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NORMAL FORMS AND HYPERBOLIC ALGEBRAIC LIMIT CYCLES FOR A CLASS OF POLYNOMIAL DIFFERENTIAL SYSTEMS

JAUME LLIBRE, CLAUDIA VALLS

Abstract. We study the normal forms of polynomial systems having a set of invariant algebraic curves with singular points. We provide sufficient conditions for the existence of hyperbolic algebraic limit cycles.

1. Introduction

The algebraic theory of integrability is a classical tool and is related with the first part of the Hilbert's 16th problem. This kind of integrability is usually called Darboux integrability, and it provides a link between the integrability of a polynomial differential system and its number of invariant algebraic curves. In this article we are interested in polynomial differential systems, integrable or not, having a given set of invariant algebraic curves more concretely, we study the normal forms of planar vector fields having a given set of invariant algebraic curves. That is, we are interested in some sense in a kind of inverse theory of the Darboux theory of integrability.

We deal with the following (planar) polynomial differential system of degree m:

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1.1}$$

where $P, Q \in \mathbb{C}_m[x, y]$, being $\mathbb{C}_m[x, y]$ the set of complex polynomials such that $\max\{\deg P, \deg Q\} = m$. The dot denotes derivative with respect to a real or complex independent variable.

Let $F(x, y) \in \mathbb{C}[x, y]$ (being $\mathbb{C}[x, y]$ the ring of polynomials in x and y). The algebraic curve $F(x, y) = 0$ of $\mathbb{C}^2$ is called an *invariant algebraic curve* of system (1.1) if

$$PF_x + QF_y = KF \tag{1.2}$$

for some complex polynomial $K(x, y)$ which is called a *cofactor* of $F = 0$. We denote by F_x and F_y the derivatives of F with respect to x and y, respectively. For simplicity we shall talk about the curve $F = 0$ by only saying the curve F, see for details [4].

Cofactors of F have degree at most $m-1$. Let $F = \prod_{l=1}^{\ell} F_i^{n_i}$ be the irreducible decomposition of F. Then F is an invariant algebraic curve with a cofactor K of

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system (1.1) if and only if F_i is an invariant algebraic curve of system (1.1) with cofactor K_i. Moreover $K = \sum_{i=1}^{\ell} n_i K_i$. For a proof see [7].

Our first main result works for complex polynomial differential systems.

Theorem 1.1. *For $i = 1, \dots, p$, let $F_i = 0$ be an irreducible invariant algebraic curves of a complex polynomial differential system, and set $r = \sum_{i=1}^{p} \deg F_i$. We assume that F_i's satisfy the following assumptions:*

- (i) *There are no points at which two curves F_i and F_j satisfy either $F_i = F_j = F_{iy} = 0$ and $F_i = F_j = F_{jy} = 0$, or $F_i = F_j = F_{ix} = 0$ and $F_i = F_j = F_{jx} = 0$. Note that when $p = 1$ then we have no condition.*
- (ii) *The highest order terms of F_i have no repeated factors.*
- (iii) *If two curves intersect at a point in the finite plane, they are transversal at this point.*
- (iv) *No more than two curves $F_i = 0$ meet at any point in the finite plane.*
- (v) *No two curves having a common factor in the highest order terms.*

Then any polynomial vector field $\mathcal{X}$ of degree m tangent to all $F_i = 0$ satisfies one of the following statements:

- (a) *If $r < m + 1$ then*
$$\mathcal{X} = \Big(\prod_{i=1}^{p} F_i\Big)\mathcal{Y} + \sum_{i=1}^{p} h_i \Big(\prod_{j=1, j\neq i}^{p} F_j\Big)\mathcal{X}_{F_i},$$
where $\chi_{F_i} = (-F_{iy}, F_{ix})$ is a Hamiltonian vector field, the h_i are polynomials of degree no more than $m - r + 1$, and $\mathcal{Y}$ is a polynomial of degree no more than $m - r$.
- (b) *If $r = m + 1$ then*
$$\mathcal{X} = \sum_{i=1}^{p} \alpha_i \Big(\prod_{j=1, j\neq i}^{p} F_j\Big)\mathcal{X}_{F_i}, \quad \alpha_i \in \mathbb{C}.$$
- (c) *If $r > m + 1$ then $\mathcal{X} = 0$.*

Theorem 1.1 is proved in section 2. When condition (i) in Theorem 1.1 is replaced by the more restrictive condition: "there are no points at which F_i and its first derivatives all vanish, i.e. the curve $F_i = 0$ does not have singular points" was proved in Theorem 1 of [8]. Note that our condition (i) allows that the curves $F_i = 0$ have singular points, but a singular point of a curve $F_i = 0$ cannot also be a singular point of the curve $F_j = 0$ if $j \neq i$.

Theorem 1 of [8] goes back to Christopher [5], and it was stated in several papers without a proof such as in [5, 6], and used in other papers, see [3, 11]. Zholadek in [14] stated a similar result with an analytical proof while in [8] the proof is algebraic. In all the cases they work with the assumption that the invariant algebraic curves $F_i = 0$ do not have singular points. The vector field of statement (b) is Darboux integrable as it was proved in [6].

Since the next two results are about limit cycles, the polynomial differential systems and the independent variables under consideration are real.

Theorem 1.2. *Let $F(x, y) = 0$ be the unique irreducible invariant algebraic curve of degree n of a real polynomial vector field $\mathcal{X}$ of degree m. Then $\mathcal{X}$ can be written as*
$$\mathcal{X} = (\lambda_3 F - \lambda_1 F_y, \lambda_2 F + \lambda_1 F_x)$$

where $\lambda_\nu = \lambda_\nu(x, y)$ for $\nu = 1, 2, 3$ are polynomials. Assume that the following conditions hold.

(I) *The intersection of the periodic orbits of $F = 0$ with the algebraic curve $\lambda_1 = 0$ is empty.*

(II) *If γ is an isolated periodic solution of $\mathcal{X}$ which does not intersect the curve $\lambda_1 = 0$ then*

$$\int_\gamma \frac{\lambda_3\,dy - \lambda_2\,dx}{\lambda_1} = \int\int_\Gamma \Big(\Big(\frac{\lambda_3}{\lambda_1}\Big)_x + \Big(\frac{\lambda_2}{\lambda_1}\Big)_y\Big)\,dx\,dy \neq 0,$$

where Γ is the bounded region limited by γ.

(III) *the polynomial $(\lambda_3\lambda_{1x} + \lambda_2\lambda_{1y})|_{\lambda_1=0}$ is not zero in $\mathbb{R}^2 \setminus \{F = 0\}$.*

Then all periodic orbits of the invariant algebraic curve $F = 0$ are hyperbolic limit cycles of $\mathcal{X}$. Furthermore $\mathcal{X}$ has no other limit cycles.

The statement of Theorem 1.2 coincides with the statement of [10, Theorem 4], but Theorem 4 has the additional assumption that the algebraic curve $F = 0$ cannot have singular points, i.e., there are no points at which $F = 0$ and all its first derivatives all vanish. So our Theorem 1.2 improves [10, Theorem 4] because it allows that $F = 0$ has singular points. But the proof of Theorem 1.2 is exactly the proof of [10, Theorem 4], the difference is that such a proof uses the next Lemma 2.3 that it works when the polynomial differential system has a unique invariant algebraic curve with singular points. Other papers related with algebraic limit cycles are [1, 2, 13].

Now we extend Theorem 1.2 to the case with two algebraic curves.

Theorem 1.3. *Let $F_\nu(x, y) = 0$ for $\nu = 1, 2$ be the two unique irreducible invariant algebraic curves of the polynomial vector field*

$$\mathcal{X} = (\lambda_4 F_1F_2 - r_1F_{1y} - r_2F_{2y}, \lambda_3F_1F_2 + r_1F_{1x} + r_2F_{2x}),$$

of degree n, where $r_1 = \lambda_1F_2$, $r_2 = \lambda_2F_1$ and $\lambda_j = \lambda_j(x, y)$ for $j = 1, 2, 3, 4$ are polynomials. Assume that the curves $F_\nu(x, y) = 0$ satisfy condition (i) *in Theorem 1.1 and that the following conditions hold:*

(I) *For $\nu, \mu = 1, 2$, the intersection of the periodic orbits of $F_\nu = 0$ with the algebraic curve $r_\mu = 0$ is empty when $\nu \neq \mu$.*

(II) *If γ is an isolated periodic solution of $\mathcal{X}$ which does not intersect the curve $r_\nu = 0$ for $\nu = 1, 2$, then*

$$I_1 = \int_\gamma \Big(\frac{\lambda_4\,dy - \lambda_3\,dx}{\lambda_1} - \frac{\lambda_2}{\lambda_1} d(\log|F_2|)\Big) \neq 0,$$
$$I_2 = \int_\gamma \Big(\frac{\lambda_4\,dy - \lambda_3\,dx}{\lambda_2} - \frac{\lambda_1}{\lambda_2} d(\log|F_1|)\Big) \neq 0.$$

(III) *The two polynomials*

$$(\lambda_4 r_{1x} + \lambda_3 r_{1y} + \lambda_2\{F_2, \lambda_1\})_{|r_1=0},$$
$$(\lambda_4 r_{2x} + \lambda_3 r_{2y} + \lambda_1\{F_1, \lambda_2\})_{|r_2=0},$$

are not zero in $\mathbb{R}^2 \setminus \{F_1F_2 = 0\}$, where $\{f, g\} = f_xg_y - f_yg_x$.

Then all periodic orbits of $F_\nu = 0$ for $\nu = 1, 2$ are hyperbolic limit cycles of $\mathcal{X}$. Furthermore $\mathcal{X}$ has no other limit cycles.

Theorem 1.3 when we additionally assume that the invariant algebraic curves have no singular points was proved in Theorem 6 of [10]. In fact, our proof is exactly the same than in Theorem 6 of [10], the difference is that we use the next Lemma 2.4, allowing the two invariant curves to have singular points, but such singular points are not shared by both curves.

2. Proof of Theorem 1.1

We will use the following well-known Hilbert's Nullstellensatz (see for instance, [9]).

Theorem 2.1. *Set $A, B_i \in \mathbb{C}[x,y]$ for $i = 1, \cdots, r$. If A vanishes in $\mathbb{C}^2$ whenever the polynomials B_i vanish simultaneously, then there exist polynomials $M_i \in \mathbb{C}[x,y]$ and a nonnegative integer n such that $A^n = \sum_{i=1}^{r} M_i B_i$. If all B_i have no common zero, then there exist polynomial M_i such that $\sum_{i=1}^{r} M_i B_i = 1$.*

In what follows if we have a polynomial A we will denote its degree by a. We will also denote by F^c the homogeneous part of degree c of the polynomial F. We shall need several auxiliary results. The first one is proved in [8].

Lemma 2.2. *If F^f has no repeated factors then $(F_x, F_y) = 1$.*

Now we consider the case in which system (1.1) has a given invariant algebraic curve. The next result improves Lemma 6 of [8] showing that the result of [8] also holds for invariant algebraic curves with singular points satisfying conditions (i) of Theorem 1.1.

Lemma 2.3. *Assume that the polynomial system* (1.1) *of degree m has an invariant algebraic curve $F = 0$ of degree f.*

(a) *If $(F_x, F_y) = 1$, then system* (1.1) *has the following normal form*
$$\dot{x} = AF - DF_y, \quad \dot{y} = BF + DF_x, \tag{2.1}$$
where A, B and D are suitable polynomials.

(b) *If F satisfies condition* (ii) *of Theorem 1.1, then system* (1.1) *has the normal form* (2.1) *with $a, b \le m - f$ and $d \le m - f + 1$. Moreover, if the highest order term F^f of F does not have the factors x and y, then $a \le p - f$, $b \le q - f$ and $d \le \min\{p, q\} - f + 1$.*

We recall that under affine changes of coordinates system (2.1) preserves its form and the degrees of the polynomials. Indeed, using the affine change $u = a_1x + b_1y + c_1$ and $v = a_2x + b_2y + c_2$ with $a_1b_2 - a_2b_1 \ne 0$, system (2.1) becomes
$$\dot{u} = (a_1A + b_1B)F - (a_1b_2 - a_2b_1)DF_v, \quad \dot{v} = (a_2A + b_2B)F + (a_1b_2 - a_2b_1)DF_u.$$

Proof. We recall that F satisfies (1.2) for some polynomial K called the cofactor.

First note that if $F = 0$ has no singular points then it follows from Lemma 6 in [8] and there is nothing to prove. By condition (ii) of Theorem 1.1 all the singular points of $F = 0$ are finite (since they are equilibrium points of the polynomial differential system and its number is $\le m^2$ (see Theorem 4 of [12])), there exists a linear polynomial L such that F, F_x, F_y and $L + K$ do not vanish simultaneously, and $L + K$ is not a constant.

By Theorem 2.1 (Hilbert's Nullstellensatz) we obtain that there exist polynomials E, G, T and H such that

$$EF_x + TF_y + GF + H(L + K) = 1 \tag{2.2}$$

From (1.2) and (2.2) we obtain

$$K(1 - H(L + K)) = (KE + GP)F_x + (KT + GQ)F_y.$$

Substituting K into (1.2) we obtain

$$\begin{aligned}&[(1 - H(L + K))P - F(KE + GP)]F_x\\ &= -[(1 - H(L + K))Q - F(KT + FQ)]F_y.\end{aligned}$$

Since $(F_x, F_y) = 1$, there exists a polynomial D such that

$$\begin{aligned}(1 - H(L + K))P - F(KE + GP) &= -DF_y,\\ (1 - H(L + K))Q - F(KT + FQ) &= DF_x.\end{aligned}$$

By scaling the time variable we can write system (1.1) as

$$\dot{x} = (1 - H(L + K))P, \quad \dot{y} = (1 - H(L + K))Q \tag{2.3}$$

Note that $1 - H(L + K)$ is not identically zero, otherwise $L + K = 1/H$ must be constant, and this is not possible by the choice of L. Hence system (2.3) has the form (2.1) with

$$A = KE + GP, \quad B = KT + FQ.$$

This proves statement (a).

The proof of statement (b) is the same as the proof of statement (b) of [8, Lemma 6]. □

The following lemma improves [8, Lemma 7] showing that it also holds for invariant algebraic curves with singular points satisfying condition (i) of Theorem 1.1.

Lemma 2.4. *Assume that $F = 0$ and $G = 0$ are different irreducible invariant algebraic curves of system* (1.1) *of degree m that satisfy the conditions* (i) *and* (iii) *of Theorem 1.1.*

(a) *If $(F_x, F_y) = 1$ and $(G_x, G_y) = 1$, then system* (1.1) *has the normal form*

$$\dot{x} = AFG - EF_yG - NCG_y, \quad \dot{y} = BFG + EF_xG + NFG_x \tag{2.4}$$

(b) *If F and G satisfy additionally conditions* (ii) *and* (v)*, then system* (1.1) *has the normal form* (2.4) *with $a, b \le m - f - g$ and $e, n \le m - f - g + 1$.*

Proof. Since $(F, G) = 1$, the curves F and G have finitely many intersection points. By assumption (i), we can assume that both systems $F = G = G_y = 0$ and $F = G = F_y = 0$ have no solutions (the case when $F = G = G_x = 0$ and $F = G = F_x = 0$ can be proved in a similar way). The rest of the proof follows the same steps than the proof of [8, Lemma 7]. □

The next lemma follows as in [8, Lemma 8].

Lemma 2.5. *Let $F_i = 0$ for $i = 1, \dots, p$ be different irreducible invariant algebraic curves of system* (1.1) *with $\deg F_i = f_i$. Assume that F_i satisfy conditions (i), (iii) and (iv) of Theorem 1.1. Then the following statements hold.*

(a) *If* $(F_{ix}, F_{iy}) = 1$ *for* $i = 1, \dots, p$ *then system* (1.1) *has the normal form*

$$\dot{x} = \Big(B - \sum_{i=1}^{p} \frac{A_i F_{iy}}{F_i}\Big) \prod_{i=1}^{p} F_i, \quad \dot{y} = \Big(C + \sum_{i=1}^{p} \frac{A_i F_{ix}}{F_i}\Big) \prod_{i=1}^{p} F_i, \tag{2.5}$$

where B, C *and* A *are suitable polynomials.*

(b) *If* F_i *satisfy additionally conditions (ii) and (v) of Theorem 1.1, then system* (1.1) *has the normal form* (2.5) *with* $b, c \le m - \sum_{i=1}^{p} f_i$ *and* $a_i \le m - \sum_{i=1}^{p} f_i + 1$.

Proof. By Lemma 2.5 we obtain statement (a) of Theorem 1.1. From the degrees of the polynomials A_i, B and C in statement (b) of Lemma 2.5 we obtain statement (b) of Theorem 1.1.

By statement (a) of Lemma 2.5, we can rewrite system (1.1) into the form (2.5). Finally, by statement (b) of Lemma 2.5 we obtain $B = 0$, $C = 0$ and $A_i = 0$. This completes the proof of statement (c) of Theorem 1.1. □

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Jaume Llibre
Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
E-mail address: `jllibre@mat.uab.cat`

Claudia Valls
Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, 1049-001 Lisboa, Portugal
E-mail address: cvalls@math.ist.utl.pt