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BOUNDARY PARTIAL HÖLDER REGULARITY FOR ELLIPTIC SYSTEMS WITH NON-STANDARD GROWTH

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ABSTRACT. We investigate regular points on the boundaries of elliptic systems with non-standard growth, in particular, so-called Orlicz growth. A regular point on the boundary in this paper is a point for which a weak solution to a system is Hölder continuous in a neighborhood. Here, we assume that the boundary of a domain and the boundary data are C^1, and that a system has VMO (vanishing mean oscillation) type coefficients.

1. INTRODUCTION

In this article, we study partial regularity on the boundaries of nonlinear elliptic systems with nonstandard Orlicz growth and the Dirichlet boundary condition. Precisely, we find a suitable condition of the boundary points to obtain Hölder continuity of the corresponding weak solution in its neighborhood for any Hölder exponent $\alpha \in (0,1)$. Here we assume that the coefficients of the systems are VMO, and that the boundaries and boundary data are C^1.

Partial regularity for general elliptic systems with 'standard' p-growth was first systematically investigated by Campanato [13, 14]; see [27, 45] for pioneering works in this direction. The main objective in this field is to obtain relations between the regularity of coefficients of systems and partial regularity of relevant weak solutions, which are naturally expected from scalar problems. For instance, if the coefficients are Hölder continuous, then the gradient of the weak solution is partially Hölder continuous, i.e., Hölder continuous except for a measure zero set. In addition, if the coefficients are merely continuous, then the weak solution is partially Hölder continuous for all Hölder exponents $\alpha \in (0,1)$. This result for general dimension $n \geq 2$ was first proved by Foss & Mingione [24], and then Beck [7] characterized the boundary points to obtain partial Hölder regularity. We remark that the actual existence of regular boundary points for systems with Hölder continuous coefficients was proved in [22, 35]. For further regularity results, concerning both systems and integral functionals, we refer to [5, 7, 8, 9, 10, 11, 12, 23, 25, 28, 29, 32, 33, 34, 36]. An extensive overview can be found in [44].

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For the last few decades, there have been a lot of research activities regarding the partial differential equations(PDEs) and functionals with non-standard growth, which was first studied by Marcellini [40, 41, 42, 43]. The most basic non-standard growth type is the so-called Orlicz growth condition, which implies that PDEs or functionals are controlled by Orlicz functions. The definition and properties of Orlicz functions and related properties will be introduced in the next section. PDEs and functionals with Orlicz growth were first investigated by Lieberman [37, 38, 39]; see also [1, 2, 15, 19] for further regularity results. In addition, partial regularity for systems or functionals with Orlicz growth have also been studied in [19, 21, 48]. In particular, in [48] the authors obtained partial Hölder regularity for elliptic systems with VMO coefficients. Finally, we would like to mention that non-autonomous problems, for instance, problems with $p(x)$-growth and double phase problems, are closely related to the Orlicz case, and we refer to recent results in [3, 4, 16, 17, 18, 47] for double phase problems and [30, 46, 48, 50, 51] for partial regularity for systems with non-autonomous growth conditions.

Here, we consider boundary partial Hölder regularity for elliptic systems with Orlicz growth, which is a natural generalization of [7] in the Orlicz setting. Let us introduce the system we mainly consider in this paper. Let $G:[0,\infty)\to[0,\infty)$ with $G(0)=0$ be C^2 and satisfy

$$1 < g_1 - 1 \le \inf_{t>0} \frac{tG''(t)}{G'(t)} \le \sup_{t>0} \frac{tG''(t)}{G'(t)} \le g_2 - 1 \tag{1.1}$$

for some $2 < g_1 \le g_2 < \infty$. Note that under these assumptions, G is convex and strictly increasing. We then consider the system

$$\begin{gathered} \operatorname{div} \mathbf{a}(x,u,Du) = 0 \quad \text{in } \Omega, \\ u = h \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

Here, $\mathbf{a} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$, $N \ge 1$, satisfies

$$\begin{gathered} |\mathbf{a}(x,\zeta,\xi)| + |\partial\mathbf{a}(x,\zeta,\xi)|(1+|\xi|) \le LG_1(1+|\xi|), \\ \partial\mathbf{a}(x,\zeta,\xi)\eta\cdot\eta \ge \nu G_2(1+|\xi|)|\eta|^2 \end{gathered} \tag{1.3}$$

for all $x \in \Omega$, $\zeta \in \mathbb{R}^N$ and $\xi, \eta \in \mathbb{R}^{nN}$ and for some $0 < \nu \le L$, where $\partial\mathbf{a}(x,\zeta,\xi) := D_\xi \mathbf{a}(x,\zeta,\xi)$,

$$G_1(t) := t^{-1}G(t) \quad \text{and} \quad G_2(t) := t^{-2}G(t). \tag{1.4}$$

We note from the second inequality in (1.3) that

$$\begin{aligned} (\mathbf{a}(x,\zeta,\xi_1) - \mathbf{a}(x,\zeta,\xi_2)) : (\xi_1 - \xi_2) \qquad & \ge \tilde{\nu}\, G_2(1+|\xi_1|+|\xi_2|)|\xi_1-\xi_2|^2 \\ \ge \frac{\tilde{\nu}}{2}\{G_2(1+|\xi_1|)|\xi_1-\xi_2|^2 + G(|\xi_1-\xi_2|)\}. & \end{aligned} \tag{1.5}$$

Then, for $h \in W^{1,G}(\Omega,\mathbb{R}^N)$, we say $u \in W^{1,G}(\Omega,\mathbb{R}^N)$ with $u - h \in W^{1,G}_0(\Omega,\mathbb{R}^N)$ is a weak solution to (1.2) if

$$\int_\Omega \mathbf{a}(x,u,Du) : D\varphi \, dx = 0 \quad \forall \varphi \in W^{1,G}_0(\Omega,\mathbb{R}^N). \tag{1.6}$$

Here, $W^{1,G}$ and $W^{1,G}_0$ are Sobolev-Orlicz spaces, which we shall introduce in Section 2, and the existence and uniqueness of weak solutions to (1.2) are a consequence of nonlinear functional analysis, see for instance [49, Chapter II.2], and the properties of the Sobolev-Orlicz spaces.

We further impose regularity assumptions on nonlinearity $\mathbf{a}$ as follows. For the first variable x ,we suppose that

$$\lim_{\rho\to 0}\mathcal{V}(\rho)=0, \quad \text{where} \quad \mathcal{V}(\rho):=\sup_{0<r\le\rho}\sup_{y\in\mathbb{R}^n}\fint_{B_r(y)\cap\Omega}V(x,B_r(y)\cap\Omega)\,dx, \tag{1.7}$$

$$V(x,U):=\sup_{\zeta\in\mathbb{R}^N}\sup_{\xi\in\mathbb{R}^{nN}}\frac{|\mathbf{a}(x,\zeta,\xi)-(\mathbf{a}(\cdot,\zeta,\xi))_U|}{G_1(1+|\xi|)}\le 2L\,. \tag{1.8}$$

Here we note that condition (1.7) implies that the coefficient factor of $\mathbf{a}$ is VMO uniformly for both ζ and ξ. For the other variables, we assume that there exists a nondecreasing and concave function $\mu:[0,\infty)\to[0,1]$ with $\mu(0)=0$ such that

$$|\mathbf{a}(x,\zeta_1,\xi)-\mathbf{a}(x,\zeta_2,\xi)|\le L\mu(|\zeta_1-\zeta_2|^2)\,G_1(1+|\xi|), \tag{1.9}$$

$$|\partial\mathbf{a}(x,\zeta,\xi_1)-\partial\mathbf{a}(x,\zeta,\xi_2)|\le L\mu\Big(\frac{|\xi_1-\xi_2|}{1+|\xi_1|+|\xi_2|}\Big)G_2(1+|\xi|) \tag{1.10}$$

for all $x\in\Omega$, $\zeta,\zeta_1,\zeta_2\in\mathbb{R}^N$ and $\xi,\xi_1,\xi_2\in\mathbb{R}^{nN}$. In this setting, we show the following result.

Theorem 1.1. *Suppose $\Omega\in C^1$, $h\in C^1(\overline{\Omega})$, $G:[0,\infty)\to[0,\infty)$ is C^2 and satisfies* (1.1), *$\mathbf{a}:\Omega\times\mathbb{R}^N\times\mathbb{R}^{nN}\to\mathbb{R}^N$ satisfies* (1.3), (1.7), (1.9) *and* (1.10). *Let $u\in W^{1,G}_h(\Omega,\mathbb{R}^N)$ be a weak solution to* (1.2). *Then a set of regular points on the boundary $\partial\Omega$ given by*

$$\partial\Omega_u:=\cap_{\alpha\in(0,1)}\Big\{x_0\in\partial\Omega: u\in C^\alpha(U_{x_0}\cap\overline{\Omega},\mathbb{R}^N)\quad\textit{for some } U_{x_0}\subset B_1\Big\},$$

where U_{x_0} is an open neighborhood of x_0, satisfies

$$\begin{aligned}\partial\Omega\setminus\partial\Omega_u\subset&\Big\{x_0\in\partial\Omega:\liminf_{r\downarrow 0}\fint_{B_r(x_0)\cap\Omega}|Du-(D_{\nu_{x_0}}u)_{B_r(x_0)\cap\Omega}\otimes\nu_{x_0}|\,dx>0\Big\}\\&\cup\Big\{x_0\in\partial\Omega:\limsup_{r\downarrow 0}\fint_{B_r(x_0)\cap\Omega}G(|D_{\nu_{x_0}}u|)\,dx=\infty\Big\},\end{aligned}$$

where ν_{x_0} is the inward unit normal vector at $x_0\subset\partial\Omega$.

Note that $\Omega\in C^1$ means that for each $y\in\partial\Omega$, there exist $r>0$ and C^1 function $\gamma_y:\mathbb{R}^{n-1}\to\mathbb{R}$ such that, in the coordinate system with the origin at y and $\nu_y=e_n$, $B_r\cap\Omega=\{x=(x',x_n)\in B_r: x_n>\gamma_y(x')\}$. Note that by the continuity of $\partial\Omega$, we can consider $r>0$ independent of y in the definition.

Now, we introduce the approach used in the proof. We consider a system on a half ball with a zero boundary condition on the flat part and characterize regular points on the flat boundary, see Theorem 4.1. This implies our main result via a flattening argument. To obtain the result in Theorem 4.1, we linearize the system with a 're-normalized' weak solution, and then compare it with an $\mathcal{A}$-harmonic map. Here we will use a flat boundary version of the $\mathcal{A}$-harmonic approximation lemma, see Lemma 2.7. We note that this technique was developed in [24] (resp. [6]) for interior (resp. boundary) partial regularity for systems with p-growth. Hence, we make use of the method presented there and modify it for the setting of the Orlicz class. In this procedure, various technical difficulties are arising. To overcome these, we take advantage of an almost convex property, see Lemma 2.2, and an additional assumption, see (3.11).

The rest of this article is organized as follows. In the next section, we present notation and auxiliary results. In Section 3, we obtain Cacciopoli type estimates, and after linearization, compare the re-normalized function of the weak solution with an $\mathcal{A}$-harmonic function using an $\mathcal{A}$-harmonic approximation lemma. In the final section, Section 4, we construct a condition for regular boundary points for systems on a half ball with the zero boundary condition on a flat boundary. Using this, we prove Theorem 1.1.

2. Preliminaries

2.1. **Notation.** Define $\mathbb{R}^n_+ := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ and $B_r(x_0)$ by a standard ball with center $x_0 \in \mathbb{R}^n$ and radius $r > 0$, $B^+_r(x_0) := B_r(x_0) \cap \mathbb{R}^n_+$, and $T_r(x_0) := \{x = (x_1, \dots, x_n) \in B_r(x_0) : x_n = 0\}$. For a locally integrable (vector valued) function f in $\mathbb{R}^n$ and a bounded open set $U \subset \mathbb{R}^n$, $(f)_U$ is denoted by the integral average of f in U such that

$$(f)_U = \fint_U f\,dx = \frac{1}{|U|}\int_U f\,dx.$$

Moreover, we abbreviate $(f)_{x_0,r} = (f)_{B_r(x_0)}$ and $(f)^+_{x_0,r} = (f)_{B^+_r(x_0)}$ if there is no confusion. Let $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{nN}$, $1 \le i \le n$ and $1 \le j \le N$, be matrices, and define the inner product of them by $A : B = \sum_{i,j} a_{ij}b_{ij}$. $P : \mathbb{R}^n \to \mathbb{R}^N$ is always an affine function, that is, $P(x) = Ax + b$ for some matrix $A \in \mathbb{R}^{nN}$ and $b \in \mathbb{R}^N$. For a given $u \in L^2(B^+_r(x_0), \mathbb{R}^N)$ with $x_0 \in \mathbb{R}^{n-1} \times \{0\}$, we define an affine function $P^+_{x_0,r}$ by the minimizer of the functional

$$P \mapsto \fint_{B^+_r(x_0)} |u - P|^2\,dx.$$

Then one can see that

$$P^+_{x_0,r}(x) = Q^+_{x_0,r}x_n, \quad \text{where } Q^+_{x_0,\rho} := \frac{n+2}{r^2}\fint_{B^+_r(x_0)} u(x)x_n\,dx.$$

We note that if the center point of a ball is clear or not important, we shall omit it in the notation, for example, $B_r = B_r(x_0)$, $B^+_r = B^+_r(x_0)$, $(f)_r = (f)_{x_0,r}$, and so on.

2.2. **Orlicz function and space.** We say that $G : [0,\infty) \to [0,\infty)$ is an N-function if G is differentiable and G' is a non-decreasing right continuous function satisfying $G'(0) = 0$ and $G'(t) > 0$ for all $t > 0$. Note that an N-function is convex. From now on, we suppose G is an N-function that satisfies

$$1 < g_1 \le \inf_{t>0} \frac{tG'(t)}{G(t)} \le \sup_{t>0} \frac{tG'(t)}{G(t)} \le g_2 < \infty \tag{2.1}$$

for some $1 < g_1 \le g_2 < \infty$. For instance, $G(t) = t^p$, $1 < p < \infty$, is an N-function and satisfies (2.1) with $g_1 = g_2 = p$. We notice that if G is C^2 and satisfies (1.1), then it is an N-function and satisfies (2.1).

We next define the complement function of G by $G^* : [0,\infty) \to [0,\infty)$ such that

$$G^*(\tau) := \sup_{t\ge 0}(\tau t - G(t)).$$

Then we have that G^* is an N-function satisfying (2.1) with g_1 and g_2 replaced by $g_2/(g_2-1)$ and $g_1/(g_1-1)$, respectively. Note that (2.1) is equivalent to G and G^* satisfying the so-called Δ_2-condition, i.e., $G(2t) \le cG(t)$ and $G^*(2t) \le cG^*(t)$

for some $c \geq 1$. We briefly state some basic properties of the Orlicz functions. We refer to [48, Proposition 2.1].

Proposition 2.1. *Suppose $G : [0, \infty) \to [0, \infty)$ is convex and satisfies* (2.1) $1 < g_1 \leq g_2 < \infty$. *Let $t, \tau > 0$, $0 < a < 1 < b < \infty$.*

(1) *$G(t)t^{-g_1}$ is increasing and $G(t)t^{-g_2}$ is decreasing. Hence we have*
$$G(at) \leq a^{g_1} G(t), \quad G(bt) \leq b^{g_2} G(t). \tag{2.2}$$
Moreover,
$$G^*(at) \leq a^{\frac{g_2}{g_2-1}} G^*(t), \quad \text{and} \quad G^*(bt) \leq b^{\frac{g_1}{g_1-1}} G^*(t). \tag{2.3}$$

(2) $G(t+\tau) \leq 2^{-1}(G(2t) + G(2\tau)) \leq 2^{g_2-1}(G(t) + G(\tau))$.

(3) *(Young's inequality) For any $\kappa \in (0, 1]$, we have*
$$t\tau \leq G(\kappa^{\frac{1}{g_1}} t) + G^*(\kappa^{-\frac{1}{g_1}} \tau) \leq \kappa G(t) + \kappa^{-\frac{1}{g_1-1}} G^*(\tau), \tag{2.4}$$
$$t\tau \leq G(\kappa^{-\frac{g_2-1}{g_2}} t) + G^*(\kappa^{\frac{g_2-1}{g_2}} \tau) \leq \kappa^{-g_2+1} G(t) + \kappa\, G^*(\tau). \tag{2.5}$$

(4) *There exists $c = c(g_1, g_2) \geq 1$ such that*
$$c^{-1} G(t) \leq G^*\left(G(t)t^{-1}\right) \leq cG(t). \tag{2.6}$$

We also introduce a condition for functions that are similar to concave functions. We refer to [48, Lemma 2.2].

Lemma 2.2. *Suppose that $\Psi : [0, \infty) \to [0, \infty)$ is non-decreasing such that the map $t \mapsto \Psi(t)/t$ is non-increasing. Then there exists a concave function $\tilde{\Psi} : [0, \infty) \to [0, \infty)$ such that*
$$\frac{1}{2}\tilde{\Psi}(t) \leq \Psi(t) \leq \tilde{\Psi}(t) \quad \text{for all } t \geq 0.$$

For a given N-function G satisfying (2.1), we denote the Orlicz space $L^G(\Omega)$ by the set of all functions f satisfying
$$\|f\|_{L^p(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega G\Big(\frac{|f|}{\lambda}\Big)\, dx \leq 1\right\} < \infty.$$
In fact, the above inequality is equivalent to
$$\int_\Omega G(|f|)\, dx < \infty.$$
Furthermore, the Orlicz-Sobolev space $W^{1,G}(\Omega)$ (resp. $W_0^{1,G}(\Omega)$) is the set of $f \in W^{1,1}(\Omega)$ (resp. $f \in W_0^{1,1}(\Omega)$) with $f, |Df| \in L^G(\Omega)$.

2.3. **Basic inequalities.** For $f \in L^G(B_r(x_0), \mathbb{R}^N)$ and $A \in \mathbb{R}^N$, from Jensen's inequality and the property of the N-function (2.1), it is well known that
$$\fint_{B_r(x_0)} G(|f - (f)_{x_0,r}|)\, dx \leq 2^{g_2} \fint_{B_r(x_0)} G(|f - A|)\, dx.$$
Furthermore, in a similar way, one can also see that for $f \in W^{1,G}(B_r(x_0), \mathbb{R}^N)$ and $A \in \mathbb{R}^N$,
$$\fint_{B_r(x_0)} G(|Df - (D_n f)_{x_0,r} \otimes e_n|)\, dx \leq c \fint_{B_r(x_0)} G(|Df - A \otimes e_n|)\, dx. \tag{2.7}$$

We next introduce a Poincaré type inequality for functions vanishing on the flat boundary in $W^{1,G}(B_r^+)$, which can be easily obtained by modifying the interior counterpart in [19, Theorem 7].

Lemma 2.3. *Suppose that $G : [0,\infty) \to [0,\infty)$ is an N-function and satisfies (2.1) for some $1 < g_1 \le g_2 < \infty$, and that $f \in W^{1,1}(B_r^+(x_0),\mathbb{R}^N)$ with $u = 0$ on $T_r(x_0)$. Then there exist $0 < d_1 < 1 < d_2$ depending only on n, N, g_1, g_2 such that*

$$\Big(\fint_{B_r^+(x_0)} \big[G(\frac{|f|}{r})\big]^{d_2}\,dx\Big)^{1/d_2} \le c\Big(\fint_{B_r^+(x_0)} [G(|Df|)]^{d_1}\,dx\Big)^{1/d_1} \tag{2.8}$$

for some $c = c(n,N,g_1,g_2) > 0$.

The next lemma implies that the gradient on the right-hand side can be replaced by the directional derivative $D_n f$.

Lemma 2.4. *Let G be an N-function satisfying (2.1) and $x_0 \in \mathbb{R}^{n-1}\times\{0\}$. For $f \in W^{1,G}(B_r^+(x_0))$ with $f = 0$ on $T_r(x_0)$, we have*

$$\int_{B_r^+(x_0)} G(\frac{|f|}{r})\,dx \le \frac{1}{g_1}\int_{B_r^+(x_0)} G(|D_n f|)\,dx. \tag{2.9}$$

Proof. The proof when $G(t) = t^p$ can be found in [5, Lemma 3.4]. We follow the argument presented there. Since $f = 0$ on $T_r(x_0)$, we have

$$f(x) = f(x', x_n) = \int_0^{x_n} D_n f(x',t)\,dt,$$

where $x' = (x_1,\dots,x_{n-1})$. Using this inequality along with Jensen's inequality and Fubini's theorem, we have

$$\begin{aligned}
&\int_{B_r^+(x_0)} G(\frac{|f(x)|}{r})\,dx \\
&= \int_{-r}^{r}\int_{-\sqrt{r^2-x_1^2}}^{\sqrt{r^2-x_1^2}}\cdots\int_0^{\sqrt{r^2-|x'|^2}} G(\frac{|f(x)|}{r})\,dx_n\dots dx_2\,dx_1 \\
&\le \int_{-r}^{r}\int_{-\sqrt{r^2-x_1^2}}^{\sqrt{r^2-x_1^2}}\cdots\int_0^{\sqrt{r^2-|x'|^2}} G\Big(\frac{x_n}{r}\fint_0^{x_n}|D_n f(x',t)|\,dt\Big)\,dx_n\dots dx_2 dx_1 \\
&\le \int_{-r}^{r}\int_{-\sqrt{r^2-x_1^2}}^{\sqrt{r^2-x_1^2}}\cdots\int_0^{\sqrt{r^2-|x'|^2}} (\frac{x_n}{r})^{g_1}\fint_0^{x_n} G(|D_n f(x',t)|)\,dt dx_n\dots dx_2 dx_1 \\
&= \int_{-r}^{r}\int_{-\sqrt{r^2-x_1^2}}^{\sqrt{r^2-x_1^2}}\cdots\int_0^{\sqrt{r^2-|x'|^2}}\int_{\sqrt{r^2-|x'|^2}}^{r}\frac{x_n^{g_1-1}}{r^{g_1}} G(|D_n f(x',t)|)\,dx_n\,dt\dots dx_2\,dx_1 \\
&\le \int_0^r \frac{x_n^{g_1-1}}{r^{g_1}}\,dx_n \int_{-r}^{r}\int_{-\sqrt{r^2-x_1^2}}^{\sqrt{r^2-x_1^2}}\cdots\int_0^{\sqrt{r^2-|x'|^2}} G(|D_n f(x',t)|)\,dt\dots dx_2\,dx_1 \\
&= \frac{1}{g_1}\int_{B_r^+(x_0)} G(|D_n f(x)|)\,dx.
\end{aligned}$$

$\square$

By the same argument as in Lemma [48, Lemma 2.3], we have the following result.

Lemma 2.5. *Let $G : [0,\infty) \to [0,\infty)$ be convex and satisfy (2.1) for some $2 < g_1 < g_2 < \infty$, and let $u \in W^{1,G}(B_r^+(x_0), \mathbb{R}^N)$ with $x_0 \in \mathbb{R}^{n-1} \times \{0\}$. Then we have*

$$G(|Q^+_{x_0,r} - Q^+_{x_0,\theta r}|) \le c \fint_{B^+_{\theta r}(x_0)} G\Big(\frac{|u - P^+_{x_0,r}|}{\theta r}\Big)\,dx, \tag{2.10}$$

and for any $\xi \in \mathbb{R}^N$,

$$G(|Q^+_{x_0,r} - \xi|) \le c \fint_{B^+_r(x_0)} G(|D_n u - \xi|)\,dx \tag{2.11}$$

for some $c = c(n, g_2) > 0$.

Proof. By [7, Lemma 2.4], we have

$$\begin{gathered} |Q^+_{x_0,r} - Q^+_{x_0,\theta r}|^2 \le c(n) \fint_{B^+_{\theta r}(x_0)} \frac{|u - P^+_{x_0,r}|^2}{(\theta r)^2}\,dx, \\ |Q^+_{x_0,r} - \xi|^2 \le c(n) \fint_{B^+_r(x_0)} |D_n u - \xi|^2\,dx. \end{gathered}$$

Using these and Jensen's inequality for the convex map $t \mapsto G(\sqrt{t})$, we obtain

$$\begin{aligned} G(|DP_{x_0,r} - DP_{x_0,\theta r}|) &\le (c(n)+1)^{g_2/2}\, G\Big(\sqrt{\fint_{B^+_{\theta r}(x_0)} \frac{|u - P^+_{x_0,r}|^2}{(\theta r)^2}\,dx}\Big) \\ &\le (c(n)+1)^{g_2/2} \fint_{B^+_{\theta r}(x_0)} G\Big(\frac{|u - P^+_{x_0,r}|}{\theta r}\Big)\,dx. \end{aligned}$$

This shows (2.10). The same argument implies inequality (2.11). $\square$

We complete this subsection stating an iteration lemma, see [26, Lemma 7.3] and [24, Lemma 2.3].

Lemma 2.6. *Let $\phi : (0,\rho] \to \mathbb{R}$ be a positive and nondecreasing function satisfying*

$$\phi(\theta^{k+1}\rho) \le \theta^\lambda \phi(\theta^k \rho) + \tilde{c}(\theta^k\rho)^n \quad \text{for every } k = 0,1,2,\dots,$$

where $\theta \in (0,1)$, $\lambda \in (0,n)$ and $\tilde{c} > 0$. Then there exists $c = c(n,\theta,\lambda) > 0$ such that

$$\phi(t) \le \tilde{c}big\{\Big(\frac{t}{\rho}\Big)^\lambda \phi(\rho) + \tilde{c}t^\lambda\Big\} \quad \text{for every } t \in (0,\rho].$$

2.4. **$\mathcal{A}$-harmonic approximation on half balls.** We introduce a flat boundary version of the $\mathcal{A}$-harmonic approximation lemma. We refer to [29, Lemma 2.3]. Suppose $\mathcal{A}$ is a bilinear form with respect to $\mathbb{R}^{nN}$ such that there exists $0 < \nu \le L$ satisfying

$$\nu|\xi|^2|\eta|^2 \le \mathcal{A}(\xi \otimes \eta) : \xi \otimes \eta \le L|\xi|^2|\eta|^2 \tag{2.12}$$

for every $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^N$. If $h \in W^{1,2}(\Omega, \mathbb{R}^N)$ satisfies

$$\int_\Omega \mathcal{A}Dh : D\varphi = 0$$

for every $\varphi \in C^1_0(\Omega, \mathbb{R}^N)$, we say that h is $\mathcal{A}$-harmonic

Lemma 2.7. *For $\epsilon > 0$, there exists small $\delta = \delta(n, N, L, \nu, \epsilon) > 0$ such that the following holds: if $w \in W^{1,2}(B_r^+(x_0), \mathbb{R}^N)$ with $w = 0$ on $T_r(x_0)$ such that*

$$\fint_{B_r^+(x_0)} |Dw|^2\,dx \le 1,$$

$$\Big|\fint_{B_r^+(x_0)} \mathcal{A}Dw : D\varphi\,dx\Big| \le \delta \|D\varphi\|_{L^\infty(B_r^+(x_0))} \quad \textit{for all } \varphi \in C_0^1(B_r^+(x_0), \mathbb{R}^N),$$

then there exists an $\mathcal{A}$-harmonic map $h \in W^{1,2}(B_r^+(x_0), \mathbb{R}^N)$ with $h = 0$ on $T_r(x_0)$ such that

$$\fint_{B_r^+(x_0)} |Dh|^2\,dx \le 1, \quad \textit{and} \quad r^{-2}\fint_{B_r^+(x_0)} |w - h|^2\,dx \le \epsilon.$$

2.5. **Some estimates for weak solutions.** We introduce energy estimates and a self-improving property for systems on a half ball. In this subsection, we shall consider the system

$$\begin{gathered} \operatorname{div} \mathbf{a}(x, u, Du) = 0 \quad \text{in } B_{2r}^+(x_0), \\ u = 0 \quad \text{on } T_{2r}(x_0), \end{gathered} \tag{2.13}$$

where $x_0 \in \mathbb{R}^{n-1} \times \{0\}$ and $\mathbf{a}$ satisfies

$$|\mathbf{a}(x, \zeta, \xi)| \le L\,G_1(s + |\xi|) \quad \text{and} \quad \mathbf{a}(x, \zeta, \xi) : \xi \ge \nu\,G(|\xi|) - \nu_0 G(s) \tag{2.14}$$

for all $x \in \Omega$, $\zeta \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{nN}$, and for some $0 < \nu \le L < \infty$, $\nu_0 > 0$ and $s \in [0, 1]$. Here $G : [0, \infty) \to [0, \infty)$ is an N-function satisfying (2.1).

We start with the energy estimates.

Lemma 2.8. *Let $u \in W^{1,G}(B_{2r}^+(x_0))$ with $u = 0$ on $T_{2r}(x_0)$ be a weak solution to* (2.13)*. Then*

$$\int_{B_r^+(x_0)} G(s + |Du|)\,dx \le c \int_{B_{2r}^+(x_0)} G(s + |D_n u|)\,dx \tag{2.15}$$

for some $c = (n, N, L, \nu, \nu_0, g_1, g_2) > 0$.

Proof. By taking $\eta^{g_2} u \in W_0^{1,G}(B_{2r}(x_0))$ as a testing function in the weak formulation of (2.13), where $\eta \in C_0^\infty(B_{2r}(x_0))$ is a cut-off function so that $0 \le \eta \le 1$, $\eta \equiv 1$ in $B_r(x_0)$ and $|D\eta| \le c(n)/r$, we have

$$\begin{aligned} \int_{B_{2r}^+(x_0)} \eta^{g_2} G(|Du|)\,dx &\le c \int_{B_{2r}^+(x_0)} \eta^{g_2} \mathbf{a}(x, u, Du) : Du\,dx + cG(s) \\ &\le c \fint_{B_{2r}^+} \eta^{g_2 - 1} G_1(s + |Du|) \frac{|u|}{r}\,dx + cG(s). \end{aligned}$$

Using (2.5) with (2.6) and (2.3),

$$\fint_{B_{2r}} \psi^{g_2} G(s + |Du|)\,dx \le \frac{1}{2} \fint_{B_{2r}} \psi^{g_2} G(s + |Du|)\,dx + c \fint_{B_{2r}} G\big(\frac{|u|}{r}\big)\,dx + cG(s).$$

Finally, applying (2.9) we obtain (2.15). □

We next state self-improving properties, which can be obtained from the previous result along with Proposition 2.8 and the interior self-improving property in [48, Theorem 3.4]. Hence, we shall omit its proof.

Lemma 2.9. *Let $u \in W^{1,G}(B_{2r}^+(x_0))$ with $u = 0$ on $T_{2r}(x_0)$ be a weak solution to* (2.13)*. Then there exists $\sigma_1 = \sigma_1(n, N, L, \nu, \nu_0, g_1, g_2) > 0$ such that $G(|Du|) \in L_{loc}^{1+\sigma_1}(\Omega)$ with the estimate that for any $\sigma \in [0, \sigma_1]$ and $B_{2r}(x_0) \Subset \Omega$,*

$$\Big(\fint_{B_r^+(x_0)} [G(s + |Du|)]^{1+\sigma}\, dx \Big)^{\frac{1}{1+\sigma}} \le c \fint_{B_{2r}^+(x_0)} G(s + |Du|)\, dx \tag{2.16}$$

for some $c = c(n, N, L, \nu, \nu_0, g_1, g_2) > 0$.

3. Linearization and excess decay estimates

From now on, we shall consider problems on upper half balls such that

$$\begin{gathered} \operatorname{div} \mathbf{a}(x, u, Du) = 0 \quad \text{in } B_r^+, \\ u = 0 \quad \text{on } T_r. \end{gathered} \tag{3.1}$$

Here, $\mathbf{a}$ is assumed to satisfy (1.3). The next lemma is a boundary version of a Caccioppoli type inequality.

Lemma 3.1. *Let $G(t)$ satisfy* (1.1)*, and $u \in W^{1,G}(B_r^+)$ be a weak solution to* (3.1)*. Then for any $B_{2\rho}(x_0)$ with $x_0 \in T_r$ and $2\rho < r - |x_0|$ and any $\xi \in \mathbb{R}^N$, we have*

$$\begin{aligned} &\fint_{B_\rho^+(x_0)} \Big[\frac{|Du - \xi \otimes e_n|^2}{(1 + |\xi|)^2} + \frac{G(|Du - \xi \otimes e_n|)}{G(1 + |\xi|)} \Big]\, dx \\ &\le c \fint_{B_{2\rho}^+(x_0)} \Big[\frac{|u - x_n\xi|^2}{(2\rho)^2(1 + |\xi|)^2} + \frac{G(|u - x_n\xi|/(2\rho))}{G(1 + |\xi|)} \Big]\, dx \\ &\quad + c\mu\Big(\fint_{B_{2\rho}^+(x_0)} |u|^2\, dx \Big) + c\mathcal{V}(2\rho) \end{aligned} \tag{3.2}$$

for some $c = c(n, N, L, \nu, g_1, g_2) > 0$, where $x = (x_1, \dots, x_n)$ and $\mathcal{V}$ is denoted in (1.7)*.*

Proof. Let us fix $x_0 \in T_r$ and $\rho > 0$ with $2\rho < r - |x_0|$. Then we simply write $B_t = B_t(x_0)$ and $B_t^+ = B_t^+(x_0)$, where $t = \rho, 2\rho$. Let $P(x) := x_n\xi$ and $\eta \in C_0^\infty(B_{2\rho})$ satisfy $0 \le \eta \le 1$, $\eta \equiv 1$ on B_ρ and $|D\eta| \le c(n)/\rho$. Then taking $\varphi = \eta^{g_2}(u - P) \in W_0^{1,G}(B_{2\rho}^+)$ as a test function in the weak formulation of (3.1), we have

$$\fint_{B_{2\rho}^+} \eta^{g_2}\, \mathbf{a}(x, u, Du) : D(u - P)\, dx = -g_2 \fint_{B_{2\rho}^+} \eta^{g_2 - 1}\, \mathbf{a}(x, u, Du) : D\eta \otimes (u - P)\, dx.$$

Setting $\overline{\mathbf{a}}(\zeta, \xi) := (\mathbf{a}(\cdot, \zeta, \xi))_{B_{2\rho}^+}$, it follows that

$$\begin{aligned} I_1 &:= \fint_{B_{2\rho}^+} \eta^{g_2} (\mathbf{a}(x, u, Du) - \mathbf{a}(x, u, DP)) : (Du - DP)\, dx \\ &= -\fint_{B_{2\rho}^+} \mathbf{a}(x, u, DP) : D\varphi\, dx \\ &\quad - g_2 \fint_{B_{2\rho}^+} \eta^{g_2 - 1} (\mathbf{a}(x, u, Du) - \mathbf{a}(x, u, DP)) : D\eta \otimes (u - P)\, dx \\ &= -\fint_{B_{2\rho}^+} (\mathbf{a}(x, u, DP) - \mathbf{a}(x, 0, DP)) : D\varphi\, dx \end{aligned}$$

$$
\begin{aligned}
&- \fint_{B_{2\rho}^+} (\mathbf{a}(x,0,DP) - \overline{\mathbf{a}}(0,DP)) : D\varphi \, dx \\
&- g_2 \fint_{B_{2\rho}^+} \eta^{g_2-1} (\mathbf{a}(x,u,Du) - \mathbf{a}(x,u,DP)) : D\eta \otimes (u-P) \, dx \\
&=: -I_2 - I_3 - I_4. \qquad (3.3)
\end{aligned}
$$

Here, $\overline{\mathbf{a}}(0,DP) := (\mathbf{a}(\cdot,0,DP))_{B_{2\rho}}$ and we have used the fact that

$$
\fint_{B_{2\rho}^+} \overline{\mathbf{a}}(0,DP) : D\varphi \, dx = 0.
$$

For I_1 and I_2, we have from (1.5) that

$$
\fint_{B_{2\rho}^+} \eta^{g_2} \Big[G(1+|DP|) \frac{|Du-DP|^2}{(1+|DP|)^2} + G(|Du-DP|) \Big] \, dx \le cI_1, \qquad (3.4)
$$

and from (1.9) and (2.4) that

$$
\begin{aligned}
|I_2| &\le c \fint_{B_{2\rho}^+} \mu\left(|u|^2\right) G_1(1+|DP|) \Big(\eta^{g_2} \, |Du-DP| + \frac{|u-P|}{\rho} \Big) \, dx \\
&\le \frac{1}{4} \fint_{B_{2\rho}^+} \Big[\eta^{g_2} \, G(|Du-DP|) + G\big(\frac{|u-P|}{\rho}\big) \Big] \, dx \qquad (3.5) \\
&\quad + cG(1+|DP|) \fint_{B_{2\rho}^+} \mu(|u|^2) \, dx.
\end{aligned}
$$

We next estimate I_3. By (1.8), (1.7) and (2.4) with (2.3) and (2.6), we have

$$
\begin{aligned}
|I_3| &\le c \fint_{B_{2\rho}^+} V(x, B_{2\rho}^+) \, G_1(1+|DP|) \Big(\eta^{g_2} |Du-DP| + \frac{|u-P|}{\rho} \Big) \, dx \\
&\le \frac{1}{4} \fint_{B_{2\rho}^+} \eta^{g_2} G(|Du-DP|) \, dx \\
&\quad + c \fint_{B_{2\rho}^+} \Big[G^* \left(V(x, B_{2\rho}^+) \, G_1(1+|DP|) \right) + G\big(\frac{|u-P|}{\rho}\big) \Big] \, dx \qquad (3.6) \\
&\le \frac{1}{4} \fint_{B_{2\rho}^+} \eta^{g_2} G(|Du-DP|) \, dx \\
&\quad + c(2L+1)^{\frac{1}{g_1-1}} G(1+|DP|) \mathcal{V}(2\rho) + c \fint_{B_{2\rho}} G\big(\frac{|u-P|}{\rho}\big) \, dx.
\end{aligned}
$$

We estimate I_4. By the first inequality in (1.3), and Young's inequalities with (2.5) and (2.3), we have

$$
\begin{aligned}
|I_4| &\le c \fint_{B_{2\rho}^+} \eta^{g_2-1} \Big(\int_0^1 |\partial \mathbf{a}(x,u,tDu+(1-t)DP)| \, dt \Big) |Du-DP| \frac{|u-P|}{\rho} \, dx \\
&\le c \fint_{B_{2\rho}} \eta^{g_2-1} \frac{G(1+|DP|+|Du-DP|)}{(1+|DP|+|Du-DP|)^2} |Du-DP| \frac{|u-P|}{\rho} \, dx \\
&\le c \fint_{B_{2\rho}^+} \frac{G(1+|DP|)}{(1+|DP|)^2} \eta^{g_2-1} |Du-DP| \frac{|u-P|}{\rho} \, dx
\end{aligned}
$$

$$
\begin{aligned}
&+ c \fint_{B_{2\rho}^+} \eta^{g_2-1} \frac{G(|Du-DP|)}{|Du-DP|} \frac{|u-P|}{\rho}\, dx \\
&\le \frac14 \fint_{B_{2\rho}^+} \Big[\eta^{g_2} \frac{G(1+|DP|)}{(1+|DP|)^2} |Du-DP|^2 + \eta^{g_2} G(|Du-DP|)\Big]\, dx \\
&\quad + c\Big(\frac{G(1+|DP|)}{(1+|DP|)^2} \fint_{B_{2\rho}^+} \frac{|u-P|^2}{\rho^2}\, dx + \fint_{B_{2\rho}^+} G\big(\frac{|u-P|}{\rho}\big)\, dx\Big).
\end{aligned} \tag{3.7}
$$

Consequently, applying Jensen's inequality to μ in (3.5), inserting (3.4)-(3.7) into (3.3), and recalling $P(x) = \xi x_n$ and $DP = \xi \otimes e_n$, we get estimate (3.2). $\square$

From now on, we fix $x_0 \in T_r$, $0 < \rho < r - |x_0|$. For $\xi \in \mathbb{R}^N$, we define

$$
C(x_0,\rho,\xi) := \fint_{B_\rho^+(x_0)} \Big[\frac{|Du-\xi\otimes e_n|^2}{(1+|\xi|)^2} + \frac{G(|Du-\xi\otimes e_n|)}{G(1+|\xi|)}\Big]\, dx, \tag{3.8}
$$

$$
E^+(x_0,\rho,\xi) := C(x_0,\rho,\xi) + \Big[\mu\Big(\fint_{B_\rho^+(x_0)} |u|^2\, dx\Big)\Big]^{1/2} + [\mathcal{V}(\rho)]^{\frac{1}{2g_2-1}} \tag{3.9}
$$

$$
\mathcal{A} := \frac{\partial \mathbf{a}(x_0,0,\xi\otimes e_n)}{G_2(1+|\xi|)}, \quad w := \frac{u-\xi x_n}{(1+|\xi|)\sqrt{E^+(x_0,\rho,\xi)}}. \tag{3.10}
$$

Note that we easily check from (1.3) that $\mathcal{A}$ satisfies the Legendre-Hadamard condition (2.12). In the next lemma, we show that one can apply the harmonic approximation lemma to $\mathcal{A}$ and w if $E^+(x_0,\rho,\xi)$.

Lemma 3.2. *Under the assumption of Lemma 3.2 together with*

$$
C(x_0,\rho,\xi) \le 1, \tag{3.11}
$$

we have that for every $\varphi \in C_0^\infty(B_\rho^+(x_0))$,

$$
\Big|\fint_{B_\rho^+(x_0)} \mathcal{A} Dw : D\varphi\, dx\Big| \le c\Big[\mu\big(\sqrt{E^+(x_0,\rho,\xi)}\big) + E^+(x_0,\rho,\xi)\Big]^{1/2} \sup_{B_\rho^+(x_0)} |D\varphi| \tag{3.12}
$$

for some $c = c(n,N,L,\nu,g_1,g_2) > 0$.

The proof of this lemma is exactly same as the one of [48, Lemma 4.2] by replacing $B_\rho(x_0)$, $C(x_0,\rho,P)$ and $E^+(x_0,\rho,P)$ by $B_\rho^+(x_0)$, $C(x_0,\rho,\xi)$ and $E^+(x_0,\rho,\xi)$, respectively. Now, we choose

$$
\xi = (D_n u)_{x_0,\rho} := (D_n u)_{B_\rho^+(x_0)}
$$

and set

$$
\begin{aligned}
C(x_0,\rho) &:= C(x_0,\rho,(D_n u)_{x_0,\rho}) \\
&= \fint_{B_\rho^+(x_0)} \Big[\frac{|Du-(D_n u)_{x_0,\rho}\otimes e_n|^2}{(1+|(D_n u)_{x_0,\rho}|)^2} \\
&\quad + \frac{G(|Du-(D_n u)_{x_0,\rho}\otimes e_n|)}{G(1+|(D_n u)_{x_0,\rho}|)}\Big]\, dx,
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
\tilde{E}^+(x_0,\rho) &:= E^+(x_0,\rho,(D_n u)_{x_0,\rho}) \\
&= C(x_0,\rho) + \Big[\mu\Big(\fint_{B_\rho^+(x_0)} |u|^2\, dx\Big)\Big]^{1/2} + [\mathcal{V}(\rho)]^{\frac{1}{2g_2-1}},
\end{aligned} \tag{3.14}
$$

$$
E^+(x_0,\rho) := C(x_0,\rho) + \big[\mu\big(M(x_0,\rho)\big)\big]^{1/2} + [\mathcal{V}(\rho)]^{\frac{1}{2g_2-1}}, \tag{3.15}
$$

where

$$M(x_0,\rho) := \rho \fint_{B_\rho^+(x_0)} |D_n u|^2\,dx. \tag{3.16}$$

Then, by Poincaré's inequality (2.9) along with the fact that $\rho<1$, we see that

$$\tilde{E}^+(x_0,\rho) \le cE^+(x_0,\rho) \tag{3.17}$$

for some $c=c(n,N)\ge 1$.

Lemma 3.3. *For $\theta\in(0,1/8)$, there exists small*

$$\epsilon_1=\epsilon_1(n,N,L,\nu,g_1,g_2,\mu(\cdot),\theta)\in(0,1)$$

such that if

$$\rho\le\theta^n \quad \text{and} \quad E^+(x_0,\rho)\le\epsilon_1, \tag{3.18}$$

then

$$C(x_0,\theta\rho)\le c_1\theta^2E^+(x_0,\rho) \tag{3.19}$$

for some $c_1=c_1(n,N,L,\nu,g_1,g_2)\ge 1$.

Proof. We omit x_0 in our notation for simplicity.

Step 1. We first estimate the integrals

$$\fint_{B_{2\theta\rho}^+} \frac{|u(x)-P_{2\theta\rho}^+|^2}{(2\theta\rho)^2}\,dx \quad \text{and} \quad \fint_{B_{2\theta\rho}^+} G\Big(\frac{|u-P_{2\theta\rho}^+|}{2\theta\rho}\Big)\,dx, \tag{3.20}$$

where the affine function $P_{2\theta\rho}^+=P_{x_0,2\theta\rho}^+$ is given in Section 2.2. Recall $\mathcal{A}$ and w from (3.10) with $\xi=(D_nu)_\rho$. Then we see that

$$w:=\frac{u-(D_nu)_\rho x_n}{(1+|(D_nu)_\rho|)\sqrt{\tilde{E}^+(x_0,\rho)}} \quad \text{and} \quad \fint_{B_\rho^+}|Dw|^2\,dx\le 1.$$

Let us take $\epsilon\in(0,1)$ such that $\epsilon=\theta^{n+4}$, for which we consider $\delta=\delta(n,N,L,\nu,\epsilon)>0$ as determined in Lemma 2.7. Then by Lemma 3.2 together with (3.18), we have

$$\Big|\fint_{B_\rho^+}\mathcal{A}Dw:D\varphi\,dx\Big|\le\delta\sup_{B_\rho^+}|D\varphi|$$

by taking sufficiently small $\epsilon_1=\epsilon_1(n,N,L,\nu,g_1,g_2,\mu(\cdot),\theta)\in(0,1)$. Therefore, in view of Lemma 2.7, there exists an $\mathcal{A}$-harmonic map h such that

$$\fint_{B_\rho^+}|Dh|^2\,dx\le 1 \quad \text{and} \quad \fint_{B_\rho^+}|w-h|^2\,dx\le\theta^{n+4}\rho^2. \tag{3.21}$$

We notice by a basic regularity theory for $\mathcal{A}$-harmonic maps, see for instance [28, Theorem 2.3], that

$$\rho^{-2}\sup_{B_{\rho/2}^+}|Dh|^2+\sup_{B_{\rho/2}^+}|D^2h|\le c\rho^{-2}\fint_{B_\rho^+}|Dh|^2\,dx\le c\rho^{-2}.$$

Moreover, the Taylor expansion of h and the fact that $h=0$ on $T_\rho(x_0)$ imply that for $\theta\in(0,1/4)$,

$$\begin{aligned}\sup_{x\in B_{2\theta\rho}^+}|h(x)-D_nh(x_0)x_n|^2&=\sup_{x\in B_{2\theta\rho}^+}|h(x)-h(x_0)-Dh(x_0)(x-x_0)|^2\\&\le c(2\theta\rho)^4\sup_{B_{2\theta\rho}^+}|D^2h|^2\end{aligned}$$

$$\le c\theta^4\rho^2.$$

This and the second inequality in (3.21) imply that

$$\fint_{B_{2\theta\rho}} \frac{|w - D_n h(x_0)x_n|^2}{(2\theta\rho)^2}\, dx \le c\theta^2,$$

hence, by the definitions of the affine function $P^+_{2\theta\rho} := P^+_{x_0,2\theta\rho}$ and w and (3.17), we obtain

$$\begin{aligned}\fint_{B_{2\theta\rho}} \frac{|u - P_{2\theta\rho}|^2}{(2\theta\rho)^2}\, dx &\le (1 + |(D_n u)_\rho|)^2 \tilde{E}^+(x_0,\rho) \fint_{B_{2\theta\rho}} \frac{|w - D_n h(x_0)x|^2}{(2\theta\rho)^2}\, dx \\ &\le c\theta^2 (1 + |(D_n u)_\rho|)^2 E^+(x_0,\rho).\end{aligned} \tag{3.22}$$

Next we estimate the second integral in (3.20). Let $t \in (0,1)$ be a number satisfying

$$\frac{1}{g_2} = (1-t) + \frac{t}{g_2 d_2},$$

where $d_2 > 1$ is given in Lemma 2.3. Then by applying Hölder's inequality, Jensen's inequality to the concave map $\tilde{\Psi}$ with $\frac{1}{2}\tilde{\Psi}(t) \le \Psi(t) := [G(t^{1/2})]^{1/g_2} \le \tilde{\Psi}(t)$ (see Lemma 2.2), (3.22), (2.8) and (2.2), we have

$$\begin{aligned}
&\fint_{B^+_{2\theta\rho}} G\Big(\frac{|u - P^+_{2\theta\rho}|}{2\theta\rho}\Big)\, dx \\
&\le \Big(\fint_{B^+_{2\theta\rho}} \tilde{\Psi}\Big(\frac{|u - P^+_{2\theta\rho}|^2}{(2\theta\rho)^2}\Big)\, dx\Big)^{(1-t)g_2} \Big(\fint_{B^+_{2\theta\rho}} \Big[G\Big(\frac{|u - P^+_{2\theta\rho}|}{2\theta\rho}\Big)\Big]^{d_2}\, dx\Big)^{t/d_2} \\
&\le c\left[\tilde{\Psi}\left(\theta^2(1 + |(D_n u)_\rho|)^2 E^+(x_0,\rho)\right)\right]^{(1-t)g_2} \Big(\fint_{B_{2\theta\rho}} G(|Du - DP^+_{2\theta\rho}|)\, dx\Big)^t \\
&\le c\Big[G\Big(\theta(1 + |(D_n u)_\rho|)\sqrt{E^+(x_0,\rho)}\Big)\Big]^{1-t} \Big(\fint_{B_{2\theta\rho}} G(|Du - DP^+_{2\theta\rho}|)\, dx\Big)^t \\
&\le c[\theta\sqrt{E^+(x_0,\rho)}]^{g_1(1-t)} [G(1 + |(D_n u)_\rho|)]^{1-t} \Big(\fint_{B_{2\theta\rho}} G(|Du - DP^+_{2\theta\rho}|)\, dx\Big)^t.
\end{aligned}$$

In addition, keeping in mind that $P^+_{x_0,r} = Q^+_{x_0,r}x_n$, from (2.10), (2.2), (2.8), (2.11) and the definition of E we have

$$\begin{aligned}
&\fint_{B^+_{2\theta\rho}} G(|Du - DP^+_{2\theta\rho}|)\, dx \\
&\le c\fint_{B^+_{2\theta\rho}} G(|Du - (D_n u)_\rho \otimes e_n|)\, dx + cG(|(D_n u)_\rho \otimes e_n - DP^+_{2\theta\rho}|) \\
&\le c\theta^{-n} \fint_{B^+_\rho} G(|Du - (D_n u)_\rho \otimes e_n|)\, dx + cG(|Q^+_{2\theta\rho} - (D_n u)_\rho|) \\
&\le c\theta^{-n} \fint_{B^+_\rho} G(|Du - (D_n u)_\rho \otimes e_n|)\, dx + c\fint_{B^+_{2\theta\rho}} G(|D_n u - (D_n u)_\rho|)\, dx \\
&\le c\theta^{-n} \fint_{B^+_\rho} G(|Du - (D_n u)_\rho \otimes e_n|)\, dx \\
&\le c\theta^{-n} G(1 + |(D_n u)_\rho|) E^+(x_0,\rho).
\end{aligned}$$

Combining the two above estimates, we obtain

$$\begin{aligned}&\fint_{B_{2\theta\rho}} G\Big(\frac{|u-P_{2\theta\rho}|}{2\theta\rho}\Big)\,dx\\&\le c\theta^{g_1-(n+g_1)t}G(1+|(D_nu)_\rho|)[E^+(x_0,\rho)]^{(\frac{g_1}{2}-1)(1-t)+1}.\end{aligned}$$

Therefore, taking $\epsilon_1>0$ sufficiently small so that

$$E^+(x_0,\rho)^{(\frac{g_1}{2}-1)(1-t)}\le \epsilon_1^{(\frac{g_1}{2}-1)(1-t)}\le \theta^{-g_1+(n+g_1)t+2},$$

we obtain

$$\fint_{B^+_{2\theta\rho}} G\Big(\frac{|u-P^+_{2\theta\rho}|}{2\theta\rho}\Big)\,dx\le c\theta^2G(1+|(D_nu)_\rho|)E^+(x_0,\rho). \tag{3.23}$$

Moreover, by a further assuming that

$$\sqrt{E^+(x_0,\rho)}\le\sqrt{\epsilon_1}\le\frac{\theta^n}{8},$$

we have

$$1+|(D_nu)_\rho|\le 2(1+|(D_nu)_{\theta\rho}|),\quad 1+|(D_nu)_{2\theta\rho}|\le 2(1+|(D_nu)_{\theta\rho}|). \tag{3.24}$$

Indeed,

$$\begin{aligned}1+|(D_nu)_\rho|&\le 1+|(D_nu)_{\theta\rho}|+|(D_nu)_{\theta\rho}-(D_nu)_\rho|\\&\le 1+|(D_nu)_{\theta\rho}|+\theta^{-n}\sqrt{E^+(x_0,\rho)}(1+|(D_nu)_\rho|)\\&\le 1+|(D_nu)_{\theta\rho}|+\frac18(1+|(D_nu)_\rho|),\end{aligned}$$

which implies the first inequality in (3.24). Similarly, using the first inequality in (3.24) with θ replaced by 2θ, the second inequality in (3.24) can be obtained such that

$$\begin{aligned}1+|(D_nu)_{2\theta\rho}|&\le 1+|(D_nu)_{\theta\rho}|+|(D_nu)_{\theta\rho}-(D_nu)_\rho|+|(D_nu)_{2\theta\rho}-(D_nu)_\rho|\\&\le 1+|(D_nu)_{\theta\rho}|+(\theta^{-n}+(2\theta)^{-n})\sqrt{E^+(x_0,\rho)}(1+|(D_nu)_\rho|)\\&\le 1+|(D_nu)_{\theta\rho}|+\frac12(1+|(D_nu)_{2\theta\rho}|).\end{aligned}$$

Therefore, inserting the first inequality in (3.24) into (3.22) and (3.23), we obtain

$$\fint_{B^+_{2\theta\rho}}\frac{|u-P^+_{2\theta\rho}|^2}{(2\theta\rho)^2}\,dx\le c\theta^2(1+|(D_nu)_{\theta\rho}|)^2E^+(x_0,\rho), \tag{3.25}$$

$$\fint_{B^+_{2\theta\rho}} G\Big(\frac{|u-P^+_{2\theta\rho}|}{2\theta\rho}\Big)\,dx\le c\theta^2G(1+|(D_nu)_{\theta\rho}|)E^+(x_0,\rho). \tag{3.26}$$

Step 2. Now we prove (3.19). Suppose that

$$E^+(x_0,\rho)\le\epsilon_1\le\theta^n. \tag{3.27}$$

Then, in view of Lemma 3.1 with ρ replaced by $\theta\rho$ and $\xi=Q^+_{2\theta\rho}$, we have

$$\fint_{B_{\theta\rho}} G_2(1+|Q^+_{2\theta\rho}|)|Du-Q^+_{2\theta\rho}\otimes e_n|^2\,dx+\fint_{B_{\theta\rho}} G(|Du-Q^+_{2\theta\rho}\otimes e_n|)\,dx$$

$$
\begin{aligned}
&\le cG_2(1+|Q^+_{2\theta\rho}|)\fint_{B_{2\theta\rho}}\frac{|u-P^+_{2\theta\rho}|^2}{(2\theta\rho)^2}\,dx + c\fint_{B_{2\theta\rho}} G\Big(\frac{|u-P^+_{2\theta\rho}|}{2\theta\rho}\Big)\,dx\\
&\quad + cG(1+|Q^+_{2\theta\rho}|)\Big\{\mu\Big(\fint_{B^+_{2\theta\rho}}|u|^2\,dx\Big)+\mathcal{V}(2\theta\rho)\Big\}.
\end{aligned}
$$

Here, we note that $\tilde G(t) := G(t^{1/2})$ is also an N-function and satisfies (2.1) with g_1 and g_2 replaced by $\frac{g_1}{2}$ and $\frac{g_2}{2}$, which are larger than 1. Therefore, in view of (3) and (4) of Proposition 2.1 with $G(t)=\tilde G(t)$, we have $G_2(t)\tau^2 \le c(G(t)+G(\tau))$. From this, (2.7) and Lemma 2.5 with (ρ,θ) replaced by $(\theta\rho,1/2)$, we have

$$
\begin{aligned}
&G_2(1+|(D_nu)_{\theta\rho}|)\fint_{B_{\theta\rho}}|Du-(D_nu)_{\theta\rho}\otimes e_n|^2\,dx\\
&\le cG_2(1+|Q^+_{2\theta\rho}|)\fint_{B_{\theta\rho}}|Du-(D_nu)_{\theta\rho}\otimes e_n|^2\,dx\\
&\quad + cG_2(|(D_nu)_{\theta\rho}-Q^+_{\theta\rho}|)\fint_{B_{\theta\rho}}|Du-(D_nu)_{\theta\rho}\otimes e_n|^2\,dx\\
&\quad + cG_2(|Q^+_{\theta\rho}-Q^+_{2\theta\rho}|)\fint_{B_{\theta\rho}}|Du-(D_nu)_{\theta\rho}\otimes e_n|^2\,dx\\
&\le cG_2(1+|Q^+_{2\theta\rho}|)\fint_{B_{\theta\rho}}|Du-Q^+_{2\theta\rho}\otimes e_n|^2\,dx + c\fint_{B_{\theta\rho}} G(|Du-(D_nu)_{\theta\rho}\otimes e_n|)\,dx\\
&\quad + cG(|(D_nu)_{\theta\rho}-Q^+_{\theta\rho}|)+cG(|Q^+_{\theta\rho}-Q^+_{2\theta\rho}|)\\
&\le cG_2(1+|Q^+_{2\theta\rho}|)\fint_{B_{\theta\rho}}|Du-Q^+_{2\theta\rho}\otimes e_n|^2\,dx + c\fint_{B_{\theta\rho}} G(|Du-(D_nu)_{\theta\rho}\otimes e_n|)\,dx\\
&\quad + c\fint_{B_{2\theta\rho}} G\Big(\frac{|u-P^+_{2\theta\rho}|}{2\theta\rho}\Big)\,dx.
\end{aligned}
$$

Using the above two estimates along with (2.7), we obtain

$$
\begin{aligned}
&G(1+|(D_nu)_{\theta\rho}|)\,C(x_0,\theta\rho)\\
&= G_2(1+|(D_nu)_{\theta\rho}|)\fint_{B_{\theta\rho}}|Du-(D_nu)_{\theta\rho}\otimes e_n|^2\,dx\\
&\quad + \fint_{B_{\theta\rho}} G(|Du-(D_nu)_{\theta\rho}\otimes e_n|)\,dx\\
&\le cG_2(1+|(D_nu)_{2\theta\rho}|)\ mint_{B_{2\theta\rho}}\frac{|u-P^+_{2\theta\rho}|^2}{(2\theta\rho)^2}\,dx + c\fint_{B_{2\theta\rho}} G\Big(\frac{|u-P^+_{2\theta\rho}|}{2\theta\rho}\Big)\,dx\\
&\quad + cG(1+|Q^+_{2\theta\rho}|)\Big\{\mu\Big(\fint_{B^+_{2\theta\rho}}|u|^2\,dx\Big)+\mathcal{V}(2\theta\rho)\Big\}.
\end{aligned}
\tag{3.28}
$$

We further estimate the right-hand side of the above inequality. Applying (2.11), (3.27) and (3.24), we see that

$$
\begin{aligned}
G(|Q^+_{2\theta\rho}|) &\le cG(|Q^+_{2\theta\rho}-(D_nu)_{2\theta\rho}|)+cG(|(D_nu)_{2\theta\rho}|)\\
&\le c\theta^{-n}\fint_{B^+_\rho} G(|D_nu-(D_nu)_\rho|)\,dx + cG(|(D_nu)_{2\theta\rho}|)
\end{aligned}
$$

$$
\begin{aligned}
&= c\theta^{-n} \fint_{B_\rho^+} G(|Du - (D_n u)_\rho \otimes e_n|)\,dx + cG(|(D_n u)_{2\theta\rho}|) \\
&\le c(\theta^{-n} E^+(x_0,\rho) + 1) G(1 + |(D_n u)_{\theta\rho}|) \\
&\le cG(1 + |(D_n u)_{\theta\rho}|).
\end{aligned}
$$

Moreover, using Poincaré's inequality and the fact that $\rho \le \theta^n$, we have

$$
\fint_{B_{2\theta\rho}^+} |u|^2\,dx \le c\theta^{-n} \fint_{B_\rho^+} |u|^2\,dx \le c\rho \fint_{B_\rho^+} |D_n u|^2\,dx.
$$

Therefore, inserting the previous two inequalities, (3.25) and (3.26) into (3.28), we obtain

$$
C(x_0, \theta\rho) \le c\theta^2 E^+(x_0,\rho) + c[E^+(x_0,\rho)]^2.
$$

Finally, assuming $E^+(x_0,\rho) \le \epsilon_1 \le \theta^2$, we prove (3.19). $\square$

4. Proof of Theorem 1.1

We first consider elliptic systems in the unit half ball with the zero boundary datum.

Theorem 4.1. *Suppose $G : [0,\infty) \to [0,\infty)$ is C^2 and satisfies (1.1), $0 < r < 1$, and $\mathbf{a} : B_r \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N$ satisfies (1.3), (1.7), (1.9) and (1.10). Let $u \in W^{1,G}(B_r, \mathbb{R}^N)$ be a weak solution to*

$$
\begin{gathered}
\operatorname{div} \mathbf{a}(x,u,Du) = 0 \quad \text{in } B_r^+, \\
u = 0 \quad \text{on } T := T_r.
\end{gathered} \tag{4.1}
$$

Then the set of regular points on T denoted by

$$
T_u := \cap_{\alpha\in(0,1)} \{x_0 \in T : u \in C^\alpha(U_{x_0} \cap \overline{B_r^+}, \mathbb{R}^N) \text{ for some } U_{x_0} \subset B_r\},
$$

where U_{x_0} is an open neighborhood of x_0, satisfies

$$
\begin{aligned}
T \setminus T_u \subset &\Big\{x_0 \in T : \liminf_{\rho\downarrow 0} \fint_{B_\rho^+(x_0)} |Du - (D_n u)_{x_0,\rho} \otimes e_n|\,dx > 0\Big\} \\
&\cup \Big\{x_0 \in T : \limsup_{\rho\downarrow 0} \fint_{B_\rho^+(x_0)} G(|D_n u|)\,dx = \infty\Big\}.
\end{aligned}
$$

Proof. **Step 1: Determination of parameters.** Fix any $\alpha \in (0,1)$, and denote

$$
\lambda := n - 2(1-\alpha) \in (n-2, n). \tag{4.2}
$$

We then determine parameters θ and ϵ_2 such that

$$
\theta = \theta(n,N,L,\nu,g_1,g_2,\alpha) := \min\Big\{\frac{1}{8}, \frac{1}{\sqrt{2c_1}}, \frac{1}{3^{1/(n-\lambda)}}\Big\}, \tag{4.3}
$$

$$
\epsilon_2 = \epsilon_2(n,N,L,\nu,g_1,g_2,\mu(\cdot),\alpha) := \min\Big\{\frac{\theta^n}{16}, \frac{\epsilon_1}{2}\Big\}, \tag{4.4}
$$

where c_1 and ϵ_1 are determined in Lemma 3.3. Furthermore, by the definitions of $\mu(\cdot)$ and $\mathcal{V}(\cdot)$, one can find $\delta_1 = \delta_1(n,N,L,\nu,g_1,g_2,\mu(\cdot),\mathcal{V}(\cdot),\alpha) > 0$ such that

$$
[\mu(r)]^{1/2} + [\mathcal{V}(r)]^{\frac{1}{2g_2-1}} \le \epsilon_2 \quad \text{for every } r \in (0,\delta_1]. \tag{4.5}
$$

We denote

$$
\rho_1 := \min\{\theta^n, \delta_1\} < 1. \tag{4.6}
$$

Step 2: Decay estimates on the boundary. Let $x_0 \in T$ and $\rho \le \min\{\rho_1, r - |x_0|\}$. Without loss of generality, we shall write for $t > 0$, $B_t^+ = B_t^+(x_0)$ and $(D_n u)_t = (D_n u)_{B_t^+(x_0)}$. We then suppose that

$$C(x_0,\rho) \le \epsilon_2 \quad \text{and} \quad M(x_0,\rho) \le \delta_1, \tag{4.7}$$

see (3.13) and (3.16) for the definitions of $C(x_0,\rho)$ and $M(x_0,\rho)$. Under this condition, we will show that for any $k = 0, 1, 2, \dots,$

$$C(x_0,\theta^k\rho) \le \epsilon_2 \quad \text{and} \quad M(x_0,\theta^k\rho) \le \delta_1. \tag{4.8}$$

For convenience, we write $(4.8)_{k,1}$ (resp. $(4.8)_{k,2}$) for the first (resp. second) inequality in (4.8).

We prove (4.8) by induction. Suppose that the inequalities in (4.8) hold for k, we then prove (4.8) with k replacing by $k+1$. We first observe from $(4.8)_{k,1}$ and Hölder's inequality that

$$\begin{aligned} \fint_{B^+_{\theta^k\rho}} |Du - (D_n u)_{\theta^k\rho} \otimes e_n|^2\,dx &\le (1 + |(D_n u)_{\theta^k\rho}|)^2 C(x_0,\theta^k\rho) \\ &\le 2\epsilon_2\Big(1 + \fint_{B^+_{\theta^k\rho}} |D_n u|^2\,dx\Big), \end{aligned} \tag{4.9}$$

and so by $(4.8)_{k,2}$,

$$\theta^k\rho \fint_{B^+_{\theta^k\rho}} |Du - (D_n u)_{\theta^k\rho} \otimes e_n|^2\,dx \le 2\epsilon_2\theta^k\rho + 2\epsilon_2\delta_1.$$

This together with (4.3), (4.4) and (4.6) imply

$$\begin{aligned} M(x_0,\theta^{k+1}\rho) &\le 2\theta^{k+1}\rho \fint_{B^+_{\theta^{k+1}\rho}} |Du - (D_n u)_{\theta^k\rho} \otimes e_n|^2\,dx + 2\theta^{k+1}\rho|(D_n u)_{\theta^k\rho}|^2 \\ &\le 2\theta^{1-n}\theta^k\rho \fint_{B_{\theta^k\rho}} |Du - (D_n u)_{\theta^k\rho} \otimes e_n|^2\,dx + 2\theta M(x_0,\theta^k\rho) \\ &\le 4\theta^{k+1-n}\epsilon_2\rho + 4\theta^{1-n}\epsilon_2\delta_1 + 2\theta\delta_1 \\ &\le 4\theta^{-n}\epsilon_2\rho + 4\theta^{-n}\epsilon_2\delta_1 + 2\theta\delta_1 \\ &\le \delta_1, \end{aligned}$$

which shows $(4.8)_{k+1,2}$. It remains to prove $(4.8)_{k+1,1}$. We notice from (3.15), $(4.8)_{k,1}$, $(4.8)_{k,2}$, the fact that $\theta^k\rho \le \rho_1 \le \delta_1$ by (4.6), and (4.5) that

$$E^+(x_0,\theta^k\rho) \le \epsilon_2 + [\mu(\delta_1)]^{1/2} + [\mathcal{V}(\delta_1)]^{\frac{1}{2g_2-1}} \le 2\epsilon_2.$$

Therefore, applying Lemma 3.3 and (4.3), we have

$$C(x_0,\theta^{k+1}\rho) \le 2c_1\theta^2\epsilon_2 \le \epsilon_2.$$

This shows $(4.8)_{k+1,1}$. Then, by induction, we prove that (4.8) holds for all $k = 0, 1, 2, \dots$.

From the previous result, we also see that (4.9) holds for all $k = 0, 1, 2, \dots$, which together with (4.4) and (4.9) implies

$$\fint_{B^+_{\theta^{k+1}\rho}} |Du|^2\,dx \le 2\fint_{B_{\theta^{k+1}\rho}} |Du - (D_n u)_{\theta^k\rho} \otimes e_n|^2\,dx + 2|(D_n u)_{\theta^k\rho}|^2$$

$$
\begin{aligned}
&\le 2\theta^{-n} \fint_{B^+_{\theta^k\rho}} |Du-(D_nu)_{\theta^k\rho}|^2\,dx + 2\fint_{B^+_{\theta^k\rho}} |D_nu|^2\,dx\\
&\le 4\theta^{-n}\epsilon_2 + (4\theta^{-n}\epsilon_2+2)\fint_{B^+_{\theta^k\rho}} |D_nu|^2\,dx\\
&\le 4\theta^{-n}\epsilon_2 + 3\fint_{B^+_{\theta^k\rho}} |Du|^2\,dx,
\end{aligned}
$$

and so by (4.3),

$$
\int_{B^+_{\theta^{k+1}\rho}} |Du|^2\,dx \le \theta^\lambda \int_{B^+_{\theta^k\rho}} |Du|^2\,dx + 2|B_1|(\theta^k\rho)^n.
$$

Applying Lemma 2.6 with $\phi(r)=\int_{B^+_r(x_0)}|Du|^2\,dx$, we have for every $r\in(0,\rho]$,

$$
\begin{aligned}
\int_{B^+_r(x_0)} |Du|^2\,dx &\le c\Big\{\big(\frac{r}{\rho}\big)^\lambda \int_{B^+_\rho(x_0)} |Du|^2\,dx + r^\lambda\Big\}\\
&\le \frac{c}{\rho^\lambda}\Big(\int_{B^+_\rho(x_0)} |Du|^2\,dx + 1\Big)r^\lambda.
\end{aligned}
\tag{4.10}
$$

Step 3: Choice of regular points on the boundary. In the last step, we have shown that if (4.7) holds then we have (4.10). Hence, in this step, we find boundary points satisfying (4.7). Suppose that $x_0\in T$ satisfies

$$
\begin{gathered}
\liminf_{\rho\downarrow 0}\fint_{B^+_\rho(x_0)} |Du-(D_nu)_{B^+_\rho(x_0)}\otimes e_n|\,dx = 0,\\
m_{x_0} := \limsup_{\rho\downarrow 0}\fint_{B^+_\rho(x_0)} G(|D_nu|)\,dx < \infty.
\end{gathered}
\tag{4.11}
$$

Then we show that $x_0\in T_u$. For simplicity, we omit writing x_0, for instance, $B^+_\rho = B^+_\rho(x_0)$ and $(D_nu)_\rho = (D_nu)_{B^+_\rho(x_0)}$.

Fix $\alpha\in(0,1)$, and set $t\in(0,1)$ such that

$$
\frac{1}{g_2} = t + \frac{(1-t)}{g_2(1+\sigma_1)},
\tag{4.12}
$$

where σ_1 is determined in Lemma 2.9. We further define

$$
s := \min\Big\{G^{-1}\Big(\Big[G\big((\frac{\epsilon_2}{2})^{1/2}\big)\frac{(m_{x_0}+2)^{t-1}}{c_2}\Big]^{\frac{1}{t}}\Big), \delta_1\Big\} < 1,
\tag{4.13}
$$

where $c_2 = c_2(n,N,L,\nu,g_1,g_2) > 0$ will be determined later. Then, in view of (4.11), one can find $\tilde\rho > 0$ with

$$
\tilde\rho_0 \le \min\Big\{\rho_1, \frac{1-|x_0|}{4}, \big[\frac{4^n(m_{x_0}+1)}{G(1)}+1\big]^{-1}\delta_1\Big\}
\tag{4.14}
$$

such that

$$
\fint_{B^+_{\tilde\rho}} |Du-(D_nu)_{\tilde\rho}\otimes e_n|\,dx < s \quad\text{and}\quad \fint_{B^+_{4\tilde\rho}} G(|D_nu|)\,dx < m_{x_0}+1.
\tag{4.15}
$$

We first observe from Hölder's inequality with (4.12) that

$$
\fint_{B^+_{\tilde\rho}} G(|Du-(D_nu)_{\tilde\rho}\otimes e_n|)\,dx \le \Big(\fint_{B^+_{\tilde\rho}} [G(|Du-(D_nu)_{\tilde\rho}\otimes e_n|)]^{\frac{1}{g_2}}\,dx\Big)^{tg_2}
$$

$$\times \Big(\fint_{B^+_{\tilde\rho}} [G(|Du - (D_n u)_{\tilde\rho} \otimes e_n|)]^{1+\sigma_1}\, dx \Big)^{\frac{1-t}{1+\sigma_1}}.$$

Then by applying Jensen's inequality to the concave function $\tilde\Psi$ with

$$\frac{1}{2}\tilde\Psi(t) \le \Psi(t) := [G(t)]^{1/g_2} \le \tilde\Psi(t)$$

(see Lemma 2.2) and (4.15), we have

$$\begin{aligned} \fint_{B^+_{\tilde\rho}} [G(|Du - (D_n u)_{\tilde\rho} \otimes e_n|)]^{\frac{1}{g_2}}\, dx &\le \tilde\Psi\Big(\fint_{B^+_{\tilde\rho}} |Du - (D_n u)_{\tilde\rho} \otimes e_n|\, dx \Big) \\ &< 2[G(s)]^{\frac{1}{g_2}}. \end{aligned}$$

On the other hand, applying Jensen's inequality to the convex map $t \mapsto [G(t)]^{1+\sigma_1}$, (2.16), (4.15) and (2.15), we have

$$\begin{aligned} \fint_{B^+_{\tilde\rho}} [G(|Du - (D_n u)_{\tilde\rho} \otimes e_n|)]^{1+\sigma_1}\, dx &\le c \fint_{B^+_{\tilde\rho}} [G(|Du|)]^{1+\sigma_1}\, dx \\ &\le c\Big(\fint_{B^+_{2\tilde\rho}} [G(|Du|) + 1]\, dx \Big)^{1+\sigma_1} \\ &\le c\Big(\fint_{B^+_{4\tilde\rho}} [G(|D_n u|) + 1]\, dx \Big)^{1+\sigma_1} \\ &\le c(m_{x_0} + 2)^{1+\sigma_1}. \end{aligned}$$

Therefore,

$$\fint_{B^+_{\tilde\rho}} G(|Du - (D_n u)_{\tilde\rho} \otimes e_n|)\, dx < c_2 [G(s)]^t (m_{x_0} + 2)^{1-t}$$

for some $c_2 = c_2(n, N, L, \nu, g_1, g_2) > 0$, and so by (4.13),

$$\fint_{B^+_{\tilde\rho}} G(|Du - (D_n u)_{\tilde\rho} \otimes e_n|)\, dx < G\Big(\Big(\frac{\epsilon_2}{2}\Big)^{1/2} \Big),$$

from which together with Jensen's inequality for the convex map $t \mapsto G(\sqrt{t})$, we have

$$\begin{aligned} C(x_0, \tilde\rho) &\le \Big[G^{-1}\Big(\fint_{B^+_{\tilde\rho}} G(|Du - (Du)_{\tilde\rho} \otimes e_n|)\, dx \Big) \Big]^2 \\ &\quad + [G(1)]^{-1} \fint_{B^+_{\tilde\rho}} G(|Du - (Du)_{\tilde\rho} \otimes e_n|)\, dx \\ &\le \frac{\epsilon_2}{2} + [G(1)]^{-1} G\Big(\Big(\frac{\epsilon_2}{2}\Big)^{1/2} \Big) \\ &\le \frac{\epsilon_2}{2} + \Big(\frac{\epsilon_2}{2}\Big)^{g_1/2} \le \epsilon_2. \end{aligned}$$

Moreover, by (4.14) and (4.15), we see that

$$M(x_0, \tilde\rho) = \tilde\rho \fint_{B^+_{\tilde\rho}} |D_n u|^2\, dx \le \tilde\rho \Big(\frac{4^n}{G(1)} \fint_{B^+_{4\tilde\rho}} G(|D_n u|)\, dx + 1 \Big) < \delta_1.$$

Therefore, in view of (4.8), we obtain

$$C(x_0,\theta^k\tilde\rho) \le \frac{\theta^n\epsilon_2}{3^{2g_2+n}}, \quad M(x_0,\theta^k\tilde\rho) \le \frac{\theta^n\delta_1}{3^n} \quad \text{for } k=0,1,2,\dots. \tag{4.16}$$

Step 4: Morrey-Campanato estimates. We first consider any two sets

$$B^+_{\rho_2}(x_2) \subset B^+_{\rho_1}(x_1) \quad \text{with } C(x_1,\rho_1) \le \frac{\theta^n}{2} \text{ and } \frac{\rho_2}{\rho_1} \in [\theta,1]. \tag{4.17}$$

Then

$$\begin{aligned}
\frac{1+|(D_nu)_{B^+_{\rho_1}(x_1)}|}{1+|(D_nu)_{B^+_{\rho_2}(x_2)}|} &\le \fint_{B^+_{\rho_2}(x_2)} \frac{|D_nu-(D_nu)_{B^+_{\rho_1}(x_1)}|}{1+|(D_nu)_{B^+_{\rho_2}(x_2)}|}\,dx + 1\\
&\le \Big(\frac{\rho_1}{\rho_2}\Big)^n C(x_1,\rho_1)\frac{1+|(D_nu)_{B^+_{\rho_2}(x_2)}|}{1+|(D_nu)_{B^+_{\rho_2}(x_2)}|} + 1\\
&\le \frac{1+|(D_nu)_{B^+_{\rho_2}(x_2)}|}{2(1+|(D_nu)_{B^+_{\rho_2}(x_2)}|)} + 1,
\end{aligned} \tag{4.18}$$

which yields

$$1+|(D_nu)_{B^+_{\rho_1}(x_1)}| \le 2(1+|(D_nu)_{B^+_{\rho_2}(x_2)}|). \tag{4.19}$$

Then, for $B_\rho(x_0)$ with $\rho\in(0,\tilde\rho]$, since $\theta^{k+1}\tilde\rho < \rho \le \theta^k\tilde\rho$ for some k, we see that (4.17) holds for $x_1=x_2=x_0$, $\rho_1=\theta^k\tilde\rho$ and $\rho_2=\rho$. Hence using (4.19) and (4.16), we have

$$\begin{aligned}
&C(x_0,\rho)\\
&\le \fint_{B^+_\rho}\Big[\frac{|Du-(D_nu)_{\theta^k\tilde\rho}\otimes e_n|^2}{(1+|(D_nu)_\rho|)^2} + \frac{G(|Du-(D_nu)_{\theta^k\tilde\rho}\otimes e_n|)}{G(1+|(D_nu)_\rho|)}\Big]\,dx\\
&\le \frac{2^{g_2}}{\theta^n}\fint_{B^+_{\theta^k\tilde\rho}}\Big[\frac{|Du-(D_nu)_{\theta^k\tilde\rho}\otimes e_n|^2}{(1+|(D_nu)_{\theta^k\tilde\rho}|)^2} + \frac{G(|Du-(D_nu)_{\theta^k\tilde\rho}\otimes e_n|)}{G(1+|(D_nu)_{\theta^k\tilde\rho}|)}\Big]\,dx\\
&\le \frac{2^{g_2}}{\theta^n}C(x_0,\theta^k\tilde\rho)\\
&\le 2^{-(g_2+n)}\epsilon_2.
\end{aligned} \tag{4.20}$$

In addition, we also have from (4.16) that

$$M(x_0,\rho) = \theta^{-n}M(x_0,\theta^k\tilde\rho) \le 3^{-n}\delta_1. \tag{4.21}$$

Now we derive Campanato-Morrey type estimates. without loss of generality, we suppose that $x_0=0$. Define $\rho_0 := \tilde\rho/6$. We then consider balls $B^+_r(y)$ with $y=(y)$ which satisfy one of following:

(i) $y\in T_{2\rho_0}$ and $0<r<4\rho_0$.
(ii) $y\in B^+_{2\rho_0}$ and $B^+_r(y)\subset B^+_{2\rho_0}$.

Case (i): Since $B^+_r(y)\subset B^+_{4\rho_0}(y)\subset B^+_{6r_0}$, $(2r_0)/(6\rho_0)=1/3$ and $C(0,6\rho_0)\le \epsilon_2\le 3^{-(n+1)}$, using the same argument as in (4.18), we see that

$$1+|(D_nu)_{y,2\rho_0}| \le 2\left(1+|(D_nu)_{0,6\rho_0}|\right),$$

which by the same way as in (4.20) yields

$$C(y,2\rho_0) \le 3^{g_2+n}C(0,6\rho_0) \le \epsilon_2.$$

Moreover, we have

$$M(y,2\rho_0)\le 2^n M(0,4\rho_0)\le \delta_1.$$

Therefore, in view of *Step 2*, we have

$$\begin{aligned}\int_{B_r^+(y)}|Du|^2\,dx &\le \frac{c}{\rho_0^\lambda}\Big(\int_{B_{2\rho_0}^+(y)}|Du|^2\,dx+1\Big)r^\lambda\\ &\le \frac{c}{\rho_0^\lambda}\Big(\int_{B_{6r_0}^+}|Du|^2\,dx+1\Big)r^\lambda.\end{aligned}\tag{4.22}$$

Moreover, by (4.8), (4.20) and (4.21), we also have for all $\rho\in(0,4\rho]$ that

$$C(y,\rho)\le 2^{g_2}\theta^{-n}C(y,\theta^k(4\rho_0))\le 2^{g_2}\theta^{-n}\epsilon_2,\tag{4.23}$$

$$M(y,\rho)\le \theta^{-n}M(y,\theta^k(4\rho_0))\le \theta^{-n}\delta_1,\tag{4.24}$$

where k is a nonnegative integer satisfying $\theta^{k+1}(4\rho_0)<\rho\le\theta^k(4\rho_0)$.

Case (ii): Observe that $B_r(y)\subset B_{y_n}(y)=B^+_{y_n}(y)\subset B^+_{2y_n}(y')$, where $y=(y_1,\dots,y_{n-1},y_n)$ and $y'=(y_1,\dots,y_{n-1},0)$. Then since $C(y',2y_n)\le 2^{g_2}\theta^{-n}\epsilon_2$, in the same manner as in Case (i), we have

$$1+|(D_nu)_{B^+_{2y_n}(y')}|\le 2\left(1+|(D_nu)_{B_{y_n}(y)}|\right)\le 2\left(1+|(Du)_{B_{y_n}(y)}|\right).$$

Then we have from (4.23) and (4.24) that

$$\begin{aligned}&C_{\rm int}(y,y_n)\\ &:=\fint_{B_{y_n}(y)}\Big[\frac{|Du-(Du)_{y,y_n}|^2}{(1+|(Du)_{y,y_n}|)^2}+\frac{G(|Du-(Du)_{y,y_n}|)}{G(1+|(Du)_{y,y_n}|)}\Big]\,dx\\ &\le 2^{n+g_2}\fint_{B_{y_n}(y)}\Big[\frac{|Du-(D_nu)_{y',2y_n}\otimes e_n|^2}{(1+|(D_nu)_{y',2y_n}|)^2}+\frac{G(|Du-(D_nu)_{y',2y_n}\otimes e_n|)}{G(1+|(D_nu)_{y',2y_n}|)}\Big]\,dx\\ &\le 2^{n+g_2}C(y',2y_n)\le 2^{n+2g_2}\theta^{-n}\epsilon_2,\end{aligned}$$

$$\begin{aligned}&M_{\rm int}(y,y_n)\\ &:=y_n\fint_{B_{y_n}(y)}|Du|^2\,dx\\ &\le 2y_n\Big(\fint_{B_{y_n}(y)}|Du-(D_nu)_{y',2y_n}\otimes e_n|^2\,dx+\fint_{B_{2y_n}(y')}|D_nu|^2\,dx\Big)\\ &\le 2y_n\Big(2^{n+1}C(y',2y_n)\Big(1+\fint_{B_{2y_n}(y')}|D_nu|^2\,dx\Big)+\fint_{B_{2y_n}(y')}|D_nu|^2\,dx\Big)\\ &\le 2^{n+1}C(y',2y_n)\,(1+M(y',2y_n))+M(y',2y_n)\\ &\le 2^{n+2}2^{g_2}\theta^{-n}\epsilon_2+\theta^{-n}\delta_1.\end{aligned}$$

Therefore, by applying the results of interior partial regularity in [48], see [48, p.752-753], we have

$$\int_{B_r(y)}|Du|^2\,dx\le\frac{c}{y_n^\lambda}\Big(\int_{B_{y_n}(y)}|Du|^2\,dx+1\Big)r^\lambda\le\frac{c}{y_n^\lambda}\Big(\int_{B^+_{2y_n}(y')}|Du|^2\,dx+1\Big)r^\lambda.$$

Here, we choose sufficiently small ϵ_2 and δ_1, so that the argument in there still holds even we replace the assumption in [48, Eq. (4.32)] with the above estimates related to $C_{\rm int}$ and $M_{\rm int}$.

Moreover, since $2y_n \le 4\rho_0$ and $|y'| < 2\rho_0$, by applying (4.22) we obtain

$$\begin{aligned} \int_{B_r(y)} |Du|^2\,dx &\le \frac{c}{y_n^\lambda}\Big(\frac{c}{\rho_0^\lambda}\Big(\int_{B^+_{6\rho_0}} |Du|^2\,dx + 1\Big)(2y_n)^\lambda + 1\Big) r^\lambda \\ &\le \frac{c}{\rho_0^\lambda}\Big(\int_{B^+_{6r_0}} |Du|^2\,dx + 1\Big) r^\lambda. \end{aligned} \tag{4.25}$$

Therefore by Morrey-Campanato type embedding, see for instance [29, Theorem2.3] along with (4.22) and (4.25), we have proved the theorem $\square$

Now we prove our main result.

Proof of Theorem 1.1. Fix $\tilde{x} \in \partial\Omega$. Since $\partial\Omega \in C^1$, there exists a C^1 function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, in the coordinate system with the origin at $\tilde{x}$ and $\nu_{\tilde{x}} = e_n$, $B_r \cap \Omega = \{x = (x', x_n) \in B_r : x_n > \gamma(x')\}$. Here, $r > 0$ is sufficiently small and will be determined later. We next define a map $T : \mathbb{R}^n \to \mathbb{R}^n$ by $y = T(x', x_n - \gamma(x'))$ and its inverse by $T^{-1}(y) = (y', y_n + \gamma(y'))$. Note that by choosing sufficiently small $r > 0$, we have $\|DT\|_{L^\infty} = \|DT^{-1}\|_{L^\infty} \le \sqrt{2}$, and for any $\rho \le \sqrt{2}r$,

$$B^+_{\rho/\sqrt{2}} \le T(\Omega \cap B_\rho) \le B^+_{\sqrt{2}\rho}.$$

Now we set

$$\begin{gathered} \tilde{u}(y) := u(T^{-1}(y)) - g(T^{-1}(y)), \\ \tilde{\mathbf{a}}(y, \zeta, \xi) := \mathbf{a}(T^{-1}(y), \zeta + g(T^{-1}(y)), D[T^{-1}(y)]\xi + D[g(T^{-1}(y))]). \end{gathered}$$

Then we see that

$$\begin{gathered} |\tilde{\mathbf{a}}(y,\zeta,\xi)| + |\partial\tilde{\mathbf{a}}(y,\zeta,\xi)|(1+|\xi|) \le \tilde{L}G_1(1+|\xi|), \\ \partial\tilde{\mathbf{a}}(y,\zeta,\xi)\eta\cdot\eta \ge \tilde{\nu}G_2(1+|\xi|)|\eta|^2, \end{gathered}$$

and

$$\begin{gathered} |\tilde{\mathbf{a}}(y,\zeta_1,\xi) - \tilde{\mathbf{a}}(y,\zeta_2,\xi)| \le \tilde{L}\mu(|\zeta_1-\zeta_2|^2)G_1(1+|\xi|), \\ |\partial\tilde{\mathbf{a}}(y,\zeta,\xi_1) - \partial\tilde{\mathbf{a}}(y,\zeta,\xi_2)| \le \tilde{L}\mu\Big(\frac{|\xi_1-\xi_2|}{1+|\xi_1|+|\xi_2|}\Big)G_2(1+|\xi|), \end{gathered}$$

where $\tilde{L}$ and $\tilde{\nu}$ depend on L, ν, g_1, g_2 and $\|Dg\|_{L^\infty}$,

$$\begin{aligned} \lim_{\tilde{\rho}\to 0} \tilde{\mathcal{V}}(\tilde{\rho}) &:= \lim_{\tilde{\rho}\to 0} \Big(\sup_{0<\tilde{r}\le\tilde{\rho}} \sup_{\tilde{y}\in\mathbb{R}^n} \fint_{B^+_{\tilde{r}}(\tilde{y})} \tilde{V}(y, B^+_{\tilde{r}}(\tilde{y}))\,dy \Big) \\ &\le c\Big\{ \lim_{\rho\to 0} \mathcal{V}(\rho) + \lim_{\tilde{\rho}\to 0} \big(\mu(c\tilde{\rho}^2) + \tau(c\tilde{\rho})\big) \Big\} = 0, \end{aligned}$$

where

$$\tilde{V}(y,U) := \sup_{\zeta\in\mathbb{R}^N} \sup_{\xi\in\mathbb{R}^{nN}} \frac{|\tilde{\mathbf{a}}(y,\zeta,\xi) - (\tilde{\mathbf{a}}(\cdot,\zeta,\xi))_U|}{G_1(1+|\xi|)} \le 2\tilde{L},$$

and $\tau(\cdot)$ is the both modulus of the continuities of Dg and DT^{-1}. Moreover, $\tilde{u}$ is a weak solution to the system

$$\begin{gathered} \operatorname{div} \tilde{\mathbf{a}}(y, \tilde{u}, D\tilde{u}) = 0 \quad \text{in } B^+_r, \\ \tilde{u} = 0 \quad \text{on } T_r. \end{gathered}$$

Finally,

$$
\begin{aligned}
\fint_{B^+_{\tilde r}} |D\tilde u - (D_n\tilde u)_{B^+_{\tilde r}} \otimes e_n|\,dy &\le c\fint_{B^+_{\tilde r}} |D\tilde u - (D_n u)_{B_{\tilde r/\sqrt2}\cap\Omega} \otimes e_n|\,dy\\
&\ge \fint_{B_{\tilde r/\sqrt2}\cap\Omega} |Du - (D_n u)_{B_{\tilde r/\sqrt2}\cap\Omega} \otimes e_n|\,dx
\end{aligned}
$$

$$
\fint_{B^+_{\tilde r}} G(|D_n u|)\,dy \le c\fint_{B_{\tilde r/\sqrt2\cap\Omega}} G(|D_n u|)\,dx.
$$

Therefore, if $\tilde x \in \partial\Omega$ satisfies

$$
\liminf_{r\to0}\fint_{B_r\cap\Omega} |Du - (D_n u)_{B_r\cap\Omega} \otimes e_n|\,dx = 0,
$$

$$
\limsup_{r\to0}\fint_{B_r\cap\Omega} G(|D_n u|)\,dx < \infty,
$$

then by Theorem 4.1 we see that $0 \in T_{\tilde u}$, which implies $\tilde x \in \partial\Omega_u$. $\square$

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