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BIFURCATION OF SOLUTIONS FROM INFINITY FOR CERTAIN NONLINEAR EIGENVALUE PROBLEMS OF FOURTH-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the global bifurcation from infinity of nonlinear eigenvalue problems for ordinary differential equations of fourth order. We prove the existence of unbounded continua of solutions emanating from asymptotically bifurcation points and intervals and having the usual nodal properties near these points and intervals.

1. Introduction

We consider the nonlinear eigenvalue problem

$$\ell y \equiv (py'')'' - (qy')' + r(x)y = \lambda \tau y + h(x, y, y', y'', y''', \lambda), \quad x \in (0, l), \tag{1.1}$$

$$\begin{gathered} y'(0)\cos\alpha - (py'')(0)\sin\alpha = 0, \quad y(0)\cos\beta + Ty(0)\sin\beta = 0, \\ y'(l)\cos\gamma + (py'')(l)\sin\gamma = 0, \quad y(l)\cos\delta - Ty(l)\sin\delta = 0, \end{gathered} \tag{1.2}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, $Ty \equiv (py'')' - qy'$, p is positive, twice continuously differentiable function on $[0, l]$, q is nonnegative, continuously differentiable function on $[0, l]$, r is real-valued continuous function on $[0, l]$, τ is positive continuous function on $[0, l]$ and $\alpha, \beta, \gamma, \delta \in [0, \pi/2]$. The nonlinear term h has the form $h = f + g$, where f and g are real-valued continuous functions on $[0, l] \times \mathbb{R}^5$, satisfying the conditions: there exists $M > 0$ and sufficiently large $c_0 > 0$ such that

$$\begin{gathered} \Big|\frac{f(x, y, s, v, w, \lambda)}{y}\Big| \le M, \\ x \in [0, l],\ y, s, v, w \in \mathbb{R},\ |y| + |s| + |v| + |w| \ge c_0,\ \lambda \in \mathbb{R}; \end{gathered} \tag{1.3}$$

for any bounded interval $\Lambda \subset \mathbb{R}$

$$g(x, y, s, v, w, \lambda) = o(|y| + |s| + |v| + |w|) \quad \text{as } |y| + |s| + |v| + |w| \to \infty, \tag{1.4}$$

uniformly for $x \in [0, l]$ and $\lambda \in \Lambda$.

In nonlinear analysis an important role is played by bifurcation theory of nonlinear eigenvalue problems. The study of bifurcation of nonlinear eigenvalue problems

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has an applied interest since problems of this type arise in almost all fields of natural science (see, for example, [3, 4, 5, 6, 12, 18]). Recently, in this direction have been obtained fundamental results for a wide class of eigenvalue problems which are reflected in [1, 2, 3, 4, 5, 6, 8, 10, 11, 12, 14, 18, 21, 22, 24, 25, 26, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 39] and some others.

In studying the global bifurcation of solutions of nonlinear eigenvalue problems for ordinary differential equations, the nodal properties of the solutions allow a more detailed analysis of the structure and behavior of connected components of a set of nontrivial solutions. The oscillatory properties for the eigenfunctions of the ordinary differential operators of the second and higher orders by various methods were investigated by Sturm [38], Kellogg [19, 20], Prüfer [27], Gantmakher and Kerin [15], Karlin [17], Levin and Stepanov [23], Elias [13], Banks and Kurowski [7].

If the continuous functions f and g on $[0,l]\times\mathbb{R}^5$ satisfy the conditions

$$\Big|\frac{f(x,y,s,v,w,\lambda)}{y}\Big| \le M, \quad x\in[0,l],\ 0<|y|\le 1,\ |s|,|v|,|w|\le 1,\ \lambda\in\mathbb{R}, \tag{1.5}$$

$$g(x,y,s,v,w,\lambda)=o(|y|+|s|+|v|+|w|) \quad \text{as} \quad |y|+|s|+|v|+|w|\to 0, \tag{1.6}$$

uniformly for $x\in[0,l]$ and $\lambda\in\Lambda$, then we can consider bifurcation from $y=0$. Similar problems for Sturm-Liouville equation have been considered before by Rabinowitz [30], Berestycki [8], Schmitt and Smith [36], Rynne [33], Ma and Dai [25]. These authors prove the existence of two families of global continua of solutions in $\mathbb{R}\times C^1$, corresponding to the usual nodal properties and bifurcating from the eigenvalues and intervals (in $\mathbb{R}\times\{0\}$, which we identify with $\mathbb{R}$) surrounding the eigenvalues of the corresponding linear problem. In [3, 4, 5, 6], some elasticity models were studied that include higher-order differential equations and nodal properties. In these papers, using the nodal properties, were obtained similar global bifurcation results for the solutions of the considered mathematical models. Similar results were also demonstrated in [35] for nonlinear eigenvalue problems for a special class of ordinary differential equations of $2m$th order. But until recently it was not possible to obtain similar results for the problem (1.1)-(1.2) under the conditions (1.5) and (1.6).

Note that in nonlinear eigenvalue problems for ordinary differential equations of fourth order, the nodal properties of the solutions need not be preserved along continua, so it is not possible to investigate in detail the structure and behavior of global continua of solutions using the techniques of [30, Theorem 2.3]. Przybycin [29], Lazer and McKenna [22], Rynne [34, 35], Ma and Thompson [26] obtained results similar to the results by Rabinowitz [30, Theorem 2.3], for nonlinear eigenvalue problems of fourth order (in the case of $f\equiv 0$). In these papers for the nonlinear term g is used the smallness condition at $y=0$ of the form $g(x,y,s,v,w,\lambda)=o(|y|)$ to obtain the preservation of nodal properties.

In recent papers by Aliyev [2], the global bifurcation of solutions of problem (1.1)-(1.2) (in the case of $r\equiv 0$) under the conditions (1.5) and (1.6) is completely investigated. To preserve the nodal properties in [2] by using an extension of the Prüfer transformation, the author constructed sets S_k^ν, $k\in\mathbb{N}$, $\nu=\{+,-\}$, of functions in Banach space $E=C^3[0,l]\cap B.C.$ with the usual norm $\|\cdot\|_3$, where $\|y\|_i=\sum_{j=0}^{i}\|y^{(j)}\|_\infty$, $i\in\mathbb{N}$, $\|y\|_\infty=\max_{x\in[0,l]}|y(x)|$, B.C. is the set of functions satisfying boundary conditions (1.2), that have the nodal properties of eigenfunctions of the linear problem (1.1)-(1.2) with $h \equiv 0$ and their derivatives [2, §3.1]. In this paper (see also [1]) the existence of two families of unbounded continua of solutions of problem (1.1)-(1.2) contained in these sets and bifurcating from the points and intervals of the line of trivial solutions is proved.

If condition (1.3) and (1.4) hold, then we can consider bifurcation from $y = \infty$. Similar problems for Sturm-Liouville equation have been considered by Toland [39], Stuart [37], Rabinowitz [31], Przbycin [28], Rynne [32, 33], Ma and Dai [25]. For such problems these authors show the existence of two families of unbounded continua of solutions bifurcating from the points and intervals in $\mathbb{R} \times \{\infty\}$ and having the usual nodal properties in the neighborhood of these points and intervals. (However, the proofs of these assertions carried out in [25, Theorems 2.2 and 2.3] and [28, Theorem 2] contain gaps. In these papers the nonlinear term f has a sublinear growth with respect to y satisfying $|y| > c_0$ and $|y'| > c_0$. It follows from proofs of these theorems that if the solution (λ, y) is near to the bifurcation interval (in $\mathbb{R} \times \{\infty\}$) corresponding to the kth eigenvalue of the linear Sturm-Liouville problem and is contained in a connected component of nontrivial solutions emanating from this interval, then the function y has exactly $k-1$ simple zeros in $(0, l)$. But it is obvious that this function y can not satisfy the conditions $|y| > c_0$ and $|y'| > c_0$ for $k > 1$.) It should be noted that only Przybycin [29] for a special class of nonlinear fourth order eigenvalue problems (in the case of $f \equiv 0$) demonstrates a similar result using the smallness condition at $y = \infty$ of the form $g(x, y, s, v, w, \lambda) = o(|y|)$ for the nonlinear term g.

The purpose of this paper is to study the bifurcation of solutions of problem (1.1)-(1.2) in the cases: (i) $f \equiv 0$ and for g only the condition (1.4) holds; (ii) $f \equiv 0$ and for g both of the conditions (1.4) and (1.6) hold; (iii) $f \not\equiv 0$ and for f and g the conditions (1.3) and (1.4) hold, respectively.

This paper is arranged as follows. In Section 2, we give some statements for the problem (1.1)-(1.2) under conditions (1.5) and (1.6), which we will need in the sequel. In Section 3 the existence of two families of unbounded continua of solutions of problem (1.1)-(1.2) with $f \equiv 0$ under the condition (1.4), bifurcating from infinity and having usual nodal properties in a neighborhood of infinity is proved. In Section 4, problem (1.1)-(1.2) with $f \equiv 0$ is considered when both conditions (1.4) and (1.5) hold. In Section 5, by extending the approximation technique from [8] and combining it with the global bifurcation results in [2, 11, 30, 33], we prove the existence of global sets of solutions of problem (1.1)-(1.2) bifurcating from intervals (in $\mathbb{R} \times \{\infty\}$) which are similar to those obtained in [31, 33].

2. Preliminary results

By [2, Theorem 1.2] the eigenvalues of the linear problem

$$\begin{gathered} \ell(y)(x) = \lambda \tau(x) y(x), \quad x \in (0, l), \\ y \in B.C., \end{gathered} \tag{2.1}$$

are real and simple and form an infinitely increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$. Moreover, for each $k \in \mathbb{N}$ the eigenfunction $y_k(x)$ corresponding to the eigenvalue λ_k lies in S_k (therefore $y_k(x)$ has $k-1$ simple nodal zeros in the interval $(0, l)$).

Lemma 2.1 ([2, Lemma 2.2])**.** *If $y \in \partial S_k^\nu$, $k \in \mathbb{N}$, $\nu \in \{+, -\}$, then $y(x)$ has at least one zero with multiplicity four on the interval* $[0, l]$.

Let $\mathcal{C} \subset \mathbb{R} \times E$ denote the set of solutions of problem (1.1)-(1.2). We say (λ, ∞) is a bifurcation point (or asymptotic bifurcation point) for problem (1.1)-(1.2) if every neighborhood of (λ, ∞) contains solutions of this problem, i.e. there exists a sequence $\{(\lambda_n, u_n)\}_{n=1}^{\infty} \subset \mathcal{C}$ such that $\lambda_n \to \lambda$ and $\|u_n\|_3 \to +\infty$ as $n \to \infty$ (we add the points $\{(\lambda, \infty) : \lambda \in \mathbb{R}\}$ to space $\mathbb{R} \times E$). Next for any $\lambda \in \mathbb{R}$, we say that a subset $D \subset \mathcal{C}$ meets (λ, ∞) (respectively, $(\lambda, 0)$) if there exists a sequence $\{(\lambda_n, u_n)\}_{n=1}^{\infty} \subset D$ such that $\lambda_n \to \lambda$ and $\|u_n\|_3 \to +\infty$ (respectively, $\|u_n\|_3 \to 0$) as $n \to \infty$. Furthermore, we will say that $D \subset \mathcal{C}$ meets (λ, ∞) (respectively, $(\lambda, 0)$) through $\mathbb{R} \times S_k^{\nu}, k \in \mathbb{N}, \nu \in \{+, -\}$, if the sequence $\{(\lambda_n, u_n)\}_{n=1}^{\infty} \subset D$ can be chosen so that $u_n \in S_k^{\nu}$ for all $n \in \mathbb{N}$ (in this case we also say that (λ, ∞) (respectively, $(\lambda, 0)$) is a bifurcation point of (1.1)-(1.2) with respect to the set $\mathbb{R} \times S_k^{\nu}$). If $I \in \mathbb{R}$ is a bounded interval we say that $D \subset \mathcal{C}$ meets $I \times \{\infty\}$ (respectively, $I \times \{0\}$) if D meets (λ, ∞) (respectively, $(\lambda, 0)$) for some $\lambda \in I$; we define $D \subset \mathcal{C}$ meets $I \times \{\infty\}$ (respectively, $I \times \{0\}$) through $\mathbb{R} \times S_k^{\nu}$, $k \in \mathbb{N}$, $\nu \in \{+, -\}$, similarly (see [33]).

We suppose that the conditions (1.5) and (1.6) hold. Then we have the following results.

Theorem 2.2. *Let $f \equiv 0$. Then for each $k \in \mathbb{N}$ and each $\nu \in \{+, -\}$ there exists a continuum $\mathfrak{L}_k^{\nu}$ of solutions of problem* (1.1)-(1.2) *in $(\mathbb{R} \times S_k^{\nu}) \cup \{(\lambda_k, 0)\}$ which meets $(\lambda_k, 0)$ and ∞ in $\mathbb{R} \times E$.*

Lemma 2.3. *For each $k \in \mathbb{N}$ and each $\nu \in \{+, -\}$ the set of bifurcation points for problem* (1.1)-(1.2) *with respect to the set $\mathbb{R} \times S_k^{\nu}$ is nonempty.*

Lemma 2.4. *If $(\lambda, 0)$ is a bifurcation point for problem* (1.1)-(1.2) *with respect to the set $\mathbb{R} \times S_k^{\nu}, k \in \mathbb{N}, \nu \in \{+, -\}$, then $\lambda \in I_k$, where $I_k = [\lambda_k - \frac{M}{\tau_0}, \lambda_k - \frac{M}{\tau_0}]$, $\tau_0 = \min_{x \in [0,l]} \tau(x)$.*

For each $k \in \mathbb{N}$ and each $\nu \in \{+, -\}$, let $\tilde{D}_k^{\nu}$ denote the union of all the connected components $D_{k,\lambda}^{\nu}$ of $\mathcal{C}$ emanating from bifurcation points $(\lambda, 0) \in I_k \times \{0\}$ with respect to $\mathbb{R} \times S_k^{\nu}$. Let $D_k^{\nu} = \tilde{D}_k^{\nu} \cup (I_k \times \{0\})$. Note that D_k^{ν} is a connected subset of $\mathbb{R} \times E$, but $\tilde{D}_k^{\nu}$ is not necessarily connected in $\mathbb{R} \times E$.

Theorem 2.5. *For each $k \in \mathbb{N}$ and each $\nu \in \{+, -\}$ the connected component D_k^{ν} of $\mathcal{C}$ lies in $(\mathbb{R} \times S_k^{\nu}) \cup (I_k \times \{0\})$ and is unbounded in $\mathbb{R} \times E$.*

The proofs of Theorem 2.2, Lemmas 2.3 and 2.4 and Theorem 2.5 are similar to those of [2, Theorem 1.1], [2, Lemmas 5.3, 5.4] and [2, Theorem 1.3], respectively, by using [2, Theorem 1.2].

3. Global bifurcation from infinity of solutions of problem (1.1)-(1.2) for $f \equiv 0$

Throughout this section we assume that only condition (1.4) holds. For any set $A \subset \mathbb{R} \times E$ we let $P_R(A)$ denote the natural projection of A onto $\mathbb{R} \times \{0\}$.

Theorem 3.1. *For each $k \in \mathbb{N}$ and each $\nu \in \{+, -\}$ there exists a connected component $\mathcal{C}_k^{\nu}$ of $\mathcal{C}$ which meets (λ_k, ∞) and has the following properties:*

(i) *there exists a neighborhood Q_k of (λ_k, ∞) in $\mathbb{R} \times E$ such that*
$$Q_k \cap (\mathcal{C}_k^{\nu} \backslash (\lambda_k, \infty)) \subset \mathbb{R} \times S_k^{\nu};$$

(ii) *either $\mathcal{C}_k^{\nu}$ meets $\mathcal{C}_{k'}^{\nu'}$ through $\mathbb{R} \times S_{k'}^{\nu'}$ for some $(k', \nu') \neq (k, \nu)$, or $\mathcal{C}_k^{\nu}$ meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$, or $P_R(\mathcal{C}_k^{\nu})$ is unbounded.*

Proof. Assume that $\lambda = 0$ is not an eigenvalue of (2.1). Then problem (1.1)-(1.2) can be converted to the equivalent integral equation

$$y(x) = \lambda \int_0^l K(x,t)\tau(t)y(t)dt + \int_0^1 K(x,t)\, g(t, y(t), y'(t), y''(t), y'''(t), \lambda)dt, \quad (3.1)$$

where $K(x,t)$ is the Green's function for the differential expression $\ell(y)$ with boundary conditions (1.2). Hence, it is sufficient to search for solution of (1.1)-(1.2) in $\mathbb{R} \times E$.

Let the operator $L : E \to E$ be defined by

$$(Ly)(x) = \int_0^l K(x,t)\tau(t)y(t)dt,$$

and $G : \mathbb{R} \times E \to E$ by

$$(G(\lambda, y))(x) = \int_0^l K(x,t)g(t, y(t), y'(t), y''(t), y'''(t), \lambda)dt.$$

Hence the problem (3.1) can be rewritten in the following form

$$y = \lambda Ly + G(\lambda, y). \quad (3.2)$$

It is clear that L is compact and linear in E and has characteristic values $\lambda_1, \dots, \lambda_k, \dots$, which are the eigenvalues of the linear problem (2.1). The map G is continuous on $\mathbb{R} \times E$. Using (1.4) and following the corresponding arguments carried out in the proof of [31, Theorem 2.4], we can show that

$$G(\lambda, y) = o(\|y\|_3) \quad \text{at } y = \infty, \quad (3.3)$$

uniformly on bounded λ-intervals and $\|y\|_3^2\, G(\lambda, \frac{y}{\|y\|_3^2}) \equiv H(\lambda, y)$ is compact in E.

For any nontrivial $(\lambda, y) \in \mathbb{R} \times E$ setting $v = \frac{y}{\|y\|_3^2}$, we have $\|v\|_3 = \frac{1}{\|y\|_3}$ and $y = \frac{v}{\|v\|_3^2}$. Dividing (3.2) by $\|y\|_3^2$ yields the equation

$$v = \lambda Lv + H(\lambda, v). \quad (3.4)$$

Let $H(\lambda, 0) = 0$. By our basic assumptions the operator $H : \mathbb{R} \times E \to E$ is continuous and satisfy

$$H(\lambda, v) = o(\|v\|_3) \quad \text{at } v = 0, \quad (3.5)$$

uniformly on bounded λ-intervals.

The transformation $(\lambda, y) \to T(\lambda, y) = (\lambda, v)$ which was used in the papers [31, 33, 37, 39] turns a bifurcation from infinity problem (3.2) into a bifurcation from zero problem (3.4). By (3.5) the global bifurcation results in [11] and [30] are applicable to problem (3.4).

Let $\tilde{\mathcal{C}} \subset \mathbb{R} \times E$ be the set of nontrivial solutions of problem (3.4). By construction, the transformation $(\lambda, y) \to T(\lambda, y)$ maps $\mathcal{C}$ into $\tilde{\mathcal{C}}$ and, heuristically, interchanges points at $y = \infty$ (respectively, $y = 0$) with points at $v = 0$ (respectively, $v = \infty$). By [11, Theorem 2] and [30, Lemmas 1.24, 1.27 and Theorem 1.40] for each $k \in \mathbb{N}$ and each $\nu \in \{+, -\}$ there exists a connected component $\tilde{\mathcal{C}}_k^\nu$ of $\tilde{\mathcal{C}}$ with meets $(\lambda_k, 0)$ and has the following properties: (a) there exists a neighborhood $\tilde{Q}_k$ of $(\lambda_k, 0)$ in $\mathbb{R} \times E$ such that $\tilde{Q}_k \cap \Big(\tilde{\mathcal{C}}_k^\nu \backslash (\lambda_k, 0)\Big) \subset \mathbb{R} \times S_k^\nu$; (b) either $\tilde{\mathcal{C}}_k^\nu$ meets $\tilde{\mathcal{C}}_{k'}^{\nu'}$ respect to $\mathbb{R} \times S_{k'}^{\nu'}$ for some $(k', \nu') \neq (k, \nu)$, or $\tilde{\mathcal{C}}_k^\nu$ is unbounded in $\mathbb{R} \times E$ (that is, there exists a sequence $(\lambda_{k,n}, v_{k,n}) \in \tilde{\mathcal{C}}_k^\nu$, $n = 1, 2, 3, \dots$, such that $|\lambda_{k,n}| + \|v_{k,n}\|_3 \to +\infty$ as $n \to \infty$). Then $\mathcal{C}_k^\nu$ and Q_k are the inverse image $T^{-1}(\tilde{\mathcal{C}}_k^\nu)$ of $\tilde{\mathcal{C}}_k^\nu$ and $T^{-1}(\tilde{Q}_k)$ of $\tilde{Q}_k$

under the transformation T respectively. Thus the statements (i) and (ii) of the theorem follows from properties (a) and (b) of $\tilde{\mathcal{C}}_k^\nu$ respectively (second and third alternatives in part (ii) of the theorem for correspond, via T, to the various ways in which $\mathcal{C}_k^\nu$ can be unbounded).

Next, using the above ideas, together with an approximation argument (see [31, p. 468]) we can show that the statements of this theorem are true also in the degenerate case in which 0 is an eigenvalue of linear problem (2.1). The proof is complete. □

Remark 3.2. Unlike in the case of bifurcation from zero in Theorem 2.2, for bifurcation from infinity it need not be the case that $\mathcal{C}_k^\nu \subset (\mathbb{R} \times S_k^\nu) \cup \{(\lambda_k, \infty)\}$, in Theorem 3.1.

Example 3.3. We consider the following nonlinear eigenvalue problem (see [29])

$$y^{(4)}(x) = \lambda(y(x) + 1), \quad x \in (0, \pi),\ y(0) = y(\pi) = y''(0) = y''(\pi) = 0. \tag{3.6}$$

The eigenvalues of the linear eigenvalue problem

$$y^{(4)}(x) = \lambda y(x)\ x \in (0, \pi), \quad y(0) = y(\pi) = y''(0) = y''(\pi) = 0 \tag{3.7}$$

are $\lambda \neq k^4$, $k \in \mathbb{N}$, and corresponding eigenfunctions are $\sin kx$, $k \in \mathbb{N}$.

For $\lambda \neq k^4$, $k \in \mathbb{N}$, the solution of problem (3.6) is unique and given by

$$y_\lambda(x) = -1 + \cos \sqrt[4]{\lambda} x + \frac{1 - \cos \sqrt[4]{\lambda}\pi}{\sin \sqrt[4]{\lambda}\pi} \sin \sqrt[4]{\lambda} x.$$

If k odd, then $y_\lambda(x) \to \infty$ as $\lambda \to k^4$, and if k is even, then $y_\lambda(x) \to -1 + \cos kx \equiv y_{k^4}(x)$. In addition to the solution (k^4, y_{k^4}), the problem (3.6) has also the family of solutions of the form $(k^4, y_{k^4} + c \sin kx), c \in \mathbb{R}$. Hence, we have

$$\mathcal{C}_1^+ = \{(\lambda, y_\lambda) : \lambda \in (0, 1)\} \cup \{(0, 0)\} \cup \{(1, \infty)\}$$

and

$$\begin{aligned}\mathcal{C}_1^- = \big\{(\lambda, y_\lambda) &: \lambda \in (1, 81)\} \cup \{(16, y_\lambda + c \sin 2x) : c \in \mathbb{R}\} \\ &\cup \{(1, \infty)\} \cup \{(16, \infty)\} \cup \{(81, \infty)\}.\end{aligned}$$

Consequently, $\mathcal{C}_1^+ \not\subset ((\mathbb{R} \times S_1^+) \cup \{(1, \infty)\})$ and $\mathcal{C}_1^- \not\subset ((\mathbb{R} \times S_1^-) \cup \{(1, \infty)\})$. Moreover, $\mathcal{C}_1^-$ meets $\mathcal{C}_2^\nu, \nu \in \{+, -\}$, as well as $\mathcal{C}_3^+$.

Remark 3.4. If for each $\lambda \in \mathbb{R}$ there is an x such that $g(x, 0, 0, 0, 0, \lambda) \neq 0$ then the second alternative in part (ii) of the Theorem 3.1 cannot hold.

If we impose some additional conditions on the function g we can obtain stronger results on the structure of the set of solutions of problem (1.1)-(1.2).

Corollary 3.5. *If additionally we assume that*

$$\begin{aligned}g(x, u, s, v, w, \lambda) = g_1&(x, u, s, v, w, \lambda)u + g_2(x, u, s, v, w, \lambda)s \\ &+ g_3(x, u, s, v, w, \lambda)v + g_3(x, u, s, v, w, \lambda)w\end{aligned}$$

where g_1, g_2, g_3 and g_4 are continuous at $(u, s, v, w) = (0, 0, 0, 0)$, then $C_k^\nu \backslash Q_k$ contains a subcontinuum lying in $\mathbb{R} \times S_k^\nu$ and which is unbounded or meets $\mathcal{R} = \mathbb{R} \times \{0\}$.

Proof. It follows from Theorem 3.1 that $(\mathcal{C}_k^+ \cap Q_k) \subset (\mathbb{R} \times S_k^+) \cup \{(\lambda_k, \infty)\}$. We denote by $\mathcal{H}_k^+$ the maximal subcontinuum of $\mathcal{C}_k^+$ lying in $\mathbb{R} \times S_k^+$. If $\mathcal{H}_k^+$ is bounded, then there exists $(\lambda, y) \in \partial\mathcal{H}_k^+ \cap (\mathbb{R} \times \partial S_k^+)$. Hence by Lemma 2.1 the function y has at least one zero of multiplicity 4. Then it follows by [2, Lemma 1.1] that $y \equiv 0$. The proof of corollary is complete. □

Example 3.6. Consider the nonlinear eigenvalue problem

$$\begin{gathered} y^{(4)}(x) = \lambda y(x) + \lambda \tilde{g}(y(x))y(x), \quad 0 < x < l, \\ y(0) = y''(0) = y(l) = y''(l) = 0, \end{gathered} \tag{3.8}$$

where $\tilde{g}(t) = -1$ if $|t| \le 1$, $\tilde{g}(t) = 0$ if $|t| \ge 2$ and $\tilde{g}(t)$ is linear if $1 < |t| < 2$. Note that problem (3.8) has no nontrivial solution (λ, y) such that $\|y\|_3 \le 1$. Hence this problem has no bifurcation points respect to the line of trivial solutions. Consequently, for each $k \in \mathbb{N}$ and each $\nu \in \{+, -\}$ the set $\mathcal{C}_k^\nu \backslash Q_k$ contains an unbounded subcontinuum lying in $\mathbb{R} \times S_k^\nu$.

The following example shows that the second alternative of Corollary 3.5 holds.

Example 3.7. Now we consider the boundary value problem

$$\begin{gathered} y^{(4)}(x) = \lambda(1 + (1 + y^2(x))^{-1})y(x), \quad 0 < x < l, \\ y(0) = y''(0) = y(l) = y''(l) = 0. \end{gathered} \tag{3.9}$$

Let $(\tilde{\lambda}, \tilde{y}(x))$ be a solution of problem (3.9). Then $(\tilde{\lambda}, \tilde{y}(x))$ is an eigenpair of the linear spectral problem

$$\begin{gathered} y^{(4)}(x) = \lambda(1 + (1 + \tilde{y}^2(x))^{-1})y(x), \quad 0 < x < l, \\ y(0) = y''(0) = y(l) = y''(l) = 0. \end{gathered} \tag{3.10}$$

By [2, Theorem 1.2] we have $\tilde{y} \in \cup_{m=1}^\infty S_m$. Let now $\tilde{y} \in S_k$. Then it follows by [2, Theorem 1.2] that $\tilde{\lambda}$ is the k-th eigenvalue of linear problem (3.10). It is obvious that $\tilde{\lambda} > 0$. By the max-min property of eigenvalues [19, Ch. 6, §4]), the k-th eigenvalue $\tilde{\lambda}_k$ of problem (3.10) is determined from the relation

$$\tilde{\lambda}_k = \max_{V^{(k-1)}} \min_{y \in B.C.} \big\{\tilde{R}[y] : \int_0^l y(x)\varphi(x)dx = 0,\, \varphi(x) \in V^{(k-1)}\big\}, \tag{3.11}$$

where $\tilde{R}(y)$ is the Rayleigh quotient

$$\tilde{R}[y] = \frac{\int_0^l \{y''^2(x) - \tilde{\lambda}\tilde{\rho}(x)y^2(x)\}dx}{\int_0^l y^2(x)dx}, \quad \tilde{\rho}(x) = \frac{1}{1 + \tilde{y}^2(x)}, \tag{3.12}$$

and $V^{(k-1)}$ is any arbitrary set of $k-1$ linearly independent functions $\varphi_j(x) \in B.C.$, $1 \le j \le k-1$.

It is obvious that the k-th eigenvalue of problem (3.7) (with π replaced by l) is characterized as

$$\lambda_k = \max_{V^{(k-1)}} \min_{y \in B.C.} \big\{R[y] : \int_0^l y(x)\varphi(x)dx = 0,\, \varphi(x) \in V^{(k-1)}\big\}, \tag{3.13}$$

where

$$R[y] = \frac{\int_0^l y''^2(x)dx}{\int_0^l y^2(x)dx}. \tag{3.14}$$

For any choice of $V^{(k-1)}$ from (3.12) and (3.14) we obtain

$$R[y]-\tilde{\lambda}\leq \tilde{R}[y]\leq R[y].$$

Hence it follows from (3.11) and (3.13) that

$$\lambda_k-\tilde{\lambda}\leq \tilde{\lambda}_k\leq \lambda_k,$$

which implies (by virtue of $\tilde{\lambda}_k=\tilde{\lambda}$) that

$$\frac{\lambda_k}{2}\leq \tilde{\lambda}\leq \lambda_k.$$

Thus, we have shown that if $(\tilde{\lambda},\tilde{y})\in\mathbb{R}\times S_k$ is a solution of problem (3.9), then $\tilde{\lambda}\in[\frac{\lambda_k}{2},\lambda_k]$. Hence $\mathcal{C}_k^\nu$ lies in $[\frac{\lambda_k}{2},\lambda_k]\times S_k^\nu$.

Let $\{(\lambda_{k,n},y_{k,n})\}_{n=1}^\infty\subset\mathbb{R}\times S_k^\nu$ be a sequence of solutions of problem (3.9) converges to $(\hat{\lambda},0)$ in $\mathbb{R}\times E$. Setting $v_n=\frac{y_{k,n}}{\|y_{k,n}\|_3}$ we obtain that v_n satisfies the relations

$$\begin{gathered}v_n^{(4)}(x)=\lambda_{k,n}(1+(1+y_{k,n}^2(x))^{-1})v_n(x),\quad 0<x<l,\\ v_n(0)=v_n''(0)=v_n(l)=v_n''(l)=0.\end{gathered}\tag{3.15}$$

Since v_n is bounded in $C^3[0,l]$, $1+\frac{1}{1+y_{k,n}^2}$ is bounded in $C[0,l]$, it follows from (3.15) that v_n is bounded in $C^4[0,l]$. Therefore, by the Arzelà-Ascoli theorem, we may assume that $v_n\to v$ in $C^3[0,l]$; $\|v\|_3=1$. Moreover, $v\in\overline{S_k^\nu}=S_k^\nu\cup\partial S_k^\nu$. Since $\|v\|_3=1$ it follows from [2, Lemma 1.1] that $v\in S_k^\nu$. Passing to the limit as $n\to\infty$ in (3.15) we obtain

$$\begin{gathered}v^{(4)}(x)=2\hat{\lambda}v(x),\quad 0<x<l,\\ v(0)=v''(0)=v(l)=v''(l)=0.\end{gathered}$$

Since $v\in S_k^\nu$ it follows from [2, Theorem 1.1] that $2\hat{\lambda}$ is a k-th eigenvalue of the linear problem

$$\begin{gathered}y^{(4)}(x)=\lambda y(x),\quad 0<x<l,\\ y(0)=y''(0)=y(l)=y''(l)=0,\end{gathered}$$

which implies that $\hat{\lambda}=\frac{\lambda_k}{2}$. Therefore, $\mathcal{C}_k^\nu$ meets $\mathcal{R}$ at $(\frac{\lambda_k}{2},0)$.

Corollary 3.8. *If g is as in Corollary 3.5 with $g_i(x,0,0,0,0,\lambda)=0, i=1,2,3,4$, and $\mathcal{C}_k^\nu$ meets $\mathcal{R}$, then it does so at $(\lambda_k,0)$.*

Proof. The point at which $\mathcal{C}_k^\nu$ meets $\mathcal{R}$ corresponds to an eigenvalue of problem (2.1). But the only point $(\lambda_m,0)$ which can be the limit of elements (λ,y) with $y\in S_k^\nu$ is $(\lambda_k,0)$. The proof of this corollary is complete. □

It should be noted that this fact is also true in the more general case (see Theorem 4.1).

4. Global bifurcation from zero and infinity of solutions of problem (1.1)-(1.2) for $f\equiv 0$

If $f\equiv 0$ and for g the conditions (1.4) and (1.6) both hold then we can improve Theorems 2.2 and 3.1 as follows.

Theorem 4.1. *Let $f \equiv 0$ and the conditions* (1.4) *and* (1.6) *both hold. Then for each $k \in \mathbb{N}$ and each $\nu \in \{+,-\}$ we have $\mathcal{C}_k^\nu \subset \mathbb{R} \times S_k^\nu$ and the first part of alternative (ii) of Theorem 3.1 cannot hold. Furthermore, if $\mathfrak{L}_k^\nu$ meets (λ, ∞) for some $\lambda \in \mathbb{R}$, then $\lambda = \lambda_k$. Similarly, if $\mathcal{C}_k^\nu$ meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$, then $\lambda = \lambda_k$.*

Proof. It follows from [2, Lemma 1.1] that if condition (1.4) holds, then $\mathcal{C} \cap (\mathbb{R} \times \partial S_k^\nu) = \emptyset$. Hence the sets $\mathcal{C} \cap (\mathbb{R} \times S_k^\nu)$ and $\mathcal{C} \backslash (\mathbb{R} \times S_k^\nu)$ are mutually separated in $\mathbb{R} \times E$ (see [40, Definition 26.4]). Thus it follows by [40, Corollary 26.6] that any component of $\mathcal{C}$ must be a subset of one or another of these sets. Since $\mathcal{C}_k^\nu$ is a component of $\mathcal{C}$ which intersect $\mathbb{R} \times S_k^\nu$, then $\mathcal{C}_k^\nu$ must be a subset of $\mathbb{R} \times S_k^\nu$, i.e. $\mathcal{C}_k^\nu \subset \mathbb{R} \times S_k^\nu$. But this shows that the first part of alternative (ii) of Theorem 3.1 cannot hold.

Now suppose that $\mathfrak{L}_k^\nu$ meets (λ, ∞) for some $\lambda \in \mathbb{R}$. Then there exists a sequence $\{(\lambda_{k,n}, y_{k,n})\}_{n=1}^\infty \subset \mathfrak{L}_k^\nu$ such that $\lambda_{k,n} \to \lambda$ and $\|y_{k,n}\|_3 \to \infty$ as $n \to \infty$ and

$$y_{k,n} = \lambda_{k,n} L y_{k,n} + G(\lambda_{k,n}, y_{k,n}).$$

Let $v_{k,n} = \frac{y_{k,n}}{\|y_{k,n}\|_3}$, so $\|v_{k,n}\|_3 = 1$. Dividing this equality by $\|y_{k,n}\|_3$ shows that $v_{k,n}$ satisfies

$$v_{k,n} = \lambda_{k,n} L v_{k,n} + \frac{G(\lambda_{k,n}, y_{k,n})}{\|y_{k,n}\|_3}.$$

Then it follows from the compactness of operator L and the condition (3.3) that there exists a subsequence of the sequence $\{(\lambda_{k,n}, v_{k,n})\}_{n=1}^\infty$ (which we will relabel as $\{(\lambda_{k,n}, v_{k,n})\}_{n=1}^\infty$) which converges in $\mathbb{R} \times E$ to (λ, v). Letting $n \to \infty$ in the above equality we obtain

$$v = \lambda L v.$$

Hence (λ, v) is eigenpair of problem (2.1) and v lies in the closure of S_k^ν. Since $\|v\|_3 = 1$ it follows from [2, Lemma 1.1] that $v \in S_k^\nu$. Then by [2, Theorem 1.2] we have $\lambda = \lambda_k$ and $v = \nu y_k$. Thus $\mathfrak{L}_k^\nu$ can only meet (λ, ∞) if $\lambda = \lambda_k$. Similarly is proved that $\mathcal{C}_k^\nu$ can only meet $(\lambda, 0)$ if $\lambda = \lambda_k$. The proof is complete. □

The naturally question arises whether or not $\mathfrak{L}_k^\nu$ intersects $\mathcal{C}_k^\nu$. The following examples show that, both cases are possible.

Example 4.2. Now we consider the boundary problem

$$\begin{gathered} y^{(4)}(x) = \lambda y(x) + \lambda f(x, y(x), y'(x), y''(x), y'''(x)) y(x), \quad 0 < x < l, \\ y(0) = y''(0) = y(l) = y''(l) = 0, \end{gathered} \tag{4.1}$$

We assume that f satisfies the following conditions:

(i) there exist positive constants K, d and θ such that

$$|f(x, u, s, v, w)| \le K(|u| + |s| + |v| + |w|)^{-\theta}$$

for all $(x, u, s, v, w) \in [0, l] \times \mathbb{R}^4$ with $|u| + |s| + |v| + |w| \ge d$;

(ii) f is continuous in $[0, l] \times \mathbb{R}^4$ and $f(x, 0, 0, 0, 0) = 0$ for $x \in [0, l]$.

These two conditions ensure that for $g(x, u, s, v, w, \lambda) = \lambda f(x, u, s, v, w)$ conditions (1.4) and (1.6) both hold.

Since $\lambda_1 > 0$ (in this case $\lambda_1 = \frac{l}{\pi}$), for the location of continua $\mathfrak{L}_1^\nu$ and $\mathcal{C}_1^\nu, \nu \in \{+,-\}$, we have the following results.

(a) If $f(x, u, s, v, w) \ge 0$ for $(x, u, s, v, w) \in [0, l] \times \mathbb{R}^4$ and $(\lambda, y) \in \mathfrak{L}_1 \cup \mathcal{C}_1$, then $0 < \lambda < \lambda_1$. Indeed, if (λ, y) be a solution of (4.1), then multiplying both sides of equation in (4.1) by y and integrating this relation from 0 to l, using the formula for the integration by parts, and taking into account boundary conditions in (4.1), we obtain

$$\int_0^l (y''(x))^2 dx = \lambda \int_0^l \big\{1 + f(x, y(x), y'(x), y''(x), y'''(x))\big\} y^2(x) dx,$$

which implies that $\lambda > 0$. If (λ, y) be a solution of (4.1) and $y \in S_1$, then

$$\begin{gathered} y^{(4)}(x) - \lambda f(x, y(x), y'(x), y''(x), y'''(x)) y(x) = \lambda y(x),\ 0 < x < l, \\ y(0) = y''(0) = y(l) = y''(l) = 0, \end{gathered} \tag{4.2}$$

which implies that λ is the first eigenvalue of the problem

$$\begin{gathered} v^{(4)}(x) - \lambda r_1(x) v(x) = \mu v(x), \quad 0 < x < l, \\ v(0) = v''(0) = v(l) = v''(l) = 0, \end{gathered} \tag{4.3}$$

where $r_1(x) = f(x, y(x), y'(x), y''(x), y'''(x)) \geq 0$. The first eigenvalue of problem (4.3) can be characterized as

$$\mu_1 = \min_{v \in B.C.} \frac{\int_0^l v''^2(x) dx - \lambda \int_0^l r_1(x) v^2(x) dx}{\int_0^l v^2(x) dx}.$$

Since $\lambda > 0$ and $r_1(x) \geq 0, x \in [0, l]$, it follows from the above equality that

$$\lambda = \mu_1 < \min_{v \in B.C.} \frac{\int_0^l v''^2(x) dx}{\int_0^l v^2(x) dx} = \lambda_1 .$$

Therefore, for $\mathcal{C}_1^\nu$, first alternative in part (ii) of Theorem 3.1 cannot hold. Hence by Theorem 4.1 the set $\mathcal{C}_1^\nu$ must be unbounded in $[0, \lambda_1] \times E$ and so must bifurcate from $(\lambda_1, 0)$. Also by Theorem 2.2 and 4.1 the set $\mathfrak{L}_1^\nu$ must approach (λ_1, ∞). Hence $\mathfrak{L}_1^\nu \cap \mathcal{C}_1^\nu \neq \emptyset$.

(b) If $f(x, u, s, v, w) \leq 0$ for $(x, u, s, v, w) \in [0, l] \times \mathbb{R}^4$ and $(\lambda, y) \in \mathfrak{L}_1 \cup \mathcal{C}_1$, then $\lambda > \lambda_1$. Indeed, in this case $r_1(x) \leq 0, x \in [0, l]$, and consequently, we have

$$\begin{aligned} \lambda = \mu_1 &= \min_{v \in B.C.} \frac{\int_0^l v''^2(x) dx - \lambda \int_0^l r_1(x) v^2(x) dx}{\int_0^l v^2(x) dx} \\ &= \frac{\int_0^l y''^2(x) dx - \lambda \int_0^l r_1(x) y^2(x) dx}{\int_0^l y^2(x) dx} \\ &> \frac{\int_0^l y''^2(x) dx}{\int_0^l y^2(x) dx} \geq \min_{v \in B.C.} \frac{\int_0^l v''^2(x) dx}{\int_0^l v^2(x) dx} = \lambda_1. \end{aligned}$$

Let $f(x, u, s, v, w) = f_1(u^2 + s^2 + v^2 + w^2)$, where $f_1(z) = -z$ if $|z| \leq 1$, $f_1(z) = -1$ if $2 < |z| < 3$, $f_1(z) = -\frac{16}{z}$ if $|z| \geq 4$ and is continuous for all z. Then $\mathcal{C}_1^\nu = \{\big(\frac{c^2}{c^2-16}, c \sin x\big) : \nu c \geq 4\}$ and $\mathfrak{L}_1^\nu = \big\{\big(\frac{1}{1-c^2}, c \sin x\big) : 0 \leq \nu c \leq 1\big\}$. Thus $\mathfrak{L}_1^\nu \cap \mathcal{C}_1^\nu = \emptyset$. Moreover, for each $\nu \in \{+, -\}$ the continua $\mathfrak{L}_1^\nu$ and $\mathcal{C}_1^\nu$ are unbounded in $\mathbb{R} \times E$ and lies in $[\lambda_1, \infty) \times S_1^\nu$.

5. Global bifurcation from infinity of solutions of problem (1.1)-(1.2)

Throughout this section we assume that $f \not\equiv 0$ and the conditions (1.3) and (1.4) are satisfied.

Recall that to study the bifurcation from infinity of the solutions of problem (1.1)-(1.2), as in the papers [31, 33, 37, 39] we use the inversion $(\lambda, y) \to T(\lambda, y) = (\lambda, \frac{y}{\|y\|_3^2})$ which transforms the bifurcation from infinity problem (1.1)-(1.2) to the corresponding bifurcation from zero problem. But in this case the set $\{y \in E : |y| + |y'| + |y''| + |y'''| \geq c_0\}$ is not transformed to the set of the form $\{v \in E : |y| + |y'| + |y''| + |y'''| \leq r_0\}$ for some sufficiently small $r_0 > 0$. Consequently, it is impossible to apply Theorem 2.5. Therefore, we need the following result to solve this problem.

Lemma 5.1. *There exists functions $f^*, g^* \in C\big([0,l] \times \mathbb{R}^5\big)$ such that h can be also represented in the form $h = f^* + g^*$, and f^*, g^* satisfy the conditions:*

$$\Big|\frac{f^*(x,u,s,v,w,\lambda)}{u}\Big| \leq M, \quad (x,u,s,v,w,\lambda) \in [0,l] \times \mathbb{R}^5,\ u \neq 0; \tag{5.1}$$

$$g^*(x,u,s,v,w,\lambda) = o(|u| + |s| + |v| + |w|), \quad \text{as } |u| + |s| + |v| + |w| \to \infty, \tag{5.2}$$

uniformly in $x \in [0,l]$ and in $\lambda \in \Lambda$, for any bounded interval $\Lambda \subset \mathbb{R}$.

Proof. Let $U = (u,s,v,w) \in \mathbb{R}^4$ and $|U| = |u| + |s| + |v| + |w|$. Suppose that $\zeta(U)$ is a continuous function in $\mathbb{R}^4$, $0 \leq \zeta \leq 1$, such that

$$\zeta(U) = \begin{cases} 0 & \text{for } |U| \leq c_0, \\ 1 & \text{for } |U| \geq c_0 + \kappa_0, \end{cases}$$

where κ_0 is a sufficiently small fixed positive number. Then we can write

$$f(x,U,\lambda) = \zeta(U) f(x,U,\lambda) + (1 - \zeta(U)) f(x,U,\lambda).$$

Hence the functions

$$f_1(x,U,\lambda) = \zeta(U) f(x,U,\lambda) \quad \text{and} \quad f_2(x,U,\lambda) = (1 - \zeta(U)) f(x,U,\lambda)$$

are continuous in $[0,l] \times \mathbb{R}^5$ and by (1.3) satisfy the following conditions:

$$\Big|\frac{f_1(x,U,\lambda)}{u}\Big| \leq M, \quad (x,U,\lambda) \in [0,l] \times \mathbb{R}^5, u \neq 0; \tag{5.3}$$

$$f_2(x,U,\lambda) = 0, \quad (x,U,\lambda) \in [0,l] \times \mathbb{R}^5,\ |U| \geq c_0 + \kappa_0. \tag{5.4}$$

We define the functions $f^*, g^* : [0,l] \times \mathbb{R}^5 \to \mathbb{R}$ as follows:

$$f^* = f_1, \quad g^* = g + f_2.$$

Then the function h is represented in the form $h = f^* + g^*$, where f^* and g^* are continuous functions in $[0,l] \times \mathbb{R}^5$ and by virtue of (1.4), (5.3), and (5.4) satisfy the conditions (5.1) and (5.2), respectively. The proof is complete. □

Recall that if 0 is not an eigenvalue of the linear problem (2.1), then the nonlinear problem (1.1)-(1.2) is reduced to the equivalent integral equation

$$\begin{aligned} y(x) = \lambda \int_0^1 K(x,t)\,\tau(t) y(t) dt + \int_0^1 K(x,t) f^*(t, y(t), y'(t), y''(t), y'''(t), \lambda) dt \\ + \int_0^1 K(x,t) g^*(t, y(t), y'(t), y''(t), y'''(t), \lambda)\, dt. \end{aligned} \tag{5.5}$$

Let

$$F^*(\lambda, y)(x)) = \int_0^1 K(x,t) f^*(t, y(t), y'(t), y''(t), y'''(t), \lambda) dt, \tag{5.6}$$

$$G^*(\lambda, y)(x) = \int_0^1 K(x,t) g^*(t, y(t), y'(t), y''(t), y'''(t), \lambda) dt. \tag{5.7}$$

Note that $F^* : \mathbb{R} \times E \to E$ is completely continuous, $G^* : \mathbb{R} \times E \to E$ is continuous and satisfies the condition

$$G^*(\lambda, y) = o(\|y\|_3) \quad \text{at } y = \infty, \tag{5.8}$$

uniformly on bounded λ-intervals. Also, the operator $H^* : (\lambda, y) \to \|y\|_3^2 \, g^*(\lambda, \frac{y}{\|y\|_3^2})$ is compact.

By Lemma 5.1 and (5.5)-(5.7), problem (1.1)-(1.2) can be rewritten in the equivalent form

$$y = \lambda L y + F^*(\lambda, y) + G^*(\lambda, y). \tag{5.9}$$

Along with (2.1) we consider the linear spectral problem

$$\begin{gathered} \ell y(x) + \varphi(x) y(x) = \lambda \tau(x) y(x), \quad x \in (0, l), \\ y \in B.C., \end{gathered} \tag{5.10}$$

where $\varphi(x) \in C[0,1]$. We need the following result which is basic in the sequel.

Lemma 5.2. *For each $k \in \mathbb{N}$ it holds*

$$|\mu_k - \lambda_k| \le K/\tau_0, \tag{5.11}$$

where μ_k is the kth eigenvalue of problem (5.10)*, $K = \sup_{x \in [0,l]} |\varphi(x)|$.*

The proof of this lemma is similar to that of [2, Lemma 4.1].

Remark 5.3. Since the class of continuous functions $C[0,1]$ is dense in $L_1[0,1]$, Lemma 5.2 also holds for $\varphi(x) \in L_1[0,1]$.

To study the bifurcation from infinity of solutions of (1.1)-(1.2), we consider the approximate problem

$$\begin{gathered} \ell y = \lambda \tau(x) y + \frac{f^*(x, \|y\|_3^\varepsilon y, \|y\|_3^\varepsilon y', \|y\|_3^\varepsilon y'', \|y\|_3^\varepsilon y''', \lambda)}{\|y\|_3^{2\varepsilon}} \\ + g^*(x, y, y', y'', y''', \lambda), \quad x \in (0, l), \\ y \in B.C., \end{gathered} \tag{5.12}$$

where $\varepsilon \in (0,1]$.

Lemma 5.4. *Let $\delta > 0$ be the sufficiently small fixed number. Then for each $k \in \mathbb{N}$ there exists sufficiently large $R_k^* > 0$ such that for given any $\varepsilon \in (0,1]$ problem* (5.12) *has no nontrivial solution (λ, y) which satisfied the conditions* $\operatorname{dist}\{\lambda, I_k\} > \delta$*, $y \in S_k^\nu, \nu \in \{+,-\}$, and $\|y\|_3 > R_k^*$.*

Proof. On the contrary assume that there exists $\varepsilon_0 \in (0,1]$ and sufficiently large $n_0 \in \mathbb{N}$ such that for any $n \ge n_0$ problem (3.3) for $\varepsilon = \varepsilon_0$ has a nontrivial solution (λ_n, y_n) satisfying $\operatorname{dist}\{\lambda_n, I_k\} > \delta$, $y_n \in S_k^\nu, \nu \in \{+,-\}$, and $\|y_n\|_3 > n$.

For each $n \geq n_0$, we have

$$\begin{aligned} \ell y_n = \lambda_n \tau(x) y_n &+ \frac{f^*(x, \|y_n\|_3^{\varepsilon_0} y_n, \|y_n\|_3^{\varepsilon_0} y_n', \|y_n\|_3^{\varepsilon_0} y_n'', \|y_n\|_3^{\varepsilon_0} y_n''', \lambda_n)}{\|y_n\|_3^{2\varepsilon_0}} \\ &+ g^*(x, y_n, y_n', y_n'', y_n''', \lambda_n), \; x \in (0, l), \\ &y_n \in B.C. \end{aligned} \tag{5.13}$$

We define a function $\varphi_n(x), n \geq n_0, x \in [0, l]$, as follows:

$$\varphi_n(x) = \begin{cases} -\frac{f^*\left(x, \|y_n\|_3^{\varepsilon_0} y_n(x), \|y_n\|_3^{\varepsilon_0} y_n'(x), \|y_n\|_3^{\varepsilon_0} y_n''(x), \|y_n\|_3^{\varepsilon_0} y_n'''(x), \lambda_n\right)}{\|y_n\|_3^{2\varepsilon_0} y_n(x)} & \text{if } y_n(x) \neq 0, \\ 0 & \text{if } y_n(x) = 0. \end{cases}$$

Then from (5.13) it follows that $(\lambda_n, y_n), n \geq n_0$ solves the nonlinear problem

$$\begin{gathered} \ell y + \varphi_n(x) y = \lambda \tau(x) y + g^*(x, y, y', y'', y''', \lambda), \; x \in (0, l), \\ y \in B.C. \end{gathered} \tag{5.14}$$

From (5.1) we have $|\varphi_n(x)| \leq \frac{M}{\|y_n(x)\|_3^{\varepsilon_0}} \leq M$, $n \geq n_0$, $x \in [0, l]$. Since $y_n(x), n \geq n_0$, has a finite number of zeros on $(0, l)$ and is bounded on the closed interval $[0, l]$, Remark 5.3 shows that the result of Lemma 5.2 also holds for the linear problem

$$\begin{gathered} \ell y + \varphi_n(x) y = \lambda \tau(x) y, \quad x \in (0, l), \\ y \in B.C. \end{gathered} \tag{5.15}$$

Then it follows from (5.11) that the k-th eigenvalue $\lambda_{k,n}$ of the linear problem (5.15) lies in I_k. By [9, Ch. 4, §3, Theorem 3.1] for each $n \geq n_0$ the point $(\lambda_{k,n}, \infty)$ is a unique asymptotic bifurcation point of (5.14) which corresponds to a continuous branch of solutions that meets this point through $\mathbb{R} \times S_k^\nu$. Hence for each sufficiently large $n > n_0$ we can assign a small $\delta_n > 0$ such that $\delta_n < \delta$ and $|\lambda_n - \lambda_{k,n}| < \delta_n$. Then it follows that $\operatorname{dist}\{\lambda_n, I_k\} < \delta$, contradicting $\operatorname{dist}\{\lambda_n, I_k\} > \delta$. The proof is complete. □

Lemma 5.5. *For any sufficiently small $\epsilon > 0$ there exists sufficiently large $\rho_\epsilon > 0$ such that for $\lambda \in \Lambda$, $\|y\|_3 > \rho_\epsilon$,*

$$|g^*(x, y, y', y'', y''', \lambda)| < \epsilon \|y\|_3, \quad x \in [0, l]. \tag{5.16}$$

Proof. It follows from (5.2) that for any sufficiently small $\epsilon > 0$ there exists sufficiently large $\varrho_\epsilon > 0$ such that for $x \in [0, l], \lambda \in \Lambda$, $(u, s, v, w) \in \mathbb{R}^4$, and $|u| + |s| + |v| + |w| > \varrho_\epsilon$ the following relation holds

$$|g^*(x, u, s, v, w, \lambda)| < \epsilon(|u| + |s| + |v| + |w|). \tag{5.17}$$

Moreover, by continuity of g^* there exists $K_\epsilon > 0$ such that for $x \in [0, l], \lambda \in \Lambda$ and $|u| + |s| + |v| + |w| \leq \varrho_\epsilon$,

$$|g^*(x, u, s, v, w, \lambda)| \leq K_\epsilon. \tag{5.18}$$

Let $\rho_\epsilon > \varrho_\epsilon$ such that $\frac{K_\epsilon}{\rho_\epsilon} < \epsilon$ and $y \in E$ such that $\|y\|_3 > \rho_\epsilon$. Introduce the sets $A_{1,\epsilon} \subset [0, l]$, $A_{2,\epsilon} \subset [0, l]$ $(A_{1,\epsilon} \cup A_{2,\epsilon} = [0, l])$ defined the following way:

$$\begin{gathered} A_{1,\epsilon} = \{x \in [0, l] : |y(x)| + |y'(x)| + |y''(x)| + |y'''(x)| \leq \varrho_\epsilon\}, \\ A_{2,\epsilon} = \{x \in [0, l] : |y(x)| + |y'(x)| + |y''(x)| + |y'''(x)| > \varrho_\epsilon\}. \end{gathered}$$

If $x \in A_{1,\epsilon}$, $\lambda \in \Lambda$, then it follows from (5.18) that

$$|g^*(x, y(x), y'(x), y''(x), y'''(x), \lambda)| \le K_\epsilon = \frac{K_\epsilon}{\rho_\epsilon}\rho_\epsilon < \epsilon \,\|y\|_3.$$

Moreover, if $x \in A_{2,\epsilon}, \lambda \in \Lambda$, then it follows from (5.17) that

$$\begin{aligned}|g^*(x, y(x), y'(x), y''(x), y'''(x), \lambda)| &< \epsilon\big(|y(x)| + |y'(x)| + |y''(x)| + |y'''(x)|\big)\\ &\le \epsilon \,\|y\|_3.\end{aligned}$$

The proof is complete. □

Let $p_0 = \min_{x\in[0,l]} p(x)$. For $k \in \mathbb{N}$ we define the numbers

$$r_k = p_0^{-1}\big\{2\|p\|_2 + \|q\|_1 + \|r\|_\infty + (|\lambda_k| + M/\tau_0 + 1)\|\tau\|_\infty + M/R_k^*\big\}.$$

Lemma 5.6. *Let $\delta > 0$ and $\epsilon_k > 0, k \in \mathbb{N}$, be a sufficiently small fixed number, and $\epsilon_k < \frac{p_0}{2le^{(r_k+1)l}}$. Then for each $k \in \mathbb{N}$ there exists a sufficiently large $R_k > \max\{R_k^*, \rho_{\epsilon_k}\}$ such that for any $R > R_k$ problem (1.1)-(1.2) has a solution $(\lambda_{R,k}^\nu, v_{R,k}^\nu)$ which satisfies conditions $\operatorname{dist}\{\lambda_{R,k}^\nu, I_k\} \le \delta$, $v_{R,k}^\nu \in S_k^\nu, \nu \in \{+,-\}$, and $\|v_{R,k}^\nu\|_3 = R$.*

Proof. Using (5.9) we can write (5.12) in an equivalent form as follows:

$$y = \lambda Ly + \|y\|_3^{-2\varepsilon} F^*(\lambda, \|y\|_3^\varepsilon\, y) + G^*(\lambda, y). \tag{5.19}$$

By (5.1) it follows from (5.6) that

$$\|F^*(\lambda, \|y\|_3^\varepsilon y)\|_3 \le C_1\|y\|_3^{1+\varepsilon}. \tag{5.20}$$

where $C_1 = c_1M$ and c_1 depends on bounds for K, K_x, K_{xx} and K_{xxx}.

In view of (5.20) we have

$$\|y\|_3^{-2\varepsilon} F^*(\lambda, \|y\|_3^\varepsilon\, y) = o(\|y\|_3) \quad \text{as } \|y\|_3 \to \infty, \tag{5.21}$$

uniformly in $\lambda \in \Lambda$. Then by (5.8) and (5.21) it follows from Theorem 3.1 that for each $k \in \mathbb{N}$ and each $\nu \in \{+,-\}$ there exists an unbounded component $C_{k,\varepsilon}^\nu$ of solutions of (5.19) (or (5.12)) which meets (λ_k, ∞) and there exists a neighborhood $\mathcal{Q}_{k,\varepsilon}$ of (λ_k, ∞) such that $\mathcal{Q}_{k,\varepsilon} \cap (C_{k,\varepsilon}^\nu \backslash (\lambda_k, \infty)) \subset \mathbb{R} \times S_k^\nu$ and either $C_{k,\varepsilon}^\nu \backslash \mathcal{Q}_{k,\varepsilon}$ is bounded in $\mathbb{R} \times E$ in which case $C_{k,\varepsilon}^\nu \backslash \mathcal{Q}_{k,\varepsilon}$ meets $\mathcal{R}$ or $C_{k,\varepsilon}^\nu \backslash \mathcal{Q}_{k,\varepsilon}$ is unbounded in $\mathbb{R} \times E$. Moreover, if $C_{k,\varepsilon}^\nu \backslash \mathcal{Q}_{k,\varepsilon}$ is unbounded and has a bounded projection on $\mathbb{R}$, then this set meets $(\lambda_{k'}^{\sigma'}, \infty)$ through $\mathbb{R} \times S_{k'}^{\nu'}$ for some $(k', \sigma') \ne (k, \sigma)$. Hence by Lemma 5.4 it follows that for any $\varepsilon \in (0, 1)$ and each $R > \max\{R_k^*,\, \rho_{\epsilon_k}\}$ there exists a solution $(\lambda_{R,k,\varepsilon}^\nu, v_{R,k,\varepsilon}^\nu) \in \mathbb{R} \times E$ of (5.12) such that $\operatorname{dist}\{\lambda_{R,k,\varepsilon}^\nu, I_k\} \le \delta$ and $\|v_{R,k,\varepsilon}^\nu\|_3 = R$. Following the proof of Lemma 5.4 one can show that there exists sufficiently large $R_k > \max\{R_k^*, \rho_{\epsilon_k}\}$ such that $v_{R,k,\varepsilon}^\nu \in S_k^\nu, \nu \in \{+,-\}$, for any $R > R_k$.

Let $R > R_k$ be fixed. Since $\{v_{R,k,\varepsilon}^\nu \in E : 0 < \varepsilon \le 1\}$ is a bounded subset of $C^3[0, 1]$, the functions f^* and g^* are continuous in $[0, 1] \times \mathbb{R}^5$, satisfying the conditions (5.1) and (5.2), and the set $\{\lambda_{R,k,\varepsilon}^\nu \in \mathbb{R} : 0 < \varepsilon \le 1\}$ is bounded in $\mathbb{R}$, it follows from (5.12) that $\{v_{R,k,\varepsilon}^\nu \in E : 0 < \varepsilon \le 1\}$ is also bounded in $C^4[0, 1]$. Hence it is precompact in E by the Arzelà-Ascoli theorem.

Let $\{\varepsilon_n\}_{n=1}^\infty \subset (0, 1)$ be a sequence such that $\varepsilon_n \to 0$ and $(\lambda_{R,k,\varepsilon_n}^\nu, v_{R,k,\varepsilon_n}^\nu) \to (\lambda_{R,k}^\nu, v_{R,k}^\nu)$ as $n \to \infty$. Taking the limit (as $n \to \infty$) in (5.12) we see that

$(\lambda_{R,k}^{\nu}, v_{R,k}^{\nu})$ is a solutions of (1.1)-(1.2), i.e. the following relations hold:

$$\begin{gathered} \ell v_{R,k}^{\nu} = \lambda_{R,k}^{\nu}\,\tau(x) v_{R,k}^{\nu} + f^*(x, v_{R,k}^{\nu}, (v_{R,k}^{\nu})', (v_{R,k}^{\nu})'', (v_{R,k}^{\nu})''', \lambda_{R,k}^{\nu}) \\ + g^*(x, v_{R,k}^{\nu}, (v_{R,k}^{\nu})', (v_{R,k}^{\nu})'', (v_{R,k}^{\nu})''', \lambda_{R,k}^{\nu}), \\ v_{R,k}^{\nu} \in B.C.. \end{gathered} \tag{5.22}$$

Since $v_{R,k,\varepsilon_n}^{\nu} \in S_k^{\nu}$ it follows that $v_{R,k}^{\nu} \in \overline{S_k^{\nu}} = S_k^{\nu} \cup \partial S_k^{\nu}$. If $v_{R,k}^{\nu} \in \partial S_k^{\nu}$, then by Lemma 2.1 there exists $\varsigma \in [0, l]$ such that $v_{R,k}^{\nu}(\varsigma) = (v_{R,k}^{\nu})'(\varsigma) = (v_{R,k}^{\nu})''(\varsigma) = (v_{R,k}^{\nu})'''(\varsigma) = 0$. Let $w_{R,k}^{\nu} = \frac{v_{R,k}^{\nu}}{\|v_{R,k}^{\nu}\|_3}$. Then we have $\|w_{R,k}^{\nu}\| = 1$. Dividing (5.22) by $\|v_{R,k}^{\nu}\|_3$ shows that $w_{R,k}^{\nu}$ satisfies the equation

$$\begin{aligned} \ell w_{R,k}^{\nu} = \lambda_{R,k}^{\nu}\,\tau(x) w_{R,k}^{\nu} &+ \frac{f^*(x, v_{R,k}^{\nu}, (v_R^{\nu})', (v_{R,k}^{\nu})'', (v_{R,k}^{\nu})''', \lambda_{R,k}^{\nu})}{\|v_{R,k}^{\nu}\|_3} \\ &+ \frac{g^*(x, v_{R,k}^{\nu}, (v_{R,k}^{\nu})', (v_{R,k}^{\nu})'', (v_{R,k}^{\nu})''', \lambda_{R,k}^{\nu})}{\|v_{R,k}^{\nu}\|_3}. \end{aligned} \tag{5.23}$$

By the relations (5.1) and (5.16) we get

$$\Big|\frac{f^*(x, v_{R,k}^{\nu}, (v_{R,k}^{\nu})', (v_{R,k}^{\nu})'', (v_R^{\nu})''', \lambda_{R,k}^{\nu})}{\|v_{R,k}^{\nu}\|_3}\Big| \le \frac{M}{R_k^*}|w_{R,k}^{\nu}|, \tag{5.24}$$

$$\Big|\frac{g^*(x, v_{R,k}^{\nu}, (v_{R,k}^{\nu})', (v_{R,k}^{\nu})'', (v_{R,k}^{\nu})''', \lambda_{R,k}^{\nu})}{\|v_{R,k}^{\nu}\|_3}\Big| < \epsilon_k. \tag{5.25}$$

In view of (5.24) and (5.25), it is easy to we see from (5.23) that

$$|(w_{R,k}^{\nu})^{(4)}| \le r_k\big(|w_{R,k}^{\nu}| + |(w_{R,k}^{\nu})'| + |(w_{R,k}^{\nu})''| + |(w_{R,k}^{\nu})'''|\big) + p_0^{-1}\epsilon_k. \tag{5.26}$$

Let the norm of $z_{R,k}^{\nu} = (w_{R,k}^{\nu}, (w_{R,k}^{\nu})', (w_{R,k}^{\nu})'', (w_{R,k}^{\nu})''')$ in $\mathbb{R}^4$ be

$$|z_{R,k}^{\nu}| = |w_{R,k}^{\nu}| + |(w_{R,k}^{\nu})'| + |(w_{R,k}^{\nu})''| + |(w_{R,k}^{\nu})'''|.$$

Then it follows from (5.26) that

$$|(z_{R,k}^{\nu})'| \le (r_k + 1)|z_{R,k}^{\nu}| + p_0^{-1}\epsilon_k.$$

Integrating both sides of this inequality from ς to x we obtain

$$\Big|\int_{\varsigma}^{x} |(z_R^{\nu})'(t)|\,dt\Big| \le (r_k + 1)\Big|\int_{\varsigma}^{x} |z_R^{\nu}(t)|\,dt\Big| + p_0^{-1} l \epsilon_k. \tag{5.27}$$

By $|z_{R,k}^{\nu}(\varsigma)| = 0$ it follows from (5.27) that

$$|z_{R,k}^{\nu}(x)| = \Big|\int_{\varsigma}^{x} (z_{R,k}^{\nu})'(t) dt\Big| \le (r_k + 1)\Big|\int_{\varsigma}^{x} |z_{R,k}^{\nu}(t)|\,dt\Big| + p_0^{-1} l \epsilon_k. \tag{5.28}$$

Using Gronwall's inequality, from (5.28) we obtain

$$|z_{R,k}^{\nu}(x)| \le p_0^{-1} l \epsilon_k e^{(r_k+1)l} < \frac{1}{2}, \quad x \in [0, l],$$

which yields the inequality $\|w_{R,k}^{\nu}\|_3 \le \frac{1}{2}$, contradicting $\|w_{R,k}^{\nu}\|_3 = 1$. Therefore, $v_{R,k}^{\nu} \notin \partial S_k^{\nu}$ which implies that $v_{R,k}^{\nu} \in S_k^{\nu}$. The proof is complete. □

Corollary 5.7. *The set of asymptotic bifurcation points of problem* (1.1)-(1.2) *with respect to the set* $\mathbb{R} \times S_k^{\nu}$ *is nonempty. Moreover, if* (λ, ∞) *is a bifurcation point for* (1.1)-(1.2) *with respect to the set* $\mathbb{R} \times S_k^{\nu}$*, then* $\lambda \in I_k$.

For each $k \in \mathbb{N}$ and each $\nu \in \{+,-\}$ we define the set $\mathcal{D}_k^\nu \subset \mathcal{C}$ to be the union of all the components of $\mathcal{C}$ which meet $I_k \times \{\infty\}$ through $\mathbb{R} \times S_k^\nu$. It follows from Corollary 5.7 that this set is nonempty. The set $\mathcal{D}_k^\nu$ may not be connected in $\mathbb{R}\times E$, but the set $\mathcal{D}_k^\nu \cup (I_k \times \{\infty\})$ is connected in $\mathbb{R} \times E$.

Remark 5.8. By [2, Lemma 1.1], if (λ, y) is a nontrivial solution of problem (1.1)-(1.2) in the case when the nonlinear terms f and g satisfies the conditions (1.5) and (1.6), respectively, and $(\lambda, y) \in \partial S_k^\nu$, then $y \equiv 0$. It is clear from the proof of Lemma 5.6 that this assertion does not hold for problem (1.1)-(1.2) under the conditions (1.3) and (1.4). Consequently, the set $\mathcal{D}_k^\nu, k \in \mathbb{N}, \nu \in \{+,-\}$, can intersect the set $\mathcal{D}_{k'}^{\nu'}$ for some $(k',\nu') \neq (k,\nu)$ outside of the set $\{(\lambda, y) \in \mathbb{R}\times E : \operatorname{dist}\{\lambda, I_k\} \leq \delta, \|y\|_3 > R_k\}$ (see Remark 5.10).

The main result of this article is the following theorem.

Theorem 5.9. *For each $k \in \mathbb{N}$ and each $\nu \in \{+,-\}$ for the set $\mathcal{D}_k^\nu$ at least one of the followings holds:*

(i) *$\mathcal{D}_k^\nu$ meets $I_{k'} \times \{\infty\}$ through $\mathbb{R} \times S_{k'}^{\nu'}$ for some $(k',\nu') \neq (k,\nu)$;*
(ii) *$\mathcal{D}_k^\nu$ meets $\mathcal{R}$ for some $\lambda \in \mathbb{R}$;*
(iii) *$P_R(\mathcal{D}_k^\nu)$ is unbounded.*

In addition, if the union $\mathcal{D}_k = \mathcal{D}_k^+ \cup \mathcal{D}_k^-$ does not satisfy (ii) or (iii) then it must satisfy (i) with $k' \neq k$.

Proof. For any $(\lambda, v) \in \mathbb{R}\times E$, $v \neq 0$, we define the functions $\tilde f(\lambda, v), \tilde g(\lambda, v) \in C[0,l]$ as follows:

$$\tilde f(\lambda,v)(x) = \begin{cases} \|v\|_3^2 f^*\big(x, \frac{v(x)}{\|v\|_3^2}, \frac{v'(x)}{\|v\|_3^2}, \frac{v''(x)}{\|v\|_3^2}, \frac{v'''(x)}{\|v\|_3^2}, \lambda\big), & \text{if } v(x) \neq 0, \\ 0 & \text{if } v(x) = 0, \end{cases}$$

$$\tilde g(\lambda,v)(x) = \begin{cases} \|v\|_3^2\, g^*\big(x, \frac{v(x)}{\|v\|_3^2}, \frac{v'(x)}{\|v\|_3^2}, \frac{v''(x)}{\|v\|_3^2}, \frac{v'''(x)}{\|v\|_3^2}, \lambda\big), & \text{if } v(x) \neq 0, \\ 0 & \text{if } v(x) = 0, \end{cases}$$

for $x \in [0,l]$. Because $f^*, g^* \in C([0,l] \times \mathbb{R}^5)$, by (5.1) and (5.17) it follows that the functions $\tilde f, \tilde g : \mathbb{R}\times E \to C[0,l]$ are continuous and satisfy the following conditions:

$$\|\tilde f(\lambda,v)\|_\infty \leq M\|v\|_\infty; \tag{5.29}$$

$$\|\tilde g(\lambda,v)\|_\infty = o(\|v\|_3), \quad \text{as } \|v\|_3 \to 0, \tag{5.30}$$

uniformly in $\lambda \in \Lambda$ for any bounded interval $\Lambda \subset \mathbb{R}$.

If $(\lambda, y) \in \mathbb{R}\times E$, $\|y\|_3 \neq 0$, then dividing (1.1)-(1.2) by $\|y\|_3^2$ and setting $v = \frac{y}{\|y\|_3^2}$ we obtain

$$\begin{gathered} \ell(v)(x) = \lambda\tau(x)v(x) + \tilde f(\lambda,v)(x) + \tilde g(\lambda,v)(x), \quad x \in (0,l), \\ v \in B.C. \end{gathered} \tag{5.31}$$

Note that $\|v\|_3 = \frac{1}{\|y\|_3}$ and $y = \frac{v}{\|v\|_3^2}$. Thus the transformation $(\lambda, y) \to T(\lambda, y) = (\lambda, v)$ turns a bifurcation from infinity problem (1.1)-(1.2) into a bifurcation from zero problem (5.31). It should be noted that Theorem 2.5 cannot be directly applied to problem (5.31) in view of Remark 5.8. As equation in (5.31) contains the nonlinear term $\tilde f$ satisfying (5.29) problem (5.31) is not always linearizable in a neighborhood of zero. Hence we also cannot immediately apply standard global bifurcation theory to prove our theorem, as was done in [11, 30]. To deal with this problem alongside (5.31) we will consider the approximating problem

$$\begin{gathered}\ell(v)(x)=\lambda\tau(x)v(x)+\tilde f(\lambda,\|v\|_3^{\varepsilon}v)(x)+\tilde g(\lambda,v)(x), x\in(0,l),\\ v\in B.C.,\end{gathered}\tag{5.32}$$

where $\varepsilon\in(0,1]$. It follows from the above definitions that (5.32) is equivalent to (5.12). For fixed $\varepsilon\in(0,1]$ by (5.29) we have

$$\|\tilde f(\lambda,\|v\|_3^{\varepsilon}v)\|_\infty=o(\|v\|_3)\quad\text{as } \|v\|_3\to 0,\tag{5.33}$$

so the global bifurcation results in [2, 11, 30] are applicable to problem (5.32). Also, from the proof of Lemma 5.6 it is obvious that (5.32) approximates (5.31) as $\varepsilon\to 0$, in a suitable sense. We now choose some fixed arbitrary $k_0\in\mathbb{N}$ and we will prove the theorem for $k=k_0$ and $\nu=+$ (the case of $\nu=-$ is considered similarly).

For any $k\in\mathbb{N},\nu\in\{+,-\}$ and $\delta,R,\varrho>0$, let

$$\begin{gathered}U_k^{\nu}(\delta,R)=\{(\lambda,y)\in\mathbb{R}\times E:\operatorname{dist}\{\lambda,I_k\}\le\delta,\ y\in S_k^{\nu},\ \|y\|_3>R\},\\ \tilde U_k^{\nu}(\delta,\varrho)=\{(\lambda,v)\in\mathbb{R}\times E:\operatorname{dist}\{\lambda,I_k\}\le\delta,\ v\in S_k^{\nu},\ \|v\|_3<\varrho\}.\end{gathered}$$

It follows from Lemma 5.6 and Corollary 5.7 that $U_k^{\nu}(0,R)\subset\mathcal{D}_k^{\nu}$ for $R=R_k$.

Let $\tilde{\mathcal{C}}\subset\mathbb{R}\times E$ be the set of nontrivial solutions of (5.31). By construction, the transformation $(\lambda,y)\to T(\lambda,y)$ maps $\mathcal{C}$ into $\tilde{\mathcal{C}}$ and $U_k^{\nu}(\delta,R)$ into $\tilde U_k^{\nu}(\delta,\varrho)$, where $\varrho=\frac{1}{R}$. Let $\tilde{\mathcal{D}}_{k_0}^{+}$ be the union of all the components of $\tilde{\mathcal{C}}$ which meet $I_{k_0}\times\{0\}$ through $\mathbb{R}\times S_{k_0}^{+}$. Then $\tilde{\mathcal{D}}_{k_0}^{+}=T^{-1}(\mathcal{D}_{k_0}^{+})$. Thus to prove the theorem it suffices to show that the set $\tilde{\mathcal{D}}_{k_0}^{+}$ either meets some interval $I_k\times\{0\}$ through $\mathbb{R}\times S_k^{\nu}$ with $(k,\nu)\ne(k_0,+)$ or is unbounded in $\mathbb{R}\times E$ (the alternatives (ii) and (iii) of this theorem for $\mathcal{D}_{k_0}^{+}$ correspond, via T, to the various ways in which $\tilde{\mathcal{D}}_{k_0}^{+}$ can be unbounded).

Now suppose that the assertion of the theorem for $\tilde{\mathcal{D}}_{k_0}^{+}$ is not true. Then $\tilde{\mathcal{D}}_{k_0}^{+}$ is bounded and hence we can choose a compact interval $\Lambda_0\subset\mathbb{R}$ such that $P_R(\tilde{\mathcal{D}}_{k_0}^{+})\cup I_{k_0}$ is in the interior of Λ_0 and Λ_0 contains only finitely many intervals I_k with $\partial\Lambda_0\cap I_k=\emptyset$.

For any $\delta,\varrho>0$, let

$$W_0^{+}(\delta,\varrho)=\cup_{(k,\nu)\ne(k_0,+)}\tilde U_k^{\nu}(\delta,\varrho)$$

The set $W_0^{+}(\delta,\varrho)$ is open in $\mathbb{R}\times E$, and we denote by $\overline{W_0^{+}(\delta,\varrho)}$ the closure of this set. Since Λ_0 contains only finitely intervals I_k it follows from Lemmas 5.4 and 5.6 that there exist $\delta_0,\varrho_0>0$ such that

$$\tilde{\mathcal{D}}_{k_0}^{+}\cap\overline{W_0^{+}(\delta_0,\varrho_0)}=\emptyset.\tag{5.34}$$

By following the arguments in [33, Theorem 3.1, p. 151] we can find a neighborhood $\tilde{\mathcal{Q}}^{+}$ of $\tilde{\mathcal{D}}_{k_0}^{+}$ such that

$$\tilde U_{k_0}^{+}(\delta_0,\varrho_0)\subset\tilde{\mathcal{Q}}^{+},\quad \tilde{\mathcal{Q}}^{+}\cap\overline{W_0^{+}(\delta_0,\varrho_0)}=\emptyset,\quad \partial\tilde{\mathcal{Q}}^{+}\cap\tilde{\mathcal{C}}=\emptyset.\tag{5.35}$$

By [30, Theorem 1.3], [11, Theorem 2] and Theorem 2.2 for each fixed $\varepsilon\in(0,1]$ there exists a component $\tilde{\mathcal{D}}_{k_0}^{+}(\varepsilon)\subset\mathbb{R}\times E$ of nontrivial solutions of problem (5.32) such that $\tilde{\mathcal{D}}_{k_0}^{+}(\varepsilon)$ meets $(\lambda_{k_0},0)$ through $\mathbb{R}\times S_{k_0}^{+}$ and is either $\tilde{\mathcal{D}}_{k_0}^{+}(\varepsilon)$ unbounded in $\mathbb{R}\times E$ or $\tilde D_{k_0}^{+}(\varepsilon)$ meets $(\lambda_k,0)$ through $\mathbb{R}\times S_k^{\nu}$ for some $(k,\nu)\ne(k_0,+)$. Then the component $\tilde{\mathcal{D}}_{k_0}^{+}(\varepsilon)$ intersects both $\tilde{\mathcal{Q}}^{+}$ and $(\mathbb{R}\times E)\backslash\tilde{\mathcal{Q}}^{+}$ which implies that

$\tilde{\mathcal{D}}_{k_0}^+(\varepsilon) \cap \partial\tilde{\mathcal{Q}}^+ \neq \emptyset$. Thus, there exists $(\lambda_\varepsilon, v_\varepsilon) \in \tilde{\mathcal{D}}_{k_0}^+(\varepsilon) \cap \partial\tilde{\mathcal{Q}}^+$ for all $\varepsilon \in (0,1]$. Since $\tilde{\mathcal{Q}}^+$ is bounded in $\mathbb{R} \times E$, problem (5.32) shows that the set $\{(\lambda_\varepsilon, v_\varepsilon) \in \mathbb{R} \times E : 0 < \varepsilon \le 1\}$ is bounded in $\mathbb{R} \times C^4[0,l]$. Therefore, we can find a sequence $\{\varepsilon_n\}_{n=1}^\infty \subset (0,1)$ such that $\varepsilon_n \to 0$ and $(\lambda_{\varepsilon_n}, v_{\varepsilon_n})$ converges in $\mathbb{R} \times E$ to a solution $(\tilde{\lambda}, \tilde{v})$ of (5.31). If $\tilde{v} = 0$, then by Theorem 3.1 it follows from the proof of Lemma 5.4 that for sufficiently large $n \in \mathbb{N}$, $(\lambda_{\varepsilon_n}, v_{\varepsilon_n}) \in W_0^+(\delta_0, \varrho_0)$ which contradicts (5.35). Hence $\tilde{v} \neq 0$, and consequently, $(\tilde{\lambda}, \tilde{v}) \in \partial\tilde{\mathcal{Q}}^+ \cap \tilde{\mathcal{C}}$ that also contradicts (5.35). The proof is complete. □

Remark 5.10. Unlike Theorem 2.5 for bifurcation from zero, it need not be case $\mathcal{D}_k^\nu \subset (\mathbb{R} \times S_k^\nu) \cup (I_k \times \{\infty\})$ in Theorem 3.1 (a counterexample for $f \equiv 0$ is given in Example 3.3).

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