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OSCILLATION FOR SECOND-ORDER DIFFERENTIAL EQUATIONS WITH DELAY

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Abstract. We establishing monotonic properties of non-oscillatory solutions, and oscillation criteria for the second-order delay differential equation

$$y''(t) + p(t)y(\tau(t)) = 0.$$

The criteria obtained fulfil the gap in the oscillation theory and essentially improves the earlier ones. The progress is illustrated via Euler's differential equation. Moreover, we provide upper and lower bounds for the non-oscillatory solutions.

1. Introduction

We consider the second-order delay differential equation

$$y''(t) + p(t)y(\tau(t)) = 0, \tag{1.1}$$

under the following assumptions:

(H1) $p \in C([t_0, \infty))$ and is positive;
(H2) $\tau \in C([t_0, \infty))$ and $\tau(t) \le t$.

By a solution of (1.1) we mean a function y in $C^2([t_0, \infty))$ that satisfies (1.1) on $[t_0, \infty)$. We consider only those solutions that satisfy $\sup\{|y(t)| : t \ge T\} > 0$ for all $T \ge t_0$. A solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

There are many papers devoted to the oscillation of (1.1) (see e.g. [1]–[15]). Various techniques have been obtained for investigation of (1.1). We mention here the pioneering work of Sturm [15] who introduced comparison principle to the oscillation theory. Later Kneser [11] contribute to the subject. Brands [4] proved that oscillation of (1.1) with bounded delay is equivalent to oscillation of ordinary differential equations. A new impetus to the investigation of oscillation was given by Mahfoud [13] who deduce oscillation of delay equations from that of ordinary equations.

Theorem 1.1. *Let $\tau'(t) > 0$. If the ordinary differential equation*

$$y''(t) + \frac{p(\tau^{-1}(t))}{\tau'(\tau^{-1}(t))} y(t) = 0$$

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is oscillatory, then so does (1.1).

This comparison result permit us to extend any oscillatory criterion from ordinary to delay differential equation. Koplatadze et al. [9] elaborated very nice technique for investigation of (1.1) and presented the following criterion.

Theorem 1.2. *Assume that*
$$\limsup_{t\to\infty}\Big\{\tau(t)\int_t^\infty p(s)\,\mathrm{d}s+\int_{\tau(t)}^t \tau(s)p(s)\,\mathrm{d}s+\frac{1}{\tau(t)}\int_{t_1}^{\tau(t)} s\tau(s)p(s)\,\mathrm{d}s\Big\}>1.$$
Then (1.1) *is oscillatory.*

The aim of this article is to establish new technique that improves criteria existing for oscillation of (1.1). This fact will be illustrated via Theorems 1.1, 1.2. Our method is based on new monotonicity properties of possible non-oscillatory solutions of (1.1).

In this article, we assume that all functional inequalities hold eventually, that is they are satisfied for all t large enough.

2. Preliminaries

For non-oscillatory solutions of (1.1), we restrict our attention to positive solutions because if y is a solution of so is $-y$. Next we recall a well-known lemma by Kiguradze (see [7, 8]) about the structure of non-oscillatory solutions.

Lemma 2.1. *If $y(t)$ is a positive solution of* (1.1), *then*
$$y'(t)>0 \quad \text{and} \quad y''(t)<0, \tag{2.1}$$
eventually.

As a preliminary, from [10, Lemma 4.1] it follows that the condition
$$\int_{t_0}^\infty \tau(s)p(s)\mathrm{d}s=\infty. \tag{2.2}$$
is necessary for the oscillation of (1.1). So in what follows, we shall assume that (2.2) holds.

Lemma 2.2. *If $y(t)$ is a positive solution of* (1.1), *then*
$$\frac{y(t)}{t}\downarrow 0 \quad \text{and} \quad ty'(t)\le y(t). \tag{2.3}$$

Proof. Assume that (1.1) possesses a positive solution $y(t)$. Then (2.1) is satisfied, let us say for $t\ge t_1$. It follows from L'Hospital's rule that
$$\lim_{t\to\infty}\frac{y(t)}{t}=\lim_{t\to\infty}y'(t).$$
We claim that (2.2) implies $\lim_{t\to\infty}y'(t)=0$. If we admit that $\lim_{t\to\infty}y'(t)=\ell>0$, then integrating (1.1) yields
$$y'(t_1)\ge\int_{t_1}^\infty p(s)y(\tau(s))\mathrm{d}s\ge\int_{t_1}^\infty \tau(s)p(s)\frac{y(\tau(s))}{\tau(s)}\mathrm{d}s\ge\ell\int_{t_1}^\infty\tau(s)p(s)\mathrm{d}s.$$
This contradicts to (2.2) and we see that $\lim_{t\to\infty}y'(t)=0$, which implies
$$y(t)=y(t_1)+\int_{t_1}^t y'(s)\mathrm{d}s\ge y(t_1)-t_1y'(t)+ty'(t)\ge ty'(t).$$

Consequently

$$\Big(\frac{y(t)}{t}\Big)' = \frac{ty'(t) - y(t)}{t^2} \le 0.$$

The proof is complete. $\square$

We recall the following comparison result, which is a particular case of [12, Theorem 2].

Lemma 2.3. *Assume that $a(t) \ge b(t) \ge 0$. If the differential inequality*

$$y''(t) + a(t)y(t) \le 0$$

has a positive solution, then the equation

$$y''(t) + b(t)y(t) = 0$$

has a positive solution.

Theorem 2.4. *Assume that there is a constant a_0 such that for $t \ge t_0$*

$$t\tau(t)p(t) \ge a_0 > \frac{1}{4}\,. \tag{2.4}$$

Then (1.1) *is oscillatory.*

Proof. On the contrary, assume that (1.1) possesses an eventually positive solution $y(t)$. Taking the monotonicity of $y(t)/t$ into account, we see that $y(t)$ is also solution of the inequality

$$y''(t) + \frac{\tau(t)}{t}p(t)y(t) \le 0. \tag{2.5}$$

Lemma 2.3 applied to (2.5) and the Euler differential equation

$$y''(t) + \frac{a_0}{t^2}y(t) = 0, \tag{2.6}$$

guarantees that (2.6) has a positive solution. This is a contradiction since (2.6) is oscillatory for $a_0 > 1/4$. $\square$

In our next considerations we improve (2.4). In what follows we shall assume that there exists a constant a_0 such that for $t \ge t_0$

$$t\tau(t)p(t) \ge a_0 > 0 \quad \text{and} \quad a_0 \le \frac{1}{4}. \tag{2.7}$$

We denote

$$\beta = \frac{1 + \sqrt{1 - 4a_0}}{2}. \tag{2.8}$$

3. Main results

In this section we derive new properties of non-oscillatory solutions of (1.1) that will be used for establishing new oscillatory criteria.

Lemma 3.1. *Assume that $y(t)$ is a positive solution of* (1.1). *Then for any $\varepsilon > 0$, the function $y(t)/t^{\beta+\varepsilon}$ is decreasing.*

Proof. Assume that $y(t) > 0$ is a solution of (1.1). Then (2.1) holds for $t \geq t_1$. Using the monotonicity of $\frac{y(t)}{t}$ into account, it is easy to verify that

$$\begin{aligned}\Big(t^{2\beta}\Big(\frac{y(t)}{t^\beta}\Big)'\Big)' &= y''(t)t^\beta - \beta(\beta-1)t^{\beta-2}y(t)\\ &= -t^\beta p(t)y(\tau(t)) - \beta(\beta-1)t^{\beta-2}y(t)\\ &\leq t^{\beta-2}y(t)\left(-t\tau(t)p(t) - \beta(\beta-1)\right)\\ &\leq t^{\beta-2}y(t)\big(-a_0 - \beta(\beta-1)\big) = 0,\end{aligned}\tag{3.1}$$

for $t \geq t_1$. Therefore, $t^{2\beta}\left(\frac{y(t)}{t^\beta}\right)'$ is decreasing. Denote

$$\begin{gathered}\bar\beta = \beta + \varepsilon \quad \text{for } \varepsilon \text{ small enough},\\ \delta = \varepsilon(2\beta-1) + \varepsilon^2.\end{gathered}$$

Since $-\beta(\beta-1) = a_0$, it is easy to verify, that $-\bar\beta(\bar\beta-1) = a_0 - \delta$. Then

$$\Big(t^{2\bar\beta}\Big(\frac{y(t)}{t^{\bar\beta}}\Big)'\Big)' \leq t^{\bar\beta-2}y(t)\left(-a_0 - \bar\beta(\bar\beta-1)\right) = -t^{\bar\beta-2}y(t)\delta < 0. \tag{3.2}$$

Since $\big(t^{2\bar\beta}\big(\frac{y(t)}{t^{\bar\beta}}\big)'\big)' < 0$, then $t^{2\bar\beta}\big(\frac{y(t)}{t^{\bar\beta}}\big)'$ is decreasing and so either

$$\Big(\frac{y(t)}{t^{\bar\beta}}\Big)' > 0 \quad \text{or} \quad \Big(\frac{y(t)}{t^{\bar\beta}}\Big)' < 0,$$

eventually.

If we admit that $\big(\frac{y(t)}{t^{\bar\beta}}\big)' > 0$, then integrating inequality (3.2) from t_1 to ∞, we have

$$t_1^{2\bar\beta}\Big(\frac{y(x)}{x^{\bar\beta}}\Big)'_{x=t_1} \geq \delta\int_{t_1}^\infty s^{2\bar\beta-2}\frac{y(s)}{s^{\bar\beta}}\,\mathrm{d}s \geq \delta\frac{y(t_1)}{t_1^{\bar\beta}}\int_{t_1}^\infty s^{2\bar\beta-2}\,\mathrm{d}s = \infty.$$

It is a contradiction and we conclude, that $\big(\frac{y(t)}{t^{\bar\beta}}\big)' > 0$ and so $y(t)/t^{\beta+\varepsilon}$ is decreasing. $\square$

Lemma 3.2. *Assume that there are constants a_1 and ε such that for $t \geq t_0$,*

$$t^2p(t)\Big(\frac{\tau(t)}{t}\Big)^{\beta+\varepsilon} \geq a_1. \tag{3.3}$$

If $a_1 > \frac14$, then (1.1) is oscillatory. If $a_1 \leq \frac14$, then for any positive solution $y(t)$ of (1.1)

$$\frac{y(t)}{t^\beta}$$

is decreasing.

Proof. Assume that $y(t) > 0$ is a solution of (1.1). The monotonicity of $\frac{y(t)}{t^{\beta+\varepsilon}}$ implies that $y(t)$ is a positive solution of the inequality

$$y''(t) + \Big(\frac{\tau(t)}{t}\Big)^{\beta+\varepsilon}p(t)y(t) \leq 0. \tag{3.4}$$

Lemma 2.3 implies that the Euler equation

$$y''(t) + \frac{a_1}{t^2}y(t) = 0,$$

has a positive solution. This contradicts the fact that the Euler equation is oscillatory for $a_1 > 1/4$ and so we conclude that (1.1) is oscillatory.

Now, we assume, that $a_1 \le \frac{1}{4}$. Denote

$$\beta_1 = \frac{1+\sqrt{1-4a_1}}{2}.$$

Let us consider $\varepsilon > 0$, such that $\beta_1 + \varepsilon \le \beta$. It is easy to see that

$$-(\beta_1+\varepsilon)(\beta_1+\varepsilon-1) = a_1 - \delta_1,$$

where $\delta_1 = \varepsilon(2\beta_1 - 1) + \varepsilon^2$.

On the other hand, the monotonicity of $y(t)/t^{\beta+\varepsilon}$ yields

$$y(\tau(t)) \ge \Big(\frac{\tau(t)}{t}\Big)^{\beta+\varepsilon} y(t).$$

Thus,

$$\begin{aligned}
\Big(t^{2\beta_1+\varepsilon}\Big(\frac{y(t)}{t^{\beta_1+\varepsilon}}\Big)'\Big)' &= -t^{\beta_1+\varepsilon}p(t)y(\tau(t)) - (\beta_1+\varepsilon)(\beta_1+\varepsilon-1)t^{\beta_1+\varepsilon-2}y(t) \\
&\le t^{\beta_1+\varepsilon-2}y(t)\Big(-t^2p(t)\Big(\frac{\tau(t)}{t}\Big)^{\beta+\varepsilon} - (\beta_1+\varepsilon)(\beta_1+\varepsilon-1)\Big) \\
&\le -t^{\beta_1+\varepsilon-2}y(t)\delta_1.
\end{aligned}$$

Proceeding similarly as in proof of Lemma 3.1, we obtain that $\frac{y(t)}{t^{\beta_1+\varepsilon}}$ is decreasing. Since

$$\beta_1 + \varepsilon \le \beta,$$

we can conclude that $\frac{y(t)}{t^{\beta}}$ is decreasing too. The proof is complete. $\square$

Now we are ready to provide the oscillatory criterion that improves Theorem 2.4.

Theorem 3.3. *Assume that there is a constant a_2 such that for $t \ge t_0$,*

$$t^{2-\beta}(\tau(t))^{\beta}p(t) \ge a_2 > \frac{1}{4}, \tag{3.5}$$

then (1.1) *is oscillatory.*

Proof. Assume to the contrary that (1.1) has a positive solution $y(t)$. The monotonicity of $\frac{y(t)}{t^{\beta}}$ implies that $y(t)$ is a solution of the differential inequality

$$y''(t) + \Big(\frac{\tau(t)}{t}\Big)^{\beta} p(t)y(t) \le 0.$$

Lemma 2.3 implies that the Euler equation

$$y''(t) + \frac{a_2}{t^2}y(t) = 0,$$

has a positive solution. This contradicts to fact that considered Euler equation is oscillatory for $a_2 > 1/4$. The proof is complete. $\square$

Remark 3.4. In contrast to results presented in [14] our oscillatory criterion is easily verifiable and does not require any auxiliary constants and functions. Unlike to [14] our results will be supported by illustrative example.

We illustrate the novelty and progress of our oscillation criterion via its application to Euler differential equations with a delay argument:

$$y''(t) + \frac{a}{t^2}y(\lambda t) = 0, \quad \lambda \in (0,1). \tag{3.6}$$

Corollary 3.5. *If*
$$\lambda^\beta a > \frac{1}{4}, \tag{3.7}$$
then (3.6) *is oscillatory.*

Remark 3.6. By Theorem 1.1, the oscillation of (3.6) follows from the oscillation of
$$y''(t) + \frac{a\lambda}{t^2} y(t) = 0, \tag{3.8}$$
which leads to the condition
$$\lambda a > \frac{1}{4}. \tag{3.9}$$
What is more, [2, Corollaries 7.5 and 7.6 , and Theorem 7.9] guarantee the oscillation of (3.6) if (3.9) holds. Evidently criterion (3.7) provides better result.

Our next considerations are intended to essentially improve Theorem 1.2. For this reason we need the monotonicity which is opposite to that in Lemma 3.2.

Lemma 3.7. *Let* (2.7) *hold and* $\alpha = \frac{1-\sqrt{1-4a_0}}{2}$. *Assume that* $y(t)$ *is a positive solution of* (1.1). *Then* $y(t)/t^\alpha$ *is increasing.*

Proof. Assume that $y(t)$ is a positive solution of (1.1). Then (2.1) is satisfied for $t \geq t_1$. Taking the monotonicity of $\frac{y(t)}{t}$ into account, it is easy to verify that
$$\begin{aligned} \Big(t^{2\alpha}\Big(\frac{y(t)}{t^\alpha}\Big)'\Big)' &= y''(t)t^\alpha - \alpha(\alpha-1)t^{\alpha-2}y(t) \\ &= -t^\alpha p(t)y(\tau(t)) - \alpha(\alpha-1)t^{\alpha-2}y(t) \\ &\leq t^{\alpha-2}y(t)\left(-t\tau(t)p(t) - \alpha(\alpha-1)\right) \leq 0. \end{aligned} \tag{3.10}$$
Therefore $t^{2\alpha}\left(\frac{y(t)}{t^\alpha}\right)'$ is decreasing. If we admit that $t^{2\alpha}\left(\frac{y(t)}{t^\alpha}\right)' < 0$ for $t \geq t_2 \geq t_1$, then there exists constant $k > 0$ such that
$$t^{2\alpha}\Big(\frac{y(t)}{t^\alpha}\Big)' < -k < 0$$
for $t > t_2$. Integrating the last inequality form t_2 to t, we have
$$\frac{y(t)}{t^\alpha} < \frac{y(t_2)}{t_2^\alpha} - k\int_{t_2}^t s^{-2\alpha}\,\mathrm{d}s \to -\infty \quad \text{for } t \to \infty.$$
This is a contradiction and we conclude that $t^{2\alpha}\big(\frac{y(t)}{t^\alpha}\big)' > 0$. The proof is complete. □

Lemmas 3.2 and 3.7 provide upper and lower bound for possible non-oscillatory solutions of (1.1).

Theorem 3.8. *Let* (2.7) *hold,* $\alpha = \frac{1-\sqrt{1-4a_0}}{2}$ *and* $\beta = \frac{1+\sqrt{1-4a_0}}{2}$. *Then every positive solution* $y(t)$ *of* (1.1) *satisfies*
$$c_1 t^\alpha \leq y(t) \leq c_2 t^\beta,$$
c_1, c_2 *are constants.*

Proof. By Lemma 3.7, the function $y(t)/t^\alpha$ is increasing and so for all $t \geq t_1$,
$$\frac{y(t)}{t^\alpha} \geq \frac{y(t_1)}{t_1^\alpha} = c_1.$$
The second part of the theorem can be proved similarly. □

Now, we present new oscillatory results using both monotonic properties of non-oscillatory solutions of (1.1) presented in Lemma 3.2 and Lemma 3.7.

Theorem 3.9. *Let* (2.7) *hold, and assume that*

$$\begin{aligned}&\limsup_{t\to\infty}\Big\{\tau^{-\beta}(t)\int_{t_1}^{\tau(t)} s\tau^{\beta}(s)p(s)\,\mathrm{d}s\\ &+\tau^{1-\beta}(t)\int_{\tau(t)}^{t}\tau^{\beta}(s)p(s)\,\mathrm{d}s+\tau^{1-\alpha}(t)\int_t^\infty \tau^{\alpha}(s)p(s)\,\mathrm{d}s\Big\}>1.\end{aligned}\tag{3.11}$$

Then (1.1) *is oscillatory.*

Proof. On the contrary, assume that (1.1) possesses a positive solution $y(t)$. Then (2.1) holds for $t\ge t_1$. Integrating (1.1) twice, we get

$$\begin{aligned}y(t)&\ge y(t_1)+\int_{t_1}^{t}\int_u^\infty p(s)y(\tau(s))\,\mathrm{d}s\,\mathrm{d}u\\ &=y(t_1)+\int_{t_1}^{t}\int_u^t p(s)y(\tau(s))\,\mathrm{d}s\,\mathrm{d}u+\int_{t_1}^{t}\int_t^\infty p(s)y(\tau(s))\,\mathrm{d}s\,\mathrm{d}u\\ &=y(t_1)+\int_{t_1}^{t}(s-t_1)p(s)y(\tau(s))\,\mathrm{d}s+(t-t_1)\int_t^\infty p(s)y(\tau(s))\,\mathrm{d}s\\ &=y(t_1)-t_1\int_{t_1}^\infty p(s)y(\tau(s))\,\mathrm{d}s+\int_{t_1}^{t}sp(s)y(\tau(s))\,\mathrm{d}s\\ &\quad+t\int_t^\infty p(s)y(\tau(s))\,\mathrm{d}s.\end{aligned}\tag{3.12}$$

On the other hand, an integration of (1.1) yields

$$y'(t)\ge\int_t^\infty p(s)y(\tau(s))\,\mathrm{d}s$$

which in view of (2.3) implies

$$y(t_1)>t_1\int_{t_1}^\infty p(s)y(\tau(s))\,\mathrm{d}s.$$

Employing the last inequality in (3.12), we see that

$$y(t)\ge\int_{t_1}^{t}sp(s)y(\tau(s))\,\mathrm{d}s+t\int_t^\infty p(s)y(\tau(s))\,\mathrm{d}s.\tag{3.13}$$

Therefore,

$$\begin{aligned}y(\tau(t))&\ge\int_{t_1}^{\tau(t)}sp(s)y(\tau(s))\,\mathrm{d}s+\tau(t)\int_{\tau(t)}^\infty p(s)y(\tau(s))\,\mathrm{d}s\\ &=\int_{t_1}^{\tau(t)}sp(s)y(\tau(s))\,\mathrm{d}s+\tau(t)\int_{\tau(t)}^{t} p(s)y(\tau(s))\,\mathrm{d}s\\ &\quad+\tau(t)\int_t^\infty p(s)y(\tau(s))\,\mathrm{d}s.\end{aligned}$$

Using that $y(t)/t^{\beta}$ is decreasing and $\frac{y(t)}{t^{\alpha}}$ is increasing, we have

$$1\ge\tau^{-\beta}(t)\int_{t_1}^{\tau(t)}s\tau^{\beta}(s)p(s)\,\mathrm{d}s+\tau^{1-\beta}(t)\int_{\tau(t)}^{t}\tau^{\beta}(s)p(s)\,\mathrm{d}s$$

$$+\tau^{1-\alpha}(t)\int_t^\infty \tau^\alpha(s)p(s)\,\mathrm{d}s.$$

Taking limit superior as $t\to\infty$ on both sides of the previous inequality, we are led to contradiction with assumptions of the theorem. The proof is complete. □

Corollary 3.10. *Let* (2.7) *hold, and assume that*

$$\limsup_{t\to\infty}\Big\{t^{-\beta}\int_{t_1}^{\lambda t}s^{1+\beta}p(s)\,\mathrm{d}s + \lambda t^{1-\beta}\int_{\lambda t}^{t}s^{\beta}p(s)\,\mathrm{d}s + t^{1-\alpha}\lambda\int_t^\infty s^\alpha p(s)\,\mathrm{d}s\Big\}>1.$$

Then

$$y''(t)+p(t)y(\lambda t)=0,\quad \lambda\in(0,1), \tag{3.14}$$

is oscillatory.

Proof. Proceeding as in proof of Theorem
refthm3 for $\tau(t)=\lambda t$ with $\lambda\in(0,1)$, we obtain the result. □

Following result is simple consequence of Corollary 3.10 for (3.6)

Corollary 3.11. *If*

$$\Big\{\frac{\lambda^\beta}{\beta}+\frac{\lambda^\beta-\lambda}{1-\beta}+\frac{\lambda}{\beta}\Big\}>1, \tag{3.15}$$

then is (3.6) *oscillatory.*

Remark 3.12. If we employ the additional condition $\tau'(t)>0$, it is easy to see that each term of (3.11) is greater than the corresponding term of the criterion presented in Theorem 1.2. Consequently Theorem 3.9 essentially improves the result of [9].

We illustrate the results obtained with example.

Example 3.13. We consider the Euler delay equation

$$y''(t)+\frac{a}{t^2}y(\lambda t)=0,\quad \lambda\in(0,1).$$

For $\lambda=0,2$ and $a=1,25$ criterion (3.15) gives $2,2361>1$. On the other hand criterion [9, (2.5)] gives $0,9024\not>1$. For $\lambda=0,8$ and $a=0,3125$ our criterion holds $(1,1180>1)$ and Koplatadze et al. [9] fails since $0,5558\not>1$. So our criterion essentially improves the known ones.

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