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QUASISTATIC THERMO-ELECTRO-VISCOELASTIC CONTACT PROBLEM WITH SIGNORINI AND TRESCA'S FRICTION

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Abstract. In this article we consider a mathematical model that describes the quasi-static process of contact between a thermo-electro-viscoelastic body and a conductive foundation. The constitutive law is assumed to be linear thermo-electro-elastic and the process is quasistatic. The contact is modelled with a Signiorini's condition and the friction with Tresca's law. The boundary conditions of the electric field and thermal conductivity are assumed to be non linear. First, we establish the existence and uniqueness result of the weak solution of the model. The proofs are based on arguments of time-dependent variational inequalities, Galerkin's method and fixed point theorem. Also we study a associated penalized problem. Then we prove its unique solvability as well as the convergence of its solution to the solution of the original problem, as the penalization parameter tends to zero.

1. Introduction

Certain crystals, such as quartz, tourmaline, Rochelle salt, when subjected to a stress, become electrically polarized (J. and P. Curie 1880) [6]. This is the simple piezoelectric effect. The deformation resulting from the application of a electric potential is the reversible effect. An elastic material with piezoelectric effect is called an electrolytic material and the discipline dealing with the study of electrolytic materials is the theory of electroelasticity. General models for elastic materials with piezoelectric effects can be found in [19] and, more recently, in [20]. The electro-elastic characteristics of piezoelectric materials have been studied extensively, and their dependence on temperature is well-established [1, 21, 22]. The models for elastic materials with thermo-piezoelectric effects can be found in [18] and, more recently, in [1]. Some theoretical results for static frictional contact models taking into account the interaction between the electric and the mechanic fields have been obtained in [14], under the assumption that the foundation is insulated, and in [15] under the assumption that the foundation is electrically conductive. The mathematical model which describes the frictional contact between a thermo-piezoelectric body and a conductive foundation is already addressed in the static case see [3, 4].

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A number of papers investigating quasi-static frictional contact problems with viscoelastic materials have recently been published for example [12]. In [3] a bilateral contact with Tresca's friction law was analyzed, while in [17] frictional contact with normal compliance was studied. Moreover, the contact problems involving elastic or viscoelastic materials have received considerable attention recently in the mathematical literature, see for instance [5, 8, 9].

This work deals with a quasistatic mathematical model which describes the frictional contact between a thermo-electro viscoelastic body and an electrically and thermally conductive rigid foundation. The novelty of this model lies in the chosen linear thermo-electro-visco-elastic behavior for the body and in the electrical and thermal conditions describing the contact, by Signorini condition, Tresca friction law and a regularized electrical and thermal conductivity condition. The variational formulation of this problem is derived and its unique weak solvability is established.

This article is structured as follows. In Section 2, we state the model of equilibrium process of the thermo-electro-viscoelastic body in frictional contact with a conductive rigid foundation, we introduce the notation and the assumptions on the problem data. We also derive the variational formulation of the problem and we present the main results concerned the existence and uniqueness of a weak solution and also the penalty problem and its convergence of the penalized solution. Finally in Section 3, we prove the existence of a weak solution of the model and its uniqueness under additional assumptions. The proof is based on an abstract result on elliptic, parabolic variational inequalities, Faedo-Galerkin, compactness method and fixed point arguments. We show also the existence and uniqueness of penalty problem and prove the solution converge as the penalty parameter ϵ vanishes.

2. Setting of the problem

2.1. **Contact problem.** We consider a body of a piezoelectric material which occupies in the reference configuration the domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ which will be supposed bounded with a smooth boundary $\partial\Omega = \Gamma$. This boundary is divided into three open disjoint parts Γ_D, Γ_N, and Γ_C, on one hand, and a partition of $\Gamma_D \cup \Gamma_N$ into two open parts Γ_a and Γ_b, on the other hand, such that $\operatorname{meas}(\Gamma_D) > 0$ and $\operatorname{meas}(\Gamma_a) > 0$. Let $[0; T]$ time interval of interest, where $T > 0$.

The body is submitted to the action of body forces of density f_0, a volume electric charge of density q_0, and a heat source of constant strength q_1. It also submitted to mechanical, electrical and thermal constant on the boundary. Indeed, the body is assumed to be clamped in Γ_D and therefore the displacement filed vanishes there. Moreover, we assume that a density of traction forces, denote by f_2, acts on the boundary part Γ_N. We also assume that the electrical potential vanishes on Γ_a, and surface electrical charge of density q_2 is prescribed on Γ_b. We assume that the temperature θ_0 is prescribed on the surface $\Gamma_D \cup \Gamma_N$.

In the reference configuration, the body may come in contact over Γ_C with an electrically-thermally conductive foundation. We assume that its potential, temperature are maintained at φ_F, θ_F. The contact is frictional, and there may be electrical charges and heat transfer on the contact surface. The normalized gap between Γ_C and the rigid foundation is denoted by g.

Everywhere below we use $\mathbb{S}^d$ to denote the space of second order symmetric tensors on $\mathbb{R}^d$ while "$\cdot$" and $|\cdot|$ will represent the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$; that is,

$$u \cdot v = u_i v_i, \quad |v| = (v.v)^{1/2}, \quad \forall u, v \in \mathbb{R}^d,$$
$$\sigma \cdot \tau = \sigma_{ij}\tau_{ij}, \quad |\tau| = (\tau.\tau)^{1/2}, \quad \forall \sigma, \tau \in \mathbb{S}^d.$$

We denote by $u : \Omega \times]0;T[\to \mathbb{R}^d$ the displacement field, $\sigma : \Omega \to S^d$ and $\sigma = (\sigma_{ij})$ the stress tensor, $\theta : \Omega \times]0;T[\to \mathbb{R}^d$ the temperature, $q : \Omega \to \mathbb{R}^d$ and $q = (q_i)$ the heat flux vector, and by $D : \Omega \to \mathbb{R}^d$ and $D = (D_i)$ the electric displacement filed. We also denote $E(\varphi) = (E_i(\varphi))$ the electric vector field, where $\varphi : \Omega \times]0;T[\to \mathbb{R}$ is the electric potential. Moreover, let $\varepsilon(u) = (\varepsilon_{ij}(u))$ denote the linearized strain tensor given by $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$, and "Div" and "div" denote the divergence operators for tensor and vector valued functions, respectively, i.e., $\operatorname{Div} \sigma = (\sigma_{ij,j})$ and $\operatorname{div} \xi = (\xi_{j,j})$. We shall adopt the usual notation for normal and tangential components of displacement vector and stress: $\upsilon_n = \upsilon \cdot n$, $\upsilon_\tau = \upsilon - \upsilon_n n$, $\sigma_n = (\sigma n) \cdot n$, and $\sigma_\tau = \sigma n - \sigma_n n$, where n denote the outward normal vector on Γ.

The equations of stress equilibrium, the equation of quasi-stationary electric field, the equation of thermic field are given by

$$\operatorname{Div} \sigma + f_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{2.1}$$
$$\operatorname{div} D = q_0 \quad \text{in } \Omega \times (0,T), \tag{2.2}$$
$$\dot{\theta} + \operatorname{div} q = q_1 \quad \text{in } \Omega \times (0,T). \tag{2.3}$$

The constitutive equation of a linear piezoelectric material can be written as

$$\sigma = \Im\varepsilon(u) - \mathcal{E}^* E(\varphi) - \theta\mathcal{M} + \mathcal{C}\epsilon(\dot{u}) \quad \text{in } \Omega \times (0,T), \tag{2.4}$$
$$D = \mathcal{E}\varepsilon(u) + \beta E(\varphi) - \theta\mathcal{P} \quad \text{in } \Omega \times (0,T), \tag{2.5}$$

where $\Im = (f_{ijkl})$, $\mathcal{E} = (e_{ijk})$, $\mathcal{M} = (m_{ij})$, $\beta = (\beta_{ij})$, $\mathcal{P} = (p_i)$, and $\mathcal{C} = (c_{ijkl})$ are respectively, elastic, piezoelectric, thermal expansion, electric permittivity, pyroelectric tensor and (fourth-order) viscosity tensor. $\mathcal{E}^*$ is the transpose of $\mathcal{E}$ given by

$$\begin{gathered} \mathcal{E}^* = (e^*_{ijk}), \quad e^*_{ijk} = e_{kij}, \\ \mathcal{E}\sigma\upsilon = \sigma\mathcal{E}^*\upsilon, \quad \forall \sigma \in \mathbb{S}^d,\ \upsilon \in \mathbb{R}^d. \end{gathered} \tag{2.6}$$

The elastic strain-displacement, the electric field-potential and the Fourier law of heat conduction are, respectively, given by

$$\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^*) \quad \text{on } \Omega \times (0,T), \tag{2.7}$$
$$E(\varphi) = -\nabla\varphi \quad \text{on } \Gamma_N \times (0,T), \tag{2.8}$$
$$q = -\mathcal{K}\nabla\theta \quad \text{in } \Omega \times (0,T), \tag{2.9}$$

where $\mathcal{K} = (k_{ij})$ denotes the thermal conductivity tensor. Next, to complete the mathematical model according to the description of the physical setting, we have the following boundary condition: The displacement conditions

$$u = 0 \quad \text{on } \Gamma_D \times (0,T), \tag{2.10}$$
$$\sigma\nu = f_2 \quad \text{on } \Gamma_N \times (0,T), \tag{2.11}$$
$$u(0,x) = u_0(x) \quad \text{in } \Omega. \tag{2.12}$$

The electric conditions

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T), \tag{2.13}$$

$$D \cdot \nu = q_b \quad \text{on } \Gamma_b \times (0,T). \tag{2.14}$$

The thermal conditions

$$\theta = 0 \quad \text{on } (\Gamma_D \cup \Gamma_N) \times (0,T), \tag{2.15}$$

$$\theta(0,x) = \theta_0(x) \quad \text{in } \Omega. \tag{2.16}$$

The contact conditions, see [13],

$$\sigma_\nu(u) \le 0, \quad u_\nu - g \le 0, \quad \sigma_\nu(u)(u_\nu - g) = 0 \quad \text{on } \Gamma_C \times (0,T). \tag{2.17}$$

The Tresca's friction conditions:

$$\begin{gathered} \|\sigma_\tau\| \le S \quad \text{on } \Gamma_C \times (0,T), \\ \|\sigma_\tau\| < S \Longrightarrow \dot{u}_\tau = 0 \quad \text{on } \Gamma_C \times (0,T), \\ \|\sigma_\tau\| = S \Longrightarrow \exists \lambda \ne 0 \quad \text{such that } \sigma_\tau = -\lambda \dot{u}_\tau \text{ on } \Gamma_C \times (0,T). \end{gathered} \tag{2.18}$$

The regularized electrical and thermal conditions, see [7, 8],

$$D \cdot \nu = \psi(u_\nu - g)\phi_L(\varphi - \varphi_F) \quad \text{on } \Gamma_C \times (0,T), \tag{2.19}$$

$$\frac{\partial q}{\partial \nu} = k_c(u_\nu - g)\phi_L(\theta - \theta_F) \quad \text{on } \Gamma_C \times (0,T), \tag{2.20}$$

such that

$$\phi_L(s) = \begin{cases} -L & \text{if } s < -L, \\ s & \text{if } -L \le s \le L, \\ L & \text{if } s > L, \end{cases} \qquad \psi(r) = \begin{cases} 0 & \text{if } r < 0, \\ k_e \delta r & \text{if } 0 \le r \le \frac{1}{\delta}, \\ k_e & \text{if } r > \frac{1}{\delta}, \end{cases} \tag{2.21}$$

where L is a large positive constant, $\delta > 0$ is a small parameter, and $k_e \ge 0$ is the electrical conductivity coefficient such that the thermal conductance function $k_c : r \to k_c(r)$ is supposed to be zero for $r < 0$ and positive otherwise, nondecreasing and Lipschitz continuous. We note that when $\psi = 0$, the equality (2.19) leads to the condition

$$D \cdot \nu = 0 \quad \text{on } \Gamma_C \times (0,T),$$

which models the case when the foundation is a perfect electric insulator. Similarly, we have:

$$\frac{\partial q}{\partial \nu} = 0 \quad \text{on } \Gamma_C \times (0,T).$$

We collect the above equations and conditions to obtain the following mathematical problem.

Problem (P). Find a displacement field $u : \Omega \times]0,T[\to \mathbb{R}^d$, an electric potential $\varphi : \Omega \times]0,T[\to \mathbb{R}$, and a temperature filed $\theta : \Omega \times]0,T[\to \mathbb{R}$ such that (2.1)-(2.20).

2.2. **Weak formulation and main results.** In this section, we establish a weak formulation of Problem (P) and we state the main results. Let X be a Banach space, T a positive real number and $1 \le p \le \infty$, denote by $L^p(0,T;X)$ and $C(0,T;X)$ the Banach spaces of all measurable function $u :]0,T[\to X$ with the norms

$$\|u\|_{L^p(0,T;X)} = \Big(\int_0^T \|u(t)\|_X^p dt \Big)^{1/p},$$

$$\|u\|_{C(0,T;X)} = \sup_{t \in [0,T]} \|u(t)\|_X, \quad \|u\|^2_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\dot{u}\|^2_{L^2(\Omega)}.$$

We also use the Hilbert spaces

$$\mathbf{L}^2(\Omega) = L^2(\Omega)^d, \quad \mathbf{H}^1(\Omega) = H^1(\Omega)^d,$$
$$\mathcal{H} = \big\{\sigma \in \mathbb{S}^d : \sigma = \sigma_{ij},\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\big\},$$

endowed with the inner products

$$(u,v)_{\mathbf{L}^2(\Omega)} = \int_\Omega u_i v_i dx, \quad (\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma_i \tau_i dx,$$
$$(u,v)_{\mathbf{H}^1(\Omega)} = (u,v)_{\mathbf{L}^2(\Omega)} + (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}.$$

Keeping in mind the boundary condition (2.10), we introduce the closed subspace of $\mathbf{H}^1(\Omega)$,

$$V = \big\{v \in \mathbf{H}^1(\Omega) : v = 0 \ \text{ on } \ \Gamma_D\big\},$$

and the set of admissible displacement

$$K = \big\{v \in V : v_\nu - g \le 0 \text{ on } \Gamma_C\big\}.$$

Here and below, we write w for the trace $\gamma(w)$ of the function $w \in \mathbf{H}^1(\Omega)$ on Γ. Since meas$(\Gamma_1) > 0$, Korn's inequality hold

$$\|\varepsilon(v)\|_{\mathcal{H}} \ge c_k \|v\|_{\mathbf{H}^1(\Omega)}, \quad \forall v \in V, \tag{2.22}$$

where c_k is a nonnegative constant depending only on Ω and Γ_D. Therefore, the space V endowed with the inner product $(u,v)_V = (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}$ is a real Hilbert space, and its associated norm $\|v\|_V = \|\varepsilon(v)\|_{\mathcal{H}}$ is equivalent on V to the usual norm $\|.\|_{\mathbf{H}^1(\Omega)}$. By Sobolev's trace theorem, there exists a constant $c_0 > 0$ which depends only on Ω, Γ_C, and Γ_D such that

$$\|v\|_{L^2(\Gamma)^d} \le c_0 \|v\|_V, \quad \forall v \in V. \tag{2.23}$$

We also introduce the function spaces

$$W = \big\{\xi \in H^1(\Omega) : \xi = 0 \text{ on } \Gamma_a\big\},$$
$$Q = \big\{\eta \in H^1(\Omega) : \eta = 0 \text{ on } \Gamma_D \cup \Gamma_N\big\},$$
$$\mathcal{W} = \big\{D = (D)_i \in \mathbf{H}^1(\Omega) : D_i \in L^2(\Omega), \operatorname{div} D \in L^2(\Omega)\big\}.$$

Similarly, we write ζ for trace $\gamma(\zeta)$ of the function $\zeta \in H^1(\Omega)$ on Γ. Since meas$(\Gamma_a) > 0$ and meas$(\Gamma_D) > 0$, it is known that W and Q are real Hilbert spaces with the inner products.

$$(\varphi,\xi)_W = (\nabla\varphi,\nabla\xi)_{\mathbf{L}^2(\Omega)}, \quad (\theta,\eta)_Q = (\nabla\theta,\nabla\eta)_{\mathbf{L}^2(\Omega)}.$$

Moreover, the associated norms $\|\xi\|_W = \|\nabla\xi\|_{\mathbf{L}^2(\Omega)}$, $\|\eta\|_Q = \|\nabla\eta\|_{\mathbf{L}^2(\Omega)}$ are equivalent on W and Q, respectively, with the usual norms $\|\cdot\|_{H^1(\Omega)}$. By Sobolev's trace theorem, there exists a constant $c_1 > 0$ which depends only on Ω, Γ_a, and Γ_C such that

$$\|\xi\|_{L^2(\Gamma_c)} \le c_1 \|\xi\|_W, \quad \forall \xi \in W, \tag{2.24}$$

and c_2 which depends only on Ω, Γ_D, Γ_N and Γ_C such that

$$\|\eta\|_{L^2(\Gamma_c)} \le c_2 \|\eta\|_Q, \quad \forall \eta \in Q. \tag{2.25}$$

The following Friedrichs-Poincaré inequalities hold on W and Q are

$$\|\nabla\xi\|_{\mathcal{W}} \ge c_{p1}\|\xi\|_W, \quad \|\nabla\eta\|_{\mathbf{L}^2(\Omega)} \ge c_{p2}\|\eta\|_Q, \quad \forall \xi \in W \text{ and } \forall \eta \in Q. \tag{2.26}$$

In the study of the mechanical Problem (P), we denote by $a : V \times V \to \mathbb{R}$, $b : W \times W \to \mathbb{R}$, $c : V \times V \to \mathbb{R}$ and $d : Q \times Q \to \mathbb{R}$ are the following bilinear and symmetric applications

$$\begin{aligned} a(u,v) &:= (\Im\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \quad b(\varphi,\xi) := (\beta\nabla\varphi, \nabla\xi)_{\mathbf{L}^2(\Omega)}, \\ c(u,v) &:= (\mathcal{C}\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \quad d(\theta,\eta) := (\mathcal{K}\nabla\theta, \nabla\eta)_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

also denote by $e : V \times W \to \mathbb{R}$, $m : Q \times V \to \mathbb{R}$ and $p : Q \times W \to \mathbb{R}$ are following bilinear applications

$$\begin{gathered} e(v,\xi) := (\mathcal{E}\varepsilon(v), \nabla\xi)_{\mathbf{L}^2(\Omega)} = (\mathcal{E}^*\nabla\xi, \varepsilon(v))_V, \\ m(\theta,v) := (\mathcal{M}\theta, \varepsilon(v))_Q, \quad p(\theta,\xi) := (\mathcal{P}\nabla\theta, \nabla\xi)_{\mathbf{L}^2(\Omega)}. \end{gathered}$$

We need the following assumptions.

(H1) The elasticity operator $\Im : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$, the electric permittivity tensor $\beta = (\beta_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, the viscosity tensor $\mathcal{C} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ and the thermal conductivity tensor $\mathcal{K} = (k_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy the usual properties of symmetry, boundedness, and ellipticity,

$$\begin{gathered} f_{ijkl} = f_{jikl} = f_{lkij} \in L^\infty(\Omega), \quad \beta_{ij} = \beta_{ji} \in L^\infty(\Omega), \\ c_{ijkl} = c_{jikl} = c_{lkij} \in L^\infty(\Omega), \quad k_{ij} = k_{ji} \in L^\infty(\Omega), \end{gathered}$$

and there exists that $m_\Im, m_\beta, m_\mathcal{C}, m_\mathcal{K} > 0$ such that

$$\begin{gathered} f_{ijkl}(x)\xi_k\xi_l \ge m_\Im\|\xi\|^2, \quad \forall \xi \in \mathbb{S}^d,\ \forall x \in \Omega, \\ c_{ijkl}(x)\xi_k\xi_l \ge m_\mathcal{C}\|\xi\|^2, \quad \forall \xi \in \mathbb{S}^d,\ \forall x \in \Omega, \\ \beta_{ij}\zeta_i\zeta_j \ge m_\beta\|\zeta\|^2, \quad k_{ij}\zeta_i\zeta_j \ge m_\mathcal{K}\|\zeta\|^2, \quad \forall \zeta \in \mathbb{R}^d. \end{gathered}$$

(H2) From (H1) we have

$$\begin{gathered} |a(u,v)| \le M_\Im\|u\|_V\|v\|_V, \quad |b(\varphi,\xi)| \le M_\beta\|\varphi\|_W\|\xi\|_W, \\ |c(u,v)| \le M_\mathcal{C}\|u\|_V\|v\|_V, \quad |d(\theta,\eta)| \le M_\mathcal{K}\|\theta\|_Q\|\eta\|_Q, \\ |e(v,\xi)| \le M_\mathcal{E}\|v\|_V\|\xi\|_W, \quad |m(\theta,v)| \le M_\mathcal{M}\|\theta\|_Q\|v\|_V, \\ |p(\theta,\xi)| \le M_\mathcal{P}\|\theta\|_Q\|\xi\|_W. \end{gathered}$$

(H3) The piezoelectric tensor $\mathcal{E} = (e_{ijk}) : \Omega \times \mathbb{S}^d \to \mathbb{R}$, the thermal expansion tensor $\mathcal{M} = (m_{ij}) : \Omega \times \mathbb{R} \to \mathbb{R}$, and the pyroelectric tensor $\mathcal{P} = (p_i) : \Omega \to \mathbb{R}^d$ satisfy

$$e_{ijk} = e_{ikj} \in L^\infty(\Omega), \quad m_{ij} = m_{ji} \in L^\infty(\Omega), \quad p_i \in L^\infty(\Omega).$$

(H4) The surface electrical conductivity $\psi : \Gamma_C \times \mathbb{R} \to \mathbb{R}^+$ and the thermal conductance $k_c : \Gamma_C \times \mathbb{R} \to \mathbb{R}^+$ satisfy for $\pi = \psi$ or k_c: There exists $M_\pi > 0$ such that $|\pi(x,u)| \le M_\pi$ for all $u \in \mathbb{R}$ and $x \in \Gamma_C$, $x \to \pi(x,u)$ is measurable on Γ_C for all $u \in \mathbb{R}$, $\pi(x,u) = 0$ for all $x \in \Gamma_C$ and $u \le 0$.

(H5) The functions $u \to \pi(x,u)$ $(\pi = \psi, k_c)$ for $\pi = \psi$ (rep k_c) are a Lipschitz function on $\mathbb{R}$ for all $x \in \Gamma_C$ and $\forall u_1, u_2 \in \mathbb{R}$, there exists $L_\pi > 0$ such that $|\pi(x,u_1) - \pi(x,u_2)| \le L_\pi|u_1 - u_2|$.

(H6) The forces, the traction, the volume, the surfaces charge densities, the strength of the heat source,

$$\begin{gathered} f_0 \in L^2\big(0,T; L^2(\Omega)^d\big), \quad f_2 \in L^2\big(0,T; L^2(\Gamma_N)^d\big), \\ q_0 \in W^{1,2}\big(0,T; L^2(\Omega)\big), \quad q_b \in W^{1,2}\big(0,T; L^2(\Gamma_b)\big), \end{gathered}$$

$$q_1 \in L^2\big(0,T;L^2(\Omega)\big).$$

The potential and temperature satisfy

$$\varphi_F \in L^2\big(0,T;L^2(\Gamma_C)\big), \quad \theta_F \in L^2\big(0,T;L^2(\Gamma_C)\big).$$

The initial conditions the friction bounded function and the gap function satisfy

$$u_0 \in K, \quad \theta_0 \in L^2(\Omega), \quad g \in L^2(\Gamma_C), \quad g \geq 0.$$

Next, using Riesz's representation theorem, we define the elements $f \in V$, $q_e \in W$ and $q_{th} \in Q$ by

$$(f(t),v)_V = \int_\Omega f_0(t)\cdot v dx + \int_{\Gamma_N} f_2(t).v da, \quad \forall v \in V, \tag{2.27}$$

$$(q_e(t),\xi)_W = \int_\Omega q_0(t).\xi dx - \int_{\Gamma_b} q_b(t).\xi da, \quad \forall \xi \in V, \tag{2.28}$$

$$(q_{th}(t),\eta)_Q = \int_\Omega q_1(t).\eta dx, \quad \forall \eta \in Q. \tag{2.29}$$

We define the mappings $j : V \to \mathbb{R}$, $\ell : V \times W^2 \to \mathbb{R}$, and $\chi : V \times Q^2 \to \mathbb{R}$, by

$$j(v) = \int_{\Gamma_C} S\|v_\tau\| da, \quad \forall v \in V, \tag{2.30}$$

$$\ell(u(t),\varphi(t),\xi) = \int_{\Gamma_C} \psi(u_\nu(t)-g)\phi_L(\varphi(t)-\varphi_F)\xi da, \quad \forall u \in V,\ \forall \varphi,\xi \in W, \tag{2.31}$$

$$\chi(u(t),\theta(t),\eta) = \int_{\Gamma_C} k_c(u_\nu(t)-g)\phi_L(\theta(t)-\theta_F)\eta da, \quad \forall u \in V,\ \forall \theta,\eta \in Q\,, \tag{2.32}$$

respectively. Now, by a standard variational technique, it is straightforward to see that if (u,φ,θ) satisfy the conditions (2.1)-(2.21), then for a.e. $t \in]0;T[$,

$$\big(\sigma(t),\varepsilon(v)-\varepsilon(\dot u(t))\big)_{\mathcal{H}} + j(v) - j(\dot u(t))) \geq \big(f(t), v-\dot u(t))\big)_V, \quad \forall v \in K, \tag{2.33}$$

$$\big(D(t),\nabla\xi\big)_{\mathbf{L}^2(\Omega)} = \ell(u(t),\varphi(t),\xi) - (q_e(t),\xi)_W, \quad \forall \xi \in W, \tag{2.34}$$

$$\big(q(t),\nabla\eta\big)_{\mathbf{L}^2(\Omega)} = (\dot\theta(t),\eta)_Q + \chi(u(t),\theta(t),\eta) - (q_{th}(t),\eta)_Q, \quad \forall \eta \in Q. \tag{2.35}$$

Using all of this assumptions, notation, and (2.8), we obtain the following variational formulation of Problem (P), in terms a displacement field, electric potential and a temperature field.

Problem (PV). Find a displacement field $u :]0;T[\to K$, an electric potential $\varphi :]0;T[\to W$ and a temperature field $\theta :]0;T[\to Q$ a.e. $t \in]0;T[$ such that

$$\begin{aligned} &a(u(t), v-\dot u(t)) + e(v-\dot u(t),\varphi(t)) - m(\theta(t), v-\dot u(t)) \\ &+ c(\dot u(t), v-\dot u(t)) + j(v) - j(\dot u(t)) \\ &\geq (f(t), v-\dot u(t)))_V, \quad \forall v \in K, \end{aligned} \tag{2.36}$$

$$b(\varphi(t),\xi) - e(u(t),\xi) - p(\theta(t),\xi) + \ell(u(t),\varphi(t),\xi) = (q_e(t),\xi)_W, \quad \forall \xi \in W, \tag{2.37}$$

$$d(\theta(t),\eta) + (\dot\theta(t),\eta)_Q + \chi(u(t),\theta(t),\eta) = (q_{th}(t),\eta)_Q, \quad \forall \eta \in Q, \tag{2.38}$$

$$u(0,x) = u_0(x), \quad \theta(0,x) = \theta_0(x). \tag{2.39}$$

Now, we to state the main result of existence and uniqueness.

Theorem 2.1. *Assume that* (H1)–(H6), (2.30)-(2.31) *and*

$$m_\beta > M_\psi c_1^2, \quad m_{\mathcal{K}} < c_2\big(M_{k_c} c_2 + L_{k_k} L c_0\big)/2$$

hold. Then Problem (PV) *has a unique solution,*

$$u \in C^1(0,T;V), \quad \varphi \in L^2(0,T;W), \quad \theta \in L^2(0,T;Q). \tag{2.40}$$

2.3. **Convergence analysis of the penalty method.** Now, we use the penalty problem, for this, let $\epsilon > 0$ the penalty parameter . We replaced the Signorini's condition (2.17) by

$$\sigma_\nu(u_\epsilon - g) = -\frac{1}{\epsilon}[u_{\epsilon_\nu} - g]^+. \tag{2.41}$$

We consider the functional $\Phi : V \times V \to \mathbb{R}$ defined by

$$\Phi(u,v) = \int_{\Gamma_C} [u_\nu]^+ v_\nu da = \langle [u_\nu]^+, v_\nu\rangle_{\Gamma_C}, \quad \forall u, v \in V. \tag{2.42}$$

We also consider, for all $\epsilon > 0$, the family of convex and differentiable functions $\Psi_\epsilon : \mathbb{R}^d \to \mathbb{R}$ given by

$$\Psi_\epsilon(v) = \sqrt{\|v\|^2 + \epsilon^2}, \quad \forall v \in \mathbb{R}, \tag{2.43}$$

it is easy to show that such a family of functions satisfies:

$$0 < \Psi_\epsilon(v) - \|v\| \le \epsilon, \tag{2.44}$$

$$\Psi'_\epsilon(v)(w) = \frac{v.w}{\sqrt{\|v\|^2 + \epsilon^2}}, \quad \forall v, w \in \mathbb{R}. \tag{2.45}$$

We then define a family of regularized frictional functional $j_\epsilon : V \to \mathbb{R}$ by

$$j_\epsilon(v) = \int_{\Gamma_C} S\Psi_\epsilon(v_\tau) da, \quad \forall v \in V. \tag{2.46}$$

The functional j_ϵ are Gâteaux-differentiable with respect to the second argument v and represent an approximation of j_ϵ, i.e., there exists a constant $C > 0$ such that

$$|j_\epsilon(v) - j(v)| \le C\epsilon, \quad \forall v \in V. \tag{2.47}$$

We denote by $j'_\epsilon : V \to V$ the derivative of j_ϵ given by

$$\langle j'_\epsilon(v), w\rangle_{V',V} = \int_{\Gamma_C} S\Psi'_\epsilon(v_\tau)(w_\tau) da, \quad \forall v, w \in V. \tag{2.48}$$

Now, we define the regularized problem associated to (2.36)-(2.39).

Problem (PV$_\epsilon$). Find a displacement field $u_\epsilon :]0;T[\to K$, an electric potential $\varphi_\epsilon :]0;T[\to W$, and a temperature field $\theta_\epsilon :]0;T[\to Q$ a.e. $t \in]0;T[$ such that

$$\begin{aligned} &c(\dot u_\epsilon(t), v) + a(u_\epsilon(t), v) + e(v, \varphi_\epsilon(t)) - m(\theta_\epsilon(t), v)\\ &+ \frac{1}{\epsilon}\Phi(u_\epsilon(t), v) + \langle j'_\epsilon(\dot u_\epsilon(t)), v\rangle\\ &= (f(t), v)_V, \quad \forall v \in V, \end{aligned} \tag{2.49}$$

$$\begin{gathered} b(\varphi_\epsilon(t), \xi) - e(u_\epsilon(t), \xi) - p(\theta_\epsilon(t), \xi) + \ell(u_\epsilon(t), \varphi_\epsilon(t), \xi) = (q_e(t), \xi)_W,\\ \forall \xi \in W, \end{gathered} \tag{2.50}$$

$$d(\theta_\epsilon(t), \eta) + (\dot\theta_\epsilon(t), \eta)_Q + \chi(u_\epsilon(t), \theta_\epsilon(t), \eta) = (q_{th}(t), \eta)_Q, \quad \forall \eta \in Q. \tag{2.51}$$

$$u_\epsilon(0,x) = u_0(x), \quad \theta_\epsilon(0,x) = \theta_0(x). \tag{2.52}$$

We recall that Problem (PV_ϵ) is well-posed see [11]. Then we have the following existence, uniqueness and convergence of penalized problem.

Theorem 2.2. *Assume the conditions stated in Theorem 2.1 and for all $\epsilon > 0$, we have*

(a) *Problem* (PV_ϵ) *admits a unique solution*

$$u_\epsilon \in C^1(0,T;V), \quad \varphi_\epsilon \in L^2(0,T;W), \quad \theta_\epsilon \in L^2(0,T;Q).$$

(b) *The solution $(u_\epsilon, \varphi_\epsilon, \theta_\epsilon)$ of penalized Problem* (PV_ϵ) *converge to a solution of Problem* (PV). *i.e.,*

$$\|u - u_\epsilon\|_V \to 0,\ \|\varphi - \varphi_\epsilon\|_W \to 0, \quad \|\theta - \theta_\epsilon\|_Q \to 0 \quad \text{as } \epsilon \to 0.$$

3. Proof of main results

3.1. **Proof of Theorem 2.1.** The proof is carried out in serval steps, and it is based on arguments of variational inequalities, Galerkin, compactness method and Banach fixed point theorem. Let $z \in C(0,T;V)$ given by

$$\big(z(t), v - \dot{u}_z(t)\big)_V = e\big(v - \dot{u}_z(t), \varphi_z(t)\big) - m\big(\theta_z(t), v - \dot{u}_z(t)\big). \tag{3.1}$$

In the first step, we prove the following existence and uniqueness result for the displacement field for this, we consider the following problem of displacement field:

Problem (PV^{dp}). Find $u_z \in K$ for a.e. $t \in]0,T[$ such that

$$\begin{gathered} c(\dot{u}_z(t), v - \dot{u}_z(t)) + a(u_z(t), v - \dot{u}_z(t)) + (z(t), v - \dot{u}_z(t))_V, \quad \forall v \in V, \\ j(v) - j(\dot{u}_z(t)) \geq (f(t), v - \dot{u}_z(t))_V, \\ u_z(0) = u_0. \end{gathered} \tag{3.2}$$

Lemma 3.1. *For all $v \in K$ and for a.e. $t \in]0,T[$, the Problem* (PV^{dp}) *has a unique solution $u_z \in C^1(0,T;V)$.*

Proof. By using the Riesz's representation theorem we define the operator

$$(f_z(t), v)_V = (f(t), v)_V - (z(t), v)_V. \tag{3.3}$$

The Problem (PV^{dp}) can be written

$$\begin{gathered} c(\dot{u}_z(t), v - \dot{u}_z(t)) + a(u_z(t), v - \dot{u}_z(t)) + j(v) - j(\dot{u}_z(t)) \geq (f_z(t), v - \dot{u}_z(t))_V, \\ u_z(0) = u_0. \end{gathered} \tag{3.4}$$

By assumptions (H1), (H2), (H6), the condition (2.30) and using the result presented in [15, P. 61-65] we obtain result. □

Remark 3.2. If the operators a and c are nonlinear, Lipschitz and monoton, we find same results of Lemma 3.1.

In the second step, we use the displacement field u_z obtained in Lemma 3.1 to obtain the following existence and uniqueness result for the temperature field θ_z of the following problem.

Problem ($\mathbf{PV}^{th}$). Find $\theta_z \in Q$ for a.e. $t \in]0,T[$ such that

$$\begin{gathered} d(\theta_z(t),\eta) + (\dot\theta_z(t),\eta)_Q + \chi(u_z(t),\theta_z(t),\eta) = (q_{th}(t),\eta)_Q, \quad \forall \eta \in Q, \\ \theta_z(0) = \theta_0. \end{gathered} \tag{3.5}$$

Lemma 3.3. *For all $\eta \in Q$ and a.e. $t \in]0,T[$, the Problem* (PV^{th}) *has a unique solution $\theta_z \in L^2(0,T;Q)$.*

To prove the above Lemma, we use the Faedo-Galerkin methods. For this, we assume the functions $w_k = w_k(t)$, $k = 1,\dots,m$ consisting of eigenfunctions of $-\Delta$ are smooth

$$\big[w_k\big]_{k=1}^{\infty} \text{ is a orthonormal basis of } H^1(\Omega). \tag{3.6}$$

Fix now a positive integer m, we will look for a function $\theta_{z_m} :]0,T[\to H^1(\Omega)$ of the form

$$\theta_{z_m} := \sum_{i=1}^{m} d_m^i(t) w_i, \tag{3.7}$$

where we hope the select the coefficients $d_m(t) = (d_m^1(t), d_m^2(t),\dots,d_m^m(t))$, $(0 < t < T)$ so that

$$d\big(\theta_{z_m}(t),w_k\big) + \big(\dot\theta_{z_m}(t),w_k\big)_Q + \chi\big(u_z(t),\theta_{z_m}(t),w_k\big) = (q_{th}(t),w_k)_Q, \tag{3.8}$$

$$d_m^k(0) = \big(\theta_0, w_k\big), \quad (k = 1,\dots,m). \tag{3.9}$$

Lemma 3.4. *For each integer $m \in \mathbb{N}$, there exists a unique θ_{z_m} of the* (3.5) *satisfying* (3.7)-(3.8).

Proof. Assuming θ_{z_m} has the structure (3.7), we first note from (3.6) that

$$\big(\dot\theta_{z_m}(t),w_k\big)_Q = d_m^{k'}(t), \tag{3.10}$$

$$d\big(\theta_{z_m}(t),w_k\big) = \mathcal{K} d_m^k(t), \tag{3.11}$$

$$\chi(u_z(t),\theta_{z_m}(t),w_k) = \chi\big(u_z(t), \sum_{k=i}^{m} d_m^i(t) w_i, w_k\big), \tag{3.12}$$

$$\big(q_{th}(t),w_k\big)_Q = q_{th}^k(t). \tag{3.13}$$

Then (3.8)-(3.9) can be written as

$$\begin{gathered} d_m^{k'}(t) + \mathcal{K} d_m^k(t) + \int_{\Gamma_C} k_c(u_{z_\nu}(t)-g)\phi_L\Big(\sum_{i=1}^{m} d_m^i(t) w_i - \theta_F\Big) w_k da = q_{th}^k(t), \\ d_m^k(0) = (\theta_0, w_k), \quad (k = 1,\dots,m). \end{gathered} \tag{3.14}$$

We pose

$$f\big(t, d_m^k(t)\big) = q_{th}^k(t) - \mathcal{K} d_m^k(t) - \int_{\Gamma_C} k_c(u_\nu(t)-g)\phi_L\Big(\sum_{i=k}^{m} d_m^i(t) w_i - \theta_F\Big) w_k da. \tag{3.15}$$

By the inequality

$$|\mathcal{K} d_{m2}^k(t) - \mathcal{K} d_{m1}^k(t)| \le M_{\mathcal{K}} \left| d_{m2}^k(t) - d_{m1}^k(t) \right|, \tag{3.16}$$

and using (H4), (H5), we find

$$\begin{aligned}
&\Big|\int_{\Gamma_C} k_c(u_{z_\nu}(t)-g)\phi_L\Big(\sum_{i=k}^{m} d^i_{m_2}(t)w_i-\theta_F\Big)w_k da\\
&-\int_{\Gamma_C} k_c(u_{z_\nu}(t)-g)\phi_L\Big(\sum_{i=k}^{m} d^i_{m_1}(t)w_i-\theta_F\Big)w_k da\Big|\\
&\le M_\psi L\,\mathrm{meas}(\Gamma_C)|d^k_{m_2}-d^k_{m_1}|.
\end{aligned}\tag{3.17}$$

Then

$$\big|f\big(t,d^k_{m2}(t)\big)-f\big(t,d^k_{m1}(t)\big)\big|\le\big(M_{\mathcal{K}}+M_\psi L\,\mathrm{meas}(\Gamma_C)\big)|d^k_{m_2}-d^k_{m_1}|.\tag{3.18}$$

There exists a unique absolutely continuous function $d_m(t)=(d^1_m(t),\dots,d^m_m(t))$ satisfying (3.17). □

Lemma 3.5 (Energy estimates)**.** *Under assumption* (H2) *and* (2.25)*, there exists a constants cs_0 and cs_1 depending only an Ω, T and the coefficient of d such that*

$$\|\theta_{z_m}\|^2_{L^2(0,T;Q)}\le cs_0\big(\|\theta_0\|^2_{L^2(\Omega)}+\|q_{th}\|^2_{L^2(0,T;Q)}\big),\tag{3.19}$$

$$\|\dot\theta_{z_m}\|^2_{L^2(0,T;Q')}\le cs_1\big(\|\theta_0\|^2_{L^2(\Omega)}+\|q_{th}\|^2_{L^2(0,T;Q)}\big).\tag{3.20}$$

Proof. Multiply (3.8) by $d^k_m(t)$, sum for $k=1,\dots,m$ and using (3.6), we obtain

$$d\big(\theta_{z_m}(t),\theta_{z_m}(t)\big)+\big(\dot\theta_{z_m}(t),\theta_{z_m}(t)\big)_Q+\chi\big(u_z(t),\theta_{z_m}(t),\theta_{z_m}(t)\big)=\big(q_{th}(t),\theta_{z_m}(t)\big)_Q.\tag{3.21}$$

We have

$$d\big(\theta_{z_m}(t),\theta_{z_m}(t)\big)\ge m_{\mathcal{K}}\|\theta_{z_m}\|^2_Q\ge\frac{m_{\mathcal{K}}}{c_2}\|\theta_{z_m}\|^2_{L^2(\Omega)},\tag{3.22}$$

$$\big(\dot\theta_{z_m},\theta_{z_m}\big)=\frac12\frac{d}{dt}\|\theta_{z_m}\|^2_Q,\tag{3.23}$$

$$|\chi(u_z,\theta_{z_m},\theta_{z_m})|\le\frac{M_1^2}{2\alpha}+\frac{\alpha c_2}{2}\|\theta_{z_m}\|^2_Q,\tag{3.24}$$

$$\big(q_{th},\theta_{z_m}\big)_Q\le\frac{1}{2\alpha}\|q_{th}\|^2_Q+\frac{\alpha}{2}\|\theta_{z_m}\|^2_Q,\tag{3.25}$$

with $M_1=M_{k_c}.M_L$ and $\alpha>0$.

Estimate for θ_{z_m}. Using (3.22), (3.25), we have

$$\frac{d}{dt}\|\theta_{z_m}\|^2_Q\le\big(\alpha(1+c_2)-2m_{\mathcal{K}}\big)\|\theta_{z_m}\|^2_Q+\frac1\alpha\big(M_1+\|q_{th}\|^2_Q\big).\tag{3.26}$$

with $m_{\mathcal{K}}<\alpha\big(1+c_2\big)/2$, $\alpha>0$. We integrate from 0 to t for almost all $t\in]0,T[$ and by Gronwall inequality we have

$$\|\theta_{z_m}\|^2_{L^2(0,T;Q)}\le cs_0\big(\|\theta_0\|^2_{L^2(\Omega)}+\|q_{th}\|^2_{L^2(0,T;Q)}\big).\tag{3.27}$$

Estimate for $\dot\theta_{z_m}$. Fix any $\eta\in Q$, with $\|\eta\|_Q\le1$, and write $\eta=\eta^1+\eta^2$, where $\eta^1\in spam[w_k]^m_{k=1}$ and $\big(\eta^2,w_k\big)=0$ $(k=1,\dots,m)$. Since the functions $[w_k]^m_{k=1}$ are orthogonal in Q,

$$\|\eta^1\|_Q\le\|\eta\|_Q\le1,$$

, using (3.8), we deduce for a.e. $0<t<T$ that

$$\big(\dot\theta_{z_m},\eta^1\big)_Q+d\big(\theta_{z_m},\eta^1\big)+\chi\big(u_z,\theta_{z_m},\eta^1\big)=\big(q_{th},\eta^1\big)_Q.\tag{3.28}$$

We have

$$|d(\theta_{z_m},\eta^1)| \le M_{\mathcal{M}}\|\theta_{z_m}\|_Q, \tag{3.29}$$

$$|(q_{th},\eta^1)_Q| \le \|q_{th}\|_Q, \tag{3.30}$$

$$|\chi(u_z,\theta_{z_m},\eta^1)| \le M_1 c_2. \tag{3.31}$$

Thus

$$\|\dot\theta_{z_m}\|_{Q^*(\Omega)} \le \|q_{th}\|_Q + M_{\mathcal{K}}\|\theta_{z_m}\|_Q + M_1 c_2. \tag{3.32}$$

We integrate from 0 to t for a.e. $t\in]0,T[$ and by Gronwall inequality and the estimate for θ_{z_m} we have

$$\|\dot\theta_{z_m}\|^2_{L^2(0,T;Q^*)} \le cs_1\big(\|\theta_0\|^2|_{L^2(\Omega)} + \|q_{th}\|^2_{L^2(0,T;Q)}\big). \tag{3.33}$$

□

Proof of Lemma 3.3.

Existence of a weak solution. We have

$$Q \subset L^2(\Omega) \subset Q^*. \tag{3.34}$$

By the previous estimates, the sequence $[\theta_{z_m}]_{m=1}^\infty$ is bounded in $L^2(0,T,Q)$, and $[\dot\theta_{z_m}]_{m=1}^\infty$ is bounded in $L^2\big(0,T,Q'\big)$. By the classical Aubin-Lions lemma [2], there exists a subsequence $[\theta_{z_{m_l}}]_{l=1}^\infty \subset [\theta_{z_m}]_{m=1}^\infty$ and a function $\theta_z \in L^2(0,T;Q)$, with $\dot\theta_z \in L^2\big(0,T;Q'\big)$ such that

$$\begin{aligned} \theta_{z_{m_l}} &\rightharpoonup \theta_z \quad \text{weakly in } L^2(0,T;Q),\\ \dot\theta_{z_{m_l}} &\rightharpoonup \dot\theta_z \quad \text{weakly in } L^2\big(0,T;Q^*\big), \end{aligned} \tag{3.35}$$

then

$$d(\theta_{z_{m_l}},\eta) \to d(\theta_z,\eta) \quad \text{in } \mathbb{R}, \tag{3.36}$$

$$(\dot\theta_{z_{m_l}},\eta) \to (\dot\theta_z,\eta) \quad \text{in } \mathbb{R}. \tag{3.37}$$

We have

$$|\chi\big(u_z,\theta_{z_m},\eta\big)| = \Big|\int_{\Gamma_c} k_c(u_\nu(t)-g)\phi_L(\theta_{z_m}-\theta_F)\eta da\Big| \le M_{k_C}L\|\eta\|_{L^2(\Gamma_c)}. \tag{3.38}$$

Then $\{\chi(u_z,\theta_{z_m},\eta)\}_{m=1}^\infty$ is bounded in $\mathbb{R}$, and so we may as well suppose upon passing to a further subsequence if necessary that. For $\eta=(\theta_{z_{m_l}}-\theta_z)$ we have

$$\begin{aligned} \big|\chi(u_z,\theta_z,\theta_z-\theta_{z_{m_l}}) - \chi(u_z,\theta_{z_{m_l}},\theta_z-\theta_{z_{m_l}})\big| &\le M_{k_c}L\|\theta_z-\theta_{z_{m_l}}\|^2_{L^2(\Gamma_C)}\\ &\le c_2 M_{k_c}L\|\theta_z-\theta_{z_{m_l}}\|^2_Q. \end{aligned} \tag{3.39}$$

Using the compactness of trace map $\gamma: Q \to L^2(\Gamma_C)$, it follows from the weak convergence of $\big(\theta_{z_{m_l}}\big)$ that

$$\big(\theta_{z_{m_l}}\big) \to \theta_z \quad \text{strongly in } L^2(0,T;L^2(\Gamma_C)),$$

then

$$\chi(u_z,\theta_{z_{m_l}},\eta) \to \chi(u_z,\theta_z,\eta) \quad \text{in } \mathbb{R}. \tag{3.40}$$

Uniqueness. Assume that θ_z and $\tilde{\theta}_z$ are two weak solutions of Problem (PVth) and let

$$\begin{aligned} &B\big(\theta_z(t),\tilde{\theta}_z(t)\big)\\ &= d\big(\theta_z(t)-\tilde{\theta}_z(t),\theta_z(t)-\tilde{\theta}_z(t)\big)+\chi\Big(u_z(t),\theta_z(t),\theta_z(t)-\tilde{\theta}_z(t)\Big)\\ &\quad -\chi\big(u_z(t),\tilde{\theta}_z(t),\theta_z(t)-\tilde{\theta}_z(t)\big). \end{aligned} \tag{3.41}$$

By (3.8),

$$(\dot{\theta}_z(t)-\dot{\tilde{\theta}}_z(t),\theta_z(t)-\tilde{\theta}_z(t))+B(\theta_z(t),\tilde{\theta}_z(t))=0. \tag{3.42}$$

Using (H2), (2.25) and (2.32), we have

$$B(\theta_z(t),\tilde{\theta}_z(t))\geq -M_{k_c}Lc_2^2\|\theta_z(t)-\tilde{\theta}_z(t)\|_Q^2, \tag{3.43}$$

$$\begin{aligned} 0&=\frac{1}{2}\frac{d}{dt}\|\theta_z(t)-\tilde{\theta}_z(t)\|_Q^2+B(\theta_z(t),\tilde{\theta}_z(t))\\ &\geq \frac{1}{2}\frac{d}{dt}\|\theta_z(t)-\tilde{\theta}_z(t)\|_Q^2-M_{k_c}Lc_2^2\|\theta_z(t)-\tilde{\theta}_z(t)\|_Q^2. \end{aligned} \tag{3.44}$$

By Gronwall inequality, we have

$$\|\theta_z(t)-\tilde{\theta}_z(t)\|_Q^2\leq 2M_{k_c}Lc_2^2\|\theta_z(t)-\tilde{\theta}_z(t)\|_Q^2. \tag{3.45}$$

Thus $\theta_z=\tilde{\theta}_z$. $\square$

In the third step, we use the displacement field u_z obtained in Lemma 3.1 and the temperature field θ_z obtained in Lemma 3.3 in the following problem of electric potential.

Problem (PV^{el}). Find $\varphi_z\in W$ for all $\xi\in W$ and a.e. $t\in]0,T[$ such that

$$\begin{gathered} b(\varphi_z(t),\xi)-e(u_z(t),\xi)-p(\theta_z(t),\xi)+\ell(u_z(t),\varphi_z(t),\xi)=(q_e(t),\xi)_W,\\ \varphi_z(0)=\varphi_0. \end{gathered} \tag{3.46}$$

Lemma 3.6. *For all $\xi\in W$ and for a.e. $t\in]0,T[$, Problem* (PV^{el}) *has a unique solution $\varphi_z\in L^2(0,T;W)$.*

The proof of this lemma is similar to those used in Lemma 3.3. We have

$$\begin{aligned} &b(\varphi_{z_m}(t),w_k)-e(u_z(t),w_k)-p(\theta_z(t),w_k)+\ell(u_z(t),\varphi_z(t),w_k)\\ &=(q_e(t),w_k)_W, \end{aligned} \tag{3.47}$$

$$d_m^k(0)=(\varphi_0,w_k),\quad (k\in\mathbb{N}). \tag{3.48}$$

with

$$\varphi_{z_m}(t):=\sum_{i=1}^m d_m^i(t)w_i.$$

To proceed further, we need the following result from [10, (p. 439].

Lemma 3.7 (Zeros of a vector field)**.** *Assume the continuous function $v:\mathbb{R}^n\to\mathbb{R}^n$ satisfies*

$$v(x)\cdot x\geq 0\quad \text{for } |x|=r, \tag{3.49}$$

for some $r>0$. Then there exists a point $x\in B(0,r)$ such that

$$v(x)=0. \tag{3.50}$$

Proof of Lemma 3.6. Let

$$v^k(d) = \beta d_m^k(t) + \ell(u_z(t), \varphi_{z_m}(t), w_k) - q_e^k(t) + \mathcal{P}\theta_z^k + \mathcal{E}\epsilon(u_z^k). \tag{3.51}$$

By the assumptions (H2) and (H4) combined with the monotonicity of the function ϕ_L, we obtain

$$v(d) \cdot d \geq \alpha_1 |d|^2 - \alpha_2, \tag{3.52}$$

with $\alpha_1 = m_\beta - \frac{3\alpha}{2} > 0$ and $\alpha_2 = M_{\mathcal{P}}^2 \|\theta_z\|_Q^2 + M_{\mathcal{E}}^2 \|u_z\|_V^2 + \|q_e\|_W^2$. We apply the Lemma 3.7 to conclude that $v(d) = 0$ for some point $d \in \mathbb{R}$. Then exists a function φ_{z_m} satisfying (3.47)-(3.48). Multiply equation (3.47) by $d_m^k(t)$, sum for $k = 1, \dots, m$ we have

$$\begin{aligned} &b(\varphi_{z_m}(t), \varphi_{z_m}(t)) - e(u_z(t), \varphi_{z_m}(t)) - p(\theta_z(t), \varphi_{z_m}(t)) \\ &+ \ell(u_z(t), \varphi_{z_m}(t), \varphi_{z_m}(t)) \\ &= (q_e(t), \varphi_{z_m}(t)). \end{aligned} \tag{3.53}$$

By assumptions (H2)–(H4), (H5) and integrating from 0 to t, for a.e. $t \in]0, T[$, we have

$$\|\varphi_{z_m}(t)\|_{L^2(0,T;W)} \leq \big(\alpha_1 + \alpha_2 \|q_e(t)\|_{L^2(0,T;W)}\big), \tag{3.54}$$

with $\alpha_1 = \alpha_2(M_{\mathcal{E}}\|u_z(t)\|_V + M_{\mathcal{P}}\|\theta_z(t)\|_Q + M_\psi M_L c_1)$ and $\alpha_2 = \frac{1}{m_\beta}$.

Existence. By (3.54) we can extract a subsequence $[\varphi_{z_{m_j}}]_{j=1}^\infty \subset [\varphi_{z_m}]_{m=1}^\infty$ and a function $\varphi_{z_{m_j}} \in L^2(0, T, W)$ such that

$$\varphi_{z_{m_j}} \rightharpoonup \varphi_z \quad \text{weakly in } L^2(0, T; W). \tag{3.55}$$

By the assumptions (H4) and (H5) we have

$$|\ell(u_z(t), \varphi_{z_m}, \xi)| \leq M_\psi L \|\xi\|_{L^2(\Gamma_C)}. \tag{3.56}$$

Then $\big\{\ell(u_z, \varphi_{z_m}, \xi)\big\}_{m=1}^\infty$ is bounded in $\mathbb{R}$. For $\xi = (\varphi_{z_{m_j}} - \varphi_z)$ and the assumptions (H4), (H5) and (2.24), we find that

$$\begin{aligned} &|\ell(u_z, \varphi_z, \varphi_z - \varphi_{z_{m_j}}) - \ell(u_z, \varphi_{z_{m_j}}, \varphi_z - \varphi_{z_{m_j}})| \\ &\leq M_\psi L \|\varphi_z - \varphi_{z_{m_j}}\|_{L^2(\Gamma_C)}^2 \\ &\leq M_\psi L c_1^2 \|\varphi_z - \varphi_{z_{m_j}}\|_{L^2(0,T;W)}^2. \end{aligned} \tag{3.57}$$

By using the compactness of trace map $\gamma : Q \to L^2(\Gamma_C)$, from the weak convergence of $\big(\varphi_{z_{m_l}}\big)$ it follows that

$$\big(\varphi_{z_{m_l}}\big) \to \varphi_z \quad \text{strongly in } L^2\big(0, T; L^2(\Gamma_C)\big).$$

Then

$$\begin{gathered} \ell(u_z, \varphi_{z_{m_l}}, \xi) \to \ell(u_z, \varphi_z, \xi) \quad \text{in } \mathbb{R}. \\ b(\varphi_{z_{m_l}}, \xi) \to b(\varphi_z, \xi) \quad \text{in } \mathbb{R}. \end{gathered} \tag{3.58}$$

Uniqueness. By Riesz's representation theorem, we define the operator $A_z(t) : W \to W$ such that

$$(A_z(t)\varphi_z, \xi) = b(\varphi_z(t), \xi) - e(u_z(t), \xi) - p(\theta_z(t), \xi) + \ell(u_z(t), \varphi_z(t), \xi). \tag{3.59}$$

For $\xi = (\varphi_z - \tilde{\varphi}_z)$, where φ_z and $\tilde{\varphi}_z$ two solution of problem $\big(PV^{el}\big)$ we have

$$(A_z(t)\varphi_z - A_z(t)\tilde{\varphi}_z, \varphi_z(t) - \tilde{\varphi}_z(t)) = 0. \tag{3.60}$$

By the monotonicity of the operator b, we have

$$\begin{aligned} &(A_z(t)\varphi_z - A_z(t)\tilde{\varphi}_z, \varphi_z(t) - \tilde{\varphi}_z(t)) \\ &\geq m_\beta \|\varphi_z(t) - \tilde{\varphi}_z(t)\|_W^2 + \ell(u_z(t), \varphi_z(t), \varphi_z(t) - \tilde{\varphi}_z(t)) \\ &\quad - \ell(u_z(t), \tilde{\varphi}_z(t), \varphi_z(t) - \tilde{\varphi}_z(t)), \end{aligned} \tag{3.61}$$

and by (H5) and the monotonicity of the function ϕ_L, we obtain

$$0 = (A_z(t)\varphi_z - A_z(t)\tilde{\varphi}_z, \varphi_z(t) - \tilde{\varphi}_z(t)) \geq m_\beta \|\varphi_z(t) - \tilde{\varphi}_z(t)\|_W^2. \tag{3.62}$$

Thus $\varphi_z = \tilde{\varphi}_z$. $\square$

In the last step, for $z \in L^2(0,T;V)$, φ_z and θ_z the functions obtained in Lemmas 3.3 and 3.6, respectively, we consider the operator $\Lambda : C(0,T;V) \to C(0,T;V)$ defined by

$$(\Lambda z(t), v)_V = e(v, \varphi_z(t)) - m(\theta_z(t), v), \tag{3.63}$$

for all $v \in V$ and for a.e. $t \in]0,T[$. We show that Λ has a unique fixed point.

Lemma 3.8. *There exists a unique $\tilde{z} \in C(0,T;V)$ such that $\Lambda\tilde{z} = \tilde{z}$.*

Proof. Let $z \in C(0,T;V)$ and $t_1, t_2 \in]0,T[$. By using the properties of operators e and m, we find that

$$\|\Lambda z(t_1) - \Lambda z(t_2)\|_V \leq c\big(\|\varphi_z(t_1) - \varphi_z(t_2)\|_W + \|\theta_z(t_1) - \theta_z(t_2)\|_Q\big). \tag{3.64}$$

Since $\varphi_z \in L^2(0,T;W)$ and $\theta_z \in L^2(0,T;Q)$, we deduce that $\Lambda z \in L^2(0,T;V)$.

Now let $z_1, z_2 \in C(0,T;V)$ and denote by u_i, φ_i and θ_i the functions obtained in Lemmas 3.1, 3.3 and 3.6. For $i = 1,2$. Let $t \in [0;T]$. Using (3.2), assumption (H2) and this inequality

$$\|u_{z_2}(t) - u_{z_1}(t)\|_V \leq \int_0^t \|\dot{u}_{z_2}(t) - \dot{u}_{z_1}(t)\|_V ds, \tag{3.65}$$

we have

$$\|u_{z_2}(t) - u_{z_1}(t)\|_V \leq \frac{M_{\Im}}{m_{\mathcal{C}}} \int_0^t \|u_{z_2}(s) - u_{z_1}(s)\|_V ds + \frac{1}{m_{\mathcal{C}}} \int_0^t \|z_2(s) - z_1(s)\|_V ds. \tag{3.66}$$

By Gronwall inequality, we obtain

$$\|u_{z_2}(t) - u_{z_1}(t)\|_V \leq ce \int_0^t \|z_2(s) - z_1(s)\|_V ds, \tag{3.67}$$

with $ce = \frac{1}{m_{\mathcal{C}}} \exp\big(\frac{TM_{\Im}}{m_{\mathcal{C}}}\big)$. Using (3.5), (H2), (H4) and (H5), we have

$$\begin{aligned} &m_{\mathcal{K}} \|\theta_{z_2}(t) - \theta_{z_1}(t)\|_Q^2 + \frac{1}{2}\frac{d}{dt}\|\theta_{z_2}(t) - \theta_{z_1}(t)\|_Q^2 \\ &\leq \beta_1 \|\theta_{z_2}(t) - \theta_{z_1}(t)\|_Q^2 + \beta_2 \|u_{z_2}(t) - u_{z_1}(t)\|_V . \|\theta_{z_2}(t) - \theta_{z_1}(t)\|_Q, \end{aligned} \tag{3.68}$$

with $\beta_1 = M_{k_c} c_2^2$ and $\beta_2 = L_{k_c} L c_0 c_2$. We integrate this inequality from 0 to t and by Gronwall inequality, we obtain

$$\|\theta_{z_2}(t) - \theta_{z_1}(t)\|_Q \le \beta_3 \int_0^t \|z_2(s) - z_1(s)\|_V ds. \tag{3.69}$$

with $\beta_3 = \big(ce^2 T L_{k_c} L c_0 c_2 \exp(\beta_1 + \beta_2 - 2m_{\mathcal{K}})\big)^{1/2}$ and the condition

$$m_{\mathcal{K}} < c_2\big(M_{k_c} c_2 + L_{k_k} L c_0\big)/2\,.$$

Using (3.49), (H2), (H4) and (H5), we have

$$\|\varphi_{z_2}(t) - \varphi_{z_1}(t)\|_W \le \beta_4 \int_0^t \|z_2(s) - z_1(s)\|_V ds. \tag{3.70}$$

with $\beta_4 = \alpha\big(M_{\mathcal{E}} + L_\psi c_0 c_1\big)/\big(m_\beta - M_\psi c_1^2\big)$ and the condition $\big(m_\beta > M_\psi c_1^2\big)$.

By (3.65), (3.67), (3.69) and (3.70), we obtain

$$\|\Lambda z_2(t) - \Lambda z_2(t)\|_V \le \beta_5 \int_0^t \|z_2(s) - z_1(s)\|_V ds, \tag{3.71}$$

with $\beta_5 = \alpha\big(\beta_3 + \beta_4\big)$, $\alpha > 0$. Iterating this inequality n times results in

$$\|\Lambda^n z_2(t) - \Lambda^n z_2(t)\|_V \le \frac{\beta_5^n}{n!}\|z_2(s) - z_1(s)\|_{C(0,T;V)}. \tag{3.72}$$

This inequality show that a sufficiently large n the operator Λ^n is a contraction on the Banach space $C(0,T;V)$, and therefore, there exists a unique element $\tilde z \in C(0,T;V)$, such that $\Lambda \tilde z = \tilde z$. $\square$

We are now ready to prove Theorem 2.1.

Existence. Let $\tilde z \in C(0,T;V)$ be the fixed point of the operator Λ and denote $\tilde x = (\tilde u_z, \tilde\varphi_z, \tilde\theta_z)$ the solution of the variational problem (PV_z), for $\tilde z = z$, the definition of Λ and problem (PV_z) prove that $\tilde x$ is a solution of problem (PV).

Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Λ.

3.2. **Proof of Theorem 2.2.** In this paragraph we prove the existence and uniqueness of Problem (PV$_\epsilon$) presented in Theorem 2.2(a) follow the same steps that Theorem 2.1, for this let $z_\epsilon \in C(0,T;V)$ such that

$$(z_\epsilon(t), v)_V = e(v, \varphi_{\epsilon z}(t)) - m(\theta_{\epsilon z}(t), v). \tag{3.73}$$

Proof of (a) in Theorem 2.2. We consider the following problem.

Problem (PV$_{\epsilon z}^{dp}$). Find $u_{\epsilon z} \in K$ such that for a.e. $t \in]0,T[$ and $v \in V$ such that

$$\begin{gathered} c(\dot u_{\epsilon z}(t), v) + a(u_{\epsilon z}(t), v) + (z_\epsilon(t), v)_V \\ + \frac{1}{\epsilon}\Phi(u_{\epsilon z}, v) + \langle j'_\epsilon(\dot u_{\epsilon z}), v\rangle (f(t), v)_V, \\ u_\epsilon(0,x) = u_0(x). \end{gathered} \tag{3.74}$$

Using the Riesz's representation theorem, we define the operator

$$\big(f_{\epsilon z}(t), v\big) = \big(f(t), v\big)_V - \big(z_\epsilon(t), v\big)_V, \tag{3.75}$$

and

$$\tilde a(u_{\epsilon z}(t), v) = a(u_{\epsilon z}(t), v) + \frac{1}{\epsilon}\Phi(u_{\epsilon z}, v). \tag{3.76}$$

Note that Problem ($\mathrm{PV}_{\epsilon z}^{dp}$) is equivalent to the Cauchy problem

$$\begin{gathered}\tilde{a}(u_{\epsilon z}(t), v) + c(\dot{u}_{\epsilon z}(t), v) + \langle j'_\epsilon(\dot{u}_{\epsilon z}), v\rangle = \big(f_z(t), v\big),\\ u_\epsilon(0,x) = u_0(x).\end{gathered} \tag{3.77}$$

By the coercivity of j_ϵ and the inequality (2.47), for all $w \in L^2(0,T;V)$, we have

$$\langle j'_\epsilon(v), w - v\rangle \le j_\epsilon(w) - j_\epsilon(v). \tag{3.78}$$

Then Problem $\big(PV_\epsilon^{dp}\big)$ can be written as

$$\tilde{a}(u_{\epsilon z}(t), v) + c(\dot{u}_{\epsilon z}(t), v) + j_\epsilon(\dot{u}_{\epsilon z}(t)) - j_\epsilon(v) \ge (f_{\epsilon z}(t), v). \tag{3.79}$$

By assumption (H6) and $z_\epsilon \in C(0,T;V)$, we have $f_{\epsilon z} \in C(0,T;V)$, and by $(h_1)-(h_2)$ the operator c is continuous and coercive. We prove now the operator $\tilde{a}$ is continuous, for this let $u, v \in L^2(0,T;V)$, it follows from the definition of $\tilde{a}$ that

$$\begin{aligned}|\tilde{a}(u,v)| &= |a(u,v) + \frac{1}{\epsilon}\Phi(u,v)|\\ &\le |a(u,v)| + \frac{1}{\epsilon}\Big|\int_{\Gamma_C}[u_\nu]^+ v_\nu da\Big|\\ &\le M_{\Im}\|u\|_V\|v\|_V + \frac{1}{\epsilon}\|u_\nu\|_{L^2(\Gamma_C)}\|v_\nu\|_{L^2(\Gamma_C)}\\ &\le \big(M_{\Im} + \frac{c_0^2}{\epsilon}\big)\|u\|_V\|v\|_V.\end{aligned} \tag{3.80}$$

By (2.46) the functional j_ϵ is proper convex and lower semicontinuous. Using now the result presented in [16, pp. 61-65], Problem ($\mathrm{PV}_{\epsilon z}^{dp}$) has a unique solution $u_{\epsilon z} \in C^1(0,T;V)$. Now we consider the following two problems:

Problem ($\mathbf{PV}_{\epsilon z}^{el}$). Find $\varphi_{\epsilon z} :]0,T[\to W$ such that for a.e. $t \in]0,T[$ and $\xi \in W$

$$b(\varphi_{\epsilon z}(t), \xi) - e(u_{\epsilon z}(t), \xi) - p(\theta_{\epsilon z}(t), \xi) + \ell(u_{\epsilon z}(t), \varphi_{\epsilon z}(t), \xi) = (q_e(t), \xi)_W, \tag{3.81}$$

Problem ($\mathbf{PV}_{\epsilon z}^{th}$). Find $\theta_{\epsilon z} :]0,T[\to Q$ such that for a.e. $t \in]0,T[$ and $\eta \in Q$

$$d(\theta_{\epsilon z}(t), \eta) + (\dot{\theta}_{\epsilon z}(t), \eta)_Q + \chi(u_{\epsilon z}(t), \theta_{\epsilon z}(t), \eta) = (q_{th}(t), \eta)_Q. \tag{3.82}$$

Similar to Lemmas 3.3 and 3.6 the previous problems have a unique solution $\varphi_{\epsilon z} \in L^2(0,T;W)$ and $\theta_{\epsilon z} \in L^2(0,T;Q)$. Finally by lemma 3.8, Problem $\big(PV_\epsilon\big)$ has a unique solution $(u_\epsilon, \varphi_\epsilon, \theta_\epsilon)$. $\square$

In the following paragraph, we provide a convergence result involving the sequences $\{u_\epsilon\}$, $\{\varphi_\epsilon\}$ and $\{\theta_\epsilon\}$.

Proof of (b) in Theorem 2.2. We need a priori estimates for passing to limit. Similar to (3.54)-(3.27) and (3.23), we find

$$\begin{gathered}\{\varphi_\epsilon\} \text{ is bounded in } L^2(0,T;W),\\ \{\theta_\epsilon\} \text{ is bounded in } L^2(0,T;Q),\\ \{\dot{\theta}_\epsilon\} \text{ is bounded in } L^2(0,T;Q').\end{gathered} \tag{3.83}$$

Estimate for u_ϵ. Setting $v = u_\epsilon$ in (2.49), we obtain

$$
\begin{aligned}
&c(\dot u_\epsilon(t), u_\epsilon(t)) + a(u_\epsilon(t), u_\epsilon(t)) + e(u_\epsilon(t), \varphi_\epsilon(t)) - m(\theta_\epsilon(t), u_\epsilon(t))\\
&+ \frac{1}{\epsilon}\Phi(u_\epsilon(t), u_\epsilon(t)) + \langle j_\epsilon'(\dot u_\epsilon), u_\epsilon(t)\rangle\\
&= (f(t), u_\epsilon(t))_V.
\end{aligned} \tag{3.84}
$$

As $\Phi(u_\epsilon(t), u_\epsilon(t)) \geq 0$ and $\langle j_\epsilon'(\dot u_\epsilon), u_\epsilon(t)\rangle \geq 0$, we find that

$$
\begin{aligned}
&a(u_\epsilon(t), u_\epsilon(t)) + c(\dot u_\epsilon(t), u_\epsilon(t)) + e(u_\epsilon(t), \varphi_\epsilon(t)) - m(\theta_\epsilon(t), u_\epsilon(t))\\
&\leq (f(t), u_\epsilon(t))_V.
\end{aligned} \tag{3.85}
$$

By assumptions (H1),(H2) and (3.83), we have

$$
m_{\Im}\|u_\epsilon(t)\|_V^2 + m_{\mathcal{C}}\frac{1}{2}\frac{d}{dt}\|u_\epsilon(t)\|_V^2 \leq s1\|u_\epsilon(t)\|_V^2, \tag{3.86}
$$

with $s1$ depend of constants $M_{\mathcal{E}}$, $M_{\mathcal{M}}$, $\|f(t)\|_V$, $\|q_e(t)\|_{L^2(0,T;W)}$, $\|q_{th}(t)\|_{L^2(0,T;Q)}$ and $\|\theta_0\|_{L^2(\Omega)}$. We integrate from 0 to t, for a.e. $t \in]0,T[$ and using Gronwall inequality we obtain

$$
\{u_\epsilon\} \text{ is bounded in } L^2(0,T;V). \tag{3.87}
$$

Estimate for $\dot u_\epsilon$. We take $v = \dot u_\epsilon$ in (2.49), we obtain

$$
\begin{aligned}
&c(\dot u_\epsilon(t), \dot u_\epsilon(t)) + a(u_\epsilon(t), \dot u_\epsilon(t)) + e(\dot u_\epsilon(t), \varphi_\epsilon(t)) - m(\theta_\epsilon(t), \dot u_\epsilon)\\
&+ \frac{1}{\epsilon}\Phi(u_\epsilon(t), \dot u_\epsilon(t)) + \langle j_\epsilon'(\dot u_\epsilon), \dot u_\epsilon(t)\rangle\\
&= (f(t), u_\epsilon(t))_V.
\end{aligned} \tag{3.88}
$$

By $\Phi(u_\epsilon(t), u_\epsilon(t)) \geq 0$, $\langle j_\epsilon'(\dot u_\epsilon), u_\epsilon(t)\rangle \geq 0$ and assumptions $(h_1)-(h_2)$, we find that

$$
m_{\mathcal{C}}\|\dot u_\epsilon(t)\|_{V'} \leq M_{\Im}\|u_\epsilon(t)\|_V + s1. \tag{3.89}
$$

Integrating from 0 to t, for a.e. $t \in]0,T[$, using Gronwall inequality and estimate for u_ϵ we obtain that

$$
\{\dot u_\epsilon\} \text{ is bounded in } L^2(0,T;V'). \tag{3.90}
$$

Estimate for $[u_{\epsilon\nu}]^+$. We have

$$
\frac{1}{\epsilon}\Phi\big([u_{\epsilon\nu}]^+, u_{\epsilon\nu}\big) = \frac{1}{\epsilon}\int_{\Gamma_C}\big([u_{\epsilon\nu}]^+ u_{\epsilon\nu}\big)da = \frac{1}{\epsilon}\|[u_{\epsilon\nu}]^+\|_{L^2(\Gamma_C)}^2 \leq s2,
$$

Integrate from 0 to t, for a.e. $t \in]0,T[$, we obtain

$$
\{[u_{\epsilon\nu}]^+\} \text{ is bounded in } L^2\big(0,T;L^2(\Gamma_C)\big). \tag{3.91}
$$

Passage to the limit in ϵ. Using now (3.83), (3.87) and (3.90) to deduce that there exists a subsequences of u_ϵ, φ_ϵ and θ_ϵ denoted again by u_ϵ, φ_ϵ and θ_ϵ such that

$$
\begin{gathered}
u_\epsilon \rightharpoonup \tilde u \quad \text{in } L^2(0,T;V), \quad \dot u_\epsilon \rightharpoonup \dot{\tilde u} \quad \text{in } L^2(0,T;V'),\\
\varphi_\epsilon \rightharpoonup \tilde\varphi \quad \text{in } L^2(0,T;W), \quad \theta_\epsilon \rightharpoonup \tilde\theta \quad \text{in } L^2(0,T;Q),\\
\dot\theta_\epsilon \rightharpoonup \dot{\tilde\theta} \quad \text{in } L^2(0,T;Q').
\end{gathered} \tag{3.92}
$$

Using the compactness of trace map $\gamma : V \times W \times Q \to L^2(\Gamma_C)^d \times L^2(\Gamma_C) \times L^2(\Gamma_C)$, we find that

$$\begin{gathered} u_\epsilon \to \tilde{u} \quad \text{in } L^2(0,T;L^2(\Gamma_C)^d), \quad \dot{u}_\epsilon \to \dot{\tilde{u}} \quad \text{in } L^2(0,T;L^2(\Gamma_C)^d), \\ \varphi_\epsilon \to \tilde{\varphi} \quad \text{in } L^2(0,T;L^2(\Gamma_C)), \quad \theta_\epsilon \to \tilde{\theta} \quad \text{in } L^2(0,T;L^2(\Gamma_C)) \\ \dot{\theta}_\epsilon \to \dot{\tilde{\theta}} \quad \text{in } L^2(0,T;L^2(\Gamma_C)). \end{gathered} \tag{3.93}$$

By (3.91), we find that

$$\lim_{\epsilon\to 0} \|[u_{\epsilon\nu}]^+\|_{L^2\big(0,T;L^2(\Gamma_C)\big)} = \|[u_\nu]^+\|_{L^2\big(0,T;L^2(\Gamma_C)\big)} = 0. \tag{3.94}$$

It results that $\|[u_\nu]^+\|_{L^2(0,T;L^2(\Gamma_C))} = 0$ and $[u_\nu]^+ = 0$ a.e. on Γ_C and $\tilde{u}_\nu \le 0$ on Γ_C; then $\tilde{u} \in K$.

Using now (3.74), (3.78), (3.81), (3.82) and $\Phi(u_\epsilon, v - \dot{u}_\epsilon) \ge 0$, we get for all $v \in K$, $\xi \in W$ and $\eta \in Q$,

$$\begin{gathered} a(u_\epsilon(t), v - \dot{u}_\epsilon(t)) + e(v - \dot{u}_\epsilon(t), \varphi_\epsilon(t)) - m(\theta_\epsilon(t), v - \dot{u}_\epsilon(t)) \\ + c(\dot{u}_\epsilon(t), v - \dot{u}_\epsilon(t)) + j_\epsilon(v) - j_\epsilon(\dot{u}_\epsilon(t)) \\ \ge (f(t), v - \dot{u}_\epsilon(t)))_V, \\ b(\varphi_\epsilon(t), \xi) - e(u_\epsilon(t), \xi) - p(\theta_\epsilon(t), \xi) + \ell(u_\epsilon(t), \varphi_\epsilon(t), \xi) = (q_e(t), \xi)_W, \\ d(\theta_\epsilon(t), \eta) + (\dot{\theta}_\epsilon(t), \eta)_Q + \chi(u_\epsilon(t), \theta_\epsilon(t), \eta) = (q_{th}(t), \eta)_Q. \end{gathered} \tag{3.95}$$

By (3.93) and the properties of ψ, k_c and ϕ_L, we have

$$\begin{gathered} j_\epsilon(v) - j_\epsilon(\dot{u}_\epsilon(t)) \to j(v) - j(\dot{\tilde{u}}(t)) \quad \text{in } \mathbb{R}, \\ \ell(u_\epsilon(t), \varphi_\epsilon(t), \xi) \to \ell(\tilde{u}(t), \tilde{\varphi}(t), \xi) \quad \text{in } \mathbb{R}, \\ \chi(u_\epsilon(t), \theta_\epsilon(t), \eta) \to \chi(\tilde{u}(t), \tilde{\theta}(t), \eta) \quad \text{in } \mathbb{R}. \end{gathered} \tag{3.96}$$

Let $w \in L^2(0,T;V)$, by the coercivity of j_ϵ and inequality (2.47) imply that

$$\langle j'_\epsilon(v), w - v\rangle_{V',V} \le j_\epsilon(w) - j_\epsilon(v) \le j(w) - j(v) + 2C\epsilon. \tag{3.97}$$

Therefore, by (3.92), (2.26) and (3.97), we find that when $\epsilon \to 0$

$$\begin{gathered} a(\tilde{u}(t), v - \dot{\tilde{u}}(t)) + e(v - \dot{\tilde{u}}(t), \tilde{\varphi}(t)) - m(\tilde{\theta}(t), v - \dot{\tilde{u}}(t)) \\ + c(\dot{\tilde{u}}(t), v - \tilde{u}(t)) + j(v) - j(\dot{\tilde{u}}(t)) \\ \ge (f(t), v - \dot{\tilde{u}}(t)))_V, \\ b(\tilde{\varphi}(t), \xi) - e(\tilde{u}(t), \xi) - p(\tilde{\theta}(t), \xi) + \ell(\tilde{u}(t), \tilde{\varphi}(t), \xi) = (q_e(t), \xi)_W, \\ d(\tilde{\theta}(t), \eta) + (\dot{\tilde{\theta}}(t), \eta)_Q + \chi(\tilde{u}(t), \tilde{\theta}(t), \eta) = (q_{th}(t), \eta)_Q, \quad \forall \eta \in Q. \end{gathered} \tag{3.98}$$

By (2.36)-(2.39), we obtain $(\tilde{u}, \tilde{\varphi}, \tilde{\theta}) = (u, \varphi, \theta)$. □

Conclusion. In this work, we present a new model of thermo-electro-viscoelasticity, we prove the existence and uniqueness of the solution of contact problem with Tresca's friction law by using Galerkin and fixed point method. The difficulty of solving this type of problem lies not only in the coupling of viscoelastic, electrical and thermal aspects, but also in the nonlinearity of the boundary conditions modeling this type of physical phenomena (contact and friction conditions), which gives us a nonlinear variational, quasi-variational inequalities and two types of nonlinear, parabolic and elliptic family variational equations. To simplify this model, it can be treated without friction or neglecting the effect of the conductivity of the foundation. We proved the existence and uniqueness of solution to the penalty problem and its convergence to the solution of the original problem. The numerical analysis by finite element or other method is an interesting direction of future research.

References

[1] M. Aouadi; *Generalized thermo-piezoelectric problems with temperature-dependent properties.* International Journal of Solids and Structures, 43(2), (2006), 6347–6358 .

[2] J. Aubin, *Un théoreme de compacité.* Acad. Sci. Paris 256 (1963), 5042–5044.

[3] H. Benaissa, E.-H. Essoufi, R. Fakhar; *Existence results for unilateral contact problem with friction of thermo-electro-elasticity.* Appl. Math. Mech.-Engl. Ed., 36(7), (2015), 911–926.

[4] H. Benaissa, E.-H. Essoufi, R. Fakhar; *A study of a Nonlinear Coupled and Frictional Thermo-Piezoelectric Contact Problem.* Journal of Advanced Research in Applied Mathematics, Vol. 00 No. 0, January 2015, pp. 1–24 .

[5] M. Cocu; *Existence of solutions of Signorini problems with friction.* Int. J. Engng. Sci. 22, 5, (1984), 567–575.

[6] P. Curie, M. Curie; *Comptes rendus.* Vol. 91, ,(1980), no. 294.

[7] Z. Lerguet, M. Shillor, M. Sofonea; *A frictional contact problem for an electro-viscoelastic body.* Electron. J. Differential Equations 2007 (2007), no. 170, 16 pp.

[8] G. Duvaut; *Free boundary problem connected with thermoelasticity and unilateral contact.* Free boundary problem, Vol. II, 1st. Naz. Alta Mat. Francesco Severi, Rome (1980).

[9] W. Han, M. Sofonea; *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity.* Studies in Advanced Mathematics 30, Americal Mathematical Society-Intl. Press, 2002.

[10] L. C. Evans; *Partial Differential Equations.* Granduate Studies in Mathematics Volume 19 American Mathematical Society.

[11] J. L. Lions; *Quelques Méthodes de Résolution des problème aux limites non linéaires.* Dunod, Paris(1969).

[12] M. Shillor, M. Sofnea, J. J. Telega; *Quasistatic Viscoelastic Contact with Friction and Wear Diffusion.* Quarterly of Applied Mathematics. Vol. 62, No. 2 (June 2004), pp. 379–399.

[13] A. Signorini; *Sopra alcune questioni di elastostatica.* Atti della Societ'a Italiana per il Progresso delle Scienze, 1933.

[14] M. Sofonea, El-H. Essoufi; *A piezoeletric contact problem with slip dependent coefficient of friction.* Mathematical Modelling and Analysis, 9, (2004), 229–242.

[15] M. Sofonea, O. Chau and W. Han, *Analysis and approximationof a viscoelastic contact problem with slip dependent friction,* Dynam. Cont. Discr. Impuls. Syst., Series B: Vol. 8, No. 2, (2001), pp. 153–174 .

[16] M. Sofonea, A. Matei; *Variational Inequalities with Application, A study of Antiolane Frictional Contact Problems.* Springer Science+Busness Media LLC (2009).

[17] M. Shillor, K.T Andrews, K.L Kuttler; *On the dynamic behavior of a thermoviscoelastic body in frictional contact.* Europ. J. Appl. Math., Vol. 8, No. 4, pp. 417–436.

[18] H. F. Tiersten; *On the non linear equation of electro-thermo-elasticity.* International Journal of Engineering Sciences 9, (1953), 587–604.

[19] R. D. Mindlin; *Elasticity, piezoelasticity and crystal lattice dynamics.* J. of Elasticity, 4, (1972) 217–280.

[20] R. C. Batra, J. S. Yang; *Saint-Venants principle in linear piezoelectricity.* Journal of Elasticity, 38, (1995) 209–218, .

[21] S. Migorski, A. Ochal, M. Sofonea; *Weak Solvability of a Piezoelectric Contact Problem.* European Journal of Applied Mathematics. 20, (2009), 145–167.

[22] T. Ikeda; *Fundamentals of Piezoelectricity.* Oxford University Press, Oxford, (1990).

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