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p-BIHARMONIC PARABOLIC EQUATIONS WITH LOGARITHMIC NONLINEARITY

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Abstract. We consider an initial-boundary-value problem for a class of p-biharmonic parabolic equation with logarithmic nonlinearity in a bounded domain. We prove that if $2 < p < q < p(1+\frac{4}{n})$ and $u_0 \in W^+$, the problem has a global weak solutions; if $2 < p < q < p(1+\frac{4}{n})$ and $u_0 \in W_1^-$, the solutions blow up at finite time. We also obtain the results of blow-up, extinction and non-extinction of the solutions when $\max\{1, \frac{2n}{n+4}\} < p \le 2$.

1. Introduction

In this article, we consider the p-biharmonic parabolic equation with the logarithmic nonlinearity,

$$\begin{gathered} u_t + \Delta(|\Delta u|^{p-2}\Delta u) = |u|^{q-2}u\log(|u|), \quad x \in \Omega, t > 0, \\ u(x,t) = \Delta u(x,t) = 0, \quad x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, p, q are positive constants, and $u_0 \in (W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega))\backslash\{0\}$. The term $\Delta(|\Delta u|^{p-2}\Delta u)$ is called a p-biharmonic operator.

In the past years, there have been many contributions devoted to the higher order equation. Liu and Guo [10] considered the following p-biharmonic parabolic initial-boundary value problem

$$\frac{\partial u}{\partial t} + \Delta(|\Delta u|^{p-2}\Delta u) + \lambda|u|^{p-2}u = 0, \quad x \in \Omega, \tag{1.2}$$

where $p > 2$ and $\lambda > 0$. By using the discrete-time method and uniform estimates, they established the existence and uniqueness of weak solutions. Hao and Zhou [6] considered a p-biharmonic parabolic equation

$$u_t + \Delta(|\Delta u|^{p-2}\Delta u) = |u|^q - \frac{1}{|\Omega|}\int_\Omega |u|dx, \tag{1.3}$$

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where $\max\{1, \frac{2n}{n+4}\} < p \le 2$, $q > 0$. Hao and Zhou obtained results on blowup, extinction and non-extinction of the solutions. The relevant equations have also been studied in [1, 9].

In this paper, we study the parabolic p-biharmonic equation with the logarithmic nonlinearity. The second order parabolic equation with the logarithmic nonlinearity is studied. Chen considered the semilinear heat equation with the logarithmic nonlinearity [3] and the semilinear pseudo-parabolic equations with the logarithmic nonlinearity [4]. Ji, Yin and Cao [8] established the existence of positive periodic solutions and discussed the instability of such solutions for the semilinear pseudo-parabolic equation with the logarithmic source. Nahn and Truong [12] studied the nonlinear equation

$$u_t - \Delta u_t - \Delta_p u = |u|^{p-2} u \log(|u|). \tag{1.4}$$

It is a pseudoparabolic type equation, where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and Δ_p is the p-Laplacian. By using the potential well method, Nahn and Truong obtained results of existence or nonexistence of global weak solutions, and proved the large time decay of global weak solutions and the finite time blow-up of weak solutions. Cao and Liu [2] considered equation (1.4). They discussed two cases: global boundedness and blowing-up at ∞. Moreover, they proved the asymptotic behavior of solutions and gave some decay estimates and growth estimates. He, Gao and Wang [7] considered the pseudo-parabolic p-Laplacian equation

$$u_t - \Delta u_t - \Delta_p u = |u|^{q-2} u \log(|u|), \tag{1.5}$$

where $2 < p < q < p(1 + \frac{2}{n})$, they derived the decay and the finite time blow-up for weak solutions.

We begin our work by introducing some notation that will be used in this paper, $u' = \frac{\partial u}{\partial t} = u_t$,

$$\|u\|_s = \|u\|_{L^s(\Omega)}, \quad \|u\|_{2,s} = \|u\|_{W_0^{2,s}(\Omega)} = (\|\Delta u\|_s^s + \|\nabla u\|_s^s + \|u\|_s^s)^{1/s},$$

for $1 < s < +\infty$. We also use notation X_0 to denote $(W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)) \backslash \{0\}$ and $W^{-2,p'}(\Omega)$ to denote the dual space of $W^{2,s}(\Omega)$, where s' is Hölder conjugate exponent of $s > 1$.

For $u \in (W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)) \backslash \{0\}$, we define the energy functional J and Nehari functional I as follows

$$J(u) = \frac{1}{p}\|\Delta u\|_p^p - \frac{1}{q}\int_\Omega |u|^q \log(|u|)dx + \frac{1}{q^2}\|u\|_q^q, \tag{1.6}$$

$$I(u) = \|\Delta u\|_p^p - \int_\Omega |u|^q \log(|u|)dx. \tag{1.7}$$

Let

$$N = \{u \in X_0 : I(u) = 0\}$$

be the Nehari manifold. In section 2, we will show that N is not empty. Thus, we can define

$$d = \inf_{u \in N} J(u). \tag{1.8}$$

In Section 2, we show that d is positive and is attained by some $u \in N$. Now as in [12], we introduce the following sets

$$W_1 = \{u \in X_0 : J(u) < d\}, \quad W_2 = \{u \in X_0 : J(u) = d\}, \quad W = W_1 \cup W_2,$$
$$W_1^+ = \{u \in W_1 : I(u) > 0\}, \quad W_2^+ = \{u \in W_2 : I(u) > 0\}, \quad W^+ = W_1^+ \cup W_2^+,$$

$$W_1^- = \{u \in W_1 : I(u) < 0\}, \quad W_2^- = \{u \in W_2 : I(u) < 0\}, \quad W^- = W_1^- \cup W_2^-.$$

Clearly, $W^+ \cap W^- = \emptyset$ and $W^+ \cup W^- = W$. We refer to W as the potential well and d as the depth of the well. The set W^+ is regarded as the good part of the well, as we will show that every weak solution exists globally in time, provided the initial data are taken from W^+. On the other hand, if the initial data are taken from a part of W^-, we will prove a blow-up result for weak solutions.

The plan of this paper is as follows. In Section 2, we collect some properties of the energy functional J and the Nehari functional I. In Section 3, we proved that the existence of the local weak solutions and the existence of the global weak solutions. In Section 4, we establish some properties of the weak solutions, such as the finite time blow-up, extinction and non-extinction of the solutions.

2. Preliminaries

In this section, we collect some properties of the energy functional J and the Nehari functional I, which following lemmas will be used for our main results.

By the Gagliardo-Nirenberg multiplicative embedding inequality that J and I are continuous. Moreover, we have

$$J(u) = \frac{1}{q} I(u) + \big(\frac{1}{p} - \frac{1}{q}\big)\|\Delta u\|_p^p + \frac{1}{q^2}\|u\|_q^q. \tag{2.1}$$

Let $u \in X_0$ and consider the real function $j : \lambda \mapsto J(\lambda u)$ for $\lambda > 0$, defined as follows

$$j(\lambda) = J(\lambda u) = \frac{\lambda^p}{p}\|\Delta u\|_p^p - \frac{\lambda^q}{q}\int_\Omega |u|^q \log(|u|)dx - \frac{\lambda^q}{q}\log\lambda\|u\|_q^q + \frac{\lambda^q}{q^2}\|u\|_q^q.$$

The following lemma shows that $j(\lambda)$ has a unique positive critical point $\lambda^* = \lambda^*(u)$.

Lemma 2.1. *Let $u \in X_0$. Then*

1. $\lim_{\lambda\to 0^+} j(\lambda) = 0$ *and* $\lim_{\lambda\to+\infty} j(\lambda) = -\infty$*;*
2. *there exists a unique $\lambda^* = \lambda^*(u) > 0$ such that $j'(\lambda^*) = 0$;*
3. *$j(\lambda)$ is increasing on $(0, \lambda^*)$, decreasing on $(\lambda^*, +\infty)$ and attains its maximum at λ^*;*
4. *$I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ for $\lambda > \lambda^*$, and $I(\lambda^* u) = 0$.*

Proof. For $u \in X_0$, by the definition of j, we have

$$j(\lambda) = \frac{\lambda^p}{p}\|\Delta u\|_p^p - \frac{\lambda^q}{q}\int_\Omega |u|^q \log(|u|)dx - \frac{\lambda^q}{q}\log\lambda\|u\|_q^q + \frac{\lambda^q}{q^2}\|u\|_q^q.$$

It is clearly that (1) holds because $2 < p < q$ and $\|u\|_q \neq 0$. Now, by straightforward calculations, we obtain

$$j'(\lambda) = \lambda^{p-1}\Big(\|\Delta u\|_p^p - \lambda^{q-p}\int_\Omega |u|^q \log(|u|)dx - \lambda^{q-p}\log\lambda\|u\|_q^q\Big). \tag{2.2}$$

Since $\lambda > 0$, let $k(\lambda) = \lambda^{1-p} j'(\lambda)$, through direct calculation, we have

$$k'(\lambda) = -\lambda^{q-p-1}\Big((q-p)\int_\Omega |u|^q \log(|u|)dx + (q-p)\log\lambda\|u\|_q^q + \|u\|_q^q\Big).$$

Hence, there exists a

$$\lambda_1 = \exp\Big(\frac{(p-q)\int_\Omega |u|^q \log(|u|)dx + \|u\|_q^q}{(q-p)\|u\|_q^q}\Big) > 0,$$

such that $k'(\lambda) > 0$ on $(0, \lambda_1)$, $k'(\lambda) < 0$ on $(\lambda_1, +\infty)$ and $k'(\lambda_1) = 0$. Therefore, $k(\lambda)$ is increasing on $(0, \lambda_1)$, decreasing on $(\lambda_1, +\infty)$. Because of $k(0) = \|\Delta u\|_p^p > 0$ and $\lim_{\lambda\to+\infty} k(\lambda) = -\infty$, there exactly exists a $\lambda^* > 0$, such that $k(\lambda^*) = 0$, i.e. $j'(\lambda^*) = 0$. So (2) holds. Then $j'(\lambda) = \lambda^{p-1}k(\lambda)$ is positive on $(0, \lambda^*)$, and negative on $(\lambda^*, +\infty)$. So (3) holds. The last property, (4), is only a simple corollary of the fact that

$$I(\lambda u) = \lambda^p \|\Delta u\|_p^p - \lambda^q \int_\Omega |u|^q \log(|u|)dx - \lambda^q \log \lambda \|u\|_q^q = \lambda j'(\lambda).$$

The proof is complete. □

Consequently the Nehari manifold N is not empty, and the number d defined by (1.8) is meaningful. The blow lemma gives us that d is positive and is attained by some $u \in N$.

Lemma 2.2. *d is positive and there is a positive function $u \in N$ such that $J(u) = d$.*

Proof. According to (2.1), we only need to prove that there exists a positive function $u \in N$ such that $J(u) = d$. Let $\{u_k\}_{k=1}^\infty \subset N$ be a minimizing sequence of J. i.e.

$$\lim_{k\to\infty} J(u_k) = d.$$

It is clearly that $\{|u_k|\}_{k=1}^\infty \subset N$ is also a minimizing sequence of J. So, without loss of generality, we assume that $u_k > 0$ a.e. for all $k \in \mathbb{N}$.

On the other hand, we have already observed that J is coercive on N which implies that $\{u_k\}_{k=1}^\infty$ is bounded in $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$. Let $\mu > 0$ is a sufficiently small such that $q + \mu < \frac{np}{n-2p}$, so the embedding $W_0^{2,p} \hookrightarrow L^{q+\mu}$ is compact, and there exists a function u and a subsequence of $\{u_k\}_{k=1}^\infty$, still denoted by $\{u_k\}_{k=1}^\infty$, such that

$$\begin{gathered} u_k \rightharpoonup u, \quad \text{weakly in } W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega), \\ u_k \to u, \quad \text{strongly in } L^{q+\mu}(\Omega), \\ u_k(x) \to u, \quad \text{a.e. in } \Omega. \end{gathered}$$

Thus, we have $u \geq 0$ a.e. in Ω. By Lebesgue dominated convergence theorem, we see that

$$\int_\Omega |u|^q \log(|u|)dx = \lim_{k\to\infty} \int_\Omega |u_k|^q \log(|u_k|)dx, \tag{2.3}$$

$$\int_\Omega |u|^q dx = \lim_{k\to\infty} \int_\Omega |u_k|^q dx. \tag{2.4}$$

The weak lower semicontinuity of $\|\cdot\|_{W^{2,p}}$ implies

$$\|\Delta u\|_p \leq \liminf_{k\to\infty} \|\Delta u_k\|_p. \tag{2.5}$$

Combining (1.6), (1.7), (2.3), (2.4) and (2.5), we deduce that

$$J(u) \leq \liminf_{k\to\infty} J(u_k) = d, \tag{2.6}$$

$$I(u) \leq \liminf_{k\to\infty} I(u_k) = 0. \tag{2.7}$$

Thanks to $u_k \in N$ one has $u_k \in X_0$ and $I(u_k) = 0$. Thus, by using the fact $\log x \leq (e\mu)^{-1}x^\mu$ for $x \geq 1$ and the Sobolev embedding inequality, we obtain

$$\|\Delta u_k\|_p^p = \int_\Omega |u_k|^q \log(|u_k|)dx$$

$$\begin{aligned}
&= \int_{\{x\in\Omega:|u_k(x)|\geq 1\}} |u_k|^q \log(|u_k|)dx + \int_{\{x\in\Omega:|u_k(x)|<1\}} |u_k|^q \log(|u_k|)dx \\
&\leq \int_{\{x\in\Omega:|u_k(x)|\geq 1\}} |u_k|^q \log(|u_k|)dx \\
&\leq (e\mu)^{-1} \int_{\{x\in\Omega:|u_k(x)|\geq 1\}} |u_k|^{q+\mu} dx \\
&\leq (e\mu)^{-1} \|u_k\|_{q+\mu}^{q+\mu} \leq C\|\Delta u_k\|_{q+\mu}^{q+\mu},
\end{aligned}$$

for some positive constant C, which implies

$$\int_\Omega |u_k|^q \log(|u_k|)dx = \|\Delta u_k\|_p^p \geq C.$$

From this inequality and (2.3), we derive

$$\int_\Omega |u|^q \log(|u|)dx \geq C.$$

Therefore, we have $u \in X_0$. We easily obtain $I(u) \leq 0$ by (2.7), now we show that $I(u) = 0$. Indeed, if it is not true, we have $I(u) < 0$, then by Lemma 2.1, there exists a λ^* such that $0 < \lambda^* < 1$ and $I(\lambda^* u) = 0$. Thus, we conclude that

$$\begin{aligned}
d \leq J(\lambda^* u) &= \big(\frac{1}{p} - \frac{1}{q}\big)\|\Delta(\lambda^* u)\|_p^p + \frac{1}{q^2}\|\lambda^* u\|_q^q \\
&\leq (\lambda^*)^p \Big(\big(\frac{1}{p} - \frac{1}{q}\big)\|\Delta u\|_p^p + \frac{1}{q^2}\|u\|_q^q\Big) \\
&\leq (\lambda^*)^p \liminf_{k\to\infty} \Big(\big(\frac{1}{p} - \frac{1}{q}\big)\|\Delta u_k\|_p^p + \frac{1}{q^2}\|u_k\|_q^q\Big) \\
&\leq (\lambda^*)^p \liminf_{k\to\infty} J(u_k) = (\lambda^*)^p d < d.
\end{aligned}$$

This is impossible, so we derive $I(u) = 0$ and $u \in N$. From (2.6) and (1.8), we have $J(u) = d$, and the proof is complete. □

3. Existence of weak solutions

In this section, we state our main results on the problem (1.1). To begin, we give the definition of the weak solution to the problem (1.1).

Definition 3.1. A function $u(t)$ is said to be a solution to problem (1.1) over $[0,T]$ if $u \in L^\infty(0,T;X_0)$ with $u' \in L^2(0,T;L^2(\Omega))$, satisfying the initial condition $u(0) = u_0(x) \in X_0$, and

$$\langle u_t, w\rangle + \langle |\Delta u|^{p-2}\Delta u, \Delta w\rangle = \int_\Omega |u|^{q-2}u \log(|u|)w dx, \tag{3.1}$$

for all $w \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$, and for a.e. $t \in [0,T]$.

Then we are concerned with the existence and uniqueness of local weak solutions to problem (1.1).

Theorem 3.2. *Let $u_0 \in X_0$, $2 < p < q < p(1 + \frac{4}{n})$. Then there exists a $T > 0$ and a unique weak solution $u(t)$ of (1.1) satisfying $u(0) = u_0$. Moreover, u satisfies the energy inequality*

$$\int_0^t \|u'(s)\|_2^2 ds + J(u(t)) \leq J(u_0), \ 0 \leq t \leq T. \tag{3.2}$$

Proof. We shall employ the Galerkin's method. The proof will be divided in 3 steps.

Step 1: Approximate problem. In the space $W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega)$, using a basis $\{\omega_j\}_{j=1}^{\infty}$ we define the finite dimensional space $V_m=\operatorname{span}\{\omega_1,\omega_2,\dots,\omega_m\}$. Let u_{0m} be an element of V_m such that

$$u_{0m}=\sum_{j=1}^{m}a_{mj}(t)\omega_j\to u_0,\quad \text{strongly in } W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega), \tag{3.3}$$

as $m\to\infty$. We find the approximate solution $u_m(x,t)$ of problem (1.1) in the form

$$u_m(x,t)=\sum_{j=1}^{m}\alpha_{mj}(t)\omega_j(x), \tag{3.4}$$

where the coefficients $\alpha_{mj}(1\le j\le m)$ satisfy the system of ordinary differential equations

$$\langle u_m',\omega_i\rangle+\langle|\Delta u_m|^{p-2},\Delta\omega_i\rangle=\int_\Omega|u_m|^{q-2}u_m\log(|u_m|)\omega_i dx, \tag{3.5}$$

for $i\in\{1,2,\dots,m\}$, with the initial conditions

$$\alpha_{mj}(0)=a_{mj},\quad j\in\{1,2,\dots,m\}. \tag{3.6}$$

The standard theory of ordinary differential equations, yields that there exists a positive T_m such that $\alpha_{mj}\in C^1[0,T_m]$, and therefore $u_m\in C^1([0,T_m];W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega))$.

Step 2: A priori estimates. Multiplying (3.5) by $\alpha_{mi}(t)$, summing for $i=1,\dots,m$, and then integrating with respect to time variable on $[0,t]$, we know that

$$S_m(t)=S_m(0)+\int_0^t\int_\Omega|u_m(x,s)|^q\log(|u_m(x,s)|)\,dx\,ds, \tag{3.7}$$

where

$$S_m(t)=\frac{1}{2}\|u_m\|_2^2+\int_0^t\|\Delta u_m(s)\|_p^p ds. \tag{3.8}$$

On the other hand, for any $\mu>0$, similarly we have

$$\int_\Omega|u_m(t)|^q\log(|u_m(t)|)dx\le(e\mu)^{-1}\|u_m(t)\|_{q+\mu}^{q+\mu}, \tag{3.9}$$

where μ is chosen such that $0<\mu<p(1+\frac{4}{n})-q$. Then by the Nirenberg inequality and Young's inequality, we obtain

$$\int_\Omega|u_m(t)|^q\log(|u_m(t)|)dx\le C\|\Delta u_m(s)\|_p^{q+\mu}\|u_m\|_2^{(1-\theta)(q+\mu)} \tag{3.10}$$

$$\le\varepsilon\|\Delta u_m(s)\|_p^p+C(\varepsilon)\|u_m\|_2^{\frac{p(1-\theta)(q+\mu)}{p-\theta(q+\mu)}}, \tag{3.11}$$

where $\varepsilon\in(0,1)$, and

$$\theta=\Big(\frac{1}{2}-\frac{1}{q+\mu}\Big)\Big(\frac{2}{n}-\frac{1}{p}+\frac{1}{2}\Big)^{-1}.$$

Here, we choose $\mu>0$ such that $0<\mu<p(1+\frac{4}{n})-q$ and $\theta(q+\mu)<p$ hold. Let

$$\alpha=\frac{p(1-\theta)(q+\mu)}{2[p-\theta(q+\mu)]}=\frac{p(2q+2\mu+n)-n(q+\mu)}{p(4+n)-n(q+\mu)},$$

then $\alpha > 1$ because $2 < p < q < p(1 + \frac{4}{n})$. Therefore, combining (3.3), (3.7), (3.8) and (3.10), we have

$$S_m(t) \le C_1 + C_2 \int_0^t S_m^\alpha(s) ds, \tag{3.12}$$

where C_1 and C_2 are positive constants independent of m. By the integral inequality of Gronwall-Bellman-Bihari type, there exists a positive constant $T < \frac{C_1^{1-\alpha}}{C_2(\alpha-1)}$ such that

$$S_m(t) \le C_T, \quad \forall t \in [0, T]. \tag{3.13}$$

Consequently, for any m, the solution of (3.5) exists on $[0, T]$.

Next, multiplying (3.5) by $\alpha'_{mi}(t)$, summing for $i = 1, \dots, m$, and then integrating with respect to time variable on $[0, t]$, we derive

$$\int_0^t \|u'_m(s)\|_2^2 ds + J(u_m(t)) = J(u_m(0)) = J(u_{0m}), \quad \forall t \in [0, T]. \tag{3.14}$$

By the continuity of the functional J and (3.3), we deduce that there exists a positive constant C such that

$$J(u_{0m}) \le C, \quad \forall m. \tag{3.15}$$

From this, it follows from (1.6), (3.10), (3.13)-(3.15) and using Hölder's inequality, we obtain

$$\begin{aligned} C \ge J(u_m(t)) &= \frac{1}{p}\|\Delta u_m\|_p^p - \frac{1}{q}\int_\Omega |u_m|^q \log(|u_m|)dx + \frac{1}{q^2}\|u_m\|_q^q \\ &\ge \big(\frac{1}{p} - \frac{\varepsilon}{q}\big)\|\Delta u_m\|_p^p - \frac{C(\varepsilon)}{q}\|u_m\|_2^{2\alpha} + \frac{1}{q^2}\|u_m\|_q^q \\ &\ge \big(\frac{1}{p} - \frac{\varepsilon}{q}\big)\|\Delta u_m\|_p^p - \frac{C(\varepsilon)}{q}2^\alpha S_m^\alpha(t) + \frac{1}{q^2}\|u_m\|_q^q \\ &\ge \big(\frac{1}{p} - \frac{\varepsilon}{q}\big)\|\Delta u_m\|_p^p + \frac{1}{q^2}\|u_m\|_q^q - C_3. \end{aligned} \tag{3.16}$$

Combining this inequality and (3.14), we obtain

$$\|u_m\|_{L^\infty(0,T;W^{2,p}(\Omega))} \le C, \quad \forall m, \tag{3.17}$$

$$\|u'_m\|_{L^2(0,T;L^2(\Omega))} \le C, \quad \forall m. \tag{3.18}$$

It follows from (3.8) and (3.13) that

$$\||\Delta u_m|^{p-2}\Delta u_m\|_{L^\infty(0,T;W_0^{-2,p'}(\Omega))} \le C, \quad \forall m. \tag{3.19}$$

Step 3: Passage to the limit. By the Kakutani and Banach-Alaoglu-Bourbaki Theorem, combining (3.17)-(3.19), there exist functions u and $\mathcal{X}$ and a subsequence of $\{u_m\}_{m=1}^\infty$ which we still denoted by $\{u_m\}_{m=1}^\infty$ such that

$$u_m \rightharpoonup u \quad \text{weakly* in } L^\infty(0, T; W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)), \tag{3.20}$$

$$u'_m \to u' \quad \text{weakly in } L^2(0, T; L^2(\Omega)), \tag{3.21}$$

$$|\Delta u_m|^{p-2}\Delta u_m \to \mathcal{X} \quad \text{weakly* in } L^\infty(0, T; W^{-2,p'}(\Omega)). \tag{3.22}$$

Because of (3.21) and (3.22), it follows from Aubin-Lions-Simon lemma (see [13, Corollary 4]) that

$$u_m \to u, \quad \text{strongly in } C([0, T]; L^2(\Omega)), \tag{3.23}$$

so, $u_m \to u$, a.e. $(x,t) \in \Omega \times (0,T)$. Clearly, this implies that

$$|u_m|^{q-2}u_m \log(|u_m|) \to |u|^{q-2}u \log(|u|), \quad \text{a.e. } (x,t) \in \Omega \times (0,T). \tag{3.24}$$

On the other side, because $2 < p < q < p(1+\frac{4}{n}) < \frac{np}{n-2p}$, we can choose $\mu > 0$ such that $(q-1+\mu)q' < \frac{np}{n-2p}$. Then by a direct calculation and using Sobolev's inequality, we have

$$\begin{aligned} &\int_\Omega |\Phi_m(x,t)|^{q'} dx \\ &= \int_{\{x\in\Omega:|u_m(x,t)|\le 1\}} |\Phi_m(x,t)|^{q'} dx + \int_{\{x\in\Omega:|u_m(x,t)|>1\}} |\Phi_m(x,t)|^{q'} dx \\ &\le (e(q-1))^{-q'}|\Omega| + (e\mu)^{-q'} \int_{\{x\in\Omega:|u_m(x,t)|>1\}} |u_m(t)|^{(q-1+\mu)q'} dx \\ &\le C_1 + C_2\|\Delta u_m(t)\|_p^{(q-1+\mu)q'} \le C, \end{aligned} \tag{3.25}$$

where $\Phi_m(x,t) = |u_m(x,t)|^{q-1}\log(|u_m(x,t)|)$, and we have used the fact that $|x^{q-1}\log x| \le (e(q-1))^{-1}$ for $0 < x < 1$ while $\log x \le (e\mu)^{-1}x^\mu$ for $x > 1, \mu > 0$. Hence, by Lions's lemma (see [13, Lemma 1.3]), it follows from (3.24) and (3.25) that

$$|u_m|^{q-2}u_m \log(|u_m|) \to |u|^{q-2}u \log(|u|), \quad \text{weakly* in } L^\infty(0,T;L^{q'}(\Omega)). \tag{3.26}$$

Passing to the limit in (3.3) and (3.5) as $m \to \infty$, by (3.20)-(3.22) and (3.24), we can show that u satisfies the initial condition $u(0) = u_0$ and

$$\int_\Omega u'(t)\omega dx + \int_\Omega \mathcal{X}(t)\Delta\omega dx = \int_\Omega |u(t)|^{q-2}u(t)\log(|u(t)|)\omega dx, \tag{3.27}$$

for all $\omega \in W_0^{2,p}(\Omega)$ and for almost every $t \in [0,T]$. Finally, by the well known arguments of the theory of monotone operators, we know that

$$\mathcal{X} = |\Delta u|^{p-2}\Delta u,$$

which implies

$$\langle u'(t), \omega\rangle + \langle |\Delta u|^{p-2}\Delta u, \Delta\omega\rangle = \int_\Omega |u(t)|^{q-2}u(t)\log(|u(t)|)\omega dx, \tag{3.28}$$

for all $\omega \in W_0^{2,p}(\Omega)$ and for almost every $t \in [0,T]$.

Step 4: Uniqueness. Firstly, as a result from (3.28), we derive that

$$\langle u'(t), v(t)\rangle + \langle |\Delta u|^{p-2}\Delta u, \Delta v(t)\rangle = \int_\Omega |u(t)|^{q-2}u(t)\log(|u(t)|)v(t) dx, \tag{3.29}$$

for all $v \in L^2(0,T;W_0^{2,p}(\Omega))$.

Now, assume there are two solutions u_1 and u_2 to the problem (1.1) with the same initial condition $u_0 \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$. Let $\omega = u_1 - u_2$, then $\omega(0) = 0$ and

$$\omega \in L^2(0,T;W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)), \quad \omega' \in L^2(0,T;L^2(\Omega)).$$

Let

$$v(s) = \begin{cases} u_1(s) - u_2(s), & s \in [0,t], \\ 0, & s \in [t,T], \end{cases}$$

then, it follows from (3.29) and the monotonicity of the operator $\Delta(|\Delta u|^{p-2}\Delta u)$ that

$$\frac{1}{2}\|\omega(t)\|_2^2 \le \int_0^t \langle F(u_1(s)) - F(u_2(s)), u_1(s) - u_2(s)\rangle ds,$$

where $F(s) = |s|^{q-2}s\log(|s|)$. As a consequence, the uniqueness is derived from the locally Lipschitz continuity of $F : \mathbb{R}^* \to \mathbb{R}$ and Gronwall's inequality.

Step 5: Energy inequality. Now we show that the solution u satisfies the energy inequality (3.2). For this, let $\delta \in C[0,T]$ is a nonnegative function. Then, it follows from (3.14) that

$$\int_0^T \delta(t)\int_0^t \|u_m'(s)\|_2^2 ds dt + \int_0^T J(u_m(t))\delta(t)dt = \int_0^T J(u_m(0))\delta(t)dt. \tag{3.30}$$

The right hand side of (3.30) converges to $\int_0^T J(u_0)\delta(t)dt$ as $m \to \infty$. The second term in the right hand side, $\int_0^T J(u_m(t))\delta(t)dt$, is lower semi-continuous with respect to the weak topology of $L^2(0,T;W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega))$. Hence

$$\int_0^T J(u(t))\delta(t)dt \le \liminf_{m\to+\infty}\int_0^T J(u_m(t))\delta(t)dt. \tag{3.31}$$

Therefore, we obtain

$$\int_0^T \delta(t)\int_0^t \|u'(s)\|_2^2 ds dt + \int_0^T J(u(t))\delta(t)dt \le \int_0^T J(u_0)\delta(t)dt.$$

Since δ is arbitrary nonnegative function, we obtain the energy inequality

$$\int_0^t \|u'(s)\|_2^2 ds + J(u(t)) \le J(u_0), \quad 0 \le t \le T.$$

The proof is complete. □

Next, we state the sufficient conditions for the global existence of weak solutions to the problem (1.1).

Theorem 3.3. *Let $u_0 \in W^+$, there exists a unique global weak solution u of (1.1) satisfying the initial condition $u(0) = u_0$. We have that $u(t) \in W^+$ holds for all $0 \le t < +\infty$, and the energy estimate*

$$\int_0^t \|u'(s)\|_2^2 ds + J(u(t)) = J(u_0), \quad 0 \le t \le +\infty. \tag{3.32}$$

Moreover, the solution decays algebraically provided $u_0 \in W_1^+$.

To prove Theorem 3.3, we need the following lemma.

Lemma 3.4 ([11])**.** *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function and σ is a positive constant such that*

$$\int_0^{+\infty} f^{1+\sigma}(s)ds \le \frac{1}{\omega}f(t), \quad \forall t \ge 0.$$

Then $f(t) \le f(0)(\frac{1+\sigma}{1+\omega\sigma t})^{\frac{1}{\sigma}}$, for all $t \ge 0$.

Proof of Theorem 3.3. To prove the existence of a global solution to (1.1), we first choose a sequence
$$\{\gamma_m\}_{m=1}^{\infty} \subset (0,1)$$
such that $\lim_{m\to\infty}\gamma_m = 1$. Since $I(u_0) \geq 0$, by Lemma 2.1, we have $I(\gamma_m u_0) > 0$ and $J(\gamma_m u_0) < J(u_0) \leq d$. Then, for every m, we can take a sequence of approximation solution $u_{mk} \in C^1([0,T_{mk}]; W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega))$ such that
$$u_{mk}(0) \to \gamma_m u_0, \quad \text{strongly in } W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega), \quad (k\to\infty), \tag{3.33}$$
and
$$\int_0^t \|u'_{mk}(s)\|_2^2 ds + J(u_{mk}(t)) = J(u_{mk}(0)), \quad 0 \leq t \leq T_{mk}, \tag{3.34}$$
where T_{mk} is the maximal existence time of $u_{mk}(t)$.

For each m, by (3.32) and the continuity of I, J, we can choose $k = k_m$ sufficiently large such that $\|u_{m_{k_m}}(0) - \gamma_m u_0\|_{W^{2,p}(\Omega)} < \frac{1}{m}$, $I(u_{m_{k_m}}(0)) > 0$, and $J(u_{m_{k_m}}(0)) < d$. For simplicity, we denote $u_{m_{k_m}}$ by u_m, $u_{m_{k_m}}(0)$ by u_{0m}, and $T_{m_{k_m}}$ by $\bar{T}_m$, respectively. Then, we conclude $u_m \in C^1(0,\bar{T}_m; W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega))$, $u_{0m} \in W_1^+$,
$$u_m(0) = u_{0m} \to u_0, \quad \text{strongly in } W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega), \quad \text{as } m\to\infty, \tag{3.35}$$
and
$$\int_0^t \|u'_m(s)\|_2^2 ds + J(u_m(t)) = J(u_{0m}),\ 0 \leq t \leq \bar{T}_m. \tag{3.36}$$
Therefore, it follows from (2.1) that
$$\big(\frac{1}{p} - \frac{1}{q}\big)\|\Delta u_m(t)\|_p^p + \frac{1}{q^2}\|u_m(t)\|_q^q < d, \quad \forall m. \tag{3.37}$$
Combining (3.35) and (3.37), and by using Hölder's inequality, we obtain
$$\begin{aligned} &\int_0^t \|u'_m(s)\|_2^2 ds + \|\Delta u_m(t)\|_p^p + \|u_m(t)\|_p^p \\ &\leq \int_0^t \|u'_m(s)\|_2^2 ds + \|\Delta u_m(t)\|_p^p + C\|u_m(t)\|_q^p \leq C. \end{aligned} \tag{3.38}$$
This implies that $\bar{T}_m = +\infty$. Then we can conclude that there is a unique global weak solution $u(t) \in W^+$ of the problem (1.1) as in the prove of Theorem 3.2, which satisfies the energy inequality
$$\int_0^t \|u'(s)\|_2^2 ds + J(u(t)) \leq J(u_0), \quad 0 \leq t < +\infty. \tag{3.39}$$

Secondly, we show that the algebraic decay results. Since $u_0 \in W^+$, i.e. $I(u_0) > 0$ and $J(u_0) < d$, we have $u(t) \in W^+$ for each t by a standard contradiction argument. It follows from (2.1) and (3.39) that
$$\big(\frac{1}{p} - \frac{1}{q}\big)\|\Delta u(t)\|_p^p + \frac{1}{q^2}\|u(t)\|_q^q \leq J(u(t)) \leq J(u_0). \tag{3.40}$$
Since $I(u_0) > 0$, there exists a $\lambda_* > 1$ such that $I(\lambda_* u(t)) = 0$. This implies that
$$\begin{aligned} d &\leq J(\lambda_* u(t)) = \big(\frac{1}{p} - \frac{1}{q}\big)\|\Delta(\lambda_* u(t))\|_p^p + \frac{1}{q^2}\|\lambda_* u(t)\|_q^q \\ &\leq \lambda_*^q\big\{\big(\frac{1}{p} - \frac{1}{q}\big)\|\Delta u(t)\|_p^p + \frac{1}{q^2}\|u(t)\|_q^q\big\}. \end{aligned} \tag{3.41}$$

It follows from (3.40) and (3.41) that

$$\lambda_* \geq \Big(\frac{d}{J(u_0)}\Big)^{1/q}. \tag{3.42}$$

On the one hand, we obtain

$$\begin{aligned} 0 = I(\lambda_* u(t)) &= \lambda_*^p \|\Delta u(t)\|_p^p - \lambda_*^q \int_\Omega |u(t)|^q \log(|u(t)|)dx - \lambda_*^q \log \lambda_* \|u(t)\|_q^q \\ &= \lambda_*^q I(u) - (\lambda_*^q - \lambda_*^p)\|\Delta u(t)\|_p^p - \lambda_*^q \log \lambda_* \|u(t)\|_q^q. \end{aligned}$$

From this inequality and (3.42), we deduce that

$$I(u) \geq \Big\{1 - \Big(\frac{d}{J(u_0)}\Big)^{\frac{p}{q}-1}\Big\}\|\Delta u(t)\|_p^p \geq C\|u(t)\|_{2,p}^p. \tag{3.43}$$

On the other hand, by the compact embedding $W^{2,p}(\Omega) \hookrightarrow L^2(\Omega)$, we see that

$$\begin{aligned} \int_t^T I(u(s))ds &= -\int_t^T \langle u'(s), u(s)\rangle ds \\ &= \frac{1}{2}\|u(t)\|_2^2 - \frac{1}{2}\|u(T)\|_2^2 \\ &\leq C\|u(t)\|_{2,p}^2. \end{aligned} \tag{3.44}$$

By (3.43) and (3.44), we obtain

$$\int_t^T I(u(s))ds \leq \frac{1}{\omega}\Big(I(u(t))\Big)^{2/p} \leq \frac{1}{\omega}\|\Delta u(t)\|_p^2 \leq \frac{1}{\omega}\|u(t)\|_{2,p}^2, \tag{3.45}$$

for all $t \in [0,T]$, and where ω is a positive constant.

Let $T \to +\infty$ in (3.45), it follows that

$$\int_t^{+\infty} \|u(s)\|_{2,p}^p ds \leq C\int_t^{+\infty} I(u(s))ds \leq \frac{1}{\omega}\|u(t)\|_{2,p}^2. \tag{3.46}$$

Since $p > 2$, we can choose $f(t) = \|u(t)\|_{2,p}^2$ and $\sigma = \frac{p}{2} - 1$ in Lemma 3.4 to obtain

$$\|u(t)\|_{2,p}^2 \leq \|u_0\|_{2,p}^2 \Big(\frac{1+\sigma}{1+\omega\sigma t}\Big)^{\frac{1}{p-2}}, \quad \forall t \geq 0.$$

The prove is complete. □

4. Blow-up and extinction of solutions

Firstly, we state the theorem for finite time blow-up for weak solution of problem (1.1) in when $2 < p < q < p(1 + \frac{4}{n})$.

Theorem 4.1. *Let $u_0 \in W_1^-$, and u is the unique weak solution to* (1.1)*. Then u blows up in the finite time.*

Proof. Since $u_0 \in W_1^-$, by Theorem 3.2, we obtain a unique local solution of (1.1) satisfying the energy inequality

$$\int_0^t \|u'(s)\|_2^2 ds + J(u(t)) \leq J(u_0), \quad 0 \leq t \leq T_{\max}, \tag{4.1}$$

where $T_{\max}$ is the maximal existence time of $u(t)$.

Next, by a contraction argument, we conclude that $u(t) \in W_1^-$ for $t \in [0, T_{\max}]$. We assume that $u(t)$ leaves W_1^- at time $t = t_0$, then there exists a sequence $\{t_n\}$

such that $t_n \to t_0$ as $n \to \infty$ and $I(u(t_n)) \le 0$. By the lower semicontinuity of $\|\cdot\|_{2,p}$, we obtain

$$I(u(t_0)) \le \liminf_{n\to\infty} I(u(t_n)) \le 0.$$

Because $u(t_0)$ leaves W_1^-, we have $I(u(t_0)) = 0$. Thus, by the variational definition of d and the energy inequality, this leads to a contraction

$$d = \inf_{u\in N} \le J(u(t_0)) < d.$$

Hence, we derive $u(t) \in W_1^-$ for $t \in [0, T_{\max}]$.

At the last, we show that the solution $u(t)$ is not global, that means, it blows up at finite time. Assume by contraction that the solution $u(t)$ is global. Then, for any $T > 0$, we consider $\Gamma : [0, T] \to \mathbb{R}^+$ defined by

$$\Gamma(t) = \int_0^t \|u(s)\|_2^2 ds. \tag{4.2}$$

Then, by direct calculations, we have

$$\Gamma'(t) - \Gamma'(0) = \|u(t)\|_2^2 - \|u_0\|_2^2 = 2\int_0^t \langle u'(s), u(s)\rangle ds, \tag{4.3}$$

$$\Gamma''(t) = 2\langle u', u\rangle = -2I(u). \tag{4.4}$$

Combining (2.1) and (4.1), we obtain

$$\begin{aligned} \Gamma''(t) =& -2I(u) = -2qJ(u) + \frac{2}{q}\|u\|_q^q + \Big(\frac{2q}{p} - 2\Big)\|\Delta u\|_p^p \\ &\ge -2qJ(u_0) + 2q\int_0^t \|u'(s)\|_2^2 ds + \frac{2}{q}\|u\|_q^q + \big(\frac{2q}{p} - 2\big)\|\Delta u\|_p^p. \end{aligned} \tag{4.5}$$

Since $u(t) \in W_1^-$ for $t \in [0, T_{\max}]$, so $I(u) < 0$, then there exist a $\lambda^* \in (0,1)$ such that $I(\lambda^* u) = 0$. Thus, by the definition of d, we have

$$\big(\frac{1}{p} - \frac{1}{q}\big)\|\Delta u\|_p^p + \frac{1}{q^2}\|u\|_q^q \ge J(\lambda^* u) \ge d. \tag{4.6}$$

It follows from (4.5) and (4.6) that

$$\Gamma''(t) \ge 2q\int_0^t \|u'(s)\|_2^2 ds + 2q(d - J(u_0)). \tag{4.7}$$

By (4.4) and $I(u) < 0$, we know $\Gamma''(t) > 0$, so we obtain

$$\Gamma'(t) > \Gamma'(0) = \|u_0\|_2^2 > 0, \quad \forall t > 0. \tag{4.8}$$

From (4.3) and Hölder's inequality, we obtain

$$\frac{1}{4}\left(\Gamma'(t) - \Gamma'(0)\right)^2 \le \Big(\int_0^t \langle u'(s), u(s)\rangle ds\Big)^2 \le \int_0^t \|u'(s)\|_2^2 ds \int_0^t \|u(s)\|_2^2 ds. \tag{4.9}$$

Combining (4.2), (4.7) and (4.9), we have

$$\begin{aligned} \Gamma(t)\Gamma''(t) &\ge \int_0^t \|u(s)\|_2^2 ds\Big(2q\int_0^t \|u'(s)\|_2^2 ds + 2q(d - J(u_0))\Big) \\ &\ge \frac{q}{2}\left(\Gamma'(t) - \Gamma'(0)\right)^2 + 2q(d - J(u_0))\Gamma(t). \end{aligned}$$

Now, fix $t_0 > 0$. The (4.8) implies

$$\Gamma(t) \geq \Gamma(t_0) = \int_0^{t_0} \|u(s)\|_2^2 ds \geq \|u_0\|_2^2 t_0 > 0, \quad \forall t \geq t_0. \tag{4.10}$$

Hence,

$$\Gamma(t)\Gamma''(t) - \frac{q}{2}\left(\Gamma'(t) - \Gamma'(0)\right)^2 \geq 2q(d - J(u_0))\|u_0\|_2^2 t_0 > 0, \quad \forall t \geq t_0. \tag{4.11}$$

We choose $T > t_0$ sufficiently large, and let

$$G(t) = \Gamma(t) + (T - t)\|u_0\|_2^2, \ \forall\ t \in [0, T].$$

Then $G(t) > \Gamma(t) > 0$, $G'(t) = \Gamma'(t) - \|u_0\|_2^2 = \Gamma'(t) - \Gamma'(0) > 0$ and $G''(t) = \Gamma''(t) > 0$. Thus, (4.11) implies

$$G(t)G''(t) - \frac{q}{2}(G'(t))^2 \geq 2q(d - J(u_0))\|u_0\|_2^2 t_0 > 0, \quad \forall t \geq t_0. \tag{4.12}$$

By setting $y(t) = (G(t))^{-(q-2)/2}$, inequality (4.12) becomes

$$y''(t) \leq -q(q-2)(d - J(u_0))\|u_0\|_2^2 t_0 (G(t))^{-\frac{q+2}{2}} < 0, \quad \forall t \in [t_0, T].$$

This inequality implies that y is a concave function in $[t_0, T]$, for each $T > t_0$. Because of $y(t_0) > 0$ and $y'(t) = -\frac{q-2}{2}(G(t))^{-\frac{q}{2}} G'(t) < 0$, for all t, there exists a finite time T_* such that $\lim_{t \to T_*^-} y(t) = 0$ if we choose T sufficiently large. Consequently, $\lim_{t \to T_*^-} G(t) = +\infty$. This implies that $\lim_{t \to T_*^-} \int_0^t \|u(s)\|_2^2 ds = +\infty$. Hence, we see that

$$\lim_{t \to T_*^-} \|u(t)\|_2^2 = +\infty$$

which contradicts the assumption of $u(t)$ being global. The proof is complete. $\square$

Next, we discuss the finite time blow-up, extinction and non-extinction of the weak solution to the problem (1.1) in the case of $\max\{1, \frac{2n}{n+4}\} < p \leq 2$, and $q > 0$.

Before showing these results, we claim that the local existence of the weak solution to the problem (1.1) can be obtained by using Galerkin approximation method. Let $u(x, t)$ be the weak solution to the problem (1.1). We introduce some functionals and notations as follows:

$$E(t) = \frac{1}{p}\|\Delta u\|_p^p - \frac{1}{q}\int_\Omega |u|^q \log(|u|)dx + \frac{1}{q^2}\|u\|_q^q, \tag{4.13}$$

$$M(t) = \frac{1}{2}\int_\Omega u^2 dx, \ H(t) = \int_0^t M(s)ds. \tag{4.14}$$

Since the embedding $W^{2,p}(\Omega) \hookrightarrow L^2(\Omega)$ holds if $\max\{1, \frac{2n}{n+4}\} < p \leq 2$, there exists an optimal embedding constant B such that

$$\|u\|_2 \leq B\|\Delta u\|_p. \tag{4.15}$$

Furthermore, it is not difficult to obtain the inequality

$$\int_0^t \int_\Omega |u_s|^2 \,dx\,ds + E(t) \leq E(0). \tag{4.16}$$

Then $E(t)$ is nonincreasing with respect t. Now, we show some lemmas, which will be used later.

Lemma 4.2. *Assume that $p < q$ and $E(0) \le 0$. Then*

$$M'(t) \ge q \int_0^t \int_\Omega |u_s|^2 \, dx \, ds. \tag{4.17}$$

Proof. Through direct calculations, we have

$$\begin{aligned} M'(t) = \int_\Omega u u_t dx &= \int_\Omega u \left(-\Delta(|\Delta u|^{p-2} \Delta u) + |u|^{q-2} u \log(|u|) \right) dx \\ &= - \int_\Omega |\Delta u|^p dx + \int_\Omega |u|^q \log(|u|) dx \\ &= -qE(t) + \big(\frac{q}{p} - 1\big) \int_\Omega |\Delta u|^p dx + \frac{1}{q} \int_\Omega |u|^q dx \\ &\ge -qE(t). \end{aligned} \tag{4.18}$$

Then, by the assumption $E(0) \le 0$ and (4.16), we obtain

$$M'(t) \ge -qE(0) + q \int_0^t \int_\Omega |u(s)|^2 \, dx \, ds \ge \int_0^t \int_\Omega |u_s|^2 \, dx \, ds,$$

the proof is complete. □

Lemma 4.3. *Assume that $q > 2$ and $E(0) \le 0$, then*

$$q \left(H'(t) - H'(0) \right)^2 \le 2H(t) H''(t). \tag{4.19}$$

Proof. By Hölder's inequality and (4.17), we obtain

$$\begin{aligned} H'(t) - H'(0) &= M(t) - M(0) \\ &= \int_0^t M'(s) ds = \int_0^t \int_\Omega u u_s \, dx \, ds \\ &\le \Big(\int_0^t \int_\Omega |u|^2 \, dx \, ds \Big)^{1/2} \Big(\int_0^t \int_\Omega |u_s|^2 \, dx \, ds \Big)^{1/2} \\ &\le \big(\frac{2}{q}\big)^{1/2} (H(t))^{1/2} (M'(t))^{1/2} \\ &= \big(\frac{2}{q}\big)^{1/2} (H(t))^{1/2} (H''(t))^{1/2}. \end{aligned}$$

Moreover, by (4.17) again, we have

$$H'(t) - H'(0) = \int_0^t M'(s) ds \ge q \int_0^t \int_0^s \int_\Omega |u_\tau|^2 dx d\tau ds \ge 0.$$

Then the conclusion follows from the two inequalities above. The proof is complete. □

Lemma 4.4 ([5, Lemma 1.2])**.** *Suppose that $\theta > 0$, $\alpha > 0$, $\beta > 0$ and $h(t)$ is a nonnegative and absolutely continuous function satisfying $h'(t) + \alpha h^\theta(t) \ge \beta$, then for $0 < t < \infty$, it holds*

$$h(t) \ge \min \big\{ h(0), \big(\frac{\alpha}{\beta}\big)^{1/\theta} \big\}.$$

Lemma 4.5 ([6, Lemma 3.2]). *Assume* $0 < l < r \leq 1$, $\alpha \geq 0$, $\beta \geq 0$ *and* $\varphi(t)$ *is a nonnegative and absolutely continuous function, which satisfies*

$$\begin{gathered}\varphi'(t) + \alpha\varphi^l(t) \leq \beta\varphi^r(t), \quad t \geq 0, \\ \varphi(0) > 0, \quad \beta\varphi^{r-l}(0) < \alpha\,. \end{gathered} \tag{4.20}$$

Then

$$\begin{gathered}\varphi(t) \leq [-\alpha_0(1-l)t + \varphi^{1-l}(0)]^{\frac{1}{1-l}}, \quad 0 < t < T_0, \\ \varphi(t) \equiv 0, \quad t \geq T_0, \end{gathered}$$

where $\alpha_0 = \alpha - \beta\varphi^{r-l}(0) > 0$ *and* $T_0 = \alpha_0^{-1}(1-l)^{-1}\varphi^{1-l}(0)$.

Theorem 4.6. *Assume that* $p < q$, $q > 2$, $E(0) \leq 0$ *and* $\|u_0\|_2 > 0$. *Then the solution to problem* (1.1) *blows up in the finite time.*

Proof. We will give the proof by contradiction. Suppose that the solution $u(x,t)$ to the problem (1.1) exists for all $t > 0$. Then by the definition of weak solution, we know that $u \in C([0,+\infty); L^2(\Omega))$. For any $t_0 > 0$, we claim that

$$\int_0^{t_0} \int_\Omega |u_s|^2 \, dx \, ds > 0. \tag{4.21}$$

Otherwise, there exists a $\hat{t}_0 > 0$ such that $\int_0^{\hat{t}_0} \int_\Omega |u_s|^2 \, dx \, ds = 0$, and hence $u_t(x,t) = 0$ for a.e. $(x,t) \in \Omega \times (0, \hat{t}_0)$. Thus it follows from (4.18) that $\int_\Omega |\Delta u|^p dx = \int_\Omega |u|^q \log(|u|) dx$ for a.e. $t \in (0, \hat{t}_0)$, and then we obtain from (4.16) that

$$E(t) = \frac{q-p}{pq} \int_\Omega |\Delta u|^p dx + \frac{1}{q^2} \int_\Omega |u|^q dx$$

for a.e. $t \in (0, \hat{t}_0)$, which combines $E(t) \leq E(0) \leq 0$ and $p < q$ implying $\int_\Omega |\Delta u|^p dx = 0$ and $\int_\Omega |u|^q dx = 0$ for a.e. $t \in (0, \hat{t}_0)$. By (4.15), we have $\|u(\cdot,t)\|_2 = 0$ for a.e. $t \in (0, \hat{t}_0)$. Furthermore, since $u \in C([0,+\infty); L^2(\Omega))$, we obtain $\|u(\cdot,t)\|_2 = 0$ for all $t \in [0, \hat{t}_0]$, especially $\|u_0\|_2 = 0$, which contradicts to the assumption $\|u_0\|_2 > 0$. Then (4.21) holds.

Now, fix $t_0 > 0$, and let $\rho = \int_0^{t_0} \int_\Omega |u_s|^2 \, dx \, ds$. By (4.21) we know that ρ is a positive constant. Integrating (4.17) over (t_0, t), we obtain

$$\begin{aligned} M(t) &\geq M(t_0) + q \int_{t_0}^t \int_0^s \int_\Omega |u_\tau|^2 dx d\tau ds \\ &\geq \int_{t_0}^t \int_0^{t_0} \int_\Omega |u_\tau|^2 dx d\tau ds \geq \rho(t - t_0). \end{aligned} \tag{4.22}$$

Hence,

$$\lim_{t \to +\infty} H'(t) = \lim_{t \to +\infty} M(t) = +\infty. \tag{4.23}$$

Combining (4.23) and the fact that $q > 2$, we have

$$\lim_{t \to +\infty} \frac{(H'(t))^2}{[H'(t) - H'(0)]^2} = 1 < \frac{4q}{3q+2}.$$

Therefore, there exists $t^* > t_0$ such that

$$\frac{3q+2}{4}(H'(t))^2 < q[H'(t) - H'(0)]^2 \quad \forall t \geq t^*.$$

Consequently, by (4.19), we obtain

$$\frac{3q+2}{4}(H'(t))^2 < 2H(t)H''(t) \quad \forall t \geq t^*.$$

Then we consider the function $z(t) = (H(t))^{-\frac{q-2}{4}}$. By direct computations, we obtain

$$\begin{aligned} z'(t) &= -\frac{q-2}{4}(H(t))^{-\frac{q-2}{4}-1}H'(t) \leq 0, \\ z''(t) &= \frac{q-2}{4}(H(t))^{\frac{-q-6}{4}}\Big(\frac{q+2}{4}(H'(t))^2 - H(t)H''(t)\Big) \\ &\leq -\frac{(q-2)^2}{32}(H(t))^{\frac{-q-6}{4}}(H'(t))^2 \leq 0, \end{aligned}$$

for all $t \geq t^*$, which imply that $z(t)$ is a decreasing concave function. Moreover, since $z(t) > 0$, we obtain $z(t)$ cannot converge to 0 as $t \to +\infty$. However, since $\lim_{t\to+\infty} H(t) = +\infty$, we obtain from the definition of $z(t)$ and that $z(t)$ is convergent to 0 as $t \to +\infty$, which is a contraction. The proof is complete. □

Theorem 4.7. *Assume that $p > q$ and $E(0) < 0$. Then the solution to problem* (1.1) *does not go extinct in finite time.*

Proof. Recall the $M(t)$ defined in (4.14). According to (4.14) and (4.16), we obtain

$$\begin{aligned} &M'(t) \\ &= -\int_\Omega |\Delta u|^p dx + \int_\Omega |u|^q \log(|u|)dx \\ &= \int_\Omega |u|^q \log(|u|)dx - pE(t) - \frac{p}{q}\int_\Omega |u|^q \log(|u|)dx + \frac{1}{q^2}\int_\Omega |u|^q dx \\ &= \frac{q-p}{q}\int_\Omega |u|^q \log(|u|)dx - pE(0) + \frac{1}{q^2}\int_\Omega |u|^q dx + p\int_0^t\int_\Omega |u_s|^2\,dx\,ds \\ &\geq \frac{q-p}{q}\int_\Omega |u|^q \log(|u|)dx - pE(0). \end{aligned} \tag{4.24}$$

Case 1: $p > q$**.** By $p \leq 2$, we obtain $q < 2$. Then there exists $\mu > 0$ such that $q + \mu < 2$, hence

$$\begin{aligned} \frac{q-p}{q}\int_\Omega |u|^q \log(|u|)dx &\geq \frac{q-p}{e\mu q}\int_\Omega |u|^{q+\mu}dx \\ &\geq \frac{q-p}{e\mu q}\Big(\int_\Omega |u|^2 dx\Big)^{\frac{q+\mu}{2}}|\Omega|^{\frac{2-q-\mu}{2}} \\ &= AM^{\frac{q+\mu}{2}}(t), \end{aligned} \tag{4.25}$$

where $A = \frac{q-p}{e\mu q}2^{\frac{q+\mu}{2}}|\Omega|^{\frac{2-q-\mu}{2}} > 0$. So, by (4.24) and (4.25), we see that

$$M'(t) \geq -AM^{\frac{q+\mu}{2}} - pE(0).$$

By Lemma 4.4 and $E(0) < 0$, we have

$$M(t) \geq \min\Big\{M(0), \Big(\frac{-pE(0)}{A}\Big)^{\frac{2}{q+\mu}}\Big\}, \quad t > 0.$$

Since $M(0) = \frac{1}{2}\|u_0\|_2^2 > 0$, $A > 0$ and $E(0) < 0$, we derive $M(t) > 0$ for all $t > 0$.

Case 2: $p = q$. Since $E(0) < 0$, from (4.24) we obtain $M'(t) \geq -pE(0) > 0$. Hence, we have

$$M(t) \geq M(0) - pE(0) > 0, \quad t > 0.$$

Again as in Case 1, we obtain $M(t) > 0$ for all $t > 0$.

The two cases above imply $\|u(\cdot,t)\|_2 = \sqrt{2M(t)} > 0$ for all $t > 0$. Then for any $s > 1$, by the interpolation inequality, we have

$$\|u\|_2 \leq \|u\|_s^{1/2}\|u\|_{s'}^{1/2},$$

where $s' = s/(s-1) > 1$. Combining the above inequality with $\|u(\cdot,t)\|_2 > 0$, we know that $\forall s > 1$, there does not exist $T^* > 0$ such that $\lim_{t\to T^*}\|u\|_s = 0$. The proof is complete. □

Theorem 4.8. *Assume that $p < q$, $q < 2$ and $0 < \|u_0\|_2^{q+\mu-p} < B^{-p}|\Omega|^{\frac{q+\mu-2}{2}}$. Then the solution to problem (1.1) must become extinct in finite time. Furthermore, we have the following estimates:*

$$\|u(t)\|_2 \leq \Big[\|u_0\|_2^{2-p} - (2-p)\Big(B^{-p} - \frac{1}{e\mu}|\Omega|^{\frac{2-q-\mu}{2}}\|u_0\|_2^{q+\mu-p}\Big)t\Big]^{\frac{1}{2-p}}, \quad 0 < t < T_*,$$
$$\|u(t)\|_2 = 0, \quad t \geq T_*,$$

where

$$T_* = \Big[(2-p)\Big(B^{-p} - \frac{1}{e\mu}|\Omega|^{\frac{2-q-\mu}{2}}\|u_0\|_2^{q+\mu-p}\Big)\Big]^{-1}\|u_0\|_2^{2-p},$$

and $\mu > 0$ is sufficiently small such that $q + \mu < 2$.

Proof. Multiplying the first equation of (1.1) by u and integrating over Ω, we have

$$\frac{1}{2}\int_\Omega u^2 dx + \int_\Omega |\Delta u|^p dx = \int_\Omega |u|^q \log(|u|)dx.$$

Recall the $M(t)$ defined in (4.14), then the above equation is equivalent to the inequality

$$M'(t) + \int_\Omega |\Delta u|^p dx \leq \int_\Omega |u|^q \log(|u|)dx. \tag{4.26}$$

Then (4.15), (4.26) and Hölder's inequality imply

$$M'(t) + 2^{p/2}B^{-p}M^{p/2}(t) \leq \frac{1}{e\mu}2^{\frac{q+\mu}{2}}|\Omega|^{\frac{2-q-\mu}{2}}M^{\frac{q+\mu}{2}}(t);$$

that is,

$$M'(t) + \alpha M^{p/2}(t) \leq \beta M^{\frac{q+\mu}{2}}(t),$$

where $\alpha = 2^{p/2}B^{-p} > 0$, $\beta = \frac{1}{e\mu}2^{\frac{q+\mu}{2}}|\Omega|^{\frac{2-q-\mu}{2}} > 0$, and $0 < \frac{p}{2} < \frac{q+\mu}{2} \leq 1$. By Lemma 4.5 and the assumption $0 < \|u_0\|_2^{q+\mu-p} < B^{-p}|\Omega|^{\frac{q+\mu-2}{2}}$, we obtain

$$M(t) \leq [-\alpha_0(1-\frac{p}{2})t + M^{1-\frac{p}{2}}(0)]^{\frac{2}{2-p}}, \quad 0 < t < T_*,$$
$$M(t) \equiv 0, \quad t \geq T_*,$$

where $\alpha_0 = \alpha - \beta M^{\frac{q+\mu-p}{2}}(0) > 0$ and $T_* = \alpha_0^{-1}\frac{2}{2-p}M^{\frac{2-p}{2}}(0)$. Then the conclusion follows by $\|u(\cdot,t)\|_2 = \sqrt{2M(t)}$. The proof is complete. □

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