Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 01, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

INTEGRABILITY OF VERY WEAK SOLUTION TO THE DIRICHLET PROBLEM OF NONLINEAR ELLIPTIC SYSTEM

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Abstract. This article concerns the higher integrability of a very weak solution $u \in \theta + W_0^{1,r}(\Omega)$ for $\max\{1, p-1\} < r < p < n$ to the Dirichlet problem of the nonlinear elliptic system

$$\begin{gathered} -D_\alpha \mathbf{A}_i^\alpha(x, Du) = \mathbf{B}_i(x, Du) \quad \text{in } \Omega, \\ u = \theta \quad \text{on } \partial\Omega, \end{gathered}$$

where $\mathbf{A}(x, Du) = \big(\mathbf{A}_i^\alpha(x, Du)\big)$ for $\alpha = 1, \dots, n$ and $i = 1, \dots, m$, and each entry of $\mathbf{B}(x, Du) = \big(\mathbf{B}_i(x, Du)\big)$ for $i = 1, \dots, m$ satisfies the monotonicity and controllable growth. If $\theta \in W^{1,q}(\Omega)$ for $q > r$, then we derive that the very weak solution u of above-mentioned problem is integrable with

$$u \in \begin{cases} \theta + L^{q^*}_{\rm weak}(\Omega) & \text{for } 1 \le q < n, \\ \theta + L^\tau(\Omega) & \text{for } q = n \text{ and } 1 < \tau < \infty, \\ \theta + L^\infty(\Omega) & \text{for } q > n, \end{cases}$$

provided that r is sufficiently close to p, where $q^* = qn/(n-q)$.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ for $n \ge 2$ be a bounded regular domain. By regular domain we understand the domain with a finite measure for which the Hodge decomposition Lemma 2.1 below is satisfied. The domains with Lipschitz and A-type boundary, for example, always are regular. The purpose of this present article is to study a global higher integrability of very weak solution to the Dirichlet problem of nonlinear elliptic system:

$$\begin{gathered} -D_\alpha \mathbf{A}_i^\alpha(x, Du) = \mathbf{B}_i(x, Du) \quad \text{in } \Omega, \\ u = \theta \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $m \ge 2$ and $\theta(x) \in W^{1,q}(\Omega, \mathbb{R}^m)$ for $q > r$ with r determined later. In the context, we let $1 < p < n$, and assume that $\mathbf{A}(x, Du) = \big(\mathbf{A}_i^\alpha(x, Du)\big)$ with $\alpha = 1, \dots, n$ and $i = 1, \dots, m$ satisfies the following monotonicity and controllable growth: there exist positive constants $0 < \lambda \le \Lambda_1, \Lambda_2$ such that

$$\begin{gathered} |\mathbf{A}_i^\alpha(x, \xi)| \le \Lambda_1(|\xi|^{p-1} + a(x)), \\ \langle \mathbf{A}(x, \xi_1) - \mathbf{A}(x, \xi_2), \xi_1 - \xi_2 \rangle \ge \lambda |\xi_1 - \xi_2|^p \quad \forall \xi_1, \xi_2 \in \mathbb{R}^n \setminus \{0\}; \end{gathered} \tag{1.2}$$

2010 *Mathematics Subject Classification.* 35D30, 35K10.
Key words and phrases. Integrability; very weak solution; nonlinear elliptic system; controllable growth.

Submitted December 16, 2017. Published January 2, 2019.

and $\mathbf{B}(x, Du) = \big(\mathbf{B}_i(x, Du)\big)$ for $i = 1, \dots, m$ satisfies

$$|\mathbf{B}_i(x,\xi)| \le \Lambda_2(|\xi|^{p-\delta} + b(x)) \tag{1.3}$$

with $1 < \delta < p$, $a(x) \in L^{\frac{q}{p-1}}(\Omega, \mathbb{R}^m)$ and $b(x) \in L^{\frac{nq}{q+np-n}}(\Omega, \mathbb{R}^m)$.

First of all, let us recall the notation of *very weak solutions* to the Dirichlet problem of (1.1). A mapping $u \in \theta + W_0^{1,r}(\Omega, \mathbb{R}^m)$ with $\max\{1, p-1\} < r < p$ is called a very weak solution to the Dirichlet problem (1.1) if

$$\int_\Omega \langle \mathbf{A}(x, Du), D\varphi\rangle dx = \int_\Omega \mathbf{B}(x, Du) \cdot \varphi dx \tag{1.4}$$

holds for all $\varphi \in W_0^{1,r/(r-p+1)}(\Omega, \mathbb{R}^m)$.

On the basis of the above definition, a crucial fact is that the integrable exponent r of u can be smaller than the natural index p, which is different from the usual hypothesis of classical weak solution $u \in \theta + W_0^{1,p}(\Omega, \mathbb{R}^m)$. Here, we would like to recall recent progresses involving the topic of very weak solution. Iwaniec [18] first put forward the concept of the so-called very weak solutions for p-harmonic tensors and weakly quasiregular mappings with the integrability of their weak derivatives being below natural exponent. Furthermore, Iwaniec-Sbordone [19] and Iwaniec-Scott-Stroffolini [20] considered a self-improving regularity for weak minima of variational integrals and weakly p-harmonic type equations with r sufficiently close to p from lower side, respectively; and got that such very weak solution for variational integrals and p-harmonic type equations is actually a weak solution in the classical sense by way of the so-called Hodge decomposition argument concerning disturbance vector field. On the other hand, Lewis [13] also obtained a self-improving integrability for the derivatives of very weak solutions to certain nonlinear elliptic systems by way of the technique of harmonic analysis which is rather different from Iwaniec's argument. Later, Lewis' harmonic technique was extended to the settings of parabolic systems of p-Laplacian [9, 10], and various elliptic and parabolic systems with non-standard growths [1, 2, 3, 14], respectively. This essentially is attained by a self-improving integrability of the weak derivatives based on the validity of the generalized reverse Hölder inequality [5]. In the following, we would like to mention that Greco and Iwaniec in [8] dealt with the nonhomogeneous p-harmonic equation

$$-\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = -\operatorname{div} f,$$

and obtained an estimate for the operator $\mathcal{H}$ which carries given vector function f into the gradient field ∇u. Later, Zheng-Fang [21] further considered a local very weak solutions for nonlinear elliptic systems (1.1) with that $\mathbf{B}$ satisfies (1.3), $\mathbf{A}(x, Du)$ satisfies (1.2) and

$$\sum_{1\le i\le m,\, 1\le\alpha\le n} \mathbf{A}_i^\alpha(x,\xi)\xi_i^\alpha \ge \lambda|\xi|^p \quad \forall \xi \in \mathbb{R}^n \setminus \{0\},$$

and obtained a self-improving integrability for the derivatives of very weak solutions on the basis of the so-called Hodge decomposition of perturbation vector fields. For more results for very weak solutions, see [4, 8, 13, 18].

The problem under consideration in this paper is global integrability property in line with the regularity of boundary data, which is important among the regularity theories of nonlinear elliptic PDEs and systems. In [6], Gao-Liang-Cui studied very weak solution to the following boundary value problems of p-Laplacian

$$\begin{gathered} -\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = 0 \quad x \in \Omega, \\ u(x) = \theta(x) \quad x \in \partial\Omega, \end{gathered}$$

and obtained a global integrability result, which shows that higher integrability of the a boundary datum θ forces the very weak solution u to have a higher integrability. For more information on this topic, we refer the readers to [11, 12].

To this end, let us recall some related notations and basic facts. The *weak L^t-spaces* or *Marcinkiewicz spaces* (see [6]) for open subset $\Omega \subset \mathbb{R}^n$ with parameter $t > 0$ is the set of all measurable functions f by requiring

$$|\{x \in \Omega : |f(x)| > s\}| \le \frac{k}{s^t}$$

for some positive constant $k = k(f)$ and every $s > 0$, where $|E|$ is the n-dimensional Lebesgue measure of E. We can denoted it by the weak L^t-space or $L^t_{\rm weak}(\Omega)$. Note that if $f \in L^t_{\rm weak}(\Omega)$ for some $t > 1$ and $|\Omega| < \infty$, then $f \in L^\tau(\Omega)$ for every $1 \le \tau < t$. Now we are ready to state the main result of this paper.

Theorem 1.1. *Let $\theta \in W^{1,q}(\Omega, \mathbb{R}^m)$ for $q > r$. Suppose that the operator $\mathbf{A}(x, Du)$ and $\mathbf{B}(x, Du)$ satisfy the structural conditions* (1.2) *and* (1.3)*. Then there exists a constant $\varepsilon_0 = \varepsilon_0(n, m, p, \Lambda_1, \Lambda_2, \lambda) > 0$, such that for every very weak solution $u \in \theta + W^{1,r}_0(\Omega, \mathbb{R}^m)$ for $\max\{1, p-1\} < r < p < n$, to the boundary value problem* (1.1)*, we have*

$$u \in \begin{cases} \theta + L^{q^*}_{\rm weak}(\Omega) & \text{for } 1 \le q < n, \\ \theta + L^\tau(\Omega) & \text{for } q = n \text{ and } 1 \le \tau < \infty, \\ \theta + L^\infty(\Omega) & \text{for } q > n, \end{cases} \tag{1.5}$$

provided that $|p - r| < \varepsilon_0$, where $q^* = \frac{qn}{n-q}$.

This article proposes a new way to obtain more properties for general elliptic problems, that than those in [21, 22]. We have restricted ourselves to the case $\max\{1, p-1\} < r < n$, otherwise any function in $W^{1,r}(\Omega)$ for $r \ge n$ is in the space $L^t(\Omega)$ for any $1 \le t < \infty$ by Sobolev embedding theorem. As above-mentioned, our proof is inspired by Gao et al and Zheng et al [6, 7, 21, 22]. Since for very weak solution one cannot take a test function by using a usual weak formulation in the boundary value problem (1.4). For this, we have to construct a suitable test function by the argument of *Hodge decomposition*. That is to say, a main key ingredient is based on choosing an appropriate test functions by the so-called Hodge decomposition [19, 21]; then we attain our aim in line with Stampacchia lemma [7].

The rest of the paper is organized as follows. In section 2, we are devoted to presenting some useful lemmas. In section 3, we focus on proving our main theorem.

2. Technical tools

In this section, we introduce some useful lemmas, which will play essential roles in proving our main result. Let us denote by $c(n, m, p, \lambda, \Lambda_1, \Lambda_2, \dots)$ a universal constant depending only on prescribed quantities and possibly varying from line to line in the following context. We first give a technical lemma called Hodge decomposition involved vector fields, see [21, Lemma 2.2].

Lemma 2.1. *Assume $v \in W_0^{1,r}(\Omega, \mathbb{R}^m)$ with $\max\{1, p-1\} < r < p$. Then there exist $\varphi \in W^{1,\frac{r}{r-p+1}}(\Omega, \mathbb{R}^m)$ and divergence free matrix field $h \in L^{\frac{r}{r-p+1}}(\Omega, \mathbb{R}^{n\times m})$ such that*

$$|\nabla v|^{r-p}\nabla v = \nabla\varphi + h;$$

moreover,

$$\|h\|_{L^{\frac{r}{r-p+1}}(\Omega)} \le c|p-r|\|\nabla v\|_{L^r(\Omega)}^{r-p+1},$$

where $c = c(n, r, \Omega)$.

An efficient tool is the well-known Stampacchia Lemma, which is presented in the following lemma, see [17, Lemma 4.1] or [7].

Lemma 2.2. *Let α, β be two positive constants. Let $\phi : [s_0, +\infty) \to [0, +\infty)$ be decreasing and such that*

$$\phi(r) \le \frac{c}{(r-s)^\alpha}[\phi(s)]^\beta$$

with constants $c > 0$ and $r > s \ge s_0$. Then, it leads to the following conclusions:

(i) *if $\beta > 1$, we have $\phi(s_0 + d) = 0$ with*

$$d = \Big(c2^{\frac{\alpha\beta}{\beta-1}}(\phi(s_0))^{\beta-1}\Big)^{1/\alpha}.$$

(ii) *if $\beta = 1$, for any $s \ge s_0$ we have*

$$\phi(s) \le \phi(s_0)e^{1-(ce)^{-\frac{1}{\alpha}}(s-s_0)}.$$

(iii) *if $\beta < 1$, for any $s \ge s_0 > 0$ we have*

$$\phi(s) \le 2^{\frac{\alpha}{(1-\beta)^2}}\Big(c^{\frac{1}{1-\beta}} + (2s_0)^{\frac{\alpha}{1-\beta}}\phi(s_0)\Big)\Big(\frac{1}{s}\Big)^{\frac{\alpha}{1-\beta}}.$$

3. Proof of Theorem 1.1

Proof. For any $L > 0$, we take

$$v = \begin{cases} u-\theta+L & \text{for } u-\theta < -L,\\ 0 & \text{for } -L \le u-\theta \le L,\\ u-\theta-L & \text{for } u-\theta > L,\end{cases} \tag{3.1}$$

such that, by our assumptions we have $v \in W_0^{1,r}(E)$ with $E = \{|u-\theta| > L\}$ and

$$\nabla v = (\nabla u - \nabla\theta)\cdot 1_{\{|u-\theta|>L\}} \quad \text{in } E. \tag{3.2}$$

Now we introduce the Hodge decomposition involving disturbance vector field $|\nabla v|^{p-2}\nabla v \in L^{r/(r-p+1)}(E)$ shown in Lemma 2.1. Accordingly,

$$|\nabla v|^{r-p}\nabla v = \nabla\varphi + h \tag{3.3}$$

with $\varphi \in W_0^{1,r/(r-p+1)}(E)$ and divergence free matrix field $h \in L^{r/(r-p+1)}(E, \mathbb{R}^{n\times m})$. Then we have

$$\|\nabla\varphi\|_{L^{r/(r-p+1)}(E)} \le C(n,p)\|\nabla v\|_{L^r(E)}^{r-p+1}, \tag{3.4}$$

$$\|h\|_{L^{r/(r-p+1)}(E)} \le C(n,p)|p-r|\|\nabla v\|_{L^r(E)}^{r-p+1}. \tag{3.5}$$

Extending φ by zero value from E to $\overline{\Omega}$, then the above-mentioned term $\varphi \in W_0^{1,r/(r-p+1)}(\Omega)$ can be used as a test function for the integral identity (1.4), which yields that

$$\int_\Omega \langle \mathbf{A}(x,Du), D\varphi\rangle dx = \int_\Omega \mathbf{B}(x,Du)\cdot \varphi dx.$$

By (1.2) and Hodge decomposition (3.3) we conclude that

$$\begin{aligned}
&\int_\Omega \langle \mathbf{A}(x,Du), D\varphi\rangle dx \\
&= \int_E \langle \mathbf{A}(x,Du), |Dv|^{r-p}Dv - h\rangle dx \\
&= \int_E \langle \mathbf{A}(x,Du), |Du-D\theta|^{r-p}(Du-D\theta)\rangle dx - \int_E \langle \mathbf{A}(x,Du), h\rangle dx \\
&= \int_E \langle \mathbf{A}(x,Du) - \mathbf{A}(x,D\theta), (Du-D\theta)\rangle |Du-D\theta|^{r-p} dx \\
&\quad + \int_E \langle \mathbf{A}(x,D\theta), (Du-D\theta)\rangle |Du-D\theta|^{r-p} dx - \int_E \langle \mathbf{A}(x,Du), h\rangle dx \\
&\geq \lambda \int_E |Du-D\theta|^r dx + \int_E \langle \mathbf{A}(x,D\theta), (Du-D\theta)\rangle |Du-D\theta|^{r-p} dx \\
&\quad - \int_E \langle \mathbf{A}(x,Du), h\rangle dx,
\end{aligned}$$

which implies

$$\begin{aligned}
\int_E |Du-D\theta|^r dx &\leq c\int_E |\mathbf{A}(x,D\theta)||Du-D\theta|^{r-p+1} dx \\
&\quad + \int_E \langle \mathbf{A}(x,Du), h\rangle dx + \int_\Omega \mathbf{B}(x,Du)\cdot\varphi\, dx \\
&:= c(I_1+I_2+I_3).
\end{aligned} \tag{3.6}$$

Using (1.2), (1.3), (3.5), Hölder inequality and Young inequality we deduce that I_1, I_2, I_3 can be estimated as follows:

$$\begin{aligned}
I_1 &\leq \int_E |\mathbf{A}(x,D\theta)||Du-D\theta|^{r-p+1} dx \\
&\leq \Lambda_1 \int_E \Big(|D\theta|^{p-1} + a(x)\Big)|Du-D\theta|^{r-p+1} dx \\
&\leq \varepsilon\cdot c\int_E |Du-D\theta|^r dx + c(\varepsilon)\int_E |D\theta|^r dx + c(\varepsilon)\int_E |a(x)|^{\frac{r}{p-1}} dx
\end{aligned} \tag{3.7}$$

with small $\varepsilon > 0$ determined later. For the estimate of I_2, we derive that

$$\begin{aligned}
I_2 &\leq \int_E |\mathbf{A}(x,Du)||h| dx \\
&\leq \Lambda_1 \int_E \Big(|Du|^{p-1} + a(x)\Big)|h| dx \\
&\leq 2^{p-2}\Lambda_1\Big(\int_E |Du-D\theta|^{p-1}|h|dx + \int_E |D\theta|^{p-1}|h|dx\Big) + \Lambda_1\int_E a(x)|h|dx \\
&\leq c\Big(\int_E |Du-D\theta|^r dx\Big)^{\frac{p-1}{r}} \Big(\int_E |h|^{\frac{r}{r-p+1}} dx\Big)^{\frac{r-p+1}{r}}
\end{aligned}$$

$$
\begin{aligned}
&+c\Big(\int_E |D\theta|^r dx\Big)^{\frac{p-1}{r}}\Big(\int_E |h|^{\frac{r}{r-p+1}}dx\Big)^{\frac{r-p+1}{r}}\\
&+c\Big(\int_E |a(x)|^{\frac{r}{p-1}}dx\Big)^{\frac{p-1}{r}}\Big(\int_E |h|^{\frac{r}{r-p+1}}dx\Big)^{\frac{r-p+1}{r}}\\
&\le c|p-r|\Big(\int_E |Du-D\theta|^r dx\Big)^{\frac{p-1}{r}}\Big(\int_E |Du-D\theta|^r dx\Big)^{\frac{r-p+1}{r}}\\
&+c|p-r|\Big(\int_E |D\theta|^r dx\Big)^{\frac{p-1}{r}}\Big(\int_E |Du-D\theta|^r dx\Big)^{\frac{r-p+1}{r}}\\
&+c|p-r|\Big(\int_E |a(x)|^{\frac{r}{p-1}}dx\Big)^{\frac{p-1}{r}}\Big(\int_E |Du-D\theta|^r dx\Big)^{\frac{r-p+1}{r}}\\
&\le c(\varepsilon)|p-r|\int_E |Du-D\theta|^r dx + c(\varepsilon)|p-r|\int_E |D\theta|^r dx\\
&+c(\varepsilon)|p-r|\int_E |a(x)|^{\frac{r}{p-1}}dx, \qquad (3.8)
\end{aligned}
$$

where $0<|p-r|<\varepsilon_0$. For the estimate of I_3, we have

$$
\begin{aligned}
I_3 &\le \int_E |\mathbf{B}(x,Du)||\varphi| dx\\
&\le \Lambda_2 \int_E \Big(|Du|^{p-\delta}+b(x)\Big)|\varphi| dx\\
&\le \Lambda_2\Big(\int_E \Big(|Du|^{p-\delta}+b(x)\Big)^{q_0}dx\Big)^{\frac{1}{q_0}}\Big(\int_E |\varphi|^{\frac{nr}{nr-r-np+n}}dx\Big)^{\frac{nr-r-np+n}{nr}}\\
&\le c\Big(\Big(\int_E |Du|^{(p-\delta)q_0}dx\Big)^{\frac{1}{q_0}}+\Big(\int_E |b(x)|^{q_0}dx\Big)^{\frac{1}{q_0}}\Big)\Big(\int_E |D\varphi|^{\frac{r}{r-p+1}}dx\Big)^{\frac{r-p+1}{r}}\\
&\le c\Big(\int_E |Du|^{(p-\delta)q_0}dx\Big)^{\frac{1}{q_0}}\Big(\int_E |Du-D\theta|^r dx\Big)^{\frac{r-p+1}{r}}\\
&\quad + c\Big(\int_E |b(x)|^{q_0}dx\Big)^{\frac{1}{q_0}}\Big(\int_E |Du-D\theta|^r dx\Big)^{\frac{r-p+1}{r}}\\
&:= c(J_1+J_2),
\end{aligned}
$$

where $q_0 = nr/(r+np-n)$. A direct calculation shows that $(p-\delta)\frac{nr}{r+np-n}<r$ with $\delta\in(1,p)$, then one gets that

$$
\begin{aligned}
J_1 &\le c|E|^{\frac{1}{q_0}-\frac{p-\delta}{r}}\Big(\int_E |Du|^r dx\Big)^{\frac{p-\delta}{r}}\Big(\int_E |Du-D\theta|^r dx\Big)^{\frac{r-p+1}{r}}\\
&\le c|E|^{\frac{1}{q_0}-\frac{p-\delta}{r}}\Big(\Big(\int_E |Du-D\theta|^r dx\Big)^{\frac{p-\delta}{r}}\\
&\quad + \Big(\int_E |D\theta|^r dx\Big)^{\frac{p-\delta}{r}}\Big)\Big(\int_E |Du-D\theta|^r dx\Big)^{\frac{r-p+1}{r}}\\
&= c|E|^{\frac{1}{q_0}-\frac{p-\delta}{r}}\Big(\Big(\int_E |Du-D\theta|^r dx\Big)^{\frac{r-\delta+1}{r}}\\
&\quad + \Big(\int_E |D\theta|^r dx\Big)^{\frac{p-\delta}{r}}\Big(\int_E |Du-D\theta|^r dx\Big)^{\frac{r-p+1}{r}}\Big)
\end{aligned}
$$

$$\begin{aligned} &\le c\cdot\varepsilon\int_E |Du-D\theta|^r dx + c(\varepsilon)|E|^{(\frac{1}{q_0}-\frac{p-\delta}{r})\frac{r}{\delta-1}} \\ &\quad + c(\varepsilon)|E|^{(\frac{1}{q_0}-\frac{p-\delta}{r})\frac{r}{p-1}}\Big(\int_E |D\theta|^r dx\Big)^{\frac{p-\delta}{p-1}}, \end{aligned}$$

and

$$J_2 \le \varepsilon\int_E |Du-D\theta|^r dx + c(\varepsilon)\Big(\int_E |b(x)|^{q_0} dx\Big)^{\frac{r}{q_0(p-1)}}.$$

Putting estimations of J_1 and J_2 together, we have

$$\begin{aligned} I_3 &\le c\cdot\varepsilon\int_E |Du-D\theta|^r dx + c(\varepsilon)|E|^{(\frac{1}{q_0}-\frac{p-\delta}{r})\frac{r}{\delta-1}} \\ &\quad + c(\varepsilon)|E|^{(\frac{1}{q_0}-\frac{p-\delta}{r})\frac{r}{p-1}}\Big(\int_E |D\theta|^r dx\Big)^{\frac{p-\delta}{p-1}} + c(\varepsilon)\Big(\int_E |b(x)|^{q_0} dx\Big)^{\frac{r}{q_0(p-1)}}. \end{aligned} \tag{3.9}$$

Therefore, by combining (3.7), (3.8) and (3.9) we obtain

$$\begin{aligned} &\int_E |Du-D\theta|^r dx \\ &\le c\cdot(\varepsilon+|p-r|)\int_E |Du-D\theta|^r dx + c(\varepsilon)(1+|p-r|)\int_E |D\theta|^r dx \\ &\quad + c(\varepsilon)|E|^{(\frac{1}{q_0}-\frac{p-\delta}{r})\frac{r}{p-1}}\Big(\int_E |D\theta|^r dx\Big)^{\frac{p-\delta}{p-1}} + c(\varepsilon)\int_E |a(x)|^{\frac{r}{p-1}} dx \\ &\quad + c(\varepsilon)\Big(\int_E |b(x)|^{q_0} dx\Big)^{\frac{r}{q_0(p-1)}} + c(\varepsilon)|E|^{(\frac{1}{q_0}-\frac{p-\delta}{r})\frac{r}{\delta-1}}. \end{aligned} \tag{3.10}$$

Since $|p-r| < \varepsilon_0$, we can take the positive constants $\varepsilon > 0$ and ε_0 sufficiently small such that $c\cdot(\varepsilon+|p-r|) \le \frac{1}{2}$. Then, the first term in the right-hand side of (3.10) can be absorbed by the left-hand side, and we obtain

$$\begin{aligned} &\int_E |Du-D\theta|^r dx \\ &\le c\int_E |D\theta|^r dx + c|E|^{(\frac{1}{q_0}-\frac{p-\delta}{r})\frac{r}{p-1}}\Big(\int_E |D\theta|^r dx\Big)^{\frac{p-\delta}{p-1}} + c\int_E |a(x)|^{\frac{r}{p-1}} dx \\ &\quad + c\Big(\int_E |b(x)|^{q_0} dx\Big)^{\frac{r}{q_0(p-1)}} + c|E|^{(\frac{1}{q_0}-\frac{p-\delta}{r})\frac{r}{\delta-1}} \\ &:= c(K_1+K_2+K_3+K_4+K_5). \end{aligned} \tag{3.11}$$

Note that $\theta \in W^{1,q}(\Omega)$ for $q > r$, then by the Hölder inequality to have

$$K_1 \le \Big(\int_E |D\theta|^q dx\Big)^{\frac{r}{q}}|E|^{1-\frac{r}{q}} \le \|D\theta\|^r_{L^q(\Omega)}|E|^{1-\frac{r}{q}} \tag{3.12}$$

and

$$\begin{aligned} K_2 &\le |E|^{(\frac{1}{q_0}-\frac{p-\delta}{r})\frac{r}{p-1}}\Big(\int_E |D\theta|^q dx\Big)^{\frac{r}{q}\frac{p-\delta}{p-1}}|E|^{(1-\frac{r}{q})\frac{p-\delta}{p-1}} \\ &\le \|D\theta\|^{r\frac{p-\delta}{p-1}}_{L^q(\Omega)}|E|^{(\frac{1}{q_0}-\frac{p-\delta}{r})\frac{r}{p-1}+(1-\frac{r}{q})\frac{p-\delta}{p-1}}. \end{aligned}$$

By considering $q_0 = \frac{nr}{r+np-n}$ for $\delta \in (1,p)$, we get

$$\Big(\frac{1}{q_0}-\frac{p-\delta}{r}\Big)\frac{r}{p-1} + \Big(1-\frac{r}{q}\Big)\frac{p-\delta}{p-1}$$

$$
\begin{aligned}
&= \frac{r}{p-1}\Big(\frac{r+np-n}{nr} - \frac{p-\delta}{r}\Big) + 1 - \frac{r}{q} + \Big(1-\frac{r}{q}\Big)\Big(\frac{p-\delta}{p-1} - 1\Big) \\
&= 1 - \frac{r}{q} + \frac{1}{p-1}\Big(\frac{r-n+n\delta}{n} + \frac{q-r}{q}(1-\delta)\Big) \\
&= 1 - \frac{r}{q} + \frac{1}{p-1}\frac{(\delta-1)rn+rq}{nq} > 1 - \frac{r}{q},
\end{aligned}
$$

which implies

$$
K_2 \le \|D\theta\|_{L^q(\Omega)}^{r\frac{p-\delta}{p-1}} |E|^{1-\frac{r}{q}} |\Omega|^{\frac{(\delta-1)rn+rq}{nq(p-1)}} \le \|D\theta\|_{L^q(\Omega)}^{r\frac{p-\delta}{p-1}} |E|^{1-\frac{r}{q}} (|\Omega|+1)^{\frac{(\delta-1)pn+pq}{nq(p-1)}}. \tag{3.13}
$$

Note that $a(x) \in L^{\frac{q}{p-1}}(\Omega)$ and $b(x) \in L^{\frac{nq}{q+np-n}}(\Omega)$, we have

$$
K_3 \le \Big(\int_E |a(x)|^{\frac{q}{p-1}}dx\Big)^{\frac{r}{q}} |E|^{1-\frac{r}{q}} \le \|a(x)\|_{L^{\frac{q}{p-1}}(\Omega)}^{r(p-1)} |E|^{1-\frac{r}{q}} \tag{3.14}
$$

and

$$
\begin{aligned}
K_4 &\le \Big(\int_E |b(x)|^{\frac{nq}{q+np-n}}dx\Big)^{\frac{(q+np-n)r}{nq(p-1)}} |E|^{\frac{r}{(p-1)q_0} - \frac{(q+np-n)r}{nq(p-1)}} \\
&\le \|b(x)\|_{L^{\frac{nq}{q+np-n}}(\Omega)}^{\frac{r}{p-1}} |E|^{\frac{r}{(p-1)q_0} - \frac{(q+np-n)r}{nq(p-1)}} \\
&= \|b(x)\|_{L^{\frac{nq}{q+np-n}}(\Omega)}^{\frac{r}{p-1}} |E|^{1-\frac{r}{q}}
\end{aligned} \tag{3.15}
$$

with $\frac{r}{(p-1)q_0} - \frac{(q+np-n)r}{nq(p-1)} = 1 - \frac{r}{q}$. Similarly, thanks to

$$
\Big(\frac{1}{q_0} - \frac{p-\delta}{r}\Big)\frac{r}{\delta-1} = \frac{r+(\delta-1)n}{(\delta-1)n} > 1 > 1 - \frac{r}{q},
$$

we obtain

$$
K_5 \le |E|^{1-\frac{r}{q}} |\Omega|^{\frac{r}{(\delta-1)n}+\frac{r}{q}} \le |E|^{1-\frac{r}{q}} (|\Omega|+1)^{\frac{p}{(\delta-1)n}+\frac{p}{q}}. \tag{3.16}
$$

Putting the estimates of K_1, K_2, K_3, K_4 and K_5 into (3.11), it follows that

$$
\begin{aligned}
\int_E |Du - D\theta|^r dx \le c\Big(&\|D\theta\|_{L^q(\Omega)}^r + \|D\theta\|_{L^q(\Omega)}^{r\frac{p-\delta}{p-1}} \\
&+ \|a(x)\|_{L^{\frac{q}{p-1}}(\Omega)}^{r(p-1)} + \|b(x)\|_{L^{\frac{nq}{q+np-n}}(\Omega)}^{\frac{r}{p-1}} + 1\Big)|E|^{1-\frac{r}{q}},
\end{aligned} \tag{3.17}
$$

where $c = c(n, m, p, q, \lambda, \Lambda_1, \Lambda_2, \delta)$.

We now turn our attention to the function $v \in W_0^{1,r}(E)$. Since $|v| = (|u-\theta| - L)$ in E, then by Sobolev embedding theorem and (3.2), we have

$$
\begin{aligned}
\Big(\int_E (|u-\theta|-L)^{r^*}dx\Big)^{1/r^*} &= \Big(\int_E |v|^{r^*}dx\Big)^{1/r^*} \\
&\le C(n,r)\Big(\int_E |Dv|^r dx\Big)^{1/r} \\
&= C(n,r)\Big(\int_E |Du - D\theta|^r dx\Big)^{1/r}.
\end{aligned} \tag{3.18}
$$

Hence, considering $\tilde{L} > L$ yields

$$\begin{aligned} \Big(\tilde{L}-L\Big)^{r^*} |\{|u-\theta|>\tilde{L}\}| &= \int_{\{|u-\theta|>\tilde{L}\}} \big(\tilde{L}-L\big)^{r^*} dx \\ &\le \int_{\{|u-\theta|>\tilde{L}\}} \big(|u-\theta|-L\big)^{r^*} dx \\ &\le \int_{\{|u-\theta|>L\}} \big(|u-\theta|-L\big)^{r^*} dx. \end{aligned} \tag{3.19}$$

By collecting (3.17), (3.18) and (3.19) with $E=\{|u-\theta|>L\}$, we deduce that

$$\begin{aligned} &\Big((\tilde{L}-L)^{r^*}|\{|u-\theta|>\tilde{L}\}|\Big)^{1/r^*} \\ &\le c_*\Big(\|D\theta\|_{L^q(\Omega)} + \|D\theta\|_{L^q(\Omega)}^{\frac{p-\delta}{p-1}} + \|a(x)\|^{p-1}_{L^{\frac{q}{p-1}}(\Omega)} + \|b(x)\|^{\frac{1}{p-1}}_{L^{\frac{nq}{q+np-n}}(\Omega)} + 1\Big) \\ &\quad \times |\{|u-\theta|>L\}|^{\frac{1}{r}-\frac{1}{q}}. \end{aligned}$$

where $c_* = c_*(n,m,p,q,\lambda,\Lambda_1,\Lambda_2,\delta)$. It actually means that

$$\begin{aligned} |\{|u-\theta|>\tilde{L}\}| \le \frac{1}{(\tilde{L}-L)^{r^*}} c_*^{r^*}\Big(&\|D\theta\|_{L^q(\Omega)} + \|D\theta\|_{L^q(\Omega)}^{\frac{p-\delta}{p-1}} + \|a(x)\|^{p-1}_{L^{\frac{q}{p-1}}(\Omega)} \\ &+ \|b(x)\|^{\frac{1}{p-1}}_{L^{\frac{nq}{q+np-n}}(\Omega)} + 1\Big)^{r^*} |\{|u-\theta|>L\}|^{r^*\left(\frac{1}{r}-\frac{1}{q}\right)}. \end{aligned} \tag{3.20}$$

Let $\phi(s) = |\{|u-\theta|>s\}|$, $\alpha = r^*$, $\beta = r^*\big(\frac{1}{r}-\frac{1}{q}\big)$,

$$C = c_*^{r^*}\Big(\|D\theta\|_{L^q(\Omega)} + \|D\theta\|_{L^q(\Omega)}^{\frac{p-\delta}{p-1}} + \|a(x)\|^{p-1}_{L^{\frac{q}{p-1}}(\Omega)} + \|b(x)\|^{\frac{1}{p-1}}_{L^{\frac{nq}{q+np-n}}(\Omega)} + 1\Big)^{r^*}$$

and $s_0 > 0$. Then, the above estimation (3.20) becomes

$$\phi(\tilde{L}) \le \frac{C}{(\tilde{L}-L)^{\alpha}}\phi(L)^{\beta}, \tag{3.21}$$

for $\tilde{L} > L > 0$. Now we are in a position to discuss settings in the three cases due to Stampacchia Lemma.

Case (i) If $1 \le q < n$, one has $\beta < 1$. In this case, if $s \ge 1$, we then get from Lemma 2.2 that

$$|\{|u-\theta|>s\}| \le C(\alpha,\beta,s_0)s^{-t},$$

where $t = \frac{\alpha}{1-\beta} = q^*$. If $0 < s < 1$, one has

$$|\{|u-\theta|>s\}| \le |\Omega| = |\Omega|s^{q^*}s^{-q^*} \le |\Omega|s^{-q^*}.$$

In summary, we conclude that $u \in \theta + L^{q^*}_{\rm weak}(\Omega)$.

Case (ii) If $q = n$, one has $\beta = 1$. For any $1 \le \tau < \infty$, it follows from (3.21) that

$$\phi\left(\tilde{L}\right) \le \frac{C}{(\tilde{L}-L)^{\alpha}}\phi(L) = \frac{C}{(\tilde{L}-L)^{\alpha}}\phi(L)^{1-\frac{\alpha}{\tau}}\phi(L)^{\frac{\alpha}{\tau}} \le \frac{C|\Omega|^{\frac{\alpha}{\tau}}}{(\tilde{L}-L)^{\alpha}}\phi(L)^{1-\frac{\alpha}{\tau}}.$$

As above, by Stampacchia Lemma we derive $u \in \theta + L^{\tau}(\Omega)$.

Case (iii) If $q > n$, one has $\beta > 1$. Lemma 2.2 implies $\phi(d) = 0$ for some constant d depending only on $\alpha, \beta, s_0, r, \|D\theta\|_{L^q}, \|a(x)\|_{L^{\frac{q}{p-1}}}$ and $\|b(x)\|_{L^{\frac{nq}{q+np-n}}}$. Thus $|\{|u-\theta|>d\}| = 0$, which means $u-\theta \le d$, a.e. Ω. Therefore $u \in \theta + L^{\infty}(\Omega)$, and the proof is complete. $\square$

Acknowledgments. This work was partially supported by the Fundamental Research Funds for the Central Universities grant no. 2018YJS167 and the National Science Foundation of China grant no. 11371050.

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