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TWO-POINT BOUNDARY-VALUE PROBLEMS WITH NONCLASSICAL ASYMPTOTICS ON THE SPECTRUM

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Abstract. In this article, we consider the spectral problem for an nth-order ordinary differential operator with degenerate boundary conditions. For even n, we construct nontrivial examples of boundary-value problems which have nonclassical asymptotics on the spectrum.

1. Introduction

Let us consider the boundary-value problem generated by the n-th order differential equation

$$u^{(n)}(x) + \sum_{m=1}^{n} p_m(x)u^{(n-m)}(x) + \lambda u(x) = 0, \tag{1.1}$$

where the complex-valued coefficients $p_m(x)$ are functions in $L_1(0,\pi)$, with the linearly independent boundary conditions

$$\sum_{k=0}^{n-1} \alpha_{i,k}u^{(k)}(0) + \beta_{i,k}u^{(k)}(\pi) = 0, \quad i = 1,\dots,n, \tag{1.2}$$

where $\alpha_{i,k}, \beta_{i,k}$ are complex numbers. It is well known that the characteristic determinant of (1.1), (1.2) is an entire analytical function of spectral parameter λ. Consequently, for operator (1.1), (1.2) we have only the following possibilities:

(a) the spectrum is absent;
(b) the spectrum is a finite nonempty set;
(c) the spectrum is a countable set without finite limit points;
(d) the spectrum fills the entire complex plane.

We say that problem (1.1), (1.2) has the classical asymptotics on the spectrum if the case (c) is realized, moreover the multiplicities of the eigenvalues are bounded by a single constant. For the Sturm-Liouville equation

$$Lu + \lambda u = 0, \tag{1.3}$$

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where $Lu = u'' - q(x)u$, with nondegenerate boundary conditions the spectrum always has the classical asymptotics [5]. For equation (1.3) with degenerate boundary conditions

$$u'(0) + du'(\pi) = 0, \quad u(0) - du(\pi) = 0, \tag{1.4}$$

another situation takes place. In particular, under the condition that $d \neq 0$ it follows from [3] that for any natural m there exist potentials $q(x)$ in the class $W_2^m(0,\pi)$ such that the root function system of problem (1.3), (1.4) contains associated functions of arbitrary high order. If $d = 0$, problem (1.3), (1.4) is the Cauchy problem which has no spectrum. Note, that for the Sturm-Liouville operator any two-point conditions are nondegenerate except (1.4). There is an enormous literature related to the spectral theory for operators with nondegenerate boundary conditions.The case of degenerate boundary conditions has been investigated much less. However, it is known [1, 2, 6] that there exist operators of high order, where any complex number is an eigenvalue. The main goal of present paper is to construct nontrivial examples of boundary value problems for high order operators such that the spectrum is absent or the spectrum is a countable set but the multiplicities of eigenvalues infinitely grow.

2. Unbounded growth of order for associated functions

For any even $n = 2\nu$ with $\nu > 1$, let us build an example of boundary-value problem (1.1), (1.2), for which the multiplicities of eigenvalues grow infinitely. Consider problem (1.3), (1.4) ($d \neq 0$) with a potential $q(x) \in W_2^m(0,\pi)$, where $m = 2\nu + 2$, providing infinite growth of the multiplicities of eigenvalues. Then by the embedding theorem $q(x) \in C^{(2\nu+1)}[0,\pi]$. Let $\{u_n(x)\}$ be the root function system of problem (1.3), (1.4) with the above-mentioned potential. Obviously, $u_n(x) \in C^{(2\nu+1)}[0,\pi]$. Let us prove that for any $j = 0, 1, \dots, 2\nu$,

$$q(0) = (-1)^j q(\pi). \tag{2.1}$$

Denote by $c(x,\mu)$, $s(x,\mu)$ ($\lambda = \mu^2$) the fundamental system of solutions to (1.3) with the initial conditions $c(0,\mu) = s'(0,\mu) = 1$, $c'(0,\mu) = s(0,\mu) = 0$. In [5] simple computations show that the characteristic equation of problem (1.3), (1.4) can be reduced to the form $\Delta(\mu) = 0$, where

$$\Delta(\mu) = \frac{d^2-1}{d} + c(\pi,\mu) - s'(\pi,\mu) = \frac{d^2-1}{d} + \int_0^\pi r(t)\frac{\sin\mu t}{\mu}dt, \tag{2.2}$$

where $r(t) \in C^{(2\nu+1)}[0,\pi]$. Let k be the least whole number ($0 \le k \le 2\nu$), provided that equality (2.1) does not hold. Integrating by parts $k+1$ times the last addend on the right-hand side of equality (2.2), from [4], we obtain

$$\Delta(\mu) = \sum_{j=1}^{k+1}\frac{\alpha_j}{\mu^{j+1}} + \frac{B_{k+1}\sin\pi\mu}{\mu^{k+2}} - \frac{1}{\mu^{k+2}}\int_0^\pi r^{(k+1)}(t)\sin\mu t dt$$

for odd k and

$$\Delta(\mu) = \sum_{j=1}^{k+1}\frac{\alpha_j}{\mu^{j+1}} + \frac{B_{k+1}\cos\pi\mu}{\mu^{k+2}} + \frac{1}{\mu^{k+2}}\int_0^\pi r^{(k+1)}(t)\cos\mu t dt$$

for even k. In both cases coefficients α_j are some numbers, and

$$B_{k+1} = (-1)^{k+1}r^{(k)}(\pi) = (-1)^{k+1}(q^{(k)}(\pi) - (-1)^k q^{(k)}(0))/2^{k+1} \neq 0.$$

Hence, it follows that problem (1.3), (1.4) is almost-regular in sense of [7], therefore, the multiplicities of eigenvalues are bounded by a single constant, i.e. we receive a contradiction, hence, equality (2.1) is valid.

Further, consider the problem

$$L^{\nu}u+(-1)^{\nu-1}\lambda^{\nu}u=0, \tag{2.3}$$

$$u^{(2\nu-j)}(0)+d(-1)^{j+1}u^{(2\nu-j)}(\pi)=0, \tag{2.4}$$

$j=1,\dots,2\nu$, where d is an arbitrary complex number ($d\neq 0$).

Lemma 2.1. *The functions $u_n(x)$ satisfy boundary conditions* (2.4).

Proof. Let us prove the lemma by induction. Obviously, equalities (2.4) hold if $j=2\nu,2\nu-1$. Suppose, that the functions $u_n(x)$ satisfy equalities (2.4) if $j=2\nu,2\nu-1,\dots,2\nu-l$, where $1\le l\le 2\nu-1$. Consider equality

$$u_n''(x)-q(x)u_n(x)+\lambda_n u_n(x)=u_{n-1}(x), \tag{2.5}$$

where $u_{n-1}(x)$ is an associated function per unit of lower order corresponding to a function $u_n(x)$. If $u_n(x)$ is an eigenfunction then the right-hand side of equality (2.5) equals zero identically. Differentiating equality (2.5) $2\nu-l-1$ times we obtain

$$\begin{aligned}&u_n^{(2\nu-l+1)}(x)-\sum_{m=0}^{2\nu-l-1}C_{2\nu-l-1}^{m}q^{(m)}(x)u_n^{(2\nu-l-1-m)}(x)+\lambda_n u_n^{(2\nu-l-1)}(x)\\&=u_{n-1}^{(2\nu-l-1)}(x).\end{aligned} \tag{2.6}$$

It follows by the inductive hypothesis, equalities (2.1) and (2.6) that

$$\begin{aligned}&u_n^{(2\nu-l+1)}(0)+d(-1)^{l}u_n^{(2\nu-l+1)}(\pi)\\
&=\sum_{m=0}^{2\nu-l-1}C_{2\nu-l-1}^{m}q^{(m)}(0)u_n^{(2\nu-l-3-m)}(0)-\lambda_n u_n^{(2\nu-l-1)}(0)\\
&\quad+u_{n-1}^{(2\nu-l-1)}(0)+d(-1)^{l}[\sum_{m=0}^{2\nu-l-1}C_{2\nu-l-1}^{m}q^{(m)}(\pi)u_n^{(2\nu-l-1-m)}(\pi)\\
&\quad-\lambda_n u_n^{(2\nu-l-1)}(\pi)+u_{n-1}^{(2\nu-l-1)}(\pi)]\\
&=\sum_{m=0}^{2\nu-l-1}C_{2\nu-l-1}^{m}(q^{(m)}(0)u_n^{(2\nu-l-1-m)}(0)\\
&\quad+d(-1)^{l}q^{(m)}(\pi)u_n^{(2\nu-l-1-m)}(\pi))\\
&\quad-\lambda_n(u_n^{(2\nu-l-1)}(0)+d(-1)^{l}u_n^{(2\nu-l-1)}(\pi))\\
&\quad+(u_{n-1}^{(2\nu-l-1)}(0)+d(-1)^{l}u_{n-1}^{(2\nu-l-1)}(\pi))\\
&=\sum_{m=0}^{2\nu-l-1}q^{(m)}(0)(C_{2\nu-l-1}^{m}(u_n^{(2\nu-l-1-m)}(0)+d(-1)^{m+l}u_n^{(2\nu-l-1-m)}(\pi))\\
&=0.\end{aligned} \tag{2.7}$$

□

Let a function $\overset{0}{u}(x)$ be an arbitrary solution of the equation

$$L\overset{0}{u}+\overset{0}{u}=0, \tag{2.8}$$

and a function $\overset{i}{u}(x)$ be an arbitrary solution of the equation

$$L\overset{i}{u}+\lambda\overset{i}{u}=\overset{i-1}{u}, \tag{2.9}$$

$i=1,2,\dots$. Formally set $\overset{i}{u}\equiv 0$ if $i=-1,-2,\dots$.

Lemma 2.2. *For any $p=1,2,\dots$ we have*

$$L^p\overset{i}{u}=\sum_{k=0}^{p}(-1)^{p-k}C_p^k\lambda^{p-k}\overset{i-k}{u}. \tag{2.10}$$

Proof. Let us prove the lemma by induction with respect to p. If $p=1$ then relations (2.8), (2.9) imply (2.10). Let the lemma be valid for a natural p. It follows by the inductive hypothesis and the properties of the binomial coefficients that

$$\begin{aligned}
L^{p+1}\overset{i}{u} &= L\Big(\sum_{k=0}^{p}(-1)^{p-k}C_p^k\lambda^{p-k}\overset{i-k}{u}\Big)\\
&= \sum_{k=0}^{p}(-1)^{p-k}C_p^k\lambda^{p-k}L(\overset{i-k}{u})\\
&= \sum_{k=0}^{p}(-1)^{p-k}C_p^k\lambda^{p-k}(\overset{i-k-1}{u}-\lambda\overset{i-k}{u})\\
&= \sum_{k=0}^{p}(-1)^{p-k}C_p^k\lambda^{p-k}\overset{i-(k+1)}{u}-\sum_{k=0}^{p}(-1)^{p-k}C_p^k\lambda^{p+1-k}\overset{i-k}{u}\\
&= \sum_{m=1}^{p+1}(-1)^{p-m+1}C_p^{m-1}\lambda^{p-m+1}\overset{i-m)}{u}-\sum_{m=0}^{p}(-1)^{p-m}C_p^m\lambda^{p+1-m}\overset{i-m}{u}\\
&= \overset{i-(p+1)}{u}+\sum_{m=1}^{p}[(-1)^{p-m+1}C_p^{m-1}\lambda^{p-m+1}-(-1)^{p-m}C_p^m\lambda^{p+1-m}]\overset{i-m}{u}\\
&\quad-(-1)^p\lambda^{p+1}\overset{i}{u}=\overset{i-(p+1)}{u}+\sum_{m=1}^{p}[(-1)^{p-m+1}\lambda^{p-m+1}(C_p^{m-1}+C_p^m]\overset{i-m}{u}\\
&\quad-(-1)^p\lambda^{p+1}\overset{i}{u}\\
&= \sum_{m=0}^{p+1}(-1)^{p-m+1}\lambda^{p-m+1}C_{p+1}^m\overset{i-m}{u}.
\end{aligned}$$

□

Denote $\Lambda=(-1)^{p-1}\lambda^p$ $(p=1,2,\dots)$.

Lemma 2.3. *Let $\lambda\neq 0$. If $\overset{i}{v}=\sum_{j=0}^{i}a_j\overset{j}{u}$, where a_j are some numbers, and $a_i\neq 0$, then there exists a function $\overset{i+1}{v}=\sum_{j=0}^{i+1}b_j\overset{j}{u}$, where b_j are some numbers, and $b_{i+1}\neq 0$ such that $L^p\overset{i+1}{v}+\Lambda\overset{i+1}{v}=\overset{i}{v}$.*

Proof. It follows by Lemma 2.2 that

$$L^p\overset{i+1}{v}+\Lambda\overset{i+1}{v}$$

$$
\begin{aligned}
&= L^p\Big(\sum_{j=0}^{i+1} b_j \overset{j}{u}\Big) + (-1)^{p-1}\lambda^p \overset{i+1}{v} \\
&= \sum_{j=0}^{i+1} b_j L^p \overset{j}{u} + (-1)^{p-1}\lambda^p \overset{i+1}{v} \\
&= \sum_{j=0}^{i+1} b_j \sum_{k=0}^{p} (-1)^{p-k} C_p^k \lambda^{p-k} \overset{j-k}{u} \\
&\quad + (-1)^{p-1}\lambda^p \overset{i+1}{v} = (-1)^p\lambda^p \sum_{j=0}^{i+1} b_j \overset{j}{u} + \sum_{j=0}^{i+1} b_j \sum_{k=1}^{p} (-1)^{p-k} C_p^k \lambda^{p-k} \overset{j-k}{u} \\
&\quad + (-1)^{p-1}\lambda^p \overset{i+1}{v} \\
&= \sum_{j=0}^{i+1} b_j \sum_{k=1}^{p} (-1)^{p-k} C_p^k \lambda^{p-k} \overset{j-k}{u} \\
&= \sum_{j=0}^{i+1} b_j \sum_{k=1}^{p} \gamma_k \overset{j-k}{u},
\end{aligned}
$$

where $\gamma_k = (-1)^{p-k} C_p^k \lambda^{p-k}$. Equating the coefficients at the functions $\overset{i-m}{u}$ in the relation

$$
\sum_{j=0}^{i+1} b_j \sum_{k=1}^{p} \gamma_k \overset{j-k}{u} = \sum_{l=0}^{i} a_l \overset{l}{u},
$$

we obtain the system of linear equations

$$
\sum_{l=0}^{m} \gamma_{l+1} b_{i+1-l} = a_{i-m}, \tag{2.11}
$$

$m = 0, \dots, i$. The matrix of system (2.11) is lower triangular, and all the elements of the principal diagonal are equal to $\gamma_1 = (-1)^{p-1} p \lambda^{p-1} \neq 0$. Therefore, system (2.11) has the unique solution. Since $a_i \neq 0$, we have $b_{i+1} = a_i/\gamma_1 \neq 0$. $\square$

Let $u_n(x)$ be an associated function of order k corresponding to an eigenvalue $\lambda_n \neq 0$, and functions $\{u_{n-j}(x)\}$ $(j = 0, \dots, k)$ form the corresponding Jordan chain, i.e.

$$
\begin{gathered}
Lu_{n-j}(x) + \lambda_n u_{n-j}(x) = u_{n-j-1}(x) \quad (j = 0, \dots, k-1), \\
Lu_{n-k}(x) + \lambda_n u_{n-k}(x) = 0.
\end{gathered}
$$

By lemma 2.1, the function $u_{n-k}(x)$ is an eigenfunction of problem (2.3), (2.4) corresponding to the eigenvalue $\Lambda_n = (-1)^{\nu-1}\lambda_n^{\nu}$. Set $v_{n-k}(x) = u_{n-k}(x)$. Then, by lemma 2.3, it follows that there exist functions

$$
v_{n-k+i}(x) = \sum_{j=1}^{i} b_{ij} u_{n-k+i}(x)
$$

$(i = 0, \dots, k)$ such that

$$
L^{\nu} v_{n-k+i}(x) + \Lambda_n v_{n-k+i}(x) = v_{n-k+i-1}(x).
$$

By lemma 2.1, all the functions $v_{n-k+i}(x)$ satisfy boundary conditions (2.4), then the functions $v_{n-k+i}(x)$ form the Jordan chain corresponding to the eigenvalue Λ_n of problem (2.3), (2.4). Thus we have that the function $v_n(x)$ is an associated function of order k of problem (2.3), (2.4). Whence, the following assertion is valid.

Theorem 2.4. *The root function system of problem* (2.3), (2.4) *contains associated functions of arbitrary high order.*

3. Empty spectrum

Consider boundary-value problem (1.1), (2.4), where $n = 2\nu$ ($\nu > 1$). Suppose that $p_m(x) = (-1)^m p_m(\pi - x)$ almost everywhere on the segment $[0, \pi]$, $m = 1, \dots, n$. We will study the spectrum of problem (1.1), (2.4).

Theorem 3.1. *If $d \neq \pm 1$ the spectrum of problem* (1.1), (2.4) *is empty.*

Proof. Let a function $\hat{u}_k(x)$ be the solution of equation (1.1) with initial conditions

$$u_k^{(j)}(\pi/2) = \delta_{k,j}, \tag{3.1}$$

where $k = 0, \dots, n-1$, $j = 0, \dots, n-1$. Denote

$$\hat{u}_-(x) = \sum_{k=0}^{\nu-1} c_{2k+1}\hat{u}_{2k+1}(x), \quad \hat{u}_+(x) = \sum_{k=0}^{\nu-2} c_{2k}\hat{u}_{2k}(x),$$

where c_i are arbitrary constants ($i = 0, \dots, n-1$). Then

$$\hat{u}_-^{(2k)}(\pi/2) = 0, \quad \hat{u}_+^{(2k+1)}(\pi/2) = 0, \quad k = 0, \dots, \nu - 1.$$

Obviously, that the functions $w_-(x) = -\hat{u}_-(\pi - x)$ and $w_+(x) = \hat{u}_+(\pi - x)$ are the solutions of equation (1.1) and satisfy the same initial conditions at the point $\pi/2$ as well as the functions $\hat{u}_-(x)$ and $\hat{u}_+(x)$, correspondingly. This, together with the uniqueness of the solution of Cauchy problem (1.1), (3.1) implies that $\hat{u}_-(x) = -\hat{u}_-(\pi - x)$ and $\hat{u}_+(x) = \hat{u}_+(\pi - x)$, if $0 \le x \le 1$. It follows that

$$\hat{u}_-^{(n-j)}(0) + (-1)^{j+1}\hat{u}_-^{(n-j)}(\pi) = 0, \quad \hat{u}_+^{(n-j)}(0) + (-1)^j\hat{u}_+^{(n-j)}(\pi) = 0 \tag{3.2}$$

($j = 1, \dots, n$). It follows from (3.2) that for any complex number λ the function $\hat{u}_-(x)$ is a solution of problem (1.1), (2.4) if $d = 1$, and for any complex number λ the function $\hat{u}_+(x)$ is a solution of problem (1.1), (2.4) if $d = -1$. Thus, we establish that if $d = \pm 1$ the spectrum of problem (1.1), (2.4) fills all complex plane. If $p_m(x) \in C^m(0, 1)$, $m = 1, \dots, n$, this assertion was proved in [6].

Assume, for a number λ a function $\tilde{u}(x)$ is a solution of problem (1.1), (2.4) if $d \neq \pm 1$. Then $\tilde{u}(x) = \hat{u}_+(x) + \hat{u}_-(x)$. We see that

$$\hat{u}_-^{(n-j)}(0) + \hat{u}_+^{(n-j)}(0) + (-1)^{j+1}d(\hat{u}_-^{(n-j)}(\pi) + \hat{u}_+^{(n-j)}(\pi)) = 0 \tag{3.3}$$

($j = 1, \dots, n$). It follows from (3.2), (3.3) that

$$\hat{u}_-^{(n-j)}(0)(1-d) + \hat{u}_+^{(n-j)}(0)(1+d) = 0$$

($j = 1, \dots, n$). From this and the definition of the functions $\hat{u}_+(x)$ $\hat{u}_-(x)$, we have

$$(1+d)\sum_{k=0}^{\nu-2} c_{2k}\hat{u}_{2k}^{(n-j)}(0) + (1-d)\sum_{k=0}^{\nu-1} c_{2k+1}\hat{u}_{2k+1}^{(n-j)}(0) = 0$$

$(j = 1, \dots, n)$, hence, the constants c_i $(i = 0, \dots, n-1)$ satisfy the system of linear equations

$$\sum_{i=0}^{n-1} c_i (1 + d(-1)^i) \hat{u}_i^{(n-j)}(0) = 0 \tag{3.4}$$

$(j = 1, \dots, n)$. The determinant of linear system (3.4) is

$$\Delta = (1 - d^2)^{\nu} \det ||\hat{u}_i^{(n-j)}(0)||.$$

Since the last determinant is the Wronskian of the fundamental system of the solutions of equation (1.1), it is nonzero. Therefore, system (3.4) has only trivial solution, i.e. the function $\tilde{u}(x) \equiv 0$. Hence, if $d \neq \pm 1$ problem (1.1), (2.4) has no eigenvalues. $\square$

Problem (1.3), (1.4) was first investigated in [8]. In particular, it was shown that under the conditions $d = \pm 1$, $q(x) \equiv 0$ any complex number is eigenvalue of the considered problem.

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