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EIGENFUNCTION EXPANSION FOR SINGULAR STURM-LIOUVILLE PROBLEMS WITH TRANSMISSION CONDITIONS

BILENDER P. ALLAHVERDIEV, HÜSEYIN TUNA

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Abstract. In this work, we show the existence of a spectral function for a singular Sturm-Liouville problem with transmission condition. Also we establish a Parseval equality and expansion formula in eigenfunctions in terms of the spectral function.

1. Introduction

Sturm-Liouville problems are one of the important research areas of mathematical physics. They arise when we apply the method of separation of variables to equations in mathematical physics. To study the problem of expanding an arbitrary function as a series of eigenfunctions, we need eigenfunction expansions theorems for which there are a lot of studies, see [13, 21].

On the other hand, the Sturm-Liouville problems with transmission conditions arise in problems of heat and mass transfer, various physical transfer problems [15], radio science [16], and geophysics [12]. Such conditions are known by various names including transmission conditions, interface conditions, jump conditions and discontinuous conditions. Regular problems were investigated in [1, 8, 9, 10, 17, 18, 19, 20, 23, 24, 25], and singular problems in [2, 3, 4, 5, 6, 7, 22]. Li et al. [14] investigated a singular Sturm-Liouville problems with transmission conditions at finitely many interior points. They gave a definition of Weyl function for such problems in the limit circle case.

In this article, we consider singular Sturm-Liouville problems with transmission conditions. We prove the existence of a spectral function, and give a Parseval equality and an expansion formula in eigenfunctions, for such problems.

2. Main results

We consider the Sturm-Liouville expression

$$l(y) := -\big(p(x)y'\big)' + q(x)y, \quad x \in (a,c) \cup (c,b),$$

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where $I_1 := [a, c)$, $I_2 := (c, b]$, $-\infty < a < c < b < +\infty$ and $I := I_1 \cup I_2$. We assume that the points a, b and c are regular for the differential expression l. p and q are real-valued, Lebesgue measurable functions on I and $1/p, q \in L^1(I_k)$, $k = 1, 2$. The point c is regular if $1/p, q \in L^1[c-\epsilon, c+\epsilon]$ for some $\epsilon > 0$.

Let us consider the Sturm-Liouville equation

$$l(y) = \lambda y, \quad x \in I, \tag{2.1}$$

with the boundary condition

$$y(a)\cos\beta + (py')(a)\sin\beta = 0, \quad \beta \in \mathbb{R} := (-\infty, \infty), \tag{2.2}$$

and transmission conditions

$$\begin{gathered} Y(c+) = CY(c-), \quad Y = \begin{pmatrix} y \\ py' \end{pmatrix}, \\ C \in M_2(\mathbb{R}), \quad \det C = \delta > 0, \end{gathered} \tag{2.3}$$

where $M_2(\mathbb{R})$ denotes the the 2×2 matrices with entries from $\mathbb{R}$.

Now, we introduce the Hilbert space $H_1 = L^2(I_1) \dot{+} L^2(I_2)$ with the inner product

$$\langle f, g\rangle_{H_1} := \int_a^c f^{(1)}\overline{g^{(1)}}dx + \gamma \int_c^b f^{(2)}\overline{g^{(2)}}dx, \quad \gamma = \frac{1}{\delta},$$

where

$$f(x) = \begin{cases} f^{(1)}(x), & x \in I_1 \\ f^{(2)}(x), & x \in I_2, \end{cases} \quad g(x) = \begin{cases} g^{(1)}(x), & x \in I_1 \\ g^{(2)}(x), & x \in I_2. \end{cases}$$

Denote by $\mathcal{D}$ the set of linear functions $y \in H_1$ such that y, py' are locally absolutely continuous functions on I, one-sided limits $y(c\pm), (py')(c\pm)$ exist and are finite and $l(y) \in H_1$. The operator L defined by $Ly = l(y)$ is called the maximal operator T on H_1.

For arbitrary functions $y, z \in \mathcal{D}$, we have Green's formula

$$\int_a^b l(y)\overline{z}dx - \int_a^b y\overline{l(z)}dx = [y, z]_{c-} - [y, z]_a + [y, z]_b - [y, z]_{c+}, \tag{2.4}$$

where $[y, z]_x = y(x)\overline{(pz')(x)} - (py')(x)\overline{z(x)}$ $(x \in I)$. Denote by

$$\phi(x, \lambda) = \begin{cases} \phi^{(1)}(x, \lambda), & x \in I_1 \\ \phi^{(2)}(x, \lambda), & x \in I_2 \end{cases}$$

the solution of (2.1) satisfying the initial conditions

$$\phi(a, \lambda) = \sin\beta, \quad (p\phi')(a, \lambda) = -\cos\beta, \tag{2.5}$$

and transmission conditions

$$\begin{gathered} \Phi(c+, \lambda) = C\Phi(c-, \lambda), \quad \Phi(x, \lambda) := \begin{pmatrix} \phi(x, , \lambda) \\ (p\phi')(x, , \lambda) \end{pmatrix}, \\ C \in M_2(\mathbb{R}), \quad \det C = \delta > 0. \end{gathered} \tag{2.6}$$

Now, to problem (2.1)-(2.3) we add the boundary condition

$$(py')(b)\sin\alpha + y(b)\cos\alpha = 0, \quad \alpha \in \mathbb{R}. \tag{2.7}$$

Then, (2.1)-(2.3), (2.7) is a regular problem for a Sturm-Liouville equation with transmission conditions.

In [8, 9, 10, 23, 24] the authors proved that the regular self-adjoint boundary-value problem (2.1)-(2.3), (2.7) with transmission conditions has a compact resolvent, so it has a purely discrete spectrum.

Let $\lambda_{m,b}$ ($m \in \mathbb{N} := \{1, 2, \dots\}$) denote the eigenvalues of this problem and

$$\phi_{m,b}(x) = \begin{cases} \phi^{(1)}_{m,b}(x), & x \in I_1 \\ \phi^{(2)}_{m,b}(x), & x \in I_2, \end{cases}$$
$$\phi_{m,b}(x) := \phi(x, \lambda_{m,b})$$

the corresponding real-valued eigenfunctions which satisfy conditions (2.2), (2.3), (2.7). If $f \in H_1$ is real-valued function with

$$f(x) = \begin{cases} f^{(1)}(x), & x \in I_1 \\ f^{(2)}(x), & x \in I_2, \end{cases}$$

and

$$\alpha^2_{m,b} = \int_a^c (\phi^{(1)}_{m,b}(x))^2 dx + \gamma \int_c^b (\phi^{(2)}_{m,b}(x))^2 dx,$$

then

$$\begin{aligned} \|f\|^2_{H_1} &= \int_a^c \Big(f^{(1)}(x)\Big)^2 dx + \gamma \int_c^b \Big(f^{(2)}(x)\Big)^2 dx \\ &= \sum_{m=1}^{\infty} \frac{1}{\alpha^2_{m,b}} \Big\{ \int_a^c f^{(1)}(x)\phi^{(1)}_{m,b}(x)dx + \gamma \int_c^b f^{(2)}(x)\phi^{(2)}_{m,b}(x)dx \Big\}^2. \end{aligned} \tag{2.8}$$

which is called the Parseval equality [9, 10].

Now, let us define a nondecreasing step function on $\mathbb{R}$,

$$\varrho_b(\lambda) = \begin{cases} -\sum_{\lambda<\lambda_{m,b}<0} \frac{1}{\alpha^2_{m,b}}, & \text{for } \lambda < 0 \\ \sum_{0\le\lambda_{m,b}<\lambda} \frac{1}{\alpha^2_{m,b}} & \text{for } \lambda \ge 0. \end{cases} \tag{2.9}$$

Then (2.8) can be written as

$$\int_a^c (f^{(1)}(x))^2 dx + \gamma \int_c^b (f^{(2)}(x))^2 dx = \int_{-\infty}^{\infty} F^2(\lambda) d\varrho_b(\lambda), \tag{2.10}$$

where

$$F(\lambda) = \int_a^c f^{(1)}(x)\phi^{(1)}(x,\lambda)dx + \gamma \int_c^b f^{(2)}(x)\phi^{(2)}(x,\lambda)dx.$$

We will show that the Parseval equality for problem (2.1)-(2.3), (2.7) can be obtained from (2.10) by letting $b \to \infty$.

A function f defined on an interval $[a, b]$ is said to be of *bounded variation* if there is a constant $C > 0$ such that

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \le C$$

for every partition $a = x_0 < x_1 < \dots < x_n = b$ of $[a, b]$.

Let f be a function of bounded variation. Then, by the *total variation* of f on $[a, b]$, denoted by

$$\mathrm{V}_a^b(f) := \sup \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|,$$

where the supremum is taken over all (finite) partitions of the interval $[a,b]$ (see [11]).

Lemma 2.1. *For any positive N, there is a positive constant $\Upsilon=\Upsilon(N)$ independent of b such that*

$$\mathrm{V}_{-N}^{N}(\varrho_b(\lambda))=\sum_{-N\le\lambda_{m,b}<N}\frac{1}{\alpha_{m,b}^2}=\varrho_b(N)-\varrho_b(-N)<\Upsilon. \tag{2.11}$$

Proof. Let $\sin\beta\neq 0$. Since $\phi(x,\lambda)$ is continuous in domain $-N\le\lambda\le N$, $a\le x\le c$, by the condition $\phi(a,\lambda)=\sin\beta$, there is a small positive number k such that, for $|\lambda|\le N$,

$$\frac{1}{k^2}\Big(\int_a^k\phi^{(1)}(x,\lambda)dx\Big)^2>\frac12\sin^2\beta. \tag{2.12}$$

Let us define

$$f_k(x)=\begin{cases}1/k, & a\le x<k\\ 0, & x\ge k.\end{cases}$$

From (2.10), (2.11) and (2.12), we obtain

$$\begin{aligned}\int_a^k f_k^2(x)dx=\frac{k-a}{k^2}&=\int_{-\infty}^{\infty}\Big(\frac1k\int_a^k\phi^{(1)}(x,\lambda)dx\Big)^2d\varrho_b(\lambda)\\ &\ge\int_{-N}^{N}\Big(\frac1k\int_a^k\phi^{(1)}(x,\lambda)dx\Big)^2d\varrho_b(\lambda)\\ &>\frac12\sin^2\beta\int_{-N}^{N}d\varrho_b(\lambda)\\ &=\frac12\sin^2\beta\{\varrho_b(N)-\varrho_b(-N)\},\end{aligned}$$

which proves the inequality (2.11).

If $\sin\beta=0$, then we define the function

$$f_k(x)=\begin{cases}1/k^2, & a\le x<k\\ 0, & x\ge k.\end{cases}$$

So, we obtain (2.11) by applying the Parseval equality. □

Now, we recall the following two well-known Helly's theorems.

Theorem 2.2 ([11]). *Let $(w_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of real nondecreasing function on a finite interval $a\le\lambda\le b$. Then there exists a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ and a nondecreasing function w such that*

$$\lim_{k\to\infty}w_{n_k}(\lambda)=w(\lambda),\quad a\le\lambda\le b.$$

Theorem 2.3 ([11]). *Assume $(w_n)_{n\in\mathbb{N}}$ is a real, uniformly bounded, sequence of nondecreasing function on a finite interval $a\le\lambda\le b$, and suppose*

$$\lim_{n\to\infty}w_n(\lambda)=w(\lambda),\quad a\le\lambda\le b.$$

If f is any continuous function on $a\le\lambda\le b$, then

$$\lim_{n\to\infty}\int_a^b f(\lambda)dw_n(\lambda)=\int_a^b f(\lambda)dw(\lambda).$$

We introduce the Hilbert space $H := L^2(I_1) \dot{+} L^2(I_3)$, ($I_1 := [a,c)$, $I_3 := (c,\infty)$) with the inner product

$$\langle f,g\rangle_H := \int_a^c f^{(1)}\overline{g^{(1)}}dx + \gamma \int_c^\infty f^{(2)}\overline{g^{(2)}}dx,$$

where

$$f(x) = \begin{cases} f^{(1)}(x), & x \in I_1 \\ f^{(2)}(x), & x \in I_3, \end{cases} \quad g(x) = \begin{cases} g^{(1)}(x), & x \in I_1 \\ g^{(2)}(x), & x \in I_3. \end{cases}$$

Let ϱ be any nondecreasing function on $-\infty < \lambda < \infty$. Denote by $L^2_\varrho(\mathbb{R})$ the Hilbert space of all functions $f : \mathbb{R} \to \mathbb{R}$ which are measurable with respect to the Lebesque-Stieltjes measure defined by ϱ and such that

$$\int_{-\infty}^\infty f^2(\lambda)d\varrho(\lambda) < \infty,$$

with the inner product

$$(f,g)_\varrho := \int_{-\infty}^\infty f(\lambda)g(\lambda)d\varrho(\lambda).$$

The main result of this article reads as follows.

Theorem 2.4. *For the Sturm-Liouville problem* (2.1)-(2.3)*, there exists a nondecreasing function* $\varrho(\lambda)$ *on* $-\infty < \lambda < \infty$ *with the following properties:*

(i) *If*

$$f(x) = \begin{cases} f^{(1)}(x), & x \in I_1 \\ f^{(2)}(x), & x \in I_3 \end{cases}$$

is real valued function and $f \in H$*, then there exist a function* $F \in L^2_\varrho(\mathbb{R})$ *such that*

$$\lim_{n\to\infty} \int_{-\infty}^\infty \Big\{ F(\lambda) - \int_a^c f^{(1)}(x)\phi^{(1)}(x,\lambda)dx - \gamma \int_c^n f^{(2)}(x)\phi^{(2)}(x,\lambda) \Big\}^2 d\varrho(\lambda) = 0, \tag{2.13}$$

and the Parseval equality

$$\|f\|_H^2 = \int_a^c \Big(f^{(1)}(x)\Big)^2 dx + \gamma \int_c^\infty (f^{(2)}(x))^2 dx = \int_{-\infty}^\infty F^2(\lambda)d\varrho(\lambda)\,. \tag{2.14}$$

(ii) *The integral* $\int_{-\infty}^\infty F(\lambda)\phi(x,\lambda)d\varrho(\lambda)$ *converge to* f *in* H*; that is,*

$$\begin{aligned} \lim_{n\to\infty} \Big\{ &\int_a^c [f^{(1)}(x) - \int_{-n}^n F(\lambda)\phi^{(1)}(x,\lambda)d\varrho(\lambda)]^2 dx \\ &+ \gamma \int_c^\infty [f^{(2)}(x) - \int_{-n}^n F(\lambda)\phi^{(2)}(x,\lambda)d\varrho(\lambda)]^2 dx \Big\} = 0. \end{aligned}$$

We note that the function ϱ is called a *spectral function* for boundary value problem (2.1)-(2.3).

Proof. Assume that

$$f_\xi(x) = \begin{cases} f_\xi^{(1)}(x), & x \in [a,c) \\ f_\xi^{(2)}(x), & x \in (c,\xi] \end{cases}$$

satisfies the following conditions.

(1) $f_\xi(x)$ vanishes outside the set $[a,c) \cup (c,\xi]$ with $\xi < b$;

(2) $(pf_\xi')(x)$ has a continuous derivative.
(3) $f_\xi(x)$ satisfy the conditions (2.2)-(2.3).

When we apply the Parseval equality (2.10) to $f_\xi(x)$, we obtain

$$\int_a^c (f_\xi^{(1)}(x))^2dx + \gamma\int_c^\xi (f_\xi^{(2)}(x))^2dx = \int_{-\infty}^{\infty} F_\xi^2(\lambda)d\varrho(\lambda), \tag{2.15}$$

where

$$F_\xi(\lambda) = \int_a^c f_\xi^{(1)}(x)\phi^{(1)}(x,\lambda)dx + \gamma\int_c^\xi f_\xi^{(2)}(x)\phi^{(2)}(x,\lambda)dx. \tag{2.16}$$

Since $\phi(x,\lambda)$ satisfies (2.1), we see that

$$\phi(x,\lambda) = \frac{1}{\lambda}\big[-(p\phi')'(x,\lambda) + q(x)\phi(x,\lambda)\big].$$

By (2.16), we obtain

$$\begin{aligned}F_\xi(\lambda) &= \frac{1}{\lambda}\int_a^c f_\xi^{(1)}(x)\big[-\big(p\phi^{(1)\prime}\big)'(x,\lambda) + q(x)\phi^{(1)}(x,\lambda)\big]dx\\
&\quad + \frac{1}{\lambda}\int_c^\xi f_\xi^{(2)}(x)\big[-\big(p\phi^{(2)\prime}\big)'(x,\lambda) + q(x)\phi^{(2)}(x,\lambda)\big]dx.\end{aligned}$$

Since $f_\xi(x)$ vanishes in a neighborhood of the point b and $f_\xi(x)$ and $\phi(x,\lambda)$ satisfy the boundary conditions (2.2), (2.3) we obtain

$$\begin{aligned}F_\xi(\lambda) &= \frac{1}{\lambda}\int_a^c \phi^{(1)}(x,\lambda)\big[-\big(pf_\xi^{(1)\prime}\big)'(x) + q(x)f_\xi^{(1)}(x)\big]dx\\
&\quad + \frac{1}{\lambda}\int_c^b \phi^{(2)}(x,\lambda)\big[-\big(pf_\xi^{(2)\prime}\big)'(x) + q(x)f_\xi^{(2)}(x)\big]dx,\end{aligned}$$

using integration by parts.

For any finite $N>0$, using (2.10), we have

$$\begin{aligned}&\int_{|\lambda|>N} F_\xi^2(\lambda)d\varrho_b(\lambda)\\
&\le \frac{1}{N^2}\int_{|\lambda|>N}\Big\{\int_a^c \phi^{(1)}(x,\lambda)\big[-\big(pf_\xi^{(1)\prime}\big)'(x) + q(x)f_\xi^{(1)}(x)\big]dx\\
&\quad + \gamma\int_c^b \phi^{(2)}(x,\lambda)\big[-\big(pf_\xi^{(2)\prime}\big)'(x) + q(x)f_\xi^{(2)}(x)\big]dx\Big\}^2 d\varrho_b(\lambda)\\
&\le \frac{1}{N^2}\int_{-\infty}^{\infty}\Big\{\int_a^c \phi^{(1)}(x,\lambda)\big[-\big(pf_\xi^{(1)\prime}\big)'(x) + q(x)f_\xi^{(1)}(x)\big]dx\\
&\quad + \gamma\int_c^b \phi^{(2)}(x,\lambda)\big[-\big(pf_\xi^{(2)\prime}\big)'(x) + q(x)f_\xi^{(2)}(x)\big]dx\Big\}^2 d\varrho_b(\lambda)\\
&= \frac{1}{N^2}\int_a^c \big[-\big(pf_\xi^{(1)\prime}\big)'(x) + q(x)f_\xi^{(1)}(x)\big]^2dx\\
&\quad + \frac{1}{N^2}\gamma\int_c^\xi \big[\big(-pf_\xi^{(2)\prime}\big)'(x) + q(x)f_\xi^{(2)}(x)\big]^2dx.\end{aligned}$$

From (2.15), we see that

$$\begin{aligned} &\Big|\int_a^c (f_\xi^{(1)}(x))^2dx+\gamma\int_c^\xi (f_\xi^{(2)}(x))^2dx-\int_{-N}^N F_\xi^2(\lambda)d\varrho_b(\lambda)\Big| \\ &< \frac{1}{N^2}\int_a^c \big[-\big(pf_\xi^{(1)\prime}\big)'(x)+q(x)f_\xi^{(1)}(x)\big]^2dx \\ &\quad+\frac{1}{N^2}\gamma\int_c^\xi \big[-\big(pf_\xi^{(2)\prime}\big)'(x)+q(x)f_\xi^{(2)}(x)\big]^2dx. \end{aligned} \tag{2.17}$$

By Lemma 2.1, the set $\{\varrho_b(\lambda)\}$ is bounded. Using Theorems 2.2 and 2.3, we can find a sequence $\{b_k\}$ ($b_k\to\infty$) such that the functions $\varrho_{b_k}(\lambda)$ converge to a monotone function $\varrho(\lambda)$. Passing to the limit with respect to $\{b_k\}$ in (2.17), we obtain

$$\begin{aligned} &\Big|\int_a^c (f_\xi^{(1)}(x))^2dx+\gamma\int_c^\xi (f_\xi^{(2)}(x))^2dx-\int_{-N}^N F_\xi^2(\lambda)d\varrho(\lambda)\Big| \\ &< \frac{1}{N^2}\int_a^c \big[-\big(pf_\xi^{(1)\prime}\big)'(x)+q(x)f_\xi^{(1)}(x)\big]^2dx \\ &\quad+\frac{1}{N^2}\gamma\int_c^\xi \big[-\big(pf_\xi^{(2)\prime}\big)'(x)+q(x)f_\xi^{(2)}(x)\big]^2dx. \end{aligned}$$

Hence, letting $N\to\infty$, we obtain

$$\int_a^c (f_\xi^{(1)}(x))^2dx+\gamma\int_c^\xi (f_\xi^{(2)}(x))^2dx=\int_{-\infty}^\infty F_\xi^2(\lambda)d\varrho(\lambda).$$

Now, let f be an arbitrary real valued function on H. It is known that there exists a sequence of functions $\{f_\xi(x)\}$ satisfying the condition 1-3 and such that

$$\lim_{\xi\to\infty}\Big\{\int_a^c (f^{(1)}(x)-f_\xi^{(1)}(x))^2dx+\gamma\int_c^\xi (f^{(2)}(x)-f_\xi^{(2)}(x))^2dx\Big\}=0.$$

Let

$$F_\xi(\lambda)=\int_a^\infty f_\xi^{(1)}(x)\phi^{(1)}(x,\lambda)dx+\gamma\int_a^\infty f_\xi^{(2)}(x)\phi^{(2)}(x,\lambda)dx.$$

Then, we have

$$\int_a^c (f_\xi^{(1)}(x))^2dx+\gamma\int_c^\xi (f_\xi^{(2)}(x))^2dx=\int_{-\infty}^\infty F_\xi^2(\lambda)d\varrho(\lambda).$$

Since

$$\int_a^c (f_{\xi_1}^{(1)}(x)-f_{\xi_2}^{(1)}(x))^2dx+\int_c^\infty (f_{\xi_1}^{(2)}(x)-f_{\xi_2}^{(2)}(x))^2dx\to 0$$

as $\xi_1,\xi_2\to\infty$, we have

$$\begin{aligned} &\int_{-\infty}^\infty (F_{\xi_1}(\lambda)-F_{\xi_2}(\lambda))^2d\varrho(\lambda) \\ &=\int_a^c (f_{\xi_1}^{(1)}(x)-f_{\xi_2}^{(1)}(x))^2dx+\int_c^\infty (f_{\xi_1}^{(2)}(x)-f_{\xi_2}^{(2)}(x))^2dx\to 0 \end{aligned}$$

as $\xi_1,\xi_2\to\infty$. Consequently, there is a limit function F which satisfies

$$\int_a^c \Big(f^{(1)}(x)\Big)^2dx+\gamma\int_c^\infty \Big(f^{(2)}(x)\Big)^2dx=\int_{-\infty}^\infty F^2(\lambda)d\varrho(\lambda),$$

by the completeness of the space $L^2_\varrho(\mathbb{R})$.

Our next goal is to show that

$$K_\xi(\lambda)=\int_a^c f^{(1)}(x)\phi^{(1)}(x,\lambda)dx+\gamma\int_c^\xi f^{(2)}(x)\phi^{(2)}(x,\lambda)dx\to F$$

as $\xi\to\infty$, in the metric of space $L^2_\varrho(\mathbb{R})$. Let g be another real-valued function in H. By a similar arguments, let $G(\lambda)$ be defined by g. It is clear that

$$\begin{aligned}&\int_a^c\Big(f^{(1)}(x)-g^{(1)}(x)\Big)^2dx+\gamma\int_c^\infty\Big(f^{(2)}(x)-g^{(2)}(x)\Big)^2d\\&=\int_{-\infty}^\infty\{F(\lambda)-G(\lambda)\}^2d\varrho(\lambda).\end{aligned}$$

Set

$$g(x)=\begin{cases}f(x), & x\in[a,c)\cup[c,\xi]\\ 0, & x\in(\xi,\infty).\end{cases}$$

Then

$$\int_{-\infty}^\infty\{F(\lambda)-K_\xi(\lambda)\}^2d\varrho(\lambda)=\gamma\int_\xi^\infty(f^{(2)}(x))^2dx\to0,\quad\text{as }\xi\to\infty,$$

which proves that K_ξ converges to F in $L^2_\varrho(\mathbb{R})$ as $\xi\to\infty$. This proves (i).

Now, we prove (ii). Suppose that the real valued functions $f,g\in H$, and $F(\lambda)$ and $G(\lambda)$ are their Fourier transforms (see (2.13)). Then $F\mp G$ are transforms of $f\mp g$. Consequently, by (2.14), we have

$$\begin{aligned}&\int_a^c\big[f^{(1)}(x)+g^{(1)}(x)\big]^2dx+\gamma\int_c^\infty\big[f^{(2)}(x)+g^{(2)}(x)\big]^2dx\\&=\int_{-\infty}^\infty[F(\lambda)+G(\lambda)]^2d\varrho(\lambda),\\&\int_a^c\big[f^{(1)}(x)-g^{(1)}(x)\big]^2dx+\gamma\int_c^\infty\big[f^{(2)}(x)-g^{(2)}(x)\big]^2dx\\&=\int_{-\infty}^\infty[F(\lambda)-G(\lambda)]^2d\varrho(\lambda).\end{aligned}$$

Subtracting the second relation from the first, we obtain

$$\int_a^c f^{(1)}(x)g^{(1)}(x)dx+\gamma\int_c^\infty f^{(2)}(x)g^{(2)}(x)dx=\int_{-\infty}^\infty F(\lambda)G(\lambda)d\varrho(\lambda)\tag{2.18}$$

which is called the generalized Parseval equality. Set

$$f_\tau^{(j)}(x)=\int_{-\tau}^\tau F(\lambda)\phi^{(j)}(x,\lambda)d\varrho(\lambda),\quad j=1,2,$$

where F is the function defined in (2.13). Let $g \in H$ be a real valued function which equals zero outside the set $[a,c) \cup (c,\mu]$. Thus, we obtain

$$\begin{aligned}
&\int_a^c f_\tau^{(1)}(x) g^{(1)}(x)dx + \gamma \int_c^\mu f_\tau^{(2)}(x) g^{(2)}(x)dx \\
&= \int_a^c \Big\{ \int_{-\tau}^{\tau} F(\lambda)\phi^{(1)}(x,\lambda)d\varrho(\lambda) \Big\} g^{(1)}(x)dx \\
&\quad + \gamma \int_c^\mu \Big\{ \int_{-\tau}^{\tau} F(\lambda)\phi^{(2)}(x,\lambda)d\varrho(\lambda) \Big\} g^{(2)}(x)dx \\
&= \int_{-\tau}^{\tau} F(\lambda) \Big\{ \int_a^c \phi^{(1)}(x,\lambda) g^{(1)}(x)dx + \gamma \int_c^\mu \phi^{(2)}(x,\lambda) g^{(2)}(x)dx \Big\} d\varrho(\lambda) \\
&= \int_{-\tau}^{\tau} F(\lambda)G(\lambda)d\varrho(\lambda).
\end{aligned} \tag{2.19}$$

Subtracting (2.18) and (2.19), we have

$$\begin{aligned}
&\int_a^c \big(f^{(1)}(x) - f_\tau^{(1)}(x)\big) g^{(1)}(x)dx + \gamma \int_c^\infty \big(f^{(2)}(x) - f_\tau^{(2)}(x)\big) g^{(2)}(x)dx \\
&= \int_{|\lambda|>\tau} F(\lambda)G(\lambda)d\varrho(\lambda).
\end{aligned}$$

Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\Big(\int_a^c (f^{(1)}(x) - f_\tau^{(1)}(x)) g^{(1)}(x)dx + \gamma \int_c^\infty (f^{(2)}(x) - f_\tau^{(2)}(x)) g^{(2)}(x)dx \Big)^2 \\
&\leq \int_{|\lambda|>\tau} F^2(\lambda)d\varrho(\lambda) \int_{|\lambda|>\tau} G^2(\lambda)d\varrho(\lambda).
\end{aligned}$$

We apply this inequality to the function

$$g(x) = \begin{cases} f_\tau(x) - f(x), & x \in [a,c) \cup (c,\mu] \\ 0, & x \in (\mu,\infty), \end{cases}$$

we obtain

$$\begin{aligned}
&\int_a^c \big(f^{(1)}(x) - f_\tau^{(1)}(x)\big)^2 dx + \gamma \int_c^\infty \big(f^{(2)}(x) - f_\tau^{(2)}(x)\big)^2 dx \\
&\leq \int_{|\lambda|>\tau} F^2(\lambda)d\varrho(\lambda).
\end{aligned}$$

Letting $\tau \to \infty$ yields the desired result, since the right-hand side does not depend on μ. □

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Bilender P. Allahverdiev
Department of Mathematics, Süleyman Demirel University, 32260 Isparta, Turkey
E-mail address: `bilenderpasaoglu@sdu.edu.tr`

Hüseyin Tuna
Department of Mathematics, Mehmet Akif Ersoy University, 15030 Burdur, Turkey
E-mail address: `hustuna@gmail.com`