Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 88, pp. 1–17.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE AND MULTIPLICITY OF SOLUTIONS TO SEMILINEAR ELLIPTIC EQUATION WITH NONLINEAR TERM OF SUPERLINEAR AND SUBCRITICAL GROWTH

XIAO-FENG KE, CHUN-LEI TANG

Communicated by Paul Rabinowitz

Abstract. This article concerns the existence and multiplicity of solutions to the superlinear elliptic problems. We introduce a new superlinear condition which is proved to be weaker than the Ambrosetti-Rabinowitz condition, the nonquadratic condition, the monotonicity conditions. As an application, positive solution and infinitely many solutions to semilinear elliptic equation with general subcritical growth are obtained, which generalize some known results.

1. Introduction and statement of main results

Consider the semilinear elliptic equation Dirichlet problem

$$\begin{gathered} -\triangle u + a(x)u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\triangle$ is the Laplacian operator, Ω is a bounded domain in $\mathbb{R}^N (N \geq 3)$ with smooth boundary $\partial\Omega$, and $a \in L^{\frac{N}{2}}(\Omega)$. The inner product and induced norm in $H_0^1(\Omega)$ are respectively given by

$$\langle u,v\rangle := \int_\Omega (\nabla u, \nabla v)dx, \quad \|u\| := \Big(\int_\Omega |\nabla u|^2 dx\Big)^{1/2}, \quad \forall u,v \in H_0^1(\Omega),$$

where $(\cdot,\cdot)$ is the Euclidean inner product. The operator $-\triangle + a : H_0^1(\Omega) \cap H^2(\Omega) \to L^2(\Omega)$ possesses a unbounded eigenvalues sequence

$$\lambda_1 < \lambda_2 \leq \dots \leq \lambda_n \to +\infty \text{ as } n \to \infty,$$

where λ_1 is simple and characterized by

$$\lambda_1 = \inf_{u \in H_0^1(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^2 + a(x)u^2 dx}{\int_\Omega u^2 dx},$$

the infimum is achieved by a positive function ϕ_1 which is exactly a eigenfunction corresponding to λ_1, and u is a eigenfunction corresponding to λ_1 if and only if $u \in H_0^1(\Omega) \setminus \{0\}$ is such that $\int_\Omega |\nabla u|^2 dx + \int_\Omega a(x)u^2 dx = \lambda_1 \int_\Omega u^2 dx$. Besides this, it is well known that the embedding mapping $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for

2010 *Mathematics Subject Classification.* 35J20, 35J61, 35D30.
Key words and phrases. Semilinear elliptic equation; new superlinear conditionl; general subcritical condition.

Submitted February 4, 2018. Published April 10, 2018.

$r \in [1, 2^*]$ and is compact for $r \in [1, 2^*)$, where $2^* := \frac{2N}{N-2}$. We denote by $|\cdot|_r$ the norm in $L^r(\Omega)$ and S_r the best constant to the corresponding embedding mapping, that is, $S_r|u|_r \leq \|u\|$, for all $u \in H_0^1(\Omega)$.

In the celebrated paper [1], Ambrosetti and Rabinowitz established the famous mountain pass theorem and applied it to obtain nontrivial solution and multiple solutions to problem (1.1) by assuming

(A1) f is Hölder continuous in $\overline{\Omega} \times \mathbb{R}$ and $f(x, 0) = 0$,

(A2) there exist positive constants a_1, a_2 and $q \in (2, 2^*)$ such that

$$|f(x, s)| \leq a_1 + a_2|s|^{q-1}$$

for $s \in \mathbb{R}$ and $x \in \Omega$,

(A3) $\lim_{s\to 0} f(x, s)/s = 0$ uniformly in $x \in \overline{\Omega}$,

(A4) $\lim_{|s|\to\infty} f(x, s)/s = +\infty$ uniformly in $x \in \overline{\Omega}$,

(A5) there exist constants $s_0' > 0$ and $\theta > 2$ such that

$$\theta F(x, s) \leq sf(x, s)$$

for $|s| \geq s_0'$ and $x \in \overline{\Omega}$, where $F(x, s) := \int_0^s f(x, t)dt$,

(A6) f is odd in s,

where (A4) shows that f is essentially superlinear at ∞. Moreover, (A4) together with (A5) leads to

(A7) there exist constants $s_1' > 0$ and $\theta > 2$ such that

$$0 < \theta F(x, s) \leq sf(x, s)$$

for $|s| \geq s_1'$ and $x \in \overline{\Omega}$,

which is hereafter called the Ambrosetti-Rabinowitz condition and plays a key role in ensuring that the Euler-Lagrange functional associated to problem (1.1) admits a mountain pass geometry and the Palais-Smale sequences are bounded. Integrating, from the continuity of f one deduces that

$$F(x, s) \geq \xi|s|^\theta \tag{1.2}$$

for $|s| \geq s_1'$ and $x \in \Omega$, where

$$\xi := \big(\frac{1}{s_1'}\big)^\theta \min\Big\{\min_{x\in\overline{\Omega}} F(x, s_1'), \min_{x\in\overline{\Omega}} F(x, -s_1')\Big\} > 0.$$

Here we note two things. Firstly, in order to obtain (1.2), one should add the assumption $\operatorname{ess\,inf}_{x\in\Omega} F(x, \pm s_1') > 0$ if (A7) is satisfied only on Ω rather than $\overline{\Omega}$ or $f(\cdot, \pm s_1') : \overline{\Omega} \to \mathbb{R}$ is discontinuous, see [11] for more details. Secondly, (1.2) eliminates many interesting superlinear functions, such as $F(x, s) = s^2 \ln(1 + |s|)$. For this reason, this technique has been subsequently improved in order to include more superlinear functions and extended to deal with more complicated variational problems by a large number of researchers, see [3, 4, 7, 8, 9, 10, 12, 13, 15, 17, 18, 19, 21, 23, 24, 25, 26] and references therein.

In [3], Costa and Magalhães replaced (A7) by

(A8) (i) there exist constants $q \in (2, 2^*)$ and $a_3 > 0$ such that

$$\limsup_{|s|\to\infty} \frac{F(x, s)}{|s|^q} \leq a_3 \text{ uniformly in a.e. } x \in \Omega,$$

(ii) there exist constants $\delta > 0$ and $\mu > \frac{N(q-2)}{2}$ such that
$$\liminf_{|s|\to\infty} \frac{sf(x,s) - 2F(x,s)}{|s|^\mu} \geq \delta$$
uniformly in a.e. $x \in \Omega$,

then a nontrivial solution was obtained provided
$$\limsup_{s\to 0} \frac{2F(x,s)}{s^2} < \lambda_1 < \liminf_{|s|\to\infty} \frac{2F(x,s)}{s^2} \quad \text{uniformly in a.e. } x \in \Omega.$$
Under these assumptions, they can deal with both superlinear situation and sublinear situation.

In [4], Ding and Luan investigated a class of Schrödinger equations with the nonlinear term satisfying

(A9) (i) $\lim_{|s|\to\infty} \frac{F(x,s)}{s^2} = +\infty$ uniformly in $x \in \Omega$,
(ii) $H(x,s) := sf(x,s) - 2F(x,s) > 0$ for $s \neq 0$,
(iii) there exist positive constants s_2', a_4 and $\sigma > N/2$ such that $\left(\frac{f(x,s)}{s}\right)^\sigma \leq a_4 H(x,s)$ for $|s| \geq s_2'$ and $x \in \Omega$,

where (A9)(iii) can be deduced from (A7) and a subcritical growth condition, see [5, Lemma 2.2].

In [23], Willem and Zou studied a class of superlinear Schrödinger equation by assuming

(A10) (i) there exist positive constants a_5, a_6 and $\nu \in (2, 2^*)$ such that $a_5|s|^\nu \leq f(x,s)s \leq a_6|s|^\nu$ for $s \in \mathbb{R}$ and $x \in \Omega$,
(ii) $sf(x,s) - 2F(x,s) > 0$ for $s \neq 0$ and $x \in \Omega$,
(iii) there exist constants $\delta > 0$ and $\mu > \frac{2^*\nu(\nu-2)}{2^*\nu - 2^* - \nu}$ such that
$$\liminf_{|s|\to\infty} \frac{sf(x,s) - 2F(x,s)}{|s|^\mu} \geq \delta \text{ uniformly in } x \in \Omega.$$

In [10], Miyagaki and Souto studied a eigenvalue problem under (S_2)(i) and

(A11) there exists constant $s_3' > 0$ such that
$$\frac{f(x,s)}{s} \text{ is increasing for } s \geq s_3' \text{ and decreasing for } s \leq -s_3', \quad \forall x \in \Omega,$$
which implies

(A12) there exists constant $s_4' > 0$ such that
$$H(x,s) \text{ is increasing for } s \geq s_4' \text{ and decreasing for } s \leq -s_4', \quad \forall x \in \Omega.$$

It is remarkable that (A12) also implies that (A11) when $f(x,s)$ is differentiable with respect to s (see [8]). Furthermore, (A12) can be generalized in two directions. One is the following generalized monotonic condition

(A13) there exists a constant $D \geq 1$ such that
$$H(x,t) \leq DH(x,s) \quad \text{for } s_4' < t < s \text{ or } s < t < -s_4',\ \forall x \in \Omega,$$
which was first introduced in [6]. The other is the following "quasi-monotonic" condition

(A14) there exists a nonnegative function $W_1 \in L^1(\Omega)$ such that
$$H(x,t) \leq H(x,s) + W_1(x) \quad \text{for } 0 < t < s \text{ or } s < t < 0,\ \forall x \in \Omega.$$
A weaker condition than (A13) and (A14) is

(A15) there exist a constant $D \geq 1$ and a nonnegative function $W_1 \in L^1(\Omega)$ such that

$$H(x,t) \leq DH(x,s) + W_1(x) \quad \text{for } 0 < t < s \text{ or } s < t < 0,\ \forall\, x \in \Omega.$$

Using (A15) instead of (A11), Lan and Tang in [7] generalized the result in [10].

In addition, Schechter and Zou in [15] established the existence of nontrivial solution for problem (1.1) provided

- $H(x,s)$ is convex in s, $\forall\, x \in \Omega$, or there are constants $a_7 > 0$, $\theta > 2$ and s_5' such that
$$\theta F(x,s) - sf(x,s) \leq a_7(s^2 + 1)$$
for $|s| \geq s_5'$.

As remarked in [10], the convexity of H in the above assumption is stronger than (A11), while the second alternative is equivalent to (A7).

Under (A7), Wang in [21] proved that problem (1.1) had at least three nontrivial solutions via the mountain pass theorem and Morse theory. By assuming (A12) holds, Liu and Wang in [9] also obtained at least three nontrivial solutions via the Nehari manifold method, and infinitely many solutions via the Ljusternik-Schnirelmann theory. Recently, Tang in [19] investigated a superlinear Schrödinger equation with the nonlinear term satisfying

- there exists $\theta_0 \in (0,1)$ such that $sf(x,s) \geq 0$ and
$$\frac{1-\theta^2}{2} sf(x,s) \geq \int_{\theta s}^{s} f(x,t)dt = F(x,s) - F(x,\theta s), \quad \forall\, \theta \in [0,\theta_0]$$
for $s \in \mathbb{R}$ and a.e. $x \in \Omega$.

Tang and Wu in [18] also introduced a new superquadratic condition to guarantee the existence of nontrivial solution to a second order Hamiltonian systems.

In this paper, we assume that $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and satisfies

(A16) for every $M > 0$, there exists a constant $L_M > 0$ such that
$$|f(x,s)| \leq L_M$$
for $|s| \leq M$ and a.e. $x \in \Omega$,

(A17) $\lim_{|s|\to\infty} \frac{f(x,s)}{|s|^{2^*-2}s} = 0$ uniformly in a.e. $x \in \Omega$,

(A18) there exist a function $m \in L^{\frac{N}{2}}(\Omega)$ and a subset $\Omega' \subset \Omega$ with $|\Omega'| > 0$ such that
$$\limsup_{s\to 0} \frac{2F(x,s)}{s^2} \leq m(x) \leq \lambda_1$$
uniformly in a.e. $x \in \Omega$, and $m < \lambda_1$ in Ω', where $F(x,s) = \int_0^s f(x,t)dt$ and $|\cdot|$ is the Lebesgue measure,

(A19) $\lim_{|s|\to\infty} \frac{F(x,s)}{s^2} = +\infty$ uniformly in a.e. $x \in \Omega$,

(A20) there exist constants $s_0 > 0$, $\alpha > 0$, $\sigma > \frac{N}{2}$ and a nonnegative function $W \in L^1(\Omega)$ such that
$$\Big(\frac{F(x,s)}{s^2}\Big)^{\sigma} \leq \alpha H(x,s) + W(x)$$
for $|s| \geq s_0$ and a.e. $x \in \Omega$, where $H(x,s) = sf(x,s) - 2F(x,s)$.

Remark 1.1. Obviously, (A16) holds when $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous. (A17) is essentially weaker than

(A21) there exist positive constants a_8, a_9 and $q \in (2, 2^*)$ such that
$$|f(x,s)| \le a_8 + a_9|s|^{q-1}$$
for $s \in \mathbb{R}$ and a.e. $x \in \Omega$.

which is equivalent to (A8)(i) when (A16) holds. Besides, if $\lambda_1 > 0$, (A18) is obviously weaker than

(A22) $\lim_{s\to 0} \frac{2F(x,s)}{s^2} = 0$ uniformly in a.e. $x \in \Omega$.

Remark 1.2. There exist functions which satisfy our conditions and do not satisfy (A7)–(A10), and (A15). For example, when $a(x) \equiv 0$ and $N = 4$, let $\Omega_0 \subset \Omega$ be such that $|\Omega_0| > 0$ and $|\Omega \setminus \Omega_0| > 0$, χ_{Ω_0} denotes the characteristic function of Ω_0, set $h : [1, +\infty) \to \mathbb{R}$ as follows
$$h(s) = \begin{cases} n^3(\frac{1}{n^2} - |s-n|) + \frac{1}{s}, & \text{if } |s-n| \le \frac{1}{n^2},\ n = 2,3,4,\dots, \\ \frac{1}{s}, & \text{otherwise,} \end{cases}$$
and
$$f(x,s) = \begin{cases} \frac{2s\int_1^s h(t)dt + s^2h(s)}{(\ln s+1)^{1/2}} - \frac{s\int_1^s h(t)dt}{2(\ln s+1)^{3/2}} + 3\chi_{\Omega_0}(x)s^2, & s \ge 1,\ x \in \Omega, \\ 2(s-\frac{1}{2})(1+3\chi_{\Omega_0}(x)), & s \in (\frac{1}{2},1),\ x \in \Omega, \\ 0, & s \le \frac{1}{2},\ x \in \Omega. \end{cases}$$
By simple calculation, we have $\frac{2N}{N-2} = 4$, $\frac{N}{2} = 2$,
$$h(n) = n + \frac{1}{n}, \quad h(n + \frac{1}{n^2}) = \frac{1}{n + \frac{1}{n^2}}, \quad n = 2,3,4,\dots,$$
$$F(x,s) = \begin{cases} \frac{s^2\int_1^s h(t)dt}{(\ln s+1)^{1/2}} + \chi_{\Omega_0}(x)s^3 + \frac{1}{4}(1-\chi_{\Omega_0}(x)), & s \ge 1,\ x \in \Omega, \\ (s-\frac{1}{2})^2(1+3\chi_{\Omega_0}(x)), & s \in (\frac{1}{2},1),\ x \in \Omega, \\ 0, & s \le \frac{1}{2},\ x \in \Omega, \end{cases}$$
and
$$sf(x,s) - 2F(x,s) = \frac{s^3h(s)}{(\ln s+1)^{1/2}} - \frac{s^2\int_1^s h(t)dt}{2(\ln s+1)^{3/2}} + \chi_{\Omega_0}(x)s^3 - \frac{1}{2}(1-\chi_{\Omega_0}(x)),$$
for $s \ge 1$, $x \in \Omega$. Besides this, for $s \ge 1$,
$$\int_1^s h(t)dt = \int_1^s \big(h(t) - \frac{1}{t}\big)dt + \int_1^s \frac{1}{t}dt = \int_1^s \big(h(t) - \frac{1}{t}\big)dt + \ln s,$$
then for $3 \le n \le s \le n+1$, one has
$$\int_1^s \Big(h(t) - \frac{1}{t}\Big)dt \le \sum_{k=2}^{n+1} \int_{k-\frac{1}{k^2}}^{k+\frac{1}{k^2}} k^3\Big(\frac{1}{k^2} - |s-k|\Big)ds = \sum_{k=2}^{n+1} \frac{1}{k},$$
$$\int_1^s \Big(h(t) - \frac{1}{t}\Big)dt \ge \sum_{k=2}^{n-1} \int_{k-\frac{1}{k^2}}^{k+\frac{1}{k^2}} k^3\Big(\frac{1}{k^2} - |s-k|\Big)ds = \sum_{k=2}^{n-1} \frac{1}{k}.$$
From the above two inequalities and $\lim_{n\to\infty} \frac{\sum_{k=1}^n \frac{1}{k}}{\ln n} = 1$ it follows that
$$\lim_{s\to+\infty} \frac{\int_1^s \big(h(t) - \frac{1}{t}\big)\,dt}{\ln s} = 1,$$

which leads to

$$\lim_{s\to+\infty}\frac{\int_1^s h(t)dt}{\ln s}=2.$$

Thus, it is easy to verify that assumptions (A16)–(A19) hold. Furthermore, (A20) holds for arbitrary $\sigma\in(2,3)$. However, we can draw the following conclusions.

(i) Condition (A7) is not satisfied. Indeed, for $\theta>2$, $x\in\Omega\setminus\Omega_0$ and $s_n:=n+\frac{1}{n^2}$,

$$s_nf(x,s_n)-\theta F(x,s_n)\le-\frac{(\theta-2)s_n^2\int_1^{s_n}h(t)dt}{(\ln s_n+1)^{1/2}}+\frac{s_n^3h(s_n)}{(\ln s_n+1)^{1/2}}\to-\infty$$

as $n\to+\infty$.

(ii) Conditions (A8) and (A10) are not satisfied. Indeed, it needs $q\ge3$ to ensure (A8)(i) holds. But for $x\in\Omega\setminus\Omega_0$ and arbitrary $\mu>\frac{4(q-2)}{2}\ge2$, one has

$$\begin{aligned}\liminf_{|s|\to\infty}\frac{sf(x,s)-2F(x,s)}{|s|^\mu}&\le\lim_{n\to\infty}\frac{s_nf(x,s_n)-2F(x,s_n)}{(s_n)^\mu}\\&\le\lim_{n\to\infty}\frac{s_n^3h(s_n)}{(\ln s_n+1)^{1/2}(s_n)^\mu}=0,\end{aligned}$$

which is in contradiction with (A8)(ii). Similarly, there are not constants $\nu\in(2,4)$ and $\mu>\frac{4\nu(\nu-2)}{4\nu-4-\nu}$ such that (A10) holds.

(iii) Condition (A9) is not satisfied. In fact, for $x\in\Omega\setminus\Omega_0$, we have

$$\frac{f(x,n)}{n}\ge\frac{n^2}{(\ln n+1)^{1/2}},\quad nf(x,n)-2F(x,n)\le\frac{2n^4}{(\ln n+1)^{1/2}}$$

for n large, so there does not exist constant $\sigma>2$ such that (A9) holds.

(iv) Condition (A15) is not satisfied. In fact, for $x\in\Omega\setminus\Omega_0$ and $s_n':=n-\frac{1}{n^2}$, it is not difficult to prove that

$$\begin{gathered}nf(x,n)-2F(x,n)-(s_nf(x,s_n)-2F(x,s_n))\to+\infty,\\nf(x,n)-2F(x,n)-(s_n'f(x,s_n')-2F(x,s_n'))\to+\infty\end{gathered}$$

as $n\to\infty$. Then there do not exist constant $D\ge1$ and nonnegative function $W_1\in L^1(\Omega)$ such that (A15) holds.

Our main results are the following theorems.

Theorem 1.3. *Assume that* (A16)–(A20) *hold, and that $a\in L^\infty(\Omega)$ and $sf(x,s)\ge0$ for $s\in\mathbb{R}$ and a.e. $x\in\Omega$. Then problem* (1.1) *has at least a positive solution and a negative solution.*

Remark 1.4. In the next section, we will prove that (A20) indeed weaker than (A7)–(A10), (A15) under the assumptions (A21) and (A19). In addition, if f satisfies (A15)–(A17), (A19), and (A22),, so does the term λf for $\lambda>0$. Therefore, Theorem 1.3 generalizes [10, Theorem 1.1] and complements [9, Theorem 2.1], [7, Theorem 1.2]. It is necessary to point out here that the integrability requirement $a\in L^\infty(\Omega)$ and sign condition $sf(x,s)\ge0$ for $s\in\mathbb{R}$ and a.e. $x\in\Omega$ are only used to obtain a positive solution, especially in order to guarantee the validity of the strong Maximum principle in [20]. In fact, $a\in L^{\frac{N}{2}}(\Omega)$ is enough to ensure the existence of nontrivial solution.

Theorem 1.5. *Assume that* (A6), (A16), (A17), (A19), (A20) *hold, then problem* (1.1) *has infinitely many solutions.*

Remark 1.6. Theorem 1.5 unifies and generalizes [22, Theorem 3.7], [17, Theorem 3.2], [24, Theorem 1.3], [25, Theorem 1.1], [13, Theorems 1.2 and 1.3]. Besides this, Theorem 1.5 complements [9, Theorem 2.3], [26, Theorem 3.1], [12, Theorem 1.4].

Remark 1.7. A condition similar to (A20) was introduced in [13]. However, compared with the description in [13], firstly, our description is more general and simple. Secondly, we point out the relations between (A20) and several famous superlinear conditions for the first time. Thirdly, we can deal with the superlinear problems with nonlinear term satisfying the general subcritical condition (A17).

2. Preliminaries

Let $E := H_0^1(\Omega)$, the Euler-Lagrange functional associated to problem (1.1) is

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega a(x)u^2 dx - \int_\Omega F(x,u)dx, \quad u \in E.$$

By (A16) and (A17), it is standard to verify that $\Phi \in C^1(E,\mathbb{R})$ and

$$\langle \Phi'(u), v\rangle = \int_\Omega (\nabla u, \nabla v)dx + \int_\Omega a(x)uv\,dx - \int_\Omega f(x,u)v\,dx,$$

for all $u, v \in E$. Moreover, the weak solutions of problem (1.1) are exactly the critical points of Φ in E. In order to obtain positive solution and negative solution, we let $\widetilde{f}(x,s) := f(x,s) - m(x)s$ and truncate $\widetilde{f}$ above or below $s = 0$, i.e., let

$$\widetilde{f}_+(x,s) := \begin{cases} \widetilde{f}(x,s), & s \ge 0,\\ 0, & s < 0, \end{cases} \quad \widetilde{f}_-(x,s) := \begin{cases} \widetilde{f}(x,s), & s \le 0,\\ 0, & s > 0, \end{cases}$$

and $\widetilde{F}_+(x,s) = \int_0^s \widetilde{f}_+(x,t)dt$, $\widetilde{F}_-(x,s) = \int_0^s \widetilde{f}_-(x,t)dt$. Under (A16) and (A17), the functionals $\widetilde{\Phi}_+$ and $\widetilde{\Phi}_-$ defined as follows

$$\widetilde{\Phi}_+(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega a(x)u^2 dx - \frac{1}{2}\int_\Omega m(x)u^2 dx - \int_\Omega \widetilde{F}_+(x,u)dx,$$
$$\widetilde{\Phi}_-(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega a(x)u^2 dx - \frac{1}{2}\int_\Omega m(x)u^2 dx - \int_\Omega \widetilde{F}_-(x,u)dx,$$

belong to $C^1(E,\mathbb{R})$ and

$$\langle \widetilde{\Phi}'_+(u), v\rangle = \int_\Omega (\nabla u, \nabla v)dx + \int_\Omega a(x)uv\,dx - \int_\Omega m(x)uv\,dx - \int_\Omega \widetilde{f}_+(x,u)v\,dx,$$
$$\langle \widetilde{\Phi}'_-(u), v\rangle = \int_\Omega (\nabla u, \nabla v)dx + \int_\Omega a(x)uv\,dx - \int_\Omega m(x)uv\,dx - \int_\Omega \widetilde{f}_-(x,u)v\,dx,$$

for all $u, v \in E$. The following lemmas show that our superlinear situation, i.e., (A19) and (A20), indeed includes all the superlinear situations implied by (A7)–(A10), (A15).

Lemma 2.1. *Under assumption* (A21)*, assumption (A7) implies* (A19), (A20)*.*

Proof. Clearly, (A19) naturally holds because of (1.2). Additionally, from $q \in (2, 2^*)$ we derive $\frac{N}{2} < \frac{q}{q-2}$. Taking arbitrarily $\sigma \in (\frac{N}{2}, \frac{q}{q-2})$, one has $q < \frac{2\sigma}{\sigma-1}$. Then (A21) leads to

$$\lim_{|s|\to\infty} \frac{F(x,s)}{|s|^{\frac{2\sigma}{\sigma-1}}} = 0 \quad \text{uniformly in a.e. } x \in \Omega.$$

From this and (A19) it follows that there exists a constant $s_1 > s_1'$ such that

$$0 < \frac{F(x,s)}{|s|^{\frac{2\sigma}{\sigma-1}}} \le (\theta-2)^{\frac{1}{\sigma-1}}$$

for $|s| \ge s_1$ and a.e. $x \in \Omega$, from this and (A7) we obtain that

$$\Big(\frac{F(x,s)}{s^2}\Big)^\sigma \le (\theta-2)F(x,s) \le sf(x,s) - 2F(x,s)$$

for $|s| \ge s_1$ and a.e. $x \in \Omega$. □

Lemma 2.2. *Under assumption* (A19)*, assumption* (A8) *implies* (A20)*.*

Proof. From (A8) and (A19) it follows that there exists constant $s_2 > 1$ such that

$$0 < \frac{F(x,s)}{|s|^q} \le a_3 + 1 \quad \text{and} \quad H(x,s) \ge \delta |s|^\mu$$

for $|s| \ge s_2$ and a.e. $x \in \Omega$. From $\mu > \frac{N(q-2)}{2}$ we deduce that $\frac{N}{2} < \frac{\mu}{q-2}$. Taking arbitrarily $\sigma \in (\frac{N}{2}, \frac{\mu}{q-2})$, one has $\sigma(q-2) < \mu$. Then

$$\Big(\frac{F(x,s)}{s^2}\Big)^\sigma = \Big(\frac{F(x,s)}{|s|^q}\Big)^\sigma |s|^{\sigma(q-2)} \le (a_3+1)^\sigma |s|^\mu \le \frac{(a_3+1)^\sigma}{\delta} H(x,s)$$

for $|s| \ge s_2$ and a.e. $x \in \Omega$. □

Lemma 2.3. *Condition* (A9) *implies* (A19) *and* (A20)*.*

Proof. (A9)(i) implies (A19). From this and (A9)(ii) it follows that there exists a constant $s_3 > s_2'$ such that

$$sf(x,s) \ge 2F(x,s) > 0$$

for $|s| > s_3$ and a.e. $x \in \Omega$, which together with (A9)(iii) leads to

$$\Big(\frac{F(x,s)}{s^2}\Big)^\sigma \le \frac{1}{2^\sigma}\Big(\frac{f(x,s)}{s}\Big)^\sigma \le \frac{a_4}{2^\sigma} H(x,s)$$

for $|s| \ge s_3$ and a.e. $x \in \Omega$. □

Lemma 2.4. *Condition* (A10) *implies* (A19) *and* (A20)*.*

Proof. (A10)(i) implies (A19). Moreover, from this and (A10)(iii) we deduce that there exists a constant $s_4 > 1$ such that

$$\frac{F(x,s)}{s^2} \ge 1 \quad \text{and} \quad f(x,s)s - 2F(x,s) \ge \delta |s|^\mu$$

for $|s| \ge s_4$ and a.e. $x \in \Omega$. Additionally, from $\nu \in (2, 2^*)$ it follows that

$$\nu - 2 > 0 \quad \text{and} \quad \frac{2^*\nu}{2^*\nu - 2^* - \nu} > \frac{N}{2}.$$

Taking arbitrarily $\sigma \in (\frac{N}{2}, \frac{2^*\nu}{2^*\nu-2^*-\nu})$, one derives from $\mu > \frac{2^*\nu(\nu-2)}{2^*\nu-2^*-\nu}$ and from (A10)(ii) that

$$\begin{aligned} f(x,s)s - 2F(x,s) &\ge \delta|s|^\mu > \delta\big(|s|^{\nu-2}\big)^{\frac{2^*\nu}{2^*\nu-2^*-\nu}} \\ &\ge \frac{\delta}{a_6^\sigma}\Big(\frac{sf(x,s)}{s^2}\Big)^\sigma \ge \frac{2\delta}{a_6^\sigma}\Big(\frac{F(x,s)}{s^2}\Big)^\sigma \end{aligned}$$

for $|s| \ge s_4$ and a.e. $x \in \Omega$. □

Lemma 2.5. *Under assumptions(A19) and* (A21)*, condition* (A15) *implies* (A20).

Proof. From (A21) and (A19) it follows that there exists constant $s_5 > 0$ such that

$$\frac{F(x,s)}{s^2} > 0 \quad \text{and} \quad \frac{F(x,s)}{|s|^q} \le \frac{a_9}{q} + 1 \tag{2.1}$$

for $|s| \ge s_5$ and $x \in \Omega$. The fact $q \in \left(2, \frac{2N}{N-2}\right)$ yields $\frac{q}{q-2} > \frac{N}{2}$. Then taking arbitrarily $\sigma \in \left(\frac{N}{2}, \frac{q}{q-2}\right)$, we have $2 > (\sigma-1)(q-2) > 0$. Let $\varsigma := 2-(\sigma-1)(q-2) > 0$, one gets

$$\begin{aligned}
\frac{d}{ds}\Big[\Big(\frac{F(x,s)}{s^2}\Big)^{\sigma}\Big] &= \sigma\Big(\frac{F(x,s)}{s^2}\Big)^{\sigma-1}\frac{sf(x,s)-2F(x,s)}{s^3} \\
&= \sigma\Big(\frac{F(x,s)}{|s|^q}\Big)^{\sigma-1}\frac{H(x,s)}{|s|^{\varsigma}s}
\end{aligned}$$

for a.e. $x \in \Omega$. From this, (2.1) and (A15) it follows that

$$\begin{aligned}
\Big(\frac{F(x,s)}{s^2}\Big)^{\sigma} - \Big(\frac{F(x,s_5)}{s_5^2}\Big)^{\sigma} &= \int_{s_5}^{s}\frac{d}{dt}\Big[\Big(\frac{F(x,t)}{t^2}\Big)^{\sigma}\Big]dt \\
&= \int_{s_5}^{s}\sigma\Big(\frac{F(x,t)}{|t|^q}\Big)^{\sigma-1}\frac{H(x,t)}{t^{\varsigma+1}}dt \\
&\le \sigma\Big(\frac{a_9}{q}+1\Big)^{\sigma-1}\big(DH(x,s)+W_1(x)\big)\int_{s_5}^{s}\frac{1}{t^{\varsigma+1}}dt \\
&\le \sigma\Big(\frac{a_9}{q}+1\Big)^{\sigma-1}\big(DH(x,s)+W_1(x)\big)\frac{s_5^{-\varsigma}}{\varsigma}
\end{aligned}$$

for $s \ge s_5$ and a.e. $x \in \Omega$, where in the last inequality we use the fact that $DH(x,s) + W_1(x) \ge 0$ for $s \ne 0$ and a.e. $x \in \Omega$ which can be deduced from condition (A15) and $H(x,0) = 0$ a.e. $x \in \Omega$. Then we have

$$\Big(\frac{F(x,s)}{s^2}\Big)^{\sigma} \le \alpha H(x,s) + W(x),$$

for $s \ge s_5$ and a.e. $x \in \Omega$, where $\alpha = \frac{\sigma D s_5^{-\varsigma}(a_9+q)^{\sigma-1}}{q^{\sigma-1}\varsigma}$, $W(x) = \frac{\sigma s_5^{-\varsigma}(a_9+q)^{\sigma-1}}{q^{\sigma-1}\varsigma}W_1(x) + \left(\frac{F(x,s_5)}{s_5^2}\right)^{\sigma}$. In a similar way, it is easy to verify that the above inequality holds for $s \le -s_5$ and a.e. $x \in \Omega$. □

3. Proof of main results

To prove Theorems 1.3 and 1.5, we recall two abstract critical point theorems, i.e., the mountain pass theorem and the symmetric mountain pass theorem under the (C) condition, the readers can refer to [2] and [14].

Theorem 3.1. *Let $(X, \|\cdot\|_X)$ be a Banach space, suppose that $\varphi \in C^1(X,\mathbb{R})$ satisfies $\varphi(0) = 0$ and*

(i) *there exist positive constants R_0 and α_0 such that*

$$\varphi(u) \ge \alpha_0 \quad \text{for all } u \in X \text{ with } \|u\|_X = R_0,$$

(ii) *there exists $e \in X$ with $\|e\|_X > R_0$ such that $\varphi(e) < 0$,*

(iii) *φ satisfies the* (C) *condition, that is, for $c \in \mathbb{R}$, every sequence $\{u_n\} \subset X$ such that*

$$\varphi(u_n) \to c, \quad \|\varphi'(u_n)\|(1 + \|u_n\|) \to 0$$

has a convergent subsequence.

Then $c := \inf_{\gamma\in\Gamma} \sup_{s\in[0,1]} \varphi(\gamma(s))$ is a critical value of φ, where

$$\Gamma := \{\gamma \in C([0, 1], X); \gamma(0) = 0, \gamma(1) = e\}.$$

Theorem 3.2. *Let $(X, \|\cdot\|_X)$ be an infinite dimensional Banach space, and let $\varphi \in C^1(X, \mathbb{R})$ be even. Suppose that φ satisfies $\varphi(0) = 0$ and*

(i) *there exist a closed subspace X^1 of X with $\operatorname{codim} X^1 < +\infty$ and positive constants R_1, α_1 such that*

$$\varphi(u) \ge \alpha_1 \quad \text{for } u \in X^1 \text{ with } \|u\|_X = R_1,$$

(ii) *for every finite dimensional subspace X^2 of X, there exists positive constant R_2 such that*

$$\varphi(u) \le 0 \quad \text{for } u \in X^2 \text{ with } \|u\|_X = R_2, ,$$

(iii) *φ satisfies the* (C) *condition in Theorem 3.1.*

Then φ possesses an unbounded sequence of critical values.

In addition, we need the following lemmas.

Lemma 3.3 ([22, Lemma 2.13])**.** *Assume that $N \ge 3$ and $\vartheta \in L^{\frac{N}{2}}(\Omega)$, then the functional*

$$\psi(u) := \int_\Omega \vartheta(x)u^2 dx, \ u \in H_0^1(\Omega)$$

is weakly continuous.

Lemma 3.4. *Assume that $m \in L^{\frac{N}{2}}(\Omega)$, and there exists a subset $\Omega' \subset \Omega$ with $|\Omega'| > 0$ such that*

$$m \le \lambda_1 \text{ in } \Omega \quad \text{and} \quad m < \lambda_1 \text{ in } \Omega',$$

then

$$d := \inf_{u\in H_0^1(\Omega),\ u\neq 0} \frac{\int_\Omega |\nabla u|^2 dx + \int_\Omega a(x)u^2 dx - \int_\Omega m(x)u^2 dx}{\int_\Omega |\nabla u|^2 dx} > 0.$$

Proof. From the characteristic of λ_1 and the assumption $m \le \lambda_1$ in Ω it follows that $d \ge 0$. The reminder is to prove that $d \neq 0$. Let

$$J(u) := \int_\Omega a(x)u^2 dx, \ u \in H_0^1(\Omega),$$
$$K(u) := \int_\Omega m(x)u^2 dx, \ u \in H_0^1(\Omega),$$
$$L(u) := \|u\|^2 + J(u) - K(u), \ u \in H_0^1(\Omega).$$

We argue by contradiction. If $d = 0$, there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that

$$\|u_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} L(u_n) = 0.$$

By the boundedness of $\{u_n\}$, up to subsequence we may assume that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$. From this, the weak continuity of J, K, and the weak lower continuity of L it follows that

$$\lim_{n\to\infty} J(u_n) = J(u), \quad \lim_{n\to\infty} K(u_n) = K(u) \tag{3.1}$$

and

$$0 \le L(u) \le \liminf_{n\to\infty} L(u_n) = \lim_{n\to\infty} L(u_n) = 0.$$

Then we have

$$L(u) = \|u\|^2 + J(u) - K(u) = \lim_{n\to\infty} L(u_n) = 0, \tag{3.2}$$

which implies

$$\|u\|^2 + J(u) = K(u) \le \lambda_1 \int_\Omega u^2 dx,$$

this together with the characteristic of λ_1 leads to

$$\|u\|^2 + J(u) = \lambda_1 \int_\Omega u^2 dx. \tag{3.3}$$

If $u = 0$, from (3.1) and (3.2) it follows that $\|u_n\| \to 0$ as $n \to \infty$, which is in contradiction with $\|u_n\| = 1$. So $u \neq 0$, then u is a eigenfunction corresponding to λ_1, so $u = l_0\phi_1$ for some $l_0 \in \mathbb{R} \setminus \{0\}$ as λ_1 is simple. Thus, from $\phi_1 > 0$, $m \le \lambda_1$ in Ω and $m < \lambda_1$ in Ω' with $|\Omega'| > 0$ it follows that

$$\begin{aligned} \|u\|^2 + J(u) = K(u) &= \int_\Omega m(x)u^2 dx \\ &= l_0^2 \int_\Omega m(x)\phi_1^2 dx < l_0^2\lambda_1 \int_\Omega \phi_1^2 dx \\ &= \lambda_1 \int_\Omega u^2 dx, \end{aligned}$$

which is in contradiction with (3.3). Hence, $d > 0$. The proof is complete. □

Lemma 3.5. *Assume that* (A16)–(A18) *hold. Then* $\widetilde{\Phi}_+$ *satisfies* (i) *of Theorem 3.1.*

Proof. By (A18), for $\varepsilon \in \left(0, \frac{dS_2^2}{2}\right)$, there exists a positive constant $M_1 < 1$ such that

$$F_+(x,s) = F(x,s^+) \le \frac{1}{2}(m(x)+\varepsilon)(s^+)^2 \tag{3.4}$$

for $|s| \le M_1$ and a.e. $x \in \Omega$, where and in what follows we denote by $s^+ := \max\{s,0\}$ and $s^- := \max\{-s,0\}$. For above ε, from (A16), (A17) and (3.4) it follows that there exists a constant $M_2 > 1$ such that

$$|f_+(x,s)| = |f(x,s^+)| \le \varepsilon(s^+)^{2^*-1} + L_{M_2} \tag{3.5}$$

and

$$F_+(x,s) \le \frac{1}{2}(m(x)+\varepsilon)(s^+)^2 + \Big(\frac{L_{M_2}M_2}{M_1^{2^*}} + \frac{\varepsilon}{2^*}\Big)(s^+)^{2^*} \tag{3.6}$$

for $s \in \mathbb{R}$ and a.e. $x \in \Omega$. From (3.6) and Lemma 3.4 we obtain

$$\begin{aligned} \widetilde{\Phi}_+(u) \ge{}& \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega a(x)u^2 dx - \frac{1}{2}\int_\Omega m(x)u^2 dx \\ &- \frac{1}{2}\int_\Omega (m(x)+\varepsilon)(u^+)^2 dx \\ &- \int_\Omega \Big(\frac{L_{M_2}M_2}{M_1^{2^*}} + \frac{\varepsilon}{2^*}\Big)(u^+)^{2^*} dx + \frac{1}{2}\int_\Omega m(x)(u^+)^2 dx \\ \ge{}& \frac{d}{2}\|u\|^2 - \frac{\varepsilon}{2S_2^2}\|u\|^2 - \Big(\frac{L_{M_2}M_2}{M_1^{2^*}} + \frac{\varepsilon}{2^*}\Big)\Big(\frac{1}{S_{2^*}}\Big)^{2^*}\|u\|^{2^*} \end{aligned}$$

$$= \|u\|^2 \Big[\frac{d}{4} - \Big(\frac{L_{M_2} M_2}{M_1^{2^*}} + \frac{\varepsilon}{2^*}\Big)\Big(\frac{1}{S_{2^*}}\Big)^{2^*} \|u\|^{2^*-2}\Big], \quad \forall u \in E.$$

Let

$$C_1 = \Big(\frac{L_{M_2} M_2}{M_1^{2^*}} + \frac{\varepsilon}{2^*}\Big)\Big(\frac{1}{S_{2^*}}\Big)^{2^*}, \quad R_0 = \Big(\frac{d}{8C_1}\Big)^{\frac{1}{2^*-2}}, \quad \alpha_0 = \frac{d}{8} R_0^2 .$$

Then $\widetilde{\Phi}_+$ satisfies (i) of Theorem 3.1. □

Lemma 3.6. *Assume that* (A16), (A19) *hold. Then* $\widetilde{\Phi}_+$ *satisfies* (ii) *of Theorem 3.1.*

Proof. From (A16) and (A19) it follows that for $\Lambda > \frac{\|\phi_1\|^2 + \int_\Omega a(x)\phi_1^2 dx}{2|\phi_1|_2^2}$, there exists a constant $M_3 > 0$ such that

$$F_+(x,s) \geq \Lambda (s^+)^2 - L_{M_3} M_3$$

for $s \in \mathbb{R}$ and a.e. $x \in \Omega$. Then for $t > 0$, one obtains

$$\widetilde{\Phi}_+(t\phi_1) \leq t^2 \Big(\frac{1}{2}\|\phi_1\|^2 + \frac{1}{2}\int_\Omega a(x)\phi_1^2 dx - \Lambda|\phi_1|_2^2\Big) + M_3 L_{M_3}|\Omega|.$$

Let $C_2 = \frac{1}{2}\big(\|\phi_1\|^2 + \int_\Omega a(x)\phi_1^2 dx\big) - \Lambda|\phi_1|_2^2 < 0$, $C_3 = M_3 L_{M_3}|\Omega| > 0$, $t_0 = \sqrt{\frac{2C_3}{-C_2}} + R_0$ and $e = t_0\phi_1$, then $\widetilde{\Phi}_+$ satisfies (ii) of Theorem 3.1. □

Lemma 3.7. *Assume that* (A16), (A17), (A19), (A20) *hold. Then* $\widetilde{\Phi}_+$ *satisfies the* (C) *condition in Theorem 3.1.*

Proof. For $c \in \mathbb{R}$ and $\{u_n\} \subset E$ such that

$$\|\widetilde{\Phi}'_+(u_n)\|(1 + \|u_n\|) \to 0 \text{ and } \widetilde{\Phi}_+(u_n) \to c \quad \text{as } n \to \infty, \tag{3.7}$$

we first prove that $\{u_n\}$ is bounded. Arguing by contradiction, if $\{u_n\}$ is unbounded, then $\|u_n\| \to +\infty$ as $n \to \infty$ after passing to a subsequence. Set $w_n = \frac{u_n}{\|u_n\|}$, then $\|w_n\| = 1$. Hence, up to subsequence, we may assume that

$$w_n \rightharpoonup w \quad \text{weakly in } E,$$

which results in

$$\begin{gathered} w_n \to w \quad \text{strongly in } L^r(\Omega) \text{ for } r \in [1,\ 2^*), \\ w_n^\pm \rightharpoonup w^\pm \quad \text{weakly in } E, \\ w_n^\pm(x) \to w^\pm(x) \quad \text{a.e. in } \Omega, \\ w_n^\pm \to w^\pm \quad \text{strongly in } L^r(\Omega) \text{ for } r \in [1,\ 2^*). \end{gathered} \tag{3.8}$$

From (A16) and (A19) it follows that there exists a constant $M_4 > \max\{M_1, s_0\}$ such that

$$|F_+(x,s)| \leq L_{M_4}(s^+) \leq L_{M_4} M_4 \tag{3.9}$$

for $|s| \leq M_4$ and a.e. $x \in \Omega$, and $F_+(x,s) \geq (s^+)^2$ for $|s| \geq M_4$ and a.e. $x \in \Omega$. Then we have

$$F_+(x,s) \geq (s^+)^2 - M_4 L_{M_4} - M_4^2 \geq -M_4 L_{M_4} - M_4^2 \tag{3.10}$$

for $s \in \mathbb{R}$ and a.e. $x \in \Omega$.

Now we claim that $w = 0$. In fact, if $w^+ \neq 0$, that is, $|\Omega_+| > 0$, where $\Omega_+ := \{x \in \Omega : w(x) > 0\}$. Then, for a.e. $x \in \Omega_+$, one has $u_n^+(x) = w_n^+(x)\|u_n\| \to +\infty$ as $n \to \infty$, which implies that

$$\lim_{n\to\infty} \frac{F(x, u_n^+(x))}{(u_n^+(x))^2} = +\infty. \tag{3.11}$$

From (3.7) and (3.10) it follows that

$$\begin{aligned}
&\frac{1}{2}\Big(1 + \int_\Omega a(x)w_n^2 dx - \int_\Omega m(x)(w_n^-)^2 dx\Big) - \frac{c + o(1)}{\|u_n\|^2} \\
&= \int_\Omega \frac{F_+(x, u_n)}{\|u_n\|^2} dx \\
&\geq \int_{\Omega_+} \frac{F(x, u_n^+)}{(u_n^+)^2}(w_n^+)^2 dx + \int_{\Omega\setminus\Omega_+} \frac{-M_4 L_{M_4} - M_4^2}{\|u_n\|^2} dx \\
&\geq \int_{\Omega_+} \frac{F(x, u_n^+)}{(u_n^+)^2}(w_n^+)^2 dx - \frac{(M_4 L_{M_4} + M_4^2)|\Omega|}{\|u_n\|^2}.
\end{aligned}$$

Then by Lemma 3.3, Fatou's lemma and (3.11), one obtains

$$\begin{aligned}
&\frac{1}{2}\Big(1 + \int_\Omega a(x)w^2 dx - \int_\Omega m(x)(w^-)^2 dx\Big) \\
&\geq \liminf_{n\to+\infty} \Big(\int_{\Omega_+} \frac{F(x, u_n^+)}{(u_n^+)^2}|w_n|^2 dx\Big) = +\infty,
\end{aligned}$$

a contradiction. Hence $|\Omega_+| = 0$, that is, $w^+ = 0$.

In addition, from (3.7) and Lemma 3.4 it follows that

$$\begin{aligned}
d\|u_n^-\|^2 &\leq \int_\Omega |\nabla(u_n^-)|^2 dx + \int_\Omega a(x)(u_n^-)^2 dx - \int_\Omega m(x)(u_n^-)^2 dx \\
&= \int_\Omega |\nabla(u_n^-)|^2 dx + \int_\Omega a(x)(u_n^-)^2 dx - \int_\Omega m(x)(u_n^-)^2 dx \\
&\quad - \int_\Omega \widetilde{f}_+(x, u_n)u_n^- dx \\
&= \langle \widetilde{\Phi}_+'(u_n), u_n^- \rangle \to 0
\end{aligned}$$

as $n \to \infty$, that is, $u_n^- \to 0$ in E as $n \to \infty$. This together with (3.8) shows $w^- = 0$. To sum up, we have $w = w^+ - w^- = 0$, so the claim is proved.

From (A16) it follows that the term $|sf_+(x, s) - 2F_+(x, s)|$ is bounded in $[0, M_4] \times \Omega$. Set

$$\varpi := \min_{(x,s)\in\Omega\times[0,M_4]} |sf_+(x, s) - 2F_+(x, s)|, \quad \Omega_n := \{x \in \Omega : u_n(x) \geq M_4\}.$$

Then from (3.9) and (A20) we have

$$\begin{aligned}
&\frac{1}{2}\Big(1 + \int_\Omega a(x)w_n^2 dx - \int_\Omega m(x)(w_n^-)^2 dx\Big) - \frac{c + o(1)}{\|u_n\|^2} \\
&= \int_{\Omega\setminus\Omega_n} \frac{F_+(x, u_n)}{\|u_n\|^2} dx + \int_{\Omega_n} \frac{F_+(x, u_n)}{\|u_n\|^2} dx \\
&\leq \int_{\Omega\setminus\Omega_n} \frac{L_{M_4} M_4}{\|u_n\|^2} dx + \Big[\int_{\Omega_n} \Big(\frac{F(x, u_n^+)}{(u_n^+)^2}\Big)^\sigma dx\Big]^{1/\sigma} \Big[\int_{\Omega_n} (w_n^+)^{\frac{2\sigma}{\sigma-1}} dx\Big]^{\frac{\sigma-1}{\sigma}}
\end{aligned}$$

$$\begin{aligned}
&\le \frac{L_{M_4}M_4|\Omega|}{\|u_n\|^2} + \Big[\int_{\Omega_n} \alpha\Big(u_n^+ f(x,u_n^+) - 2F(x,u_n^+)\Big) + W(x)dx\Big]^{1/\sigma} |w_n^+|^2_{\frac{2\sigma}{\sigma-1}} \\
&\le \frac{L_{M_4}M_4|\Omega|}{\|u_n\|^2} + \Big[\alpha\Big(2\widetilde{\Phi}_+(u_n) - \widetilde{\Phi}'_+(u_n)u_n\Big) + \alpha\varpi|\Omega| + |W|_1\Big]^{\frac{1}{\sigma}} |w_n^+|^2_{\frac{2\sigma}{\sigma-1}}.
\end{aligned}$$

Since $\sigma > \frac{N}{2}$, one has $\sigma > 1$ and $\frac{2\sigma}{\sigma-1} \in (1, 2^*)$. By (3.7) and (3.8), letting $n \to \infty$ in the above inequality gives the contradiction $1/2 \le 0.$, Hence $\{u_n\}$ is bounded, that is, $\|u_n\| \le C_4$ for all n, where C_4 is a positive constant independent of n. Hence, up to subsequence, there exists a $u \in E$ such that

$$\begin{gathered}
u_n \rightharpoonup u \quad \text{weakly in } E, \\
u_n \to u \quad \text{strongly in } L^r(\Omega) \text{ for } r \in [1,\ 2^*),
\end{gathered} \tag{3.12}$$

Then by the weak lower semicontinuity of norm, we have $\|u\| \le \liminf_{n\to\infty} \|u_n\| \le C_4$, which implies that $\|u_n - u\| \le \|u_n\| + \|u\| \le 2C_4$.

Additionally, for ε in (3.5), from (3.12) there exists a positive constant $N(\varepsilon)$ such that

$$|u_n - u|_1 < \varepsilon \quad \text{for } n > N(\varepsilon),$$

from this and (3.5) it follows that for $n > N(\varepsilon)$,

$$\begin{aligned}
\Big|\int_\Omega f_+(x,u_n)(u_n-u)dx\Big| &\le \int_\Omega \Big(\varepsilon(u_n^+)^{2^*-1} + L_{M_2}\Big)|u_n - u|dx \\
&\le \varepsilon|u_n|_{2^*}^{2^*-1}|u_n-u|_{2^*} + L_{M_2}|u_n-u|_1 \\
&\le \varepsilon 2\Big(\frac{C_4}{S_{2^*}}\Big)^{2^*} + \varepsilon L_{M_2},
\end{aligned}$$

that is, $\int_\Omega f_+(x,u_n)(u_n-u)dx \to 0$ as $n \to \infty$. From this, (3.7), (3.12), and Lemma 3.3 it follows that

$$\int_\Omega (\nabla u_n, \nabla(u_n - u))dx \to 0$$

as $n \to \infty$. Then one has $\|u_n - u\| \to 0$ as $n \to \infty$. □

Proof of Theorem 1.3. By Lemmas 3.5, 3.6 and 3.7, $\widetilde{\Phi}_+$ has a nontrivial critical point u via Theorem 3.1, that is, for any $v \in E$,

$$\langle \widetilde{\Phi}'_+(u), v\rangle = \int_\Omega (\nabla u, \nabla v)dx + \int_\Omega a(x)uv\,dx - \int_\Omega m(x)uv\,dx - \int_\Omega \widetilde{f}_+(x,u)v\,dx = 0.$$

Letting $v = u^-$ in the above equation gives $\|u^-\| = 0$, so $u = u^+ \ge 0$. Then u is also a critical point of Φ_+; that is,

$$\langle \Phi'_+(u), v\rangle = \int_\Omega (\nabla u, \nabla v)dx + \int_\Omega a(x)uv\,dx - \int_\Omega f_+(x,u)v\,dx = 0,\ \forall\ v \in E.$$

In addition, from (A16), (A17) and $a \in L^\infty(\Omega)$ it follows that there exists positive constant C_ε such that

$$|-a(x)u + f(x,u)| \le C_\varepsilon\left(1 + |u|^{2^*-1}\right)$$

for $s \in \mathbb{R}$ and a.e. $x \in \Omega$. Let $b(x) := \frac{-a(x)u(x)+f(x,u(x))}{1+|u(x)|}$, then $b \in L^{\frac{N}{2}}(\Omega)$ and

$$-\triangle u = b(x)(1 + |u|).$$

[16, Lemma B.3] shows $u \in L^p(\Omega)$ for any $p < \infty$, which implies that $f(x,u) \in L^p(\Omega)$ for any $p < \infty$. By [16, Lemma B.2], we have $u \in H^{2,p}(\Omega) \cap H^1_0(\Omega)$ for any $p < \infty$. Therefore, $u \in C^{1,\beta}(\Omega)$ for some $\beta \in (0,1)$ by the Sobolev embedding theorem. Moreover, from $sf(x,s) \geq 0$ it follows that

$$\triangle u = a(x)u - f(x,u) \leq |a|_\infty u := \zeta(u),$$

where $\zeta : [0,+\infty) \to \mathbb{R}$ is continuous and nondecreasing, and satisfies $\zeta(0) = 0$, $\zeta(s) > 0$ for all $s > 0$, and $\int_0^1 (\zeta(s)s)^{-\frac{1}{2}} ds = +\infty$. Then we can conclude that $u > 0$ in Ω by [20, Theorem 5]. In a similar way, we can obtain a negative solution for problem (1.1) by treating with $\widetilde{\Phi}_-$. □

Proof of Theorem 1.5. Without loss of generality, we assume that

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_{k_0} \leq 0 < \lambda_{k_0+1} \leq \dots \leq \lambda_k \dots$$

and e_k is eigenfunction corresponding to λ_k. Set $E_k = \operatorname{span}\{e_1, e_2, \dots, e_k\}$ and $E_k^\perp$ be the orthogonal complement of E_k in E. Then one has

$$\int_\Omega |\nabla u|^2 dx + \int_\Omega a(x)u^2 dx \leq \lambda_k \int_\Omega u^2 dx, \quad \forall u \in E_k,$$
$$\int_\Omega |\nabla u|^2 dx + \int_\Omega a(x)u^2 dx \geq \lambda_{k+1} \int_\Omega u^2 dx, \quad \forall u \in E_k^\perp.$$

Hence in $E_k^\perp$ with $k \geq k_0$, $\|u\|_\star := \big\{\int_\Omega |\nabla u|^2 dx + \int_\Omega a(x)u^2 dx\big\}^{1/2}$ is also a norm and is equivalent to $\|u\|$. Hence for $k \geq k_0$, there exists a positive constant C_5 such that

$$\|u\|_\star \geq \sqrt{C_5}\|u\|, \quad \forall u \in E_k^\perp.$$

Similar to (3.5), from (A16) and (A17) it follows that

$$|f(x,s)| \leq \varepsilon |s|^{2^*-1} + L_{M_2}$$

for $s \in \mathbb{R}$ and a.e. $x \in \Omega$. Set $\varrho_k := \sup_{u \in E_k^\perp, \|u\|=1} |u|_1$. It was shown in [22, Lemma 3.8] that $\varrho_k \to 0$ as $k \to \infty$. Let $X^1 = E_k^\perp$ with $k \geq k_0$ such that $\varrho_k < \frac{C_5}{8L_{M_2}}$,

$$R_1 = \Big(\frac{2^* S_{2^*}^{2^*}}{4\varepsilon} C_5\Big)^{\frac{1}{2^*-1}} > 0,$$

for $u \in E_k^\perp$ with $\|u\| = R_1$, we have

$$\begin{aligned}
\Phi(u) &\geq \frac{1}{2}\|u\|_\star^2 - \frac{\varepsilon}{2^*}|u|_{2^*}^{2^*} - L_{M_2}|u|_1 \\
&\geq \|u\| \Big(\frac{C_5}{2}\|u\| - \frac{\varepsilon}{2^* S_{2^*}^{2^*}}\|u\|^{2^*-1} - \varrho_k L_{M_2}\Big) \\
&\geq \frac{1}{8} C_5 R_1.
\end{aligned}$$

then Φ satisfies (i) of Theorem 3.2 with $\alpha_1 = \frac{1}{8}C_5 R_1 > 0$.

For every E_k, there exists a positive constant C_6 such that

$$\|u\| \leq \sqrt{C_6}|u|_2, \quad \forall\ u \in E_k,$$

because all the norms on the finite dimension space E_k are equivalent. From (A19), there exists a positive constant C_7 such that

$$F(x,s) \geq \Big(\frac{|\lambda_k|}{2} + 1\Big)s^2 - C_7$$

for $s \in \mathbb{R}$ and a.e. $x \in \Omega$. Set $R_2 = \sqrt{C_6 C_7 |\Omega|}$, for $u \in E_k$ with $\|u\| = R_2$,

$$\Phi(u) \leq \frac{\lambda_k}{2}|u|_2^2 - \Big(\frac{|\lambda_k|}{2} + 1\Big)|u|_2^2 + C_7|\Omega| \leq -\frac{1}{C_6}\|u\|^2 + C_7|\Omega| \leq 0,$$

then Φ satisfies (ii) of Theorem 3.2.

Lastly, in a way similar to treat with $\widetilde{\Phi}_+$ in Lemma 3.7, we can prove that Φ satisfies the (C) condition. Therefore Theorem 3.2 shows that Φ has a unbounded sequence of critical values. □

Acknowledgments. This research was supported by National Natural Science Foundation of China (No. 11471267, No. 11226118),by the Fundamental Research Funds for the Central Universities(XDJK2014C161), by the Doctoral Fund of Southwest University (SWU111060).

The authors express their gratitude to the reviewers for the valuable comments and helpful suggestions which lead to the improvement of the manuscript.

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Xiao-Feng Ke
School of Mathematics and Statistics, Southwest University, Chongqing 400715, China
E-mail address: kexf@swu.edu.cn

Chun-Lei Tang (corresponding author)
School of Mathematics and Statistics, Southwest University, Chongqing 400715, China
E-mail address: tangcl@swu.edu.cn