^q\, dy \Big) \end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \Big(\frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N} |v_n(y)|^2\,dy\Big) - \frac{1}{q}\int_{\mathbb{R}^N} |v(y)|^q\,dy \\
&\geq \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N} |v(y)|^2\,dy - \frac{1}{q}\int_{\mathbb{R}^N} |v(y)|^q\,dy.
\end{aligned}
$$

Thus,

$$
I_0 \geq I(v). \tag{3.23}
$$

Furthermore, we will prove that $\langle I'(v), v\rangle = 0$, i.e.

$$
\int_{\mathbb{R}^N} |v|^q\,dx = \iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \int_{\mathbb{R}^N} |v|^2\,dx. \tag{3.24}
$$

Bearing in mind that

$$
\int_{\mathbb{R}^N} |v_n|^q\,dx = \iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \int_{\mathbb{R}^N} |v_n|^2\,dx,
$$

we have

$$
\int_{\mathbb{R}^N} |v|^q\,dx \geq \iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \int_{\mathbb{R}^N} |v|^2\,dx.
$$

Denote

$$
a := -\langle I'(v), v\rangle = \int_{\mathbb{R}^N} |v|^q\,dx - \iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy - \int_{\mathbb{R}^N} |v|^2\,dx.
$$

If $a > 0$, then we obtain

$$
\begin{aligned}
&\lim_{n\to\infty} \Big(\iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \int_{\mathbb{R}^N} |v_n|^2\,dx\Big) \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^q\,dx = \int_{\mathbb{R}^N} |v|^q\,dx \\
&= \iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \int_{\mathbb{R}^N} |v|^2\,dx + a.
\end{aligned} \tag{3.25}
$$

Note that

$$
\begin{aligned}
&I(v_n) - I(v) - \frac{a}{2} \\
&= \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N} |v_n|^2\,dx - \frac{1}{q}\int_{\mathbb{R}^N} |v_n|^q\,dx \\
&\quad - \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy - \frac{1}{2}\int_{\mathbb{R}^N} |v|^2\,dx + \frac{1}{q}\int_{\mathbb{R}^N} |v|^q\,dx - \frac{a}{2},
\end{aligned}
$$

which implies that $I_0 - I(v) - a/2 = 0$. Thus $I_0 = I(v) + a/2 \geq I_{-a} > I_0$. That is a contradiction.

It follows from (3.23) and (3.24) that $v \in \mathcal{N}$ and satisfies $I(v) = \inf_{u\in\mathcal{N}} I(u)$. Thus, the proof is complete. □

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