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POSITIVE SOLUTIONS FOR THE ONE-DIMENSIONAL STURM-LIOUVILLE SUPERLINEAR p-LAPLACIAN PROBLEM

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Abstract. We prove the existence of positive classical solutions for the p-Laplacian problem

$$-(r(t)\phi(u'))' = f(t,u), \quad t \in (0,1),$$
$$au(0) - b\phi^{-1}(r(0))u'(0) = 0,\ cu(1) + d\phi^{-1}(r(1))u'(1) = 0,$$

where $\phi(s) = |s|^{p-2}s$, $p > 1$, $f : (0,1) \times [0,\infty) \to \mathbb{R}$ is a Carathéodory function satisfying

$$\limsup_{z\to 0^+} \frac{f(t,z)}{z^{p-1}} < \lambda_1 < \liminf_{z\to\infty} \frac{f(t,z)}{z^{p-1}}$$

uniformly for a.e. $t \in (0,1)$, where λ_1 denotes the principal eigenvalue of $-(r(t)\phi(u'))'$ with Sturm-Liouville boundary conditions. Our result extends a previous work by Manásevich, Njoku, and Zanolin to the Sturm-Liouville boundary conditions with more general operator.

1. Introduction

Consider the one-dimensional p-Laplacian problem

$$\begin{gathered} -(r(t)\phi(u'))' = f(t,u) \quad \text{a.e. on } (0,1), \\ au(0) - b\phi^{-1}(r(0))u'(0) = 0, \quad cu(1) + d\phi^{-1}(r(1))u'(1) = 0, \end{gathered} \tag{1.1}$$

where $\phi(s) = |s|^{p-2}s$, $p > 1$, a, b, c, d are nonnegative constants with $ac+ad+bc > 0$, $r : [0,1] \to (0,\infty)$ and $f : (0,1) \times [0,\infty) \to \mathbb{R}$.

We are interested in positive classical solution of (1.1), that is, solutions $u \in C^1[0,1]$ with $u > 0$ on $(0,1)$, $\phi(u')$ absolutely continuous on $[0,1]$ and satisfying (1.1).

Let us look at the literature on problem (1.1) with Dirichlet boundary conditions i.e. $b = d = 0$. In the sublinear case, Lan, Yang, and Yang [14] proved the existence of a classical positive solution to (1.1) when $r(t) \equiv 1$ and f is nonnegative with

$$\limsup_{z\to\infty} \frac{f(t,z)}{z^{p-1}} < \lambda_1 < \liminf_{z\to 0^+} \frac{f(t,z)}{z^{p-1}} \le \infty \tag{1.2}$$

uniformly for a.e. $t \in (0,1)$, where $\lambda_1 = 2^p(p-1)(\int_0^1 \frac{ds}{(1-s^p)^{1/p}})^p$ is the principal eigenvalue of $-(\phi(u'))'$ with zero boundary conditions (see [4, 5]). In particular,

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when $p = 2$ and $f : [0, \infty) \to [0, \infty)$ is continuous, (1.2) becomes

$$\limsup_{z\to\infty} \frac{f(z)}{z} < \pi^2 < \liminf_{z\to 0^+} \frac{f(z)}{z} \le \infty,$$

which was used by Webb and Lan [18] to obtain nonnegative solutions to (1.1) with $\phi(s) = s$. In fact, [18] gave a general method with covered many boundary conditions including nonlocal ones and included both sublinear and superlinear types of conditions. In the superlinear case, Manásevich, Njoku, and Zanolin [15] used time-mapping estimates to prove the existence of a classical positive solution to (1.1) with Dirichlet boundary conditions when $r(t) \equiv 1$,

$$\limsup_{z\to 0^+} \frac{f(t,z)}{z^{p-1}} < \lambda_1 < \liminf_{z\to\infty} \frac{f(t,z)}{z^{p-1}} \le \infty \tag{1.3}$$

and $\liminf_{z\to 0^+} \frac{f(t,z)}{z^{p-1}} > -\infty$ uniformly for a.e. $t \in (0,1)$, which improves a previous result by Kaper, Knapp, and Kwong [11] where the stronger condition

$$\lim_{z\to 0^+} \frac{f(t,z)}{z^{p-1}} = l \le 0 \quad \text{and} \quad \lim_{z\to\infty} \frac{f(t,z)}{z^{p-1}} = \infty$$

uniformly for $t \in (0,1)$ was used. Note that when $p = 2$ and f is independent of t, condition (1.3) together with $f(0) = 0$ and $f \ge 0$ was used in [8] to show the existence of a positive solution to the PDE problem

$$-\Delta u = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Wang [19] showed the existence of a positive solution to (1.1) under nonlinear boundary conditions that include the Sturm-Liouville one when f is nonnegative and satisfies either the sublinear condition

$$\lim_{z\to 0^+} \frac{f(z)}{z^{p-1}} = \infty \quad \text{and} \quad \lim_{z\to\infty} \frac{f(z)}{z^{p-1}} = 0,$$

or the superlinear one

$$\lim_{z\to 0^+} \frac{f(z)}{z^{p-1}} = 0 \quad \text{and} \quad \lim_{z\to\infty} \frac{f(z)}{z^{p-1}} = \infty,$$

which extended a previous result by Erbe and Wang [7] when $p = 2$. Similar results were established in [9] for singular Sturm-Liouville boundary value problems. Note that the conditions in [7, 9, 19] do not involve the principal eigenvalue of the corresponding operator. Existence results in the PDE version of (1.1) involving the principal eigenvalue of the p-Laplacian operator for $p \ge 2$ was studied in [3]. In particular, the existence of a nontrivial nonnegative weak solution $u \in W_0^{1,p}(\Omega)$ to the problem

$$\begin{gathered} -\Delta_p u = f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

was established for f satisfying $|f(z)|((1 + z^{p-1})^{-1}$ bounded on $[0, \infty)$ and either

$$-\infty < \lim_{z\to 0^+} \frac{f(z)}{z^{p-1}} < \lambda_1 < \lim_{z\to\infty} \frac{f(z)}{z^{p-1}} < \infty,$$

or

$$-\infty < \lim_{z\to\infty} \frac{f(z)}{z^{p-1}} < \lambda_1 < \lim_{z\to 0^+} \frac{f(z)}{z^{p-1}} < \infty$$

holds. The approach used in [3] was via the Granas fixed point index (see [6]). In this paper, we shall extend the result in [15] to include the general Sturm-Liouville boundary conditions with more general operator e.g. allowing the case $r \not\equiv 1$. Note that the proof in [15] does not apply to this general context. Since we do not require that f be non-negative but that there exists $\eta \in L^1(0,1)$ with $\eta \geq 0$ such that $\liminf_{z\to 0^+} \frac{f(t,z)}{z^{p-1}} \geq -\eta(t)$ uniformly for a.e. $t \in (0,1)$, our result also improves a corresponding result in [12]. In addition, some estimates on the principal eigenvalue λ_1 for $p > 1$ are provided (see Lemma 2.7 below). We refer to [10, 13, 16, 20] for existence results related to (1.1) under suitable sublinear or superlinear conditions. Our approach is based on a Krasnoselskii type fixed point theorem in a Banach space.

We shall make the following assumptions:

- (A1) $r : [0,1] \to (0,\infty)$ is continuous.
- (A2) $f : (0,1) \times [0,\infty)$ is a Carathéodory function, that is $f(\cdot, z)$ is measurable for each $z \geq 0$ and $f(t,\cdot)$ is continuous for a.e. $t \in (0,1)$.
- (A3) For each $k > 0$, there exists $\gamma_k \in L^1(0,1)$ such that
$$|f(t,z)| \leq \gamma_k(t)$$
for a.e. $t \in (0,1)$ and $z \in [0,k]$.
- (A4) There exists $\eta \in L^1(0,1)$ with $\eta \geq 0$ such that
$$\liminf_{z\to 0^+} \frac{f(t,z)}{z^{p-1}} \geq -\eta(t)$$
uniformly for a.e. $t \in (0,1)$.
- (A5)
$$\limsup_{z\to 0^+} \frac{f(t,z)}{z^{p-1}} < \lambda_1 < \liminf_{z\to\infty} \frac{f(t,z)}{z^{p-1}}$$
uniformly for a.e. $t \in (0,1)$.

Our main result reads as follows.

Theorem 1.1. *Let* (A1)–(A5) *hold. Then* (1.1) *has a positive classical solution* u *with* $\inf_{t\in(0,1)} \frac{u(t)}{p(t)} > 0$, *where* $p(t) = \min(at + b, d + c(1-t))$.

In particular, when f is independent of t, we obtain the following result.

Corollary 1.2. *Let* r *satisfy* (A1) *and let* $f : [0,\infty) \to \mathbb{R}$ *be continuous with*
$$-\infty < \lim_{z\to 0^+} \frac{f(z)}{z^{p-1}} < \lambda_1 < \lim_{z\to\infty} \frac{f(z)}{z^{p-1}} \leq \infty.$$
Then (1.1) *has a positive classical solution* u *with* $\inf_{t\in(0,1)} \frac{u(t)}{p(t)} > 0$.

2. Preliminaries

Let $AC^1[0,1] = \{u \in C^1[0,1] : u' \text{ is absolutely continuous on } [0,1]\}$. We shall denote the norm in $L^q(0,1)$ and $C^1[0,1]$ by $\|\cdot\|_q$ and $|\cdot|_{C^1}$ respectively. Let λ_1 be the principal eigenvalue of $-(r(t)\phi(u'))'$ on $(0,1)$ with Sturm-Liouville boundary conditions, and let ϕ_1 be the corresponding positive, normalized eigenfunction, i.e. $-(r(t)|\phi_1'|^{p-2}\phi_1')' = \lambda_1\phi_1^{p-1}$ a.e. on $(0,1), \phi_1 > 0$ on $(0,1)$, $\|\phi_1\|_\infty = 1$ and ϕ_1 satisfies the Sturm-Liouville boundary conditions in (1.1) (see [2, Theorem 3.1]). Note that $\lambda_1 > 0$. We recall the following fixed point theorem of Krasnoselskii type in a Banach space (see Amann [1, Theorem 12.3]).

Lemma 2.1. *Let E be a Banach space and $A : E \to E$ be a completely continuous operator. Suppose there exist $h \in E, h \neq 0$ and positive constants r, R with $r \neq R$ such that*

(a) *If $y \in E$ satisfies $y = \theta Ay$ for some $\theta \in (0, 1]$ then $\|y\| \neq r$,*
(b) *If $y \in E$ satisfies $y = Ay + \xi h$ for some $\xi \geq 0$ then $\|y\| \neq R$.*

Then A has a fixed point $y \in E$ with $\min(r, R) < \|y\| < \max(r, R)$.

Lemma 2.2. *Let t_0, t_1, α, β be constants with $0 \leq t_0 < t_1 \leq 1$, and $h \in L^1(t_0, t_1)$. Then the problem*

$$
\begin{gathered}
-(r(t)\phi(u'))' = h \quad \text{a.e. on } (t_0, t_1), \\
au(t_0) - b\phi^{-1}(r(t_0))u'(t_0) = \alpha, \quad cu(t_1) + d\phi^{-1}(r(t_1))u'(t_1) = \beta
\end{gathered}
\tag{2.1}
$$

has a unique solution $u = Th \in AC^1[t_0, t_1]$. Furthermore $T : L^1(t_0, t_1) \to C[t_0, t_1]$ is completely continuous.

Proof. By integrating, it follows that (2.1) has a unique solution $u \in AC^1[t_0, t_1]$ given by

$$
u(t) = C + \int_{t_0}^{t} \phi^{-1}\Big(\frac{D - \int_{t_0}^{s} h}{r(s)}\Big) ds,
$$

where C and D are constants satisfying

$$
\begin{gathered}
aC - b\phi^{-1}(D) = \alpha, \\
c\Big(C + \int_{t_0}^{t_1} \phi^{-1}(\frac{D - \int_{t_0}^{s} h}{r(s)}) ds\Big) + d\phi^{-1}\Big(D - \int_{t_0}^{t_1} h\Big) = \beta.
\end{gathered}
\tag{2.2}
$$

In what follows, we shall see, in particular, that C, D are uniquely determined. We shall denote by $K_i, i = 0, 1, 2, \dots,$ positive constants independent of u and h.

Case 1: $a = 0$**.** Then $b, c > 0$, $D = -\phi(\alpha/b)$ and

$$
C = \frac{\beta - d\phi^{-1}\big(D - \int_{t_0}^{t_1} h\big)}{c} - \int_{t_0}^{t_1} \phi^{-1}\Big(\frac{D - \int_{t_0}^{s} h}{r(s)}\Big) ds.
$$

Using the inequality

$$
(x + y)^q \leq m(x^q + y^q) \text{ for } x, y \geq 0, q > 0,
\tag{2.3}
$$

where $m = 2^{(q-1)^+}$, we deduce that $|C| \leq K_1 + K_2\phi^{-1}(\|h\|_1)$, which implies

$$
\|u\|_\infty \leq K_3 + K_4\phi^{-1}(\|h\|_1).
$$

Case 2: $a > 0$**.** Then (2.2) is equivalent to $C = \frac{\alpha + b\phi^{-1}(D)}{a}$, where D is the solution of

$$
\gamma(D) \equiv \frac{cb\phi^{-1}(D)}{a} + c\int_{t_0}^{t_1} \phi^{-1}\Big(\frac{D - \int_{t_0}^{s} h}{r(s)}\Big) ds + d\phi^{-1}\Big(D - \int_{t_0}^{t_1} h\Big) = \beta - \frac{\alpha c}{a}.
$$

Note that D is uniquely determined since $\gamma(D)$ is increasing in D, $\lim_{D\to\infty} \gamma(D) = \infty$ and $\lim_{D\to-\infty} \gamma(D) = -\infty$.

If $c = 0$ then $d > 0$ and it follows that $|D| \leq \|h\|_1 + \phi(|\beta|/d)$, while if $c > 0$ then

$$
|D| \leq \|h\|_1 + \|r\|_\infty \phi\Big(\frac{1}{c(t_1 - t_0)}|\beta - \frac{\alpha c}{a}|\Big).
$$

Hence in both cases,

$$|u|_{C^1[t_0,t_1]} = \|u\|_\infty + \|u'\|_\infty \le K_5 + K_0\phi^{-1}(\|h\|_1).$$

i.e. T maps bounded sets in $L^1(t_0,t_1)$ into bounded sets in $C^1[t_0,t_1]$. To show that T is continuous, let $\varepsilon > 0$, $h_i \in L^1(t_0,t_1)$ and $u_i = Th_i$, $i = 1,2$. We shall show that there exists a constant $\delta > 0$ depending on ε and an upper bound of $\|h_i\|_{L^1(t_0,t_1)}$, $i = 1,2$, such that

$$\|h_1 - h_2\|_{L^1(t_0,t_1)} < \delta \Longrightarrow |u_1 - u_2|_{C^1[t_0,t_1]} < \varepsilon. \tag{2.4}$$

Note that

$$u_i(t) = C_i + \int_{t_0}^t \phi^{-1}\Big(\frac{D_i - \int_{t_0}^s h_i}{r(s)}\Big)ds,$$

and from the above calculation we obtain

$$|D_i| \le \max_{i=1,2} \|h_i\|_{L^1(t_0,t_1)} + K \equiv M_0$$

for $i = 1,2$, where $K > 0$ independent of u_i and h_i. This implies

$$|D_i - \int_{t_0}^s h_i|,\ |\frac{D_i - \int_{t_0}^s h_i}{r(s)}| \le 2M_0 \max(r_0^{-1}, 1) \equiv M$$

for all $s \in [t_0,t_1], i = 1,2$, where $r_0 = \min_{[0,1]} r > 0$. Since ϕ^{-1} is uniformly continuous on $I = [-M, M]$, it follows from the formulas for C_i, D_i, and the fact that $|D_1 - D_2| \le \|h_1 - h_2\|_{L^1(t_0,t_1)}$ that there exists a constant $\delta > 0$ such that (2.4) holds. This completes the proof. □

Remark 2.3. If $\alpha = \beta = 0$ then Lemma 2.2 is reduced to [9, Lemma 3.1]. Note that in this case $K_5 = 0$ in the above proof i.e. $|u|_{C^1[t_0,t_1]} \le K_0\phi^{-1}(\|h\|_1)$ for all u satisfying (2.1).

Lemma 2.4. *Let t_0, t_1, α, β be constants with $0 \le t_0 < t_1 \le 1$, and $\gamma, h \in L^1(t_0,t_1)$ with $\gamma \ge 0$. Then the problem*

$$\begin{gathered} -(r(t)\phi(u'))' + \gamma(t)\phi(u) = h(t) \quad \text{a.e. on } (t_0,t_1), \\ au(t_0) - b\phi^{-1}(r(t_0))u'(t_0) = \alpha, \quad cu(t_1) + d\phi^{-1}(r(t_1))u'(t_1) = \beta \end{gathered} \tag{2.5}$$

has a unique solution $u \equiv T_0 h \in AC^1[t_0,t_1]$. Furthermore $T_0 : L^1(t_0,t_1) \to C[t_0,t_1]$ is completely continuous.

Proof. Let $E = C[t_0,t_1]$ be equipped with norm $\|u\| = \sup_{[t_0,t_1]} |u|$. By Lemma 2.2, for each $v \in E$, the problem

$$\begin{gathered} -(r(t)\phi(u'))' = h(t) - \gamma(t)\phi(v) \quad \text{a.e. on } (t_0,t_1), \\ au(t_0) - b\phi^{-1}(r(t_0))u'(t_0) = \alpha, \quad cu(t_1) + d\phi^{-1}(r(t_1))u'(t_1) = \beta \end{gathered}$$

has a unique solution $u = Sv \in AC^1[t_0,t_1]$ and $S : E \to E$ is completely continuous. Let $u \in E$ satisfy $u = \theta Su$ for some $\theta \in (0,1]$. Then

$$\begin{gathered} -(r(t)\phi(u'))' + \theta^{p-1}\gamma(t)\phi(u) = \theta^{p-1}h(t) \quad \text{a.e. on } (t_0,t_1), \\ au(t_0) - b\phi^{-1}(r(t_0))u'(t_0) = \theta\alpha, \quad cu(t_1) + d\phi^{-1}(r(t_1))u'(t_1) = \theta\beta \end{gathered} \tag{2.6}$$

By integrating (2.6), we obtain

$$\phi(u'(t)) = \frac{r(t_1)\phi(u'(t_1)) + \theta^{p-1}\int_t^{t_1}(h - \gamma\phi(u))ds}{r(t)} \tag{2.7}$$

for $t \in [t_0, t_1]$. Multiplying the equation in (2.6) by u and integrating gives

$$-r(t_1)\phi(u'(t_1))u(t_1) + r(t_0)\phi(u'(t_0))u(t_0) + \int_{t_0}^{t_1} r(t)|u'|^p \le \int_{t_0}^{t_1} |hu|. \tag{2.8}$$

We shall consider two cases.

Case 1. $b = 0$ **or** $d = 0$**.** Without loss of generality, we suppose $b = 0$. Then $u(t_0) = \theta\alpha/a \equiv \theta\alpha_0$. By the mean value theorem,

$$\|u\| \le |\alpha_0| + \int_{t_0}^{t_1} |u'|. \tag{2.9}$$

Suppose first that $d = 0$. Then $u(t_1) = \theta\beta/c \equiv \theta\beta_0$. Let $\xi(t) = \theta(At + B)$, where A, B are constants such that $\xi(t_0) = \theta\alpha_0, \xi(t_1) = \theta\beta_0$ i.e. $A = \frac{\beta_0-\alpha_0}{t_1-t_0}, B = \frac{\alpha_0 t_1-\beta_0 t_0}{t_1-t_0}$. In what follows, we shall denote by R_i, $i = 0, 1, \dots$, positive constants independent of u and θ.

Multiplying the equation in (2.6) by $(u - \xi)$ and integrating, we obtain

$$\begin{aligned} r_0 \int_{t_0}^{t_1} |u'|^p &\le |A|\|r\|_\infty \int_{t_0}^{t_1} |u'|^{p-1} + (|A| + |B|)|\Big(\int_{t_0}^{t_1} \gamma\Big)\|u\|^{p-1} \\ &\quad + (\|u\| + A + B)\int_{t_0}^{t_1} h. \end{aligned}$$

This, together with (2.9), implies $\int_{t_0}^{t_1} |u'|^p \le R_0$.

Suppose next that $d > 0$. Then from the boundary condition at t_1, we obtain $u'(t_1) = \frac{\theta\beta - cu(t_1)}{d\phi^{-1}(r(t_1))}$. Hence if $c = 0$ then $u'(t_1) = \frac{\theta\beta}{d\phi^{-1}(r(t_1))} \equiv \theta\beta_1$ from which (2.7) and (2.9) imply

$$\|u'\| \le R_1\Big(1 + \int_{t_0}^{t_1} |u'|\Big). \tag{2.10}$$

Consequently, (2.8) gives

$$\int_{t_0}^{t_1} r(t)|u'|^p \le \|r\|_\infty \ (|\beta_1|^{p-1}|\|u\| + |\alpha_0|\|u'\|^{p-1}) + \Big(\int_{t_0}^{t_1} |h|\Big)\|u\|,$$

which, together with (2.9) and (2.10), implies that $\int_{t_0}^{t_1} |u'|^p \le R_2$.

If $c > 0$, then

$$\begin{aligned} &-r(t_1)\phi(u'(t_1))u(t_1) \\ &= r(t_1)\phi\Big(\frac{cu(t_1) - \theta\beta}{d\phi^{-1}(r(t_1))}\Big)u(t_1) \\ &= r(t_1)\phi\Big(\frac{cu(t_1) - \theta\beta}{d\phi^{-1}(r(t_1))}\Big)\Big(\Big(\frac{cu(t_1) - \theta\beta}{d\phi^{-1}(r(t_1))}\Big)\Big(\frac{d\phi^{-1}(r(t_1))}{c}\Big) + \frac{\theta\beta}{c}\Big) \\ &\ge R_2\Big|\frac{cu(t_1) - \theta\beta}{d\phi^{-1}(r(t_1))}\Big|^p - R_3. \end{aligned} \tag{2.11}$$

By (2.7) and (2.9),

$$|\phi(u'(t_0)| \le \frac{1}{r_0}\Big(\|r\|_\infty|\frac{cu(t_1) - \theta\beta}{d\phi^{-1}(r(t_1))}|^{p-1} + \int_{t_0}^{t_1} |h| \ + (\int_{t_0}^{t_1} \gamma)\|u\|^{p-1}\Big). \tag{2.12}$$

Using (2.9), (2.11) and (2.12) together with $u(t_0)=\theta\alpha_0$ in (2.8), we deduce that $\int_{t_0}^{t_1}|u'|^p\le R_4$. Hence in either case $\int_{t_0}^{t_1}|u'|^p\le R_5$, where $R_5=\max(R_0,R_2,R_4)$ and so $\|u\|\le|\alpha_0|+R_5^{1/p}$.

Case 2. $b>0,d>0$**.** Then $u'(t_0)=\frac{\alpha u(t_0)-\theta\alpha}{b\phi^{-1}(r(t_0))}$ and $u'(t_1)=\frac{\theta\beta-cu(t_1)}{d\phi^{-1}(r(t_1))}$. Hence (2.8) and (2.9) give

$$\begin{aligned} &r(t_1)\phi\Big(\frac{cu(t_1)-\theta\beta}{d\phi^{-1}(r(t_1))}\Big)u(t_1)+r(t_0)\phi\Big(\frac{\alpha u(t_0)-\theta\alpha}{b\phi^{-1}(r(t_0))}\Big)u(t_0)+\int_{t_0}^{t_1}r(t)|u'|^p\\ &\le\Big(\int_{t_0}^{t_1}|h|\Big)\|u\|. \end{aligned}\tag{2.13}$$

Since $a+c>0$, we can assume without loss of generality that $c>0$. Then

$$\begin{aligned} \|u\|&\le|u(t_1)|+\int_{t_0}^{t_1}|u'|\\ &\le\frac{d\phi^{-1}(r(t_1))}{c}\Big|\frac{cu(t_1)-\theta\beta}{d\phi^{-1}(r(t_1))}\Big|+\frac{|\beta|}{c}+\int_{t_0}^{t_1}|u'|, \end{aligned}$$

which, together with (2.11) and (2.13), imply

$$\Big|\frac{cu(t_1)-\theta\beta}{d\phi^{-1}(r(t_1))}\Big|^p+\int_{t_0}^{t_1}|u'|^p\le R_6.$$

Consequently, $\|u\|<R_8$. Thus, we have shown that in both cases that $\|u\|$ is bounded by a constant independent of u and θ. By the Leray-Schauder fixed point theorem, S has a fixed point u, which is a solution of (2.5) in $AC^1[t_0,t_1]$. To show uniqueness, let u,v be solutions of (2.5). Then

$$-(r(t)(\phi(u')-\phi(v'))'+\gamma(t)(\phi(u)-\phi(v))=0\quad\text{a.e. on } (t_0,t_1).\tag{2.14}$$

We claim that $(\phi(u'(t_0))-\phi(v'(t_0))(u(t_0-v(t_0)\ge0$. This is true when $b=0$ since $u(t_0)=\alpha/a=v(t_0)$ in this case. If $b>0$ then $u'(t_0)=\frac{au(t_0)-\alpha}{b\phi^{-1}(r(t_0))}$, $v'(t_0)=\frac{av(t_0)-\alpha}{b\phi^{-1}(r(t_0))}$, which implies

$$\begin{aligned} &(\phi(u'(t_0))-\phi(v'(t_0))(u(t_0)-v(t_0)\\ &=\Big(\phi\Big(\frac{au(t_0)-\alpha}{b\phi^{-1}(r(t_0))}\Big)-\phi\Big(\frac{av(t_0)-\alpha}{b\phi^{-1}(r(t_0))}\Big)\Big)(u(t_0-v(t_0)\ge0. \end{aligned}$$

Similarly, $(\phi(u'(t_1))-\phi(v'(t_1))(u(t_1-v(t_1)\le0$. Hence, multiplying (2.14) by $u-v$ and integrating, we get

$$\int_{t_0}^{t_1}r(t)(\phi(u')-\phi(v'))(u'-v')dt\le0,$$

which implies $u'=v'$ on (t_1,t_2). Hence there exists a constant k such that $u(t)=v(t)+k$ for all $t\in[t_1,t_2]$. The boundary conditions then give $ak=ck=0$. Hence $k=0$, which completes the proof. □

Next, we prove a comparison principle, which extends [9, Lemma 3.2] to the case $\gamma\ge0$, $\gamma\not\equiv0$.

Lemma 2.5. *Let $\gamma, h_i \in L^1(t_0, t_1)$, $i = 1, 2$, with $\gamma \geq 0$ and $h_1 \geq h_2$. Let $u_i \in AC^1[t_0, t_1]$, $i = 1, 2$ satisfy*

$$
\begin{gathered}
-(r(t)\phi(u_i'))' + \gamma(t)\phi(u_i) = h_i \quad \text{a.e. on } (t_0, t_1), \\
au_1(t_0) - b\phi^{-1}(r(t_0))u_1'(t_0) \geq au_2(t_0) - b\phi^{-1}(r(t_0))u_2'(t_0), \\
cu_1(t_1) + d\phi^{-1}(r(t_1))u_1'(t_1) \geq cu_2(t_1) + d\phi^{-1}(r(t_1))u_2'(t_1).
\end{gathered}
$$

Then $u_1 \geq u_2$ on $[t_0, t_1]$.

Proof. Suppose on the contrary that there exists $\tilde{t} \in (t_0, t_1)$ such that $u_1(\tilde{t}) < u_2(\tilde{t})$. Let $(\alpha, \beta) \subset (t_0, t_1)$ be the largest open interval containing $\tilde{t}$ such that $u_1 < u_2$ on (α, β). Hence

$$
(r(t)(\phi(u_1') - \phi(u_2'))' \leq 0 \quad \text{a.e. on } (\alpha, \beta), \tag{2.15}
$$

Case 1. $u_1(\alpha) = u_2(\alpha)$ or $u_1(\beta) = u_2(\beta)$. Suppose $u_1(\alpha) = u_2(\alpha)$. Then $u_1'(\alpha) \leq u_2'(\alpha)$. Hence (2.15) implies $u_1' \leq u_2'$ on (α, β). If $u_1(\beta) = u_2(\beta)$ then this gives $u_1 \geq u_2$ on (α, β), a contradiction. If $u_1(\beta) < u_2(\beta)$ then $\beta = t_1$ and from the boundary condition at t_1, we get $d(u_2'(t_1) - u_1'(t_1)) \leq 0$. Hence if $d > 0$ we get $u_2'(t_1) \leq u_1'(t_1)$ from which (2.15) gives $u_1' \geq u_2'$ on (α, β) and so $u_1 \geq u_2$ on (α, β), a contradiction. On the other hand, if $d = 0$ then $c(u_1(t_1) - u_2(t_1)) \geq 0$, which implies $u_1(t_1) \geq u_2(t_1)$, a contradiction. Similarly, we get a contradiction if $u_1(\beta) = u_2(\beta)$.

Case 2. $u_1 < u_2$ on $[\alpha, \beta]$ i.e. $\alpha = t_0$ and $\beta = t_1$. Suppose $\min_{[\alpha,\beta]}(u_1 - u_2) = u_1(\tau) - u_2(\tau) < 0$ for some $\tau \in [\alpha, \beta]$. If $\tau \in (t_0, t_1)$ then $u_1'(\tau) = u_2'(\tau)$ and it follows from (2.15) that there exists a constant $k < 0$ such that $u_1 = u_2 + k$ on $[t_0, t_1]$. Using the boundary conditions, we deduce that $ak, ck \geq 0$, a contradiction. Suppose $\tau = t_0$. Then

$$
a(u_1(t_0) - u_2(t_0)) \geq b\phi^{-1}(r(t_0))(u_1'(t_0) - u_2'(t_0)) \geq 0,
$$

which implies $a = 0$. Hence $b > 0$ and the boundary condition at t_0 imply $u_1'(t_0) - u_2'(t_0) \leq 0$, from which (2.15) gives $u_1' \leq u_2'$ on (t_0, t_1). Consequently, $u_1 = u_2 + \tilde{k}$ on (t_0, t_1) for some constant $\tilde{k} < 0$, a contradiction. Similarly, we reach a contradiction when $\tau = t_1$, which completes the proof. □

The next result plays an important role in the proof of the main result. When $\gamma \equiv 0$, it was obtained in [9, Lemma 3.4] but the proof there does not apply to the case $\gamma \not\equiv 0$.

Lemma 2.6. *Let $\gamma \in L^1(0, 1)$ with $\gamma \geq 0$ and let $u \in AC^1[0, 1]$ satisfy*

$$
\begin{gathered}
-(r(t)(\phi(u'))' + \gamma(t)\phi(u) \geq 0 \quad \text{a.e. on } (0, 1), \\
au(0) - b\phi^{-1}(r(0))u'(0) \geq 0, \quad cu(1) + d\phi^{-1}(r(1))u'(1) \geq 0.
\end{gathered}
$$

Then there exists a constant $\kappa > 0$ independent of u such that for all $t \in [0, 1]$,

$$
u(t) \geq \kappa \|u\|_\infty p(t).
$$

Proof. By Lemma 2.5, $u \geq 0$ on $[0, 1]$. Suppose $\|u\|_\infty = u(\tau)$ for some $\tau \in (0, 1)$. By Lemma 2.4, the problem

$$
\begin{gathered}
-(r(t)\phi(z'))' + \gamma(t)\phi(z) = 0 \quad \text{a.e. on } (0, \tau), \\
az(0) - b\phi^{-1}(r(0))z'(0) = 0, \quad z(\tau) = \|u\|_\infty
\end{gathered}
$$

has a unique solution $z \in AC^1[0,\tau]$. By Lemma 2.5, $u \geq z \geq 0$ on $[0,\tau]$, from which the boundary condition on z at 0 gives $z'(0) \geq 0$. Note that

$$z(t) = z(0) + \int_0^t \phi^{-1}\Big(\frac{r(0)\phi(z'(0)) + \int_0^s \gamma(\xi)\phi(z)d\xi}{r(s)}\Big)ds,$$

from which (2.3) gives

$$z(t) \leq z(0) + m_0\Big(z'(0) + \phi^{-1}(\int_0^t \gamma(s)\phi(z)ds)\Big),$$

where $m_0 > 0$ is a constant independent of u. Hence using (2.3) again, it follows that

$$\phi(z(t)) \leq m_1\Big(\phi(z(0) + z'(0)) + \int_0^t \gamma(s)\phi(z)ds\Big)$$

for $t \in [0,\tau]$, where $m_1 > 0$ is a constant independent of u. By Gronwall's inequality,

$$\phi(z(t) \leq m_1\phi\big(z(0) + z'(0)\big)e^{m_1\int_0^t \gamma(s)ds}$$

for $t \in [0,\tau]$. In particular when $t = \tau$, we obtain

$$z(0) + z'(0) \geq \kappa_0\|u\|_\infty, \tag{2.16}$$

where $\kappa_0 = (e^{-m_1\|\gamma\|_1}/m_1)^{1/(p-1)}$. Since $(r(t)\phi(z'))' = \gamma(t)\phi(z) \geq 0$ on $(0,\tau)$, it follows that $r(t)\phi(z') \geq r(0)\phi(z'(0))$, which implies

$$z'(t) \geq (r(0)/\|r\|_\infty)^{1/(p-1)}z'(0).$$

If $b = 0$ then $z(0) = 0$ and (2.16) give

$$z(t) = \int_0^t z' \geq \Big(\frac{r(0)}{\|r\|_\infty}\Big)^{\frac{1}{p-1}}\kappa_0\|u\|_\infty t = \kappa_1(at + b)\|u\|_\infty \tag{2.17}$$

for $t \in [0,\tau]$, where $\kappa_1 = a^{-1}(r(0)/\|r\|_\infty)^{1/(p-1)}\kappa_0$.

On the other hand, if $b > 0$ then $z'(0) = \frac{a}{b\phi^{-1}(r(0))}z(0)$ and (2.16) becomes $z(0) \geq \tilde\kappa_1\|u\|_\infty$, where $\tilde\kappa_1 = \kappa_0(1 + \frac{a}{b\phi^{-1}(r(0))})^{-1}$. Hence

$$z(t) \geq z(0) \geq \tilde\kappa_1\|u\|_\infty \geq \kappa_2(at + b)\|u\|_\infty \tag{2.18}$$

for $t \in [0,\tau]$, where $\kappa_2 = \tilde\kappa_1/(a + b)$. Combining (2.17) and (2.18), we obtain $z(t) \geq \kappa_3(at + b)\|u\|_\infty$ for $t \in [0,\tau]$, where $\kappa_3 > 0$ is independent of u, λ, h.

Next, let $w \in AC^1[\tau, 1]$ be the unique solution of

$$\begin{gathered} -(r(t)\phi(w'))' + \gamma(t)\phi(w) = 0 \quad \text{a.e. on } (\tau, 1), \\ w(\tau) = \|u\|_\infty, \quad cw(1) + d\phi^{-1}(r(1))w'(1) = 0. \end{gathered}$$

Then $u \geq w \geq 0$ on $[\tau, 1]$ and the boundary condition on w at 1 gives $w'(1) \leq 0$. Using the integral formula

$$w(t) = w(1) - \int_t^1 \phi^{-1}\Big(\frac{r(1)\phi(w'(1)) - \int_s^1 \gamma(\xi)\phi(w)d\xi}{r(s)}\Big)ds$$

for $t \in [\tau, 1]$ and using similar arguments as above, we obtain $w(t) \geq \kappa_4(d + c(1 - t))\|u\|_\infty$ for $t \in [\tau, 1]$, where $\kappa_4 > 0$ is a constant independent of u. If $\tau = 0$ then $u \geq w$ on $[0, 1]$ while if $\tau = 1$ then $u \geq z$ on $[0, 1]$. Thus $u(t) \geq \kappa\|u\|_\infty p(t)$ for $t \in [0, 1]$, where $\kappa = \min(\kappa_3, \kappa_4)$, which completes the proof. □

The next result provides some estimates on λ_1 for $p > 1$.

Lemma 2.7. *Suppose $b+d>0$ and $r\equiv 1$. If $d>0$ then*
$$\frac{\min(A_1,1)}{2^{(p-1)^+}}\le \lambda_1\le (A_1+(m_1+2)^p e^{m_1p})(2p+1), \tag{2.19}$$
where $A_1=(c/d)^{p-1}, m_1=(c+2d)/d$, while if $b>0$, then
$$\frac{\min(B_1,1)}{2^{(p-1)^+}}\le \lambda_1\le (B_1+(m_2+2)^p e^{m_2p}(2p+1), \tag{2.20}$$
where $B_1=(a/b)^{p-1}$, $m_2=(a+2b)/b$.

Proof. Using the Rayleigh quotient, we obtain
$$\lambda_1=\inf_{u\in V}\frac{\phi(u'(0))u(0)-\phi(u'(1))u(1)+\int_0^1|u'|^p dt}{\int_0^1|u|^p dt} \tag{2.21}$$
where $V=\{u\in C^1[0,1]: au(0)-bu'(0)=0, cu(1)+du'(1)=0\}$.

Suppose $d>0$. Then $u'(1)=-(c/d)u(1)$ and $\phi(u'(0)u(0)\ge 0$ for $u\in V$. Hence
$$\begin{aligned}\lambda_1&=\inf_{u\in V}\frac{\phi(u'(0))u(0)+A_1|u(1)|^p+\int_0^1|u'|^p dt}{\int_0^1|u|^p dt}\\ &\ge \inf_{u\in V}\frac{A_1|u(1)|^p+\int_0^1|u'|^p dt}{\int_0^1|u|^p dt}.\end{aligned} \tag{2.22}$$
Let $u\in V$. Then
$$|u(t)|\le |u(1)|+\int_0^1|u'|dt,$$
which implies
$$\begin{aligned}\int_0^1|u|^p dt&\le 2^{(p-1)^+}\Big(|u(1)|^p+\int_0^1|u'|^p dt\Big)\\ &\le \frac{2^{(p-1)^+}}{\min(A_1,1)}\Big(A_1|u(1)|^p+\int_0^1|u'|^p dt\Big).\end{aligned}$$
Consequently, (2.22) gives $\lambda_1\ge \frac{\min(A_1,1)}{2^{(p-1)^+}}$.

Next, we choose $u(t)=t^2e^{m_1(1-t)}$, where $m_1=(c+2d)/d$. Then $u\in V$ and
$$u(t)\ge t^2,$$
$$|u'(t)|=te^{m_1(1-t)}|2-m_1t|\le (m_1+2)e^{m_1}$$
for $t\in[0,1]$. Hence
$$\int_0^1|u|^p dt\ge \frac{1}{2p+1},\quad \int_0^1|u'|^p dt\le (m_1+2)^p e^{m_1p}. \tag{2.23}$$
Since $u(0)=0, u(1)=1$, it follows from (2.23) and the equality in (2.22) that
$$\lambda_1\le (A_1+(m_1+2)^p e^{m_1p})(2p+1)$$
i.e. (2.19) holds. Suppose next that $b>0$. Then
$$\begin{aligned}\lambda_1&=\inf_{u\in V}\frac{B_1|u(0)|^p-\phi(u'(1))u(1)+\int_0^1|u'|^p dt}{\int_0^1|u|^p dt}\\ &\ge \inf_{u\in V}\frac{B_1|u(0)|^p+\int_0^1|u'|^p dt}{\int_0^1|u|^p dt}.\end{aligned} \tag{2.24}$$

Using the inequality

$$|u(t)| \le |u(0)| + \int_0^1 |u'| dt,$$

it follows that

$$\int_0^1 |u|^p dt \le \frac{2^{(p-1)^+}}{\min(B_1, 1)} \Big(B_1 |u(0)|^p + \int_0^1 |u'|^p dt \Big),$$

from which (2.24) implies $\lambda_1 \ge \frac{\min(B_1,1)}{2^{(p-1)^+}}$. By choosing $u(t) = (1-t)^2 e^{m_2 t}$, where $m_2 = (a+2b)/b$, we see that $u \in V$ and the equality in (2.24) gives

$$\lambda_1 \le (B_1 + (m_2+2)^p e^{m_2 p})(2p+1),$$

which establishes (2.20). This completes the proof. □

Example 2.8. It follows from (2.19) that the principal eigenvalue λ_1 of $-(\phi(u'))'$ with boundary conditions $u(0) - u'(0) = 0 = u(1) + u'(1)$ satisfies

$$\frac{1}{2^{(p-1)^+}} \le \lambda_1 \le (1 + 5^p e^{3p})(2p+1)$$

3. Proof of main results

Proof of Theorem 1.1. In view of (A2)–(A5), there exist constants $r, r_1, \bar\lambda > 0$ with $r < r_1$ and $\bar\lambda < \lambda_1$ such that for a.e. $t \in (0,1)$,

$$f(t,z) \le \bar\lambda z^{p-1}, \quad f(t,z) + (\eta(t)+1) z^{p-1} \ge 0 \tag{3.1}$$

for $z \le r$;

$$|f(t,z)| \le \gamma_{r_1}(t) \le \gamma_{r_1}(t)(z/r)^{p-1}$$

for $r < z < r_1$, and $f(t,z) > 0$ for $z > r_1$ and a.e. t. Hence

$$f(t,z) + \gamma(t) z^{p-1} \ge 0$$

for a.e. $t \in (0,1)$ and all $z \ge 0$, where $\gamma(t) = \max(\eta(t)+1, \gamma_{r_1}(t)/r^{p-1})$. For $v \in E = C[0,1]$, we have $f(t,|v|) + \gamma(t)|v|^{p-1} \in L^1(0,1)$ in view of (A3). Hence by Lemma 2.4, the problem

$$\begin{gathered} -(r(t)\phi(u'))' + \gamma(t)\phi(u) = f(t,|v|) + \gamma(t)|v|^{p-1} \quad \text{a.e. on } (0,1), \\ au(0) - b\phi^{-1}(r(0))u'(0) = 0, \quad cu(1) + d\phi^{-1}(r(1))u'(1) = 0, \end{gathered}$$

has a unique solution $u = Av \in C^1[0,1]$. Since $A = T_0 \circ S_0$, where $S_0 : C[0,1] \to L^1(0,1)$ is defined by $(S_0 v)(t) = f(t,|v|) + \gamma(t)|v|^{p-1}$ and T_0 is defined in Lemma 2.4 with $\alpha = \beta = 0$, we see that $A : E \to E$ is completely continuous. We shall verify that

(i) $u = \theta A u$, $\theta \in (0,1] \Longrightarrow \|u\|_\infty \ne r$.

Indeed, let $u \in E$ satisfy $u = \theta A u$ for some $\theta \in (0,1]$ and suppose $\|u\|_\infty = r$. Then $u \in AC^1[0,1]$ and

$$-(r(t)\phi(u'))' + \gamma(t)\phi(u) = \theta^{p-1}(f(t,|u|) + \gamma(t)|u|^{p-1}) \ge 0 \quad \text{a.e. on } (0,1),$$

which implies $u \ge 0$ on $(0,1)$ by Lemma 2.6. Hence

$$-(r(t)\phi(u'))' = \theta^{p-1} f(t,u) - (1-\theta^{p-1})\gamma(t) u^{p-1} \le \theta^{p-1} f(t,u) \tag{3.2}$$

a.e. on $(0,1)$.

By [10, Lemma 2.1], there exists a constant $k_0 > 0$ such that $|z(t)| \le k_0|z|_{C^1}p(t)$ for all $t \in [0,1]$ and $z \in C^1[0,1]$ satisfying the Sturm-Liouville boundary conditions in (1.1). In particular, $\sup_{t\in(0,1)} \frac{u(t)}{p(t)} < \infty$. Since

$$-(r(t)\phi(\phi_1')') = \lambda_1 \phi_1^{p-1} > 0 \quad \text{a.e. on } (0,1),$$

it follows from Lemma 2.6 (with $\gamma \equiv 0$) that $\inf_{t\in(0,1)} \frac{\phi_1(t)}{p(t)} > 0$. Hence there exists a smallest positive constant δ_0 such that $u \le \delta_0\phi_1$ on $[0,1]$. Then it follows from (3.1) and (3.2) that

$$-(r(t)\phi(u'))' \le \bar{\lambda} u^{p-1} \le \bar{\lambda}\delta_0^{p-1}\phi_1^{p-1} \quad \text{a.e. on } (0,1),$$

from which the weak comparison principle (see [9, Lemma 3.2], [17, Lemma A2]) gives

$$u \le (\bar{\lambda}\delta_0^{p-1}/\lambda_1)^{\frac{1}{p-1}}\phi_1$$

on $[0,1]$, a contradiction with the definition of δ_0. Thus $\|u\|_\infty \ne r$ i.e. (i) holds.

Next, we claim that

(ii) There exists a constant $R > r$ such that $u = Au + \xi$, $\xi \ge 0$ implies $\|u\|_\infty \ne R$.

Let $u \in E$ satisfy $u = Au + \xi$ for some $\xi \in [0,\infty)$. Then $u - \xi = Au$ and therefore

$$-(r(t)\phi(u'))' + \gamma(t)\phi(u-\xi) = f(t,|u|) + \gamma(t)|u|^{p-1} \quad \text{a.e. on } (0,1),$$

which implies

$$-(r(t)\phi(u'))' + \gamma(t)\phi(u) \ge f(t,|u|) + \gamma(t)|u|^{p-1} \ge 0 \text{ a.e. on } (0,1). \tag{3.3}$$

Since $\liminf_{z\to\infty} \frac{f(t,z)}{z^{p-1}} > \lambda_1$ uniformly for a.e. $t \in (0,1)$, there exist positive constants $L, \tilde{\lambda}, \lambda_0$ with $\tilde{\lambda} > \lambda_0 > \lambda_1$ such that $f(t,z) \ge \tilde{\lambda} z^{p-1}$ for a.e. $t \in (0,1)$ and $z > L$.

Let $\varepsilon = (k_0 l)^{-1}\Big((\tilde{\lambda}/\lambda_1)^{\frac{1}{p-1}} - (\lambda_0/\lambda_1)^{\frac{1}{p-1}}\Big)$, where $l = \sup_{t\in(0,1)} \frac{p(t)}{\phi_1(t)} \in (0,\infty)$, and let δ be given by (2.4).Choose $I = [\alpha,\beta] \subset [0,1]$ such that

$$\int_{[0,1\setminus I} (\tilde{\lambda} + \gamma_L(t)) < \delta,$$

where γ_L is defined by (A3). Let $R > \max(r, \dfrac{1}{\kappa l_0}, \dfrac{L}{\kappa \min_{[\alpha,\beta]} p})$, where $l_0 = \inf_{(0,1)} \frac{p}{\phi_1} > 0$ and κ is defined in Lemma 2.6. We claim that $\|u\|_\infty \ne R$. Indeed, suppose $\|u\|_\infty = R$. Then it follows from (3.3) and Lemma 2.6 that $u(t) \ge \kappa\|u\|_\infty p(t)$ for $t \in (0,1)$. In particular, (3.3) becomes

$$-(r(t)\phi(u'))' \ge f(t,u) \quad \text{on } (0,1), \tag{3.4}$$

and

$$u(t) \ge \kappa R p(t) \ge \kappa R \min_{[\alpha,\beta]} p > L$$

for $t \in I$. Hence $f(t,u) \ge \tilde{\lambda} u^{p-1}$ for a.e. $t \in I$. Let δ_1 be the largest positive number such that $u \ge \delta_1\phi_1$ on $(0,1)$. Then $\delta_1 \ge \kappa l_0 R > 1$ and

$$-\Big(r(t)\phi(\frac{u'}{\delta_1})\Big)' \ge \begin{cases} \tilde{\lambda}{\phi_1}^{p-1} & \text{if } t \in I, \\ -\gamma_L(t) & \text{if } t \notin I. \end{cases}$$

Let $u_1, u_2 \in AC^1[0,1]$ satisfy

$$\begin{aligned} -(r(t)\phi(u_1'))' &= \begin{cases} \tilde{\lambda}{\phi_1}^{p-1} & \text{if } t \in I, \\ -\gamma_L(t) & \text{if } t \notin I \end{cases} \\ &\equiv h_1 \quad \text{a.e. on } (0,1), \end{aligned}$$

and

$$-(r(t)\phi(u_2'))' = \tilde{\lambda}{\phi_1}^{p-1} \equiv h_2 \quad \text{a.e. on } (0,1).$$

with Sturm-Liouville boundary conditions. Note that $u_2 = (\tilde{\lambda}/\lambda_1)^{\frac{1}{p-1}}\phi_1$ and $u \geq \delta_1 u_1$ on $(0,1)$. Since

$$\|h_1 - h_2\|_1 \leq \int_{[0,1]\setminus I} (\tilde{\lambda} + \gamma_L(t)) < \delta,$$

it follows from (2.4) that $|u_1 - u_2|_{C^1} < \varepsilon$. Hence

$$\begin{aligned} u_1 &\geq u_2 - k_0\varepsilon p \geq u_2 - k_0 l\varepsilon\phi_1 \\ &= \Big(\tilde{\lambda}/\lambda_1\Big)^{\frac{1}{p-1}}\phi_1 - \Big((\tilde{\lambda}/\lambda_1)^{\frac{1}{p-1}} - (\lambda_0/\lambda_1)^{\frac{1}{p-1}}\Big)\phi_1 \\ &= (\lambda_0/\lambda_1)^{\frac{1}{p-1}}\phi_1 \quad \text{on } (0,1), \end{aligned}$$

and consequently, $u \geq \delta_1(\lambda_0/\lambda_1)^{\frac{1}{p-1}}\phi_1$ on $(0,1)$, a contradiction with the definition of δ_1. Thus $\|u\|_\infty \neq R$, as claimed i.e. (ii) holds.

By Lemma 2.1, operator A has a fixed point $u \in E$ with $\|u\|_\infty > r$, which is a classical positive solution of (1.1) in view of Lemmas 2.4 and 2.6. This completes the proof. □

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