Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 101, pp. 1–35.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

GEOMETRY OF THE TRIPLE JUNCTION BETWEEN THREE FLUIDS IN EQUILIBRIUM

IVAN BLANK, ALAN ELCRAT, RAYMOND TREINEN

Abstract. We present an approach to the problem of the blow up at the triple junction of three fluids in equilibrium. Although many of our results can already be found in the literature, our approach is almost self-contained and uses the theory of sets of finite perimeter without making use of more advanced topics within geometric measure theory. Specifically, using only the calculus of variations we prove two monotonicity formulas at the triple junction for the three-fluid configuration, and show that blow up limits exist and are always cones. We discuss some of the geometric consequences of our results.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary smooth enough that the interior sphere condition holds. Then consider a partition of Ω into three sets E_j, $j = 0, 1, 2$. Each E_j will represent a fluid, and we assume that the three fluids are immiscible and are in equilibrium with respect to the energy functional

$$\mathcal{F}_{SWP}(\{E_j\}) := \sum_{j=0}^{2} \Big(\alpha_j \int_{\Omega} |D\chi_{E_j}| + \beta_j \int_{\partial\Omega} \chi_{E_j} \, d\mathcal{H}^{n-1} + \rho_j g \int_{E_j} z \, dV \Big) \tag{1.1}$$

where g is determined by the force of gravity, and where the constants α_j, β_j, and ρ_j are determined by constitutive properties of our fluids. It will make the most sense to consider sets with finite perimeter, as this functional is infinite otherwise, and accordingly, we will work within the framework afforded to us by functions of bounded variation. We will define this functional more carefully and state some assumptions that we will make on the constitutive constants in Section 3 below. Two common physical situations where this mathematical model arise include first, if there is a double sessile drop of two distinct immiscible fluids resting on a surface with air above, and second, if a drop of a light fluid is floating on the top of a heavier fluid and below a lighter fluid as would be the case when oil floats on water and below air. See Figure 1 for an example of the first situation, and Figure 2 (found within Section 3) for an example of the second situation. The terms in the energy functional given above arise from (in the order in which they appear) surface tension forces, wetting energy, and the gravitational potential.

2010 *Mathematics Subject Classification.* 76B45, 35R35, 35B65.
Key words and phrases. Floating drops; capillarity; regularity; blow up.

Submitted February 14, 2019. Published August 27, 2019.

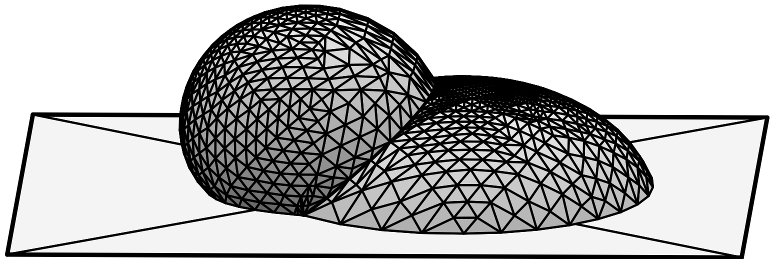


FIGURE 1. A double sessile drop.

In this work we will study the local micro-structure of the triple junction between the fluids. We prove two monotonicity formulas, one with a volume constraint, and one without the volume constraint, but which is sharp in some sense. Both of these formulas can be compared to the classical Allard monotonicity formula [1], and although the formulas we give are obviously not as broad in applicability, they are proven using only tools that are basic within the calculus of variations and the theory of sets of finite perimeter. We use the monotonicity formula to show that blow up limits of the energy minimizing configurations must be cones, and thus that they are determined completely by their values on the "blow up sphere." We then study the implications of minimizing on the blow up sphere for the minimizers in the tangent plane to the blow up sphere given that the point of tangency is at a triple point. The consequences are geometric restrictions on the energy minimizing configurations in the blow up sphere. Our results can be summarized in the following theorem.

Theorem 1.1. *Assuming that the triple* $\{E_j\}$ *minimizes the functional* (1.1) *and assuming that* $x_0 \in \partial E_0 \cap \partial E_1 \cap \partial E_2$*, there exists a blowup limit where the* ∂E_j *will converge to half-planes containing* x_0*, and the angles between the half-planes along any blowup limit satisfy the Neumann Angle Condition:*

$$\frac{\sin \gamma_{01}}{\sigma_{01}} = \frac{\sin \gamma_{02}}{\sigma_{02}} = \frac{\sin \gamma_{12}}{\sigma_{12}}. \tag{1.2}$$

Here γ_{ij} *is the angle at the triple point measured within* E_k *(where* $\{i, j, k\} = \{0, 1, 2\}$*), and* σ_{ij} *is the surface tension at the interface of* E_i *and* E_j*.*

This theorem can also be mostly constructed from the work of Morgan and his students and co-authors who use advanced topics within the field of geometric measure theory, and we will give a more thorough comparison in a paragraph below after we first turn to some of the historical background of this problem.

The study of the floating drop problem goes back at least to 1806 when Laplace [12] formulated the problem with the assumption of symmetry, and of course, the regularity of the interfaces between the fluids and also the regularity of the triple junction curve. In 2004 Elcrat, Neel, and Siegel [6] showed the existence (and, under some assumptions, uniqueness) of solutions for Laplace's formulation, and they still assumed the same conditions of symmetry and regularity. In the time between these results there were obviously great advancements in the regularity theory involving both the space of functions of bounded variation and geometric measure theory. It is with these tools that we will work, and so a quick survey may be of use for the reader, and so we will provide a very short one in Section 2 below.

The study of soap film clusters began in earnest in the 1970's, and this problem has many connections with the current work, so a comparison is in order. In the soap film problem a region of space is partitioned by sets, and the soap film is modeled by the boundaries of the sets, and the surface areas of these surfaces are minimized under some volume constraint. The energy is similar to ours, although it is simpler in some ways. In particular, there are no weights to the surface tensions (one can set those to unity), there is no gravitational potential, and there is no wetting term. The wetting term is the easiest by far to address, and even the gravitational potential can be dealt with by observing how the surface tension term becomes much more important in blow up limits, but the fact that in our energy the surface tension terms vary with each fluid creates considerable new difficulties. Jean Taylor [25] classified the structure of the singularities of soap film clusters, and among other results was able to show that at triple junction points the surfaces meet at $120°$. Frank Morgan and collaborators worked on various other aspects of soap bubble clusters, including showing that the standard double bubble is the unique energy minimizer in a collaboration with Hutchings, Ritoré, and Ros [11]. (See also his book *Geometric Measure Theory* [24] and many references therein.)

It is with this approach that Morgan, White, and others study the problem of three immiscible fluids. Lawlor and Morgan worked on paired calibrations with immiscible fluids [13], White used Fleming's flat chains in order to show the existence of least-energy configurations [27], and then Morgan was able to show regularity in $\mathbb{R}^2$ and for some cases in $\mathbb{R}^3$ [22] and he used Allard's monotonicity formula for varifolds in order to obtain blowup limits. More recently Morgan returned to the problem in $\mathbb{R}^2$ and showed under some conditions that a planar minimizer with finite boundary and with prescribed areas consists of finitely many constant-curvature arcs [23]. Although the work just described would yield most of the conclusions of our main theorem, it is difficult to follow or inaccessible to all but experts within the field of geometric measure theory.

Our approach is mostly limited to the formulation using functions of bounded variation. The framework we use is based on the work of Giusti [10], where he studies the regularity of minimal surfaces, but it is in a paper by Massari [18] that our problem is first formulated. Massari showed the existence of energy minimizers, and commented that Giusti's theory would apply in any region away from a junction of multiple fluids. Massari and Tamanini studied a related problem involving optimal segmentations using an approach similar to ours and obtained a different but analogous monotonicity formula [20]. Leonardi [15] proved a very useful elimination theorem about solutions to this problem which roughly states that if the volume of some fluids is small enough in a ball, then those fluids must not appear in a ball of half the radius. Two other references that may be helpful are by Massari and Miranda [19] and Leonardi [14]. Lastly, Maggi [17] recently published a book that treats some aspects of this problem, including a different proof of Leonardi's Elimination Theorem.

Finally, we give an outline of our paper. In Section 2 we collect results on the space of functions of bounded variation, distilling facts we need from much longer works on the subject. In Section 3 we carefully define our problem and some closely related problems, and we discuss some results by Almgren, Leonardi, and Massari that will be crucial to our work. In Section 4 we show that in the blow up limit it suffices to consider the energy functional that ignores any wetting energy and any gravitational potential. In Section 5 we prove a monotonicity formula centered about a triple point for the case with volume constraints. In Section 6 we drop the volume constraints and we are able to achieve a sharper monotonicity formula. At the end of Section 6 we give a comparison between our monotonicity formulas and some of the monotonicity formulas that have already appeared. In Section 7 we use our first monotonicity formula to show that any blow up limit must be a configuration consisting of cones. Section 8 connects these cones to the blow up sphere. We then consider the tangent plane to a triple point on the blow up sphere, and we are able to show that energy minimizers in the tangent plane must also be cones. Finally, in Section 9 we show that those fluids in the tangent plane must be connected and satisfy the same angle condition as was derived in [6], but we use different methods from them.

2. Background on bounded variation

In the process of studying the two fluid problem, we discovered that some theorems that we needed were either scattered in different sources, or embedded within the proof of an existing theorem, but not stated explicitly. For these reasons we have gathered together the theorems that we need here. Our main sources here were [4], [7], and [10].

We assume that $\Omega \subset \mathbb{R}^n$ is an open set with a differentiable boundary. We define $BV(\Omega)$ to be the subset of $L^1(\Omega)$ with bounded variation, measured by

$$\int_\Omega |Df| = \sup\Big\{ \int_\Omega f \operatorname{div}\phi : \phi \in C_c^1(\Omega;\mathbb{R}^n),\ |\phi| \le 1\Big\},$$

with the corresponding definition of $BV_{\rm loc}(\Omega)$. We assume some familiarity with these spaces, including, for example, the basic structure theorem which asserts that the weak derivative of a BV function can be understood as a vector-valued Radon measure. (See for example [7, pp. 166-167].)

Theorem 2.1 (Density Theorem I). *Let $f \in BV(\Omega)$. Then there exists $\{f_j\} \subset C^\infty(\Omega)$ such that*

1. $\|f_j - f\|_{L^1(\Omega)} \to 0$,
2. $\int_\Omega |Df_j|\,dx \to \int_\Omega |Df|$,
3. *for any* $g \in C_c^0(\overline{\Omega};\mathbb{R}^n)$ *we have* $\int_\Omega g\cdot Df_j\,dx \to \int_\Omega g \cdot Df$.

Remark 2.2 (Not $W^{1,1}$ convergence, but quite close). In any treatment on BV functions care is always taken to emphasize that one does *not* have

$$\int_\Omega |D(f_j - f)| \to 0,$$

in the theorem above. In particular, Characteristic functions of smooth sets are in BV but not in $W^{1,1}$, and so BV is genuinely larger than $W^{1,1}$. On the other hand, the second part of the theorem above can be "localized" in some useful ways which are not clear from the statement above by itself.

Theorem 2.3 (Density Theorem II). *Let f and the $\{f_j\}$ be taken to satisfy the hypotheses and the conclusions of the theorem above. Let $\Omega' \Subset \Omega$ be an open Lipschitz set with*

$$\int_{\partial\Omega'} |Df| = 0\,. \tag{2.1}$$

Then

$$\int_{\Omega'} |Df_j|\,dx \to \int_{\Omega'} |Df|\,.$$

Furthermore, although simply convolving f (or f extended to be zero outside of Ω) with a standard mollifier is insufficient to produce a sequence of $\{f_j\}$ with the properties given in the previous theorem, they will all hold on every $\Omega' \Subset \Omega$ satisfying 2.1.

Remark 2.4 (There are lots of good sets)**.** The usefulness of this theorem is unclear until we show the existence of many such Ω' which satisfy 2.1. This fact follows from the following theorem found within [10, Remark 2.13].

Theorem 2.5 (Two-sided traces)**.** *Let $\Omega' \Subset \Omega$ be an open Lipschitz set and let $f \in BV(\Omega)$. Then $f|_{\Omega'}$ and $f|_{\Omega'^c}$ have traces on $\partial\Omega'$ which we call $f^-_{\Omega'}$ and $f^+_{\Omega'}$ respectively, and these traces satisfy*

$$\int_{\partial\Omega'} |f^+_{\Omega'} - f^-_{\Omega'}|\,d\mathcal{H}^{n-1} = \int_{\partial\Omega'} |Df| \tag{2.2}$$

and even $Df = (f^+_{\Omega'} - f^-_{\Omega'})\nu d\mathcal{H}^{n-1}$ where ν is the unit outward normal. Now by taking $\Omega' = B_\rho(x_0)$ with $x_0 \in \Omega$ then for almost every ρ such that $B_\rho(x_0) \subset \Omega$ we will have

$$\int_{\partial B_\rho(x_0)} |Df| = 0 \tag{2.3}$$

and therefore $f^-_{\Omega'}(x) = f^+_{\Omega'}(x) = f(x)$ for $\mathcal{H}^{n-1}$ almost every $x \in \partial B_\rho(x_0)$.

From the proof of [10, Lemma 2.4], we extract the following result.

Theorem 2.6. *Let $\tilde{\mathcal{B}}_R$ denote the ball in $\mathbb{R}^{n-1}$ centered at 0 with radius R. Let $C^+_R = \tilde{\mathcal{B}}_R \times (0,R)$ and $f \in BV(C^+_R)$. Let $0 < \epsilon' < \epsilon < R$, and set $Q_{\epsilon,\epsilon'} = \tilde{\mathcal{B}}_R \times (\epsilon', \epsilon)$. Then*

$$\int_{\tilde{\mathcal{B}}_R} |f_\epsilon - f_{\epsilon'}|\,d\mathcal{H}^{n-1} \le \int_{Q_{\epsilon,\epsilon'}} |D_n f|\,dx. \tag{2.4}$$

We will need the following lemma.

Lemma 2.7. *Let $f \in BV(B_R)$ and $0 < \rho < r < R$. Then*

$$\begin{gathered}\int_{\partial B_1} |f^-(rx) - f^-(\rho x)|\,d\mathcal{H}^{n-1} \le \int_{B_r \setminus B_\rho} \big|\big\langle \frac{x}{|x|^n}, Df\big\rangle\big|,\\ \int_{\partial B_1} |f^+(rx) - f^+(\rho x)|\,d\mathcal{H}^{n-1} \le \int_{\overline{B_r}\setminus\overline{B_\rho}} \big|\big\langle \frac{x}{|x|^n}, Df\big\rangle\big|\,.\end{gathered} \tag{2.5}$$

We conclude with Helly's Selection Theorem which is the standard BV compactness theorem.

Theorem 2.8 (Helly's Selection Theorem)**.** *Given $U \subset \mathbb{R}^n$ and a sequence of functions $\{f_j\}$ in $BV_{\rm loc}(U)$ such that for any $W \Subset U$ there is a constant $C < \infty$ depending only on W which satisfies*

$$\|f_j\|_{BV(W)} := \|f_j\|_{L^1(W)} + \int_W |Df_j| \le C \tag{2.6}$$

then there exists a subsequence $\{f_{j_k}\}$ and a function $f \in BV_{\rm loc}(U)$ such that on every $W \Subset U$ we have

$$\|f_{j_k} - f\|_{L^1(W)} \to 0, \tag{2.7}$$

$$\int_W |Df| \le \liminf \int_W |Df_{j_k}| \,. \tag{2.8}$$

3. Definitions, notation, and more background

We denote the surface tension at the interface between E_i and E_j with σ_{ij}, we use β_i as the coefficient that determines the wetting energy of E_i on the boundary of the container, we let ρ_i be the density of the i^{th} fluid, and we use g as the gravitational constant. The domain Ω is the container, and we assume $B_1 \Subset \Omega \subset \mathbb{R}^n$. We define

$$\begin{aligned} \alpha_0 &:= \frac{1}{2}(\sigma_{01} + \sigma_{02} - \sigma_{12}) \\ \alpha_1 &:= \frac{1}{2}(\sigma_{01} + \sigma_{12} - \sigma_{02}) \\ \alpha_2 &:= \frac{1}{2}(\sigma_{02} + \sigma_{12} - \sigma_{01})\,, \end{aligned} \tag{3.1}$$

and we will assume

$$\alpha_j > 0, \quad \text{for all } j \tag{3.2}$$

throughout this article, and refer to this condition as the strict triangle inequality. Note that this condition is frequently called the strict triangularity hypothesis. (See [15] for example.)

Definition 3.1 (Permissible configurations)**.** The triple of open sets $\{E_j\}$ is said to be a permissible configuration or more simply "permissible" if

(1) The E_j are sets of finite perimeter.
(2) The E_j are disjoint.
(3) The union of their closures is $\overline{\Omega}$.

In a case where volumes are prescribed, in order for sets to be *V-permissible* we will add to this list a fourth item:

(4) The volumes are prescribed: $|E_j| = v_j$ for $j = 0, 1, 2$.

See Figure 2.

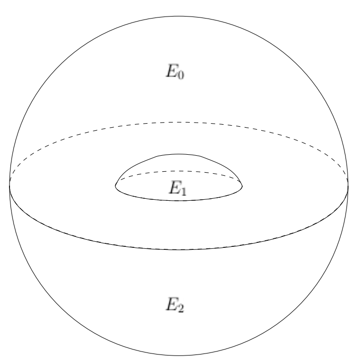


Figure 2. Permissible sets $\{E_j\}$.

The full energy functional which sums surface tension, wetting energy, and potential energy due to gravity is given by

$$\mathcal{F}_{SWP}(\{E_j\}) := \sum_{j=0}^{2} \Big(\alpha_j \int_{\Omega} |D\chi_{E_j}| + \beta_j \int_{\partial\Omega} \chi_{E_j} \, d\mathcal{H}^{n-1} + \rho_j g \int_{E_j} z \, dV \Big), \quad (3.3)$$

As we scale inward we can eliminate the wetting energy entirely and view our solution restricted to an interior ball as a minimizer of an energy given by

$$\mathcal{F}_{SP}(\{E_j\}) := \sum_{j=0}^{2} \Big(\alpha_j \int_{\Omega} |D\chi_{E_j}| + \rho_j g \int_{E_j} z \, dV \Big). \quad (3.4)$$

Of course this energy we will frequently consider on subdomains, so for $\Omega' \Subset \Omega$ we define

$$\mathcal{F}_{SP}(\{E_j\}, \Omega') := \sum_{j=0}^{2} \Big(\alpha_j \int_{\Omega'} |D\chi_{E_j}| + \rho_j g \int_{E_j \cap \Omega'} z \, dV \Big). \quad (3.5)$$

Massari showed that this energy functional is lower semicontinuous in [18] under certain assumptions on the constants. (In fact he showed it for $\mathcal{F}_{SWP}$, but where $\beta_j \equiv 0$ is allowed.) The lower semicontinuity of $\mathcal{F}_{SP}$ ensures that this Dirichlet problem is well-posed, although it does not guarantee that the Dirichlet data is attained in the usual sense. In fact, a minimizer can actually have any Dirichlet data, but if it does not match up with the given data, then it must pay for an interface at the boundary. Summarizing these statements from [18] we can say the following.

Theorem 3.2 (Massari's Existence Theorem). *If*

$$\alpha_j \geq 0, \quad \alpha_i + \alpha_j \geq |\beta_i - \beta_j|, \quad (3.6)$$

for $i, j = 0, 1, 2$, if $v_0 + v_1 + v_2 = |\Omega|$, and if Ω satisfies an interior sphere condition, then there exists a minimizer to $\mathcal{F}_{SWP}$ among permissible triples $\{E_j\}$ with $|E_j| = v_j$. The same statement is true if $\mathcal{F}_{SWP}$ is replaced by either $\mathcal{F}_{SP}$ or $\mathcal{F}_S$. ($\mathcal{F}_S$ is defined below.) Assuming that we allow a two sided trace of our BV characteristic functions on the boundary of our domain, and making the same assumptions as above, then there will also exist minimizers which satisfy given Dirichlet data. (Of course one should refer to the discussion above regarding the nature of Dirichlet data for this problem.)

Remark 3.3 (Appropriate Problems). It seems worthwhile to observe here the necessity of prescribing Dirichlet data in any problem without a volume constraint. Indeed, without a volume constraint or Dirichlet data, one expects two of the three fluids to vanish in any minimizer. On the other hand, once you have a volume constraint, you can study the minimizers both with and without Dirichlet data.

At this point, we standardize our language for the type of minimizer that we are considering in order to prevent language from becoming too cumbersome.

Definition 3.4 (Types of Minimizers). We will use the syntax:

[Qualifier(s)]-minimizer of [functional] in [set] .

The "qualifiers" we will use are "D" and/or "V" to indicate a Dirichlet or a volume constraint respectively. So a typical appearance might look like: $\{E_j\}$ is a V-minimizer of $\mathcal{F}_{SP}$ in B_3, which means that $\{E_j\}$ are V-permissible and minimize

$\mathcal{F}_{SP}$ in B_3 among all V-permissible sets. If the set is not specified, then we will assume that the minimization happens on Ω. If the functional is not specified, then we assume that $\mathcal{F}_S$ is the functional being minimized. The set given will typically be bounded, but when it is not bounded we will assume that anything which we call any kind of minimizer will minimize the given functional when restricted to any compact subset of the unbounded domain.

Remark 3.5 (On restrictions and rescalings)**.** It is also worth remarking that after restricting and rescaling, a triple which used to V-minimize some functional will still V-minimize some functional in the new set, but except in the case of the three cones, the new sets will typically be competing against V-permissible triples with different restrictions on the volume of each set from the restrictions at the outset.

Remark 3.6 (Reversal of inclusions)**.** We also observe that the inclusions of types of minimizers are also reversed from what one might assume before thinking about it. In typical set inclusions of this sort, one assumes that more constraints lead to a smaller set. Here, because it is the competitors which are being constrained, the inclusions work in reverse. Indeed, the set of all DV-minimizers contains both the set of V-minimizers and the set of D-minimizers insofar as if you take the DV-minimizer where you take the Dirichlet data to be rather "wiggly" then you only compete against other configurations with similarly wiggly boundary data. Thus, you are automatically the DV-minimizer by construction, but you are not likely to be a V-minimizer, as any V-minimizer would prefer less wiggly boundary data.

Since we intend to study the local microstructure at triple points which are in the interior of Ω, it will be useful to study the simplified energy functional which ignores the wetting energy and the potential energy. By scaling in toward a triple point, we can be sure that the forces of surface tension are much stronger than the gravitational forces in our local picture, and at the same time the wetting energy will become totally irrelevant, as the boundary of Ω can be scaled away altogether if we zoom in far enough. So, with these ideas in mind we define the simplified energy functional by

$$\mathcal{F}_S(\{E_j\}) := \sum_{j=0}^{2} \Big(\alpha_j \int_\Omega |D\chi_{E_j}|\Big), \quad \mathcal{F}_S(\{h_j\}) := \sum_{j=0}^{2} \Big(\alpha_j \int_\Omega |Dh_j|\Big) . \tag{3.7}$$

The energy on $\Omega' \Subset \Omega$ is

$$\mathcal{F}_S(\{E_j\}, \Omega') := \sum_{j=0}^{2} \Big(\alpha_j \int_{\Omega'} |D\chi_{E_j}|\Big), \tag{3.8}$$

$$\mathcal{F}_S(\{h_j\}, \Omega') := \sum_{j=0}^{2} \Big(\alpha_j \int_{\Omega'} |Dh_j|\Big) . \tag{3.9}$$

Let $\Omega' \Subset \Omega$, let $\{E_j\}$ be permissible. Using "spt" for "support", we define

$$\begin{aligned} \Upsilon(\{E_j\}, \Omega') := \inf \Big\{\mathcal{F}_S(\{\tilde{E}_j\}) : \operatorname{spt}(\chi_{E_j} - \chi_{\tilde{E}_j}) \subset \Omega' \\ \text{and } \{\tilde{E}_j\} \text{ is perm.}\Big\}, \end{aligned} \tag{3.10}$$

$$\Psi(\{E_j\}, \Omega') := \mathcal{F}_S(\{E_j\}, \Omega') - \Upsilon(\{E_j\}, \Omega') . \tag{3.11}$$

Now assume further that $\{E_j\}$ is V-permissible. Then we define

$$\Upsilon_V(\{E_j\},\Omega') := \inf\Big\{\mathcal{F}_S(\{\tilde{E}_j\}) : \operatorname{spt}(\chi_{E_j}-\chi_{\tilde{E}_j})\subset\Omega' \text{ and } \{\tilde{E}_j\} \text{ is V-perm.}\Big\}, \tag{3.12}$$

$$\Psi_V(\{E_j\},\Omega') := \mathcal{F}_S(\{E_j\},\Omega') - \Upsilon_V(\{E_j\},\Omega')\,. \tag{3.13}$$

So Υ and Υ_V give the value of the minimal energy configuration with the same boundary data, while Ψ and Ψ_V give the amount that $\{E_j\}$ deviates from minimal. Notice that we are minimizing over the class of sets of finite perimeter, not over all of BV.

Of course the existence theorem does not address any of the regularity questions near a triple point and the regularity questions near the boundary of only two of the fluids is already well-understood. On the other hand, in order to understand the microstructure of triple points which are not located on the boundary of Ω it should suffice to study minimizers of the simplified energy functional, $\mathcal{F}_S$, as we have described above. We make this heuristic argument rigorous in Section 4, but we still need two more tools from the background literature.

The first tool we need is a very nice observation due to F. Almgren which allowed him to virtually ignore volume constraints when studying the regularity of minimizers of surface area under these restrictions. Since our energy is bounded from above and below by a constant times surface area, we can adapt his result to our situation immediately.

Lemma 3.7 (Almgren's Volume Adjustment Lemma)**.** *Given any permissible triple $\{E_j\}$, there exists a $C>0$, such that very small volume adjustments can be made at a cost to the energy which is not more than C times the volume adjustment. Stated quantitatively,*

$$\Delta\mathcal{F}_S \le C\sum_{j=0}^{2}|\Delta V_j|, \tag{3.14}$$

where ΔV_j is the volume change of E_j.

This result can be found in [2, V1.2(3)] and [21, Lemma 2.2]. The next tool we need is an "elimination theorem" which in our setting is due to Leonardi. (See [15, Theorem 3.1].)

Theorem 3.8 (Leonardi's Elimination Theorem)**.** *Under the assumptions above, including the strict triangle inequality (Equation (3.2)), if $\{E_j\}$ is a V-minimizer, then $\{E_j\}$ has the elimination property. Namely, there exists a constant $\eta>0$, and a radius r_0 such that if $0<\rho<r_0$, $B_{r_0}\subset\Omega$, and*

$$|E_i\cap B_\rho(x)| \le \eta\rho^n\,, \tag{3.15}$$

then

$$|E_i\cap B_{\rho/2}(x)| = 0\,. \tag{3.16}$$

4. Restrictions and rescalings

We start with a rather trivial observation: If $\{E_j\}$ is a V-minimizer of $\mathcal{F}_{SWP}$ among V-permissible triples, and $B_r\Subset\Omega$, then the triple: $\{E_j\cap B_r\}$ DV-minimizes $\mathcal{F}_{SP}$ in B_r among V-permissible triples with Dirichlet data given by the traces of the $\{E_j\}$ on the outer boundary of B_r, and whose volumes are prescribed to be the volume of each E_j intersected with B_r. If this statement were false, then we would immediately get an improvement to our V-minimizer of $\mathcal{F}_{SWP}$, by replacing things within B_r.

Recalling that $B_1 \Subset \Omega$, we wish to define rescalings of our triples and study their properties in the hopes of producing blowup limits. For $\lambda \in \mathbb{R}^+$ we define λE_j to be the dilation of E_j by λ. In particular,

$$x \in \lambda E_j \iff \frac{x}{\lambda} \in E_j \,.$$

Now assume that $\{E_j\}$ is a D-minimizer of $\mathcal{F}_{SWP}$ in Ω, and fix $0 < \lambda < 1$. By the fact that $\{E_j\}$ is a D-minimizer of $\mathcal{F}_{SP}$ in B_λ, we can scale our triple $\{E_j\}$ to the triple $\{\lambda^{-1}E_j\}$, and easily verify that the new triple is a D-minimizer of the functional

$$\mathcal{F}_{SP\lambda}(\{A_j\}, B_1) := \sum_{j=0}^{2} \Big(\alpha_j \int_{B_1} |D\chi_{A_j}| + \lambda\rho_j g \int_{A_j \cap B_1} z\, dV\Big). \tag{4.1}$$

From here, after observing that it is immediate that the characteristic functions corresponding to the triple $\{\lambda^{-1}E_j\}$ will be uniformly bounded in $BV(B_1)$, we can apply Helly's selection theorem (given above as Theorem 2.8) to guarantee the existence of a blow up limit in BV. More importantly, the blowup limit will be a minimizer of $\mathcal{F}_S$. For convenience, define $\chi_{E_{j,\lambda_i}} := \chi_{\lambda_i^{-1}E_j}$.

Theorem 4.1 (Existence of blowup limits). *Assume that $\{E_j\}$ is a D-minimizer or a V-minimizer of $\mathcal{F}_{SP}$ in Ω. In either case, there exists a configuration (which we will denote by $\{E_{j,0}\}$) and a sequence of $\lambda_i \downarrow 0$ such that for each j:*

$$\|\chi_{E_{j,\lambda_i}} - \chi_{E_{j,0}}\|_{L^1(B_1)} \to 0 \quad \text{and} \quad D\chi_{E_{j,\lambda_i}} \overset{*}{\rightharpoonup} D\chi_{E_{j,0}} \,. \tag{4.2}$$

Furthermore, the triple $\{E_{j,0}\}$ is a D-minimizer of $\mathcal{F}_S$ for whatever Dirichlet data it has in the first case or a V-minimizer of $\mathcal{F}_S$ for whatever volume constraints it satisfies in the second case.

Proof. Based on the discussion preceding the statement of the theorem, it remains to show that $\{E_{j,0}\}$ is a minimizer of $\mathcal{F}_S$ under the appropriate constraints. Lower semicontinuity of the BV norm implies that

$$\mathcal{F}_S(\{E_{j,0}\}) \le \liminf_{j\to\infty} \mathcal{F}_S(\{E_{j,\lambda_i}\}).$$

While on the other hand

$$\begin{aligned} \mathcal{F}_{SP\lambda_i}(\{E_{j,\lambda_i}\}) &= \min\{\mathcal{F}_{SP\lambda_i}(\{A_j\}) : \{A_j\} \text{ is permissible}\} \\ &\le \mathcal{F}_{SP\lambda_i}(\{E_{j,0}\}) \end{aligned} \tag{4.3}$$

since $\{E_{j,\lambda_i}\}$ is a minimizer. Because the gravitational term is going to zero, it is clear that

$$\mathcal{F}_S(\{E_{j,0}\}) = \lim_{i\to\infty} \mathcal{F}_S(\{E_{j,\lambda_i}\})\,, \tag{4.4}$$

and for the same reason, for any $\epsilon > 0$, if λ is sufficiently small and i is sufficiently large, then we must have

$$|\mathcal{F}_{SP\lambda}(\{E_{j,0}\}) - \mathcal{F}_{SP\lambda}(\{E_{j,\lambda_i}\})| < \epsilon\,. \tag{4.5}$$

Now if $\{E_{j,0}\}$ is not a D-minimizer or V-minimizer (according to the case we are in), then there exists a D or V-minimizing triple $\{\tilde{E}_{j,0}\}$ and a $\gamma > 0$, such that $\mathcal{F}_S(\{E_{j,0}\}) - \gamma = \mathcal{F}_S(\{\tilde{E}_{j,0}\})$. In this case, for all sufficiently small λ, we will automatically have

$$\mathcal{F}_{SP\lambda}(\{E_{j,0}\}) - \gamma/2 \geq \mathcal{F}_{SP\lambda}(\{\tilde{E}_{j,0}\}), \tag{4.6}$$

but then by using (4.3) and (4.5) we will get a contradiction by observing that for small enough λ_i we will have

$$\mathcal{F}_{SP\lambda_i}(\{\tilde{E}_{j,0}\}) < \mathcal{F}_{SP\lambda_i}(\{E_{j,\lambda_i}\}). \tag{4.7}$$

□

5. Monotonicity of scaled energy (Part I)

Theorem 5.1. *Suppose $\{E_j\} \in BV(B_R)$ is V-permissible and $0 < \rho < r < R$ with $0 \in \cap_{j=0}^{2} \partial E_j$. Then there exists a constant C such that*

$$\begin{aligned}
&\sum_{j=0}^{2} \alpha_j \Big\{ \int_{B_r \setminus B_\rho} \big| \big\langle \frac{x}{|x|^n}, D\chi_{E_j} \big\rangle \big| \, dx \Big\}^2 \\
&\leq 2 \sum_{j=0}^{2} \int_{B_r \setminus B_\rho} |x|^{1-n} |D\chi_{E_j}| \, dx \Big\{ r^{1-n} \mathcal{F}_S(\{E_j\}, B_r) \\
&\quad - \rho^{1-n} \mathcal{F}_S(\{E_j\}, B_\rho) + (n-1) \int_\rho^r t^{-n} \Psi_V(\{E_j\}, B_t) \, dt \\
&\quad - \sum_{j=0}^{2} \frac{\alpha_j}{8} \int_{B_r \setminus B_\rho} |x|^{1-n} \big\langle \frac{x}{|x|}, \frac{D\chi_{E_j}}{|D\chi_{E_j}|} \big\rangle^4 |D\chi_{E_j}| \, dx + C(r - \rho) \Big\}.
\end{aligned} \tag{5.1}$$

This estimate and the argument below should be compared with [20, Lemma 5] and [10, Chapter 5].

Proof. Let $t \in (0, R)$ be such that $0 < \rho \leq t \leq r < R$. By [7, Theorem 2, p. 172] (or similar) there exist smooth functions $f_j(x;\epsilon)$ so that if $\epsilon \to 0$, then $f_j(x;\epsilon) \to \chi_{E_j}(x)$ in $L^1(B_R)$ and

$$\int_{B_R} |D\chi_{E_j}| = \lim_{\epsilon \to 0} \int_{B_R} |Df_j(x;\epsilon)| \, dx.$$

Then we define the conical projection on these smooth functions:

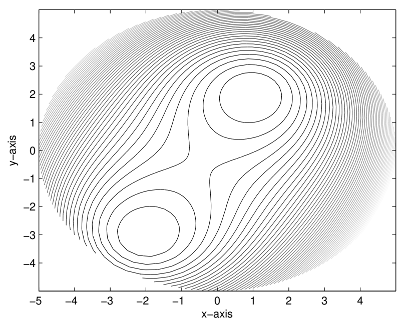
$$f_{j,t} = f_j(x;\epsilon,t) = \begin{cases} f_j(x;\epsilon) & |x| \geq t \\ f_j\big(\frac{tx}{|x|};\epsilon\big) & |x| < t\,. \end{cases} \tag{5.2}$$

An example of this process can be seen in Figure 3.

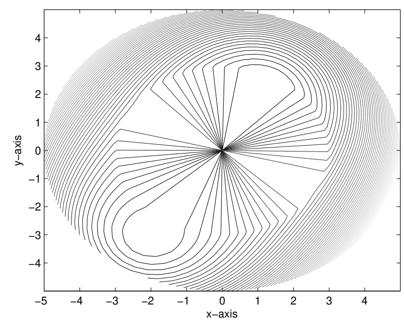
With these conical functions we have

$$\int_{B_t} |Df_{j,t}| \, dx = \frac{t}{n-1} \int_{\partial B_t} |Df_j| \Big\{ 1 - \frac{\langle x, Df_j \rangle^2}{|x|^2 |Df_j|^2} \Big\}^{1/2} d\mathcal{H}^{n-1} \text{ a.e. } t \in (0, R). \tag{5.3}$$

Then $\{E_j\}$ V-permissible implies if $\epsilon \to 0$, then $f_{j,t}(x;\epsilon,t) \to \chi_{\tilde{E}_j}$ for some *set* $\tilde{E}_j$ for $j = 0, 1, 2$. It follows from the V-permissibility of $\{E_j\}$ that $\{\tilde{E}_j\}$ have the



(a) Level curves for f_j



(b) Level curves for $f_{j,3}$

FIGURE 3. An example: $f_j(x,y) := [(x-1)^2 + (y-2)^2] \cdot [(x+2)^2 + (y+3)^2]$

properties that $\tilde{E}_j \cap \tilde{E}_i = \emptyset$ for $i \neq j$ and that $\cup$ closure $(\tilde{E}_j) = B_R$. It remains to show that each $\tilde{E}_j$ is a set of finite perimeter. Notice that

$$
\begin{aligned}
\int_{B_t} |Df_{j,t}|\,dx &= \frac{t}{n-1}\int_{\partial B_t} |Df_j| \Big\{1 - \frac{\langle x, Df_j\rangle^2}{|x|^2|Df_j|^2}\Big\}^{1/2} d\mathcal{H}^{n-1} \\
&\le \frac{t}{n-1}\int_{\partial B_t} |Df_j|\, d\mathcal{H}^{n-1} < \infty \quad \text{a.e. in } t,
\end{aligned} \tag{5.4}
$$

then Theorem 2.8 states that there is a subsequence $f_j(x;\epsilon_k,t)$ converging in L^1 to $\tilde{f}_j(x;t) \in BV(B_R)$ where the total variations converge as well. Thus $\tilde{E}_j$ are sets of finite perimeter, and $\{\tilde{E}_j\}$ is permissible, but the volume constraints which will be off by an amount controlled by Ct^n. Thus, by applying Almgren's Volume Adjustment Lemma (see Lemma 3.7), we obtain

$$
\Upsilon_V(\{E_j\}, B_t) \le \mathcal{F}_S\Big(\{\tilde{E}_j\}, B_t\Big) + Ct^n = \lim_{\epsilon\to 0} \mathcal{F}_S(\{f_j(x;\epsilon,t)\}, B_t) + Ct^n.
$$

Then by using

$$
\Upsilon_V(\{E_j\}, B_t) = \mathcal{F}_S(\{E_j\}, B_t) - \Psi_V(\{E_j\}, B_t),
$$

with (5.3) and the Taylor series for $\sqrt{1-x}$ at 0 with $x > 0$ small, we obtain

$$
\begin{aligned}
&\mathcal{F}_S(\{E_j\}, B_t) - \Psi_V(\{E_j\}, B_t) \\
&\le \mathcal{F}_S\Big(\{\tilde{E}_j\}, B_t\Big) + Ct^n \\
&\le \lim_{\epsilon\to 0} \sum_{j=0}^{2} \frac{t\alpha_j}{n-1}\Big(\int_{\partial B_t} |Df_j|\, d\mathcal{H}^{n-1} - \frac{1}{2}\int_{\partial B_t} \frac{\langle x, Df_j\rangle^2}{|x|^2|Df_j|}\, d\mathcal{H}^{n-1} \\
&\quad - \frac{1}{8}\int_{\partial B_t} \frac{\langle x, Df_j\rangle^4}{|x|^4|Df_j|^3}\, d\mathcal{H}^{n-1}\Big) + Ct^n.
\end{aligned} \tag{5.5}
$$

Then by rearranging terms and multiplying through by $(n-1)t^{-n}$ we obtain

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\sum_{j=0}^{2}\alpha_j\frac{t^{1-n}}{2}\int_{\partial B_t}\frac{\langle x, Df_j\rangle^2}{|x|^2|Df_j|}\,d\mathcal{H}^{n-1}\\
&\le -(n-1)t^{-n}\mathcal{F}_S(\{E_j\},B_t)+(n-1)t^{-n}\Psi_V(\{E_j\},B_t)\\
&\quad+\lim_{\epsilon\to 0}t^{1-n}\sum_{j=0}^{2}\alpha_j\int_{\partial B_t}|Df_j|\,d\mathcal{H}^{n-1}\\
&\quad-\lim_{\epsilon\to 0}\sum_{j=0}^{2}\frac{\alpha_j}{8}t^{1-n}\int_{\partial B_t}\frac{\langle x, Df_j\rangle^4}{|x|^4|Df_j|^3}\,d\mathcal{H}^{n-1}+C\\
&=\lim_{\epsilon\to 0}\Big[-(n-1)t^{-n}\mathcal{F}_S\left(\{f_j\},B_t\right)+t^{1-n}\sum_{j=0}^{2}\alpha_j\int_{\partial B_t}|Df_j|\,d\mathcal{H}^{n-1}\Big]\\
&\quad+(n-1)t^{-n}\Psi_V\left(\{E_j\},B_t\right)\\
&\quad-\lim_{\epsilon\to 0}\sum_{j=0}^{2}\frac{\alpha_j}{8}\int_{\partial B_t}|x|^{1-n}\Big\langle\frac{x}{|x|},\frac{Df_j}{|Df_j|}\Big\rangle^4|Df_j|\,d\mathcal{H}^{n-1}+C\\
&=\lim_{\epsilon\to 0}\Big\{\frac{d}{dt}[t^{1-n}\mathcal{F}_S(\{f_j\},B_t)]-\sum_{j=0}^{2}\frac{\alpha_j}{8}\int_{\partial B_t}|x|^{1-n}\Big\langle\frac{x}{|x|},\frac{Df_j}{|Df_j|}\Big\rangle^4|Df_j|\,d\mathcal{H}^{n-1}\Big\}\\
&\quad+(n-1)t^{-n}\Psi_V\left(\{E_j\},B_t\right)+C\quad\text{a.e. } t\in(0,R)\,.
\end{aligned}
$$

Integrating with respect to t between ρ and r, we have

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\sum_{j=0}^{2}\frac{\alpha_j}{2}\int_{B_r\setminus B_\rho}\frac{\langle x, Df_j\rangle^2}{|x|^{n+1}|Df_j|}\,dx\\
&\le r^{1-n}\mathcal{F}_S\left(\{E_j\},B_r\right)-\rho^{1-n}\mathcal{F}_S\left(\{E_j\},B_\rho\right)\\
&\quad+(n-1)\int_\rho^r t^{-n}\Psi_V\left(\{E_j\},B_t\right)\,dt\\
&\quad-\lim_{\epsilon\to 0}\frac{\alpha_j}{8}\int_{B_r\setminus B_\rho}|x|^{1-n}\Big\langle\frac{x}{|x|},\frac{Df_j}{|Df_j|}\Big\rangle^4|Df_j|\,dx+C(r-\rho)\,.
\end{aligned}
\tag{5.6}
$$

Finally, the Schwartz inequality implies

$$
\begin{aligned}
&\sum_{j=0}^{2}\Big\{\lim_{\epsilon\to 0}\alpha_j\int_{B_r\setminus B_\rho}\big|\big\langle\frac{x}{|x|^n},Df_j\big\rangle\big|\,dx\Big\}^2\\
&\le\lim_{\epsilon\to 0}\Big\{\sum_{j=0}^{2}\alpha_j\int_{B_r\setminus B_\rho}|x|^{1-n}|Df_j|\,dx\int_{B_r\setminus B_\rho}\frac{\langle x, Df_j\rangle^2}{|x|^{n+1}|Df_j|}\,dx\Big\}\\
&\le 2\Big(\lim_{\epsilon\to 0}\sum_{j=0}^{2}\int_{B_r\setminus B_\rho}|x|^{1-n}|Df_j|\,dx\Big)\Big(\lim_{\epsilon\to 0}\sum_{j=0}^{2}\frac{\alpha_j}{2}\int_{B_r\setminus B_\rho}\frac{\langle x, Df_j\rangle^2}{|x|^{n+1}|Df_j|}\,dx\Big).
\end{aligned}
$$

The result follows by combining the preceding with (5.6) and the application of [7, Theorem 3, p. 175]. □

Corollary 5.2. *Suppose* $\{E_j\} \in BV(B_R)$ *is V-permissible and is made up of sets of finite perimeter and* $0 < \rho < r < R$. *Further, suppose* $\Psi_V(\{E_j\}) \equiv 0$. *Then*

$$\begin{aligned} &\rho^{1-n}\mathcal{F}_S(\{E_j\}, B_\rho) + C\rho + \sum_{j=0}^{2} \frac{\alpha_j}{8}\int_{B_r\setminus B_\rho} |x|^{1-n}\Big\langle \frac{x}{|x|}, \frac{D\chi_{E_j}}{|D\chi_{E_j}|}\Big\rangle^4 |D\chi_{E_j}|\,dx \\ &\le r^{1-n}\mathcal{F}_S(\{E_j\}, B_r) + Cr. \end{aligned} \tag{5.7}$$

6. Monotonicity of scaled energy (Part II)

In this section we temporarily abandon the volume constraint and produce a sharp formula for monotonicity of scaled energy.

Theorem 6.1. *Let* $d = \operatorname{dist}(0, \partial\Omega)$. *If* $\{E_j\}$ *is a D-minimizer in* Ω *and* $0 \in \Omega \cap (\cap_j \partial E_j)$, *then for a.e.* $r \in (0, d)$,

$$\frac{d}{dr}\left(r^{1-n}\cdot\mathcal{F}_S(\{E_j\}, B_r)\right) = \frac{d}{dr}\sum_{j=0}^{2}\alpha_j\int_{B_r\cap\partial^*E_j} \frac{(\nu_{E_j}(x)\cdot x)^2}{|x|^{n+1}}\,d\mathcal{H}^{n-1}(x). \tag{6.1}$$

Proof. We follow Maggi [17, Theorem 28.9]. Given any $\varphi \in C^\infty(\mathbb{R}; [0,1])$ with $\varphi = 1$ on $(-\infty, 1/2)$, $\varphi = 0$ on $(1, \infty)$ and $\varphi' \le 0$ on $\mathbb{R}$, we define the following associated functions

$$\Phi(r) = \sum_{j=0}^{2}\alpha_j\int_{\partial^*E_j}\varphi\Big(\frac{|x|}{r}\Big)\,d\mathcal{H}^{n-1}(x), \quad r \in (0,d), \tag{6.2}$$

$$\Psi(r) = \sum_{j=0}^{2}\alpha_j\int_{\partial^*E_j}\varphi\Big(\frac{|x|}{r}\Big)\frac{(x\cdot\nu_{E_j}(x))^2}{|x|^2}\,d\mathcal{H}^{n-1}(x), \quad r \in (0,d). \tag{6.3}$$

Note that

$$\Phi'(r) = -\sum_{j=0}^{2}\alpha_j\int_{\partial^*E_j}\varphi'\Big(\frac{|x|}{r}\Big)\frac{|x|}{r^2}\,d\mathcal{H}^{n-1}(x), \quad r \in (0,d), \tag{6.4}$$

$$\Psi'(r) = -\sum_{j=0}^{2}\alpha_j\int_{\partial^*E_j}\varphi'\Big(\frac{|x|}{r}\Big)\frac{|x|}{r^2}\frac{(x\cdot\nu_{E_j}(x))^2}{|x|^2}\,d\mathcal{H}^{n-1}(x), \quad r \in (0,d). \tag{6.5}$$

Define

$$T_r \in C_c^1(\Omega; \mathbb{R}^n), \quad T_r(x) = \varphi\Big(\frac{|x|}{r}\Big)x, \quad x \in \mathbb{R}^n, \tag{6.6}$$

and observe the identities

$$\nabla T_r = \varphi\Big(\frac{|x|}{r}\Big)\operatorname{Id} + \frac{|x|}{r}\varphi'\Big(\frac{|x|}{r}\Big)\frac{x}{|x|}\otimes\frac{x}{|x|}, \quad \forall x \in \mathbb{R}^n \tag{6.7}$$

$$\operatorname{div} T_r = n\varphi\Big(\frac{|x|}{r}\Big) + \frac{|x|}{r}\varphi'\Big(\frac{|x|}{r}\Big), \quad \forall x \in \mathbb{R}^n \tag{6.8}$$

$$\nu_E\cdot\nabla T_r\nu_E = \varphi\Big(\frac{|x|}{r}\Big) + \frac{|x|}{r}\varphi'\Big(\frac{|x|}{r}\Big)\frac{(x\cdot\nu_E(x))^2}{|x|^2}, \quad \forall x \in \partial^*E \tag{6.9}$$

$$\begin{aligned} \operatorname{div}_E T_r &= \operatorname{div} T_r - \nu_E\cdot\nabla T_r\nu_E \\ &= (n-1)\varphi\Big(\frac{|x|}{r}\Big) + \frac{|x|}{r}\varphi'\Big(\frac{|x|}{r}\Big)\Big(1 - \frac{(x\cdot\nu_E(x))^2}{|x|^2}\Big). \end{aligned} \tag{6.10}$$

Now we quote [17, Theorem 17.5].

Theorem 6.2 (First variation of perimeter). *Suppose $A \subset \mathbb{R}^n$ is open, E is a set of locally finite perimeter, and $\{f_t\}_{|t|<\epsilon}$ is a local variation in A. Then*

$$\int_A |D\chi_{f_t(E)}| = \int_A |D\chi_E| + t\int_{\partial^* E} \mathrm{div}_E T \, d\mathcal{H}^{n-1}(x) + O(t^2)\,, \tag{6.11}$$

where T is the initial velocity of $\{f_t\}_{|t|<\epsilon}$ and $\mathrm{div}_E T : \partial^* E \to \mathbb{R}$ is given above. (T is the initial velocity of $\{f_t\}_{|t|<\epsilon}$ means

$$\frac{\partial}{\partial t} f(t,x) = T(x)$$

when f is evaluated at $t = 0$.)

In the same way that Maggi proves [17, Corollary 17.14] from this statement, we can show the following.

Corollary 6.3 (Vanishing sums of mean curvature). *A permissible triple $\{E_j\}$ is stationary for $\mathcal{F}_S$ in Ω if and only if*

$$\sum_{j=0}^{2} \alpha_j \int_{\partial^* E_j} \mathrm{div}_{E_j} T \, d\mathcal{H}^{n-1}(x) = 0\,, \quad \forall T \in C_c^1(\Omega;\mathbb{R}^n). \tag{6.12}$$

Returning to our proof of Theorem 6.1, we compute

$$\begin{aligned}
(n-1)\Phi(r) - r\Phi'(r) &= (n-1)\sum_{j=0}^{2} \alpha_j \int_{\partial^* E_j} \varphi\Big(\frac{|x|}{r}\Big)\, d\mathcal{H}^{n-1}(x) \\
&\quad + \sum_{j=0}^{2} \alpha_j \int_{\partial^* E_j} \varphi'\Big(\frac{|x|}{r}\Big)\frac{|x|}{r}\, d\mathcal{H}^{n-1}(x) \\
&= \sum_{j=0}^{2} \alpha_j \int_{\partial^* E_j} \varphi'\Big(\frac{|x|}{r}\Big)\frac{|x|}{r} \cdot \frac{(x\cdot \nu_{E_j}(x))^2}{|x|^2}\, d\mathcal{H}^{n-1}(x) \\
&= -r\Psi'(r),
\end{aligned}$$

or

$$\frac{\Phi'(r)}{r^{n-1}} - (n-1)\frac{\Phi(r)}{r^n} = \frac{\Psi'(r)}{r^{n-1}} \quad \text{a.e. } r \in (0,d). \tag{6.13}$$

Next, for $\epsilon \in (0,1)$, define Lipschitz functions $\varphi_\epsilon : \mathbb{R} \to [0,1]$ as

$$\varphi_\epsilon(s) = \chi_{(-\infty,1-\epsilon)}(s) + \frac{1-s}{\epsilon}\chi_{(1-\epsilon,1)}(s), \quad s \in \mathbb{R}, \tag{6.14}$$

and define $\Phi_\epsilon(r)$ and $\Psi_\epsilon(r)$ by replacing φ with φ_ϵ in the definitions of $\Phi(r)$ and $\Psi(r)$ respectively. Then, by approximation using [7, Theorem 2, p. 172] or something similar, for $\epsilon \in (0,1)$ and $\varphi = \varphi_\epsilon$ in (6.2) and (6.3) we obtain

$$\frac{\Phi_\epsilon'(r)}{r^{n-1}} - (n-1)\frac{\Phi_\epsilon(r)}{r^n} = \frac{\Psi_\epsilon'(r)}{r^{n-1}} \quad \text{a.e. } r \in (0,d). \tag{6.15}$$

Define $\Phi_0(r) = \mathcal{F}_S(\{E_j\}, B_r)$ and

$$\gamma(r) = \sum_{j=0}^{2} \alpha_j \int_{B_r \cap \partial^* E_j} \frac{(\nu_{E_j}(x)\cdot x)^2}{|x|^{n+1}}\, d\mathcal{H}^{n-1}(x), \quad r \in (0,d). \tag{6.16}$$

For $r \in (0, d)$, the Lebesgue Dominated Convergence Theorem implies

$$\Phi_\epsilon \to \sum_{j=0}^{2} \alpha_j \int_{\partial^* E_j \cap B_r} d\mathcal{H}^{n-1}(x) = \Phi_0, \quad \text{as } \epsilon \to 0. \tag{6.17}$$

Claim 6.4. For a.e. $r \in (0, d)$,

$$\Phi_\epsilon'(r) \to \Phi_0'(r), \quad \Psi_\epsilon'(r) \to r^{n-1}\gamma'(r), \tag{6.18}$$

as $\epsilon \to 0$. In particular, this holds for every $r \in (0, d)$ where Φ_0 and γ are differentiable.

Proof of the claim. Upon examining, we write

$$\Phi_\epsilon(r) = \sum_{j=0}^{2} \alpha_j \Big(\int_{\partial^* E_j \cap B_{r(1-\epsilon)}} d\mathcal{H}^{n-1}(x) + \int_{\partial^* E_j \cap (B_r \setminus B_{r(1-\epsilon)})} \Big(\frac{1}{\epsilon} - \frac{|x|}{\epsilon r}\Big)\, d\mathcal{H}^{n-1}(x) \Big).$$

We wish to differentiate the term in the parentheses above. We can express that term as

$$\begin{aligned} I_1(r) + I_2(r) := &\int_{\partial^* E_j \cap B_{r(1-\epsilon)}} \Big(1 - \frac{1}{\epsilon} + \frac{|x|}{\epsilon r}\Big)\, d\mathcal{H}^{n-1}(x) \\ &+ \int_{\partial^* E_j \cap B_r} \Big(\frac{1}{\epsilon} - \frac{|x|}{\epsilon r}\Big)\, d\mathcal{H}^{n-1}(x)\,. \end{aligned} \tag{6.19}$$

Then

$$\begin{aligned} I_1'(r) &= \int_{\partial^* E_j \cap \partial B_{r(1-\epsilon)}} 0\, d\mathcal{H}^{n-1}(x) - \int_{\partial^* E_j \cap B_{r(1-\epsilon)}} \Big(\frac{|x|}{\epsilon r^2}\Big) d\mathcal{H}^{n-1}(x) \\ &= - \int_{\partial^* E_j \cap B_{r(1-\epsilon)}} \Big(\frac{|x|}{\epsilon r^2}\Big) d\mathcal{H}^{n-1}(x)\,, \end{aligned} \tag{6.20}$$

and

$$\begin{aligned} I_2'(r) &= \int_{\partial^* E_j \cap \partial B_r} 0\, d\mathcal{H}^{n-1}(x) + \int_{\partial^* E_j \cap B_r} \Big(\frac{|x|}{\epsilon r^2}\Big)\, d\mathcal{H}^{n-1}(x) \\ &= \int_{\partial^* E_j \cap B_r} \Big(\frac{|x|}{\epsilon r^2}\Big)\, d\mathcal{H}^{n-1}(x)\,. \end{aligned} \tag{6.21}$$

Then it follows that

$$\Phi_\epsilon'(r) = \frac{1}{\epsilon r} \sum_{j=0}^{2} \alpha_j \int_{\partial^* E_j \cap (B_r \setminus B_{r(1-\epsilon)})} \frac{|x|}{r}\, d\mathcal{H}^{n-1}(x), \quad \text{a.e. } r \in (0, d), \tag{6.22}$$

and we estimate to obtain

$$\begin{aligned} &(1-\epsilon)\frac{\mathcal{F}_S(\{E_j\}, B_r) - \mathcal{F}_S(\{E_j\}, B_{r-\epsilon r})}{\epsilon r} \\ &\le \Phi_\epsilon'(r) \le \frac{\mathcal{F}_S(\{E_j\}, B_r) - \mathcal{F}_S(\{E_j\}, B_{r-\epsilon r})}{\epsilon r}. \end{aligned} \tag{6.23}$$

Thus, if $\Phi_0(r)$ is differentiable at r, then $\Phi_\epsilon'(r) \to \Phi_0'(r)$ as $\epsilon \to 0^+$.

Next, upon examining Ψ_ϵ, we write

$$\begin{aligned} \Psi_\epsilon(r) = \sum_{j=0}^{2} \alpha_j \Big(&\int_{\partial^* E_j \cap B_{r(1-\epsilon)}} \frac{(x \cdot \nu_{E_j}(x))^2}{|x|^2}\, d\mathcal{H}^{n-1}(x) \\ &+ \int_{\partial^* E_j \cap (B_r \setminus B_{r(1-\epsilon)})} \Big(\frac{1}{\epsilon} - \frac{|x|}{\epsilon r}\Big) \frac{(x \cdot \nu_{E_j}(x))^2}{|x|^2}\, d\mathcal{H}^{n-1}(x) \Big)\,. \end{aligned} \tag{6.24}$$

Once again we wish to differentiate this term, so we express the term within the parentheses as

$$\begin{aligned}\tilde{I}_1(r)+\tilde{I}_2(r) := \int_{\partial^* E_j\cap B_{r(1-\epsilon)}} &\Big(1-\frac{1}{\epsilon}+\frac{|x|}{\epsilon r}\Big)\frac{(x\cdot\nu_{E_j}(x))^2}{|x|^2}\,d\mathcal{H}^{n-1}(x)\\ &+\int_{\partial^* E_j\cap B_r}\Big(\frac{1}{\epsilon}-\frac{|x|}{\epsilon r}\Big)\frac{(x\cdot\nu_{E_j}(x))^2}{|x|^2}\,d\mathcal{H}^{n-1}(x)\,.\end{aligned}\tag{6.25}$$

Then

$$\begin{aligned}\tilde{I}_1'(r) &= \int_{\partial^* E_j\cap\partial B_{r(1-\epsilon)}} 0\,d\mathcal{H}^{n-1}(x)\\ &\quad -\int_{\partial^* E_j\cap B_{r(1-\epsilon)}}\Big(\frac{|x|}{\epsilon r^2}\Big)\frac{(x\cdot\nu_{E_j}(x))^2}{|x|^2}d\mathcal{H}^{n-1}(x)\\ &= -\int_{\partial^* E_j\cap B_{r(1-\epsilon)}}\Big(\frac{|x|}{\epsilon r^2}\Big)\frac{(x\cdot\nu_{E_j}(x))^2}{|x|^2}d\mathcal{H}^{n-1}(x)\,,\end{aligned}\tag{6.26}$$

and

$$\begin{aligned}\tilde{I}_2'(r) &= \int_{\partial^* E_j\cap\partial B_r} 0\,d\mathcal{H}^{n-1}(x)+\int_{\partial^* E_j\cap B_r}\Big(\frac{|x|}{\epsilon r^2}\Big)\frac{(x\cdot\nu_{E_j}(x))^2}{|x|^2}d\mathcal{H}^{n-1}(x)\\ &= \int_{\partial^* E_j\cap B_r}\Big(\frac{|x|}{\epsilon r^2}\Big)\frac{(x\cdot\nu_{E_j}(x))^2}{|x|^2}d\mathcal{H}^{n-1}(x)\,,\end{aligned}\tag{6.27}$$

implying

$$\frac{\Psi_\epsilon'(r)}{r^{n-1}} = \frac{1}{\epsilon r}\sum_{j=0}^{2}\alpha_j\int_{\partial^* E_j\cap(B_r\setminus B_{r(1-\epsilon)})}\Big(\frac{|x|}{r}\Big)^n\frac{(x\cdot\nu_{E_j}(x))^2}{|x|^{n+1}}\,d\mathcal{H}^{n-1}(x)\tag{6.28}$$

a.e. $r\in(0,d)$. As before, it follows that

$$(1-\epsilon)^n\frac{\gamma(r)-\gamma(r-\epsilon r)}{\epsilon r}\le\frac{\Psi_\epsilon'(r)}{r^{n-1}}\le\frac{\gamma(r)-\gamma(r-\epsilon r)}{\epsilon r},\tag{6.29}$$

and if $\gamma(r)$ is differentiable at r, then $\Psi_\epsilon'(r)\to r^{n-1}\gamma_0'(r)$ as $\epsilon\to 0^+$. Therefore the claim holds. □

From (6.15), (6.17) and (6.18) we find

$$\frac{\Phi_0'(r)}{r^{n-1}}-(n-1)\frac{\Phi_0(r)}{r^n}=\gamma'(r)\,,\tag{6.30}$$

and this proves (6.1). □

As we have mentioned, there are other related monotonicity formulas. Our first monotonicity formula (5.7) is based off of work found in Guisti's monograph, however, we sharpened it by including an explicit increment in the difference in the scaled energies for two different radii. This increment is measuring how far a configuration deviates from a cone. Maggi completely characterized the monotonicity for the problem that Guisti considered, insofar as he produced a formula with an equality, and his scaled energy is constant as a function of radius when the configuration is a cone. We based our second approach on his methods, and our generalization is found in Theorem 6.1, although like Maggi we do not consider a volume constraint in obtaining this result.

Morgan [24] defines a mass ratio $\Theta(T,a,r)$ that is equivalent to Guisti's formulation of scaled energy (which is the formulation used in this paper). Then Morgan goes on to prove a monotonicity result (credited to Federer [8]) saying that $\Theta(T,a,r)$ is a monotonically increasing function of r. This result corresponds to what Guisti and Maggi wrote about, but is apparently not as sharp as Maggi's result. On [24, page 108], Morgan describes Allard's results [1] in that

> *Integral varifolds of bounded first variation include surfaces of constant or bounded mean curvature and soap bubble clusters. They satisfy a weakened versions of the monotonicity ... the area ratio times e^{Cr} is monotonically increasing, where C is a bound on the first variation or mean curvature* [emphasis in original].

Because the value of C can be taken to be zero in the case where there is no volume constraint, we have reproduced, but not improved on this result. In our formula in the case with no volume constraint, the derivative of our scaled energy is given as an explicit positive function which shows exactly how much the energy increases as the radius increases.

In both sections 5 and 6, an additional result that we were unable to prove was of the uniqueness of the blowup limit. The introduction of Almgren's Big Regularity Paper [3] discusses this difficulty, and some examples of slowly rotating configurations are in Leonardi [16]. In fact, Leonardi gives an example which is spiral but which always blows up to the same conical formation. (See [16][Example 4.7]. This sort of behavior (i.e. a unique type of blowup limit, but no uniqueness of the limit because of the necessity to get a convergent subsequence) can also be found in a paper by the first author [5].) In one related setting there has been success in showing that the tangent cone is unique: See White [26]. To summarize, although we eventually have specific angle conditions satisfied by the blowup limits, we cannot prove that the actual minimizers do not have some rotation that becomes slower and slower that prevents the existence of multiple blowup limits. (We do certainly conjecture that the blowup limit will be unique.)

7. Minimal cones

We begin with the following result estimating the minimal energies by their Dirichlet data.

Lemma 7.1 (Extension of [10, Lemma 5.6])**.** *Suppose that $\{E_j\}$ and $\{\hat{E}_j\}$ are V-permissible for the same volume constraints in B_R and are identical in B_ρ^c. Suppose further that ρ is small enough to guarantee that any perturbation to $\{E_j\}$ or to $\{\hat{E}_j\}$ within B_ρ gives us something to which Almgren's Volume Adjustment Lemma applies. (See Lemma 3.7.) Then*

$$|\Upsilon_V(\{E_j\},\rho)-\Upsilon_V(\{\hat{E}_j\},\rho)| \leq \sum_{j=0}^{2}\alpha_j\int_{\partial B_\rho}|\chi_{E_j}^- -\chi_{\hat{E}_j}^-|\,d\mathcal{H}^{n-1}+C\sum_{j=0}^{2}|\Delta V_j|\,, \tag{7.1}$$

where ΔV_i is the symmetric difference $E_i\Delta\hat{E}_i$. If instead of "V-permissible" we have "permissible," then for any positive $\rho<R$ we have

$$|\Upsilon(\{E_j\},\rho)-\Upsilon(\{\hat{E}_j\},\rho)| \leq \sum_{j=0}^{2}\alpha_j\int_{\partial B_\rho}|\chi_{E_j}^- -\chi_{\hat{E}_j}^-|\,d\mathcal{H}^{n-1}\,. \tag{7.2}$$

Proof. The proof for (7.2) is almost identical to the proof for (7.1), but it is a little bit easier, so we will only prove (7.1). Given $\epsilon > 0$, we can choose $\{E_j^{\varphi}\}$ V-permissible so that we satisfy two relations:

(1) $\operatorname{spt}(\chi_{E_j^{\varphi}} - \chi_{E_j}) \Subset B_\rho$, and
(2) $\mathcal{F}_S(\{E_j^{\varphi}\}, \rho) \leq \epsilon + \Upsilon_V(\{E_j\}, \rho)$.

Let $\rho_k \uparrow \rho$ be taken such that

$$\int_{\partial B_{\rho_k}} |D\chi_{E_j}| = \int_{\partial B_{\rho_k}} |D\chi_{\hat{E}_j}| = 0\,,$$

and $\operatorname{spt}(\chi_{E_j^{\varphi}} - \chi_{E_j}) \Subset B_{\rho_k}$ for all $k \in \mathbb{N}$ and $j \in \{0, 1, 2\}$. For any j we define the set $\hat{E}_{j,k}$ by taking the union of $B_{\rho_k} \cap E_j^{\varphi}$ and $\{B_R \setminus B_{\rho_k}\} \cap \hat{E}_j$. Now observe that $\{\hat{E}_{j,k}\}$ is permissible up to the volume constraint violation. We then use Almgren's Lemma 3.7 to compute

$$\begin{aligned}
&\Upsilon_V(\{\hat{E}_j\}, \rho)\\
&\leq \sum_{j=0}^{2} \alpha_j \int_{B_\rho} |D\chi_{\hat{E}_{j,k}}| + C \sum_{j=0}^{2} |\Delta V_j|\\
&= \sum_{j=0}^{2} \alpha_j \Big(\int_{B_{\rho_k}} |D\chi_{E_j^{\varphi}}| + \int_{B_R \setminus B_{\rho_k}} |D\chi_{\hat{E}_j}| + \int_{\partial B_{\rho_k}} |\chi_{E_j} - \chi_{\hat{E}_j}| \Big) + C \sum_{j=0}^{2} |\Delta V_j|\\
&\leq \sum_{j=0}^{2} \alpha_j \Big(\int_{B_{\rho}} |D\chi_{E_j^{\varphi}}| + \int_{B_R \setminus B_{\rho_k}} |D\chi_{\hat{E}_j}| + \int_{\partial B_{\rho_k}} |\chi_{E_j} - \chi_{\hat{E}_j}| \Big) + C \sum_{j=0}^{2} |\Delta V_j|\\
&\leq \epsilon + \Upsilon_V(\{E_j\}, \rho) + \sum_{j=0}^{2} \alpha_j \Big(\int_{B_R \setminus B_{\rho_k}} |D\chi_{\hat{E}_j}| + \int_{\partial B_{\rho_k}} |\chi_{E_j} - \chi_{\hat{E}_j}| \Big) + C \sum_{j=0}^{2} |\Delta V_j|\\
&\to \epsilon + \Upsilon_V(\{E_j\}, \rho) + \sum_{j=0}^{2} \alpha_j \Big(\int_{\partial B_{\rho}} |\chi_{E_j}^{-} - \chi_{\hat{E}_j}^{-}| \Big) + C \sum_{j=0}^{2} |\Delta V_j|\,.
\end{aligned}$$

Now by using the fact that $\epsilon > 0$ is arbitrary and by the symmetry of the equation that we are trying to prove, we are done. □

Lemma 7.2 (Analogue of [10, Lemma 9.1])**.** *Let $\Omega \subset \mathbb{R}^n$ be open, let $\{E_{j,k}\}$ be a sequence of sets that DV-minimize $\mathcal{F}_S$ over Ω i.e. $\{E_{j,k}\}$ are taken such that*

$$\Psi_V(\{E_{j,k}\}, A) = 0\,, \quad \forall A \Subset \Omega \tag{7.3}$$

(with potentially different Dirichlet data and volume constraints for each k). Suppose there exists a triple $\{E_j\}$ such that

$$\chi_{E_{j,k}} \to \chi_{E_j} \quad \text{in } L^1_{\rm loc}(\Omega), \quad j = 0, 1, 2 \tag{7.4}$$

Then $\{E_j\}$ is a DV-minimizer of $\mathcal{F}_S$ over Ω:

$$\Psi_V(\{E_j\}, A) = 0\,, \quad \forall A \Subset \Omega. \tag{7.5}$$

Moreover, if $L \Subset \Omega$ is any open set such that

$$\int_{\partial L} |D\chi_{E_j}| = 0, \quad j = 0, 1, 2, \tag{7.6}$$

then we have

$$\lim_{k\to\infty} \mathcal{F}_S(\{E_{j,k}\}, L) = \mathcal{F}_S(\{E_j\}, L). \tag{7.7}$$

Remark 7.3 (Weakness of some of the hypotheses)**.** Equation (7.4) can be guaranteed by Helly's Selection Theorem as long as all of the configurations have uniformly bounded energy.

Proof. Let $A \Subset \Omega$. We may suppose that ∂A is smooth, so that for every k,

$$\mathcal{F}_S(\{E_{j,k}\}, A) \le \Big(\mathcal{H}^{n-1}(\partial A) + \frac{1}{2}\omega_{n-1}\Big(\frac{\text{max. diam. of } A}{2}\Big)^{n-1}\Big)\sum_{j=0}^{2}\alpha_j, \tag{7.8}$$

which follows by covering ∂A with all three values, and bounding the minimal energy of $\{E_j\}$ by a (standard) competitor on a possibly larger domain. Then lower semicontinuity implies the same inequality holds with $\{E_{j,k}\}$ replaced with $\{E_j\}$.

For $t > 0$, let

$$A_t = \{x \in \Omega : \operatorname{dist}(x, A) < t\}. \tag{7.9}$$

We have

$$\lim_{k\to\infty} \int_{A_t} |\chi_{E_{j,k}} - \chi_{E_j}|\, dx = 0, \quad j = 0, 1, 2 \tag{7.10}$$

and therefore there exists a subsequence $\{E_{j,k_i}\}$ such that for almost every t close to 0,

$$\lim_{k\to\infty} \int_{\partial A_t} |\chi_{E_{j,k_i}} - \chi_{E_j}|\, d\mathcal{H}^{n-1} = 0, \quad j = 0, 1, 2. \tag{7.11}$$

From Lemma 7.1, as $\sum_{j=0}^{2} |\Delta V_{j,k}| \to 0$, we have for those t,

$$\lim_{k\to\infty} \Upsilon_V(\{E_{j,k_i}, A_t) = \Upsilon_V(\{E_j, A_t) \tag{7.12}$$

and by lower semicontinuity,

$$\Psi_V(\{E_j\}, A_t) = 0. \tag{7.13}$$

Thus (7.5) holds. Now let $L \Subset \Omega$ be such that $\int_{\partial L} |D\chi_{E_j}| = 0$ for $j = 0, 1, 2$, and let A be a smooth open set such that $L \Subset A \Subset \Omega$. Let $\{F_{j,k}\}$ be any subsequence of $\{E_{j,k}\}$. Repeating the same argument as above, there is a set A_t and a subsequence $\{F_{j,k_i}\}$ such that

$$\lim_{k\to\infty} \Upsilon_V(\{F_{j,k_i}\}, A_t) = \Upsilon_V(\{F_j\}, A_t). \tag{7.14}$$

Since $\Psi_V(\{F_{j,k_i}\}, A_t) = \Psi_V(\{E_j\}, A_t) = 0$ we have

$$\lim_{k\to\infty} \mathcal{F}_S(\{F_{j,k_i}\}, A_t) = \mathcal{F}_S(\{F_j\}, A_t); \tag{7.15}$$

thus from Theorem 2.3 (with A_t playing the role of Ω),

$$\lim_{k\to\infty} \mathcal{F}_S(\{F_{j,k}\}, L) = \mathcal{F}_S(\{E_j\}, L)\,. \tag{7.16}$$

This completes the proof. □

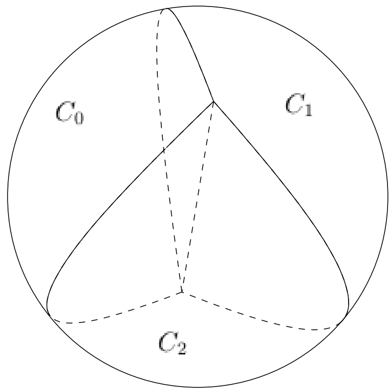


FIGURE 4. Limiting configuration as cones.

Theorem 7.4 (Analogue of [10, Theorem 9.3]). *Suppose $\{E_j\}$ is a V-minimizer of $\mathcal{F}_S$ in B_1, that is $\Psi_V(\{E_j\}, B_1) = 0$, such that $0 \in \partial E_0 \cap \partial E_1 \cap \partial E_2$. For each $t > 0$, let*

$$E_{j,t} = \{x \in \mathbb{R}^n : tx \in E_j\}, \quad j = 0, 1, 2. \tag{7.17}$$

Then for every sequence $\{t_i\}$ tending to zero there exists a subsequence $\{s_i\}$ such that E_{j,s_i} converges locally in $\mathbb{R}^n$ to permissible sets C_j. Moreover, $\{C_j\}$ are cones with positive density at the origin (the vertex of the cones) satisfying

$$\Psi_V(\{C_j\}, A) = 0 \quad \forall A \Subset \mathbb{R}^n. \tag{7.18}$$

See Figure 4. Note that this is a tiny bit more than an analogue of [10, Theorem 9.3] because we can use the Elimination Theorem to get the statement about the positive density of the cones at the origin. In light of this result, we define a triple $\{E_j\}$ to V-minimize $\mathcal{F}_S$ over $\mathbb{R}^n$ if

$$\Psi_V(\{E_j\}, A) = 0 \quad \forall A \Subset \mathbb{R}^n. \tag{7.19}$$

Proof. Let $t_i \to 0$. The first step is to show that for every $R > 0$ there exists a subsequence $\{\sigma_i\}$ such that $\{E_{j,\sigma_i}\}$ converges in B_R. We have

$$\mathcal{F}_S(\{E_{j,t}\}, B_R) = t^{1-n}\mathcal{F}_S(\{E_j\}, B_{Rt}) \tag{7.20}$$

and so choosing t sufficiently small (so that $Rt < 1$) we have that $E_{j,t}$ is a V-minimizer of $\mathcal{F}_S$ over B_R and

$$\begin{aligned} \mathcal{F}_S(\{E_{j,t}\}, B_R) &= t^{1-n}\mathcal{F}_S(\{E_j\}, B_{Rt}) \\ &< \Big(\mathcal{H}^{n-1}(\partial B_1) + \frac{1}{2}\omega_{n-1}\Big) R^{n-1} \sum_{j=0}^{2} \alpha_j. \end{aligned} \tag{7.21}$$

Hence, by Helly's Selection Theorem (see Theorem 2.8), a subsequence $\{E_{j,\sigma_i}\}$ converges to the triple of sets $\{C_{j,R}\}$ in B_R. Taking a sequence $R_t \to \infty$ we obtain, by a diagonal process, the triple of sets $\{C_j\} \subseteq \mathbb{R}^n$ and a sequence $\{s_i\}$ such that $\{E_{j,s_i}\} \to \{C_j\}$ locally. Now, applying Lemma 7.2, we see that $\{C_j\}$ is a V-minimizer of $\mathcal{F}_S$ over $\mathbb{R}^n$ in the sense that

$$\Psi_V(\{C_j\}, A) = 0 \quad \forall A \Subset \mathbb{R}^n. \tag{7.22}$$

The positive density of the C_j at the origin follows immediately by applying the Elimination Theorem (see Theorem 3.8). If we assume the opposite, then we can use the Elimination Theorem to show that 0 was not a triple point at the outset. It remains to show that the C_j are cones.

By Lemma 7.2 we have that, for almost all $R > 0$,

$$\mathcal{F}_S(\{E_{j,s_i}\}, B_R) \to \mathcal{F}_S(\{C_j\}, B_R). \tag{7.23}$$

Hence, if we define

$$p(t) = t^{1-n}\mathcal{F}_S(\{E_j\}, B_t) + Ct = \mathcal{F}_S(\{E_{j,t}\}, B_1) + Ct, \tag{7.24}$$

where C is the constant from Almgren's Volume Adjustment Lemma (see Lemma 3.7), we have, for almost all $R > 0$,

$$\lim_{i\to\infty} p(s_i R) = R^{1-n}\mathcal{F}_S(\{C_j\}, B_R), \tag{7.25}$$

as $i \to \infty$. (We must have $s_i \to 0$ as $i \to \infty$.) Also, from Equation (5.7), $p(t)$ is increasing in t.

If $\rho < R$, then for every i there exists an $m_i > 0$ such that

$$s_i\rho > s_{i+m_i}R. \tag{7.26}$$

Then

$$p(s_{i+m_i}R) \le p(s_i\rho) \le p(s_i R) \tag{7.27}$$

so that

$$\lim_{i\to\infty} p(s_i\rho) = \lim_{i\to\infty} p(s_i R) = R^{1-n}\mathcal{F}_S(\{C_j\}, B_R) \tag{7.28}$$

Thus we have proved that

$$\rho^{1-n}\mathcal{F}_S(\{C_j\}, B_\rho) \tag{7.29}$$

is independent of ρ, and so from Lemma 2.7 we have

$$\begin{aligned}\sum_{j=0}^{2}\alpha_j\int_{\partial B_1}|\chi^-_{C_j}(rx) - \chi^-_{C_j}(\rho x)|\,d\mathcal{H}^{n-1} &\le \sum_{j=0}^{2}\alpha_j\int_{B_r\setminus B_\rho}\big|\langle\frac{x}{|x|^n}, D\chi_{C_j}\rangle\big| \\ &\le r^{1-n}\mathcal{F}_S(\{C_j\}, B_r) - \rho^{1-n}\mathcal{F}_S(\{C_j\}, B_\rho) \\ &= 0\end{aligned}$$

for almost all $r, \rho > 0$. Hence the sets C_j differ only on a set of measure zero from cones with vertices at the origin. □

8. Tangent plane to the blow-up sphere

Theorem 8.1 (See [10, Proposition 9.6])**.** *Suppose $\{C_j\}$ are blowup cones resulting from the limit process in Theorem 7.4, and let $x_0 \in \partial C_0 \cap \partial C_1 \cap \partial C_2 \setminus \{0\}$. For $t > 0$, let*

$$\{C_{j,t}\} = \{x \in \mathbb{R}^n : x_0 + t(x - x_0) \in \{C_j\}\}. \tag{8.1}$$

Then there exists a sequence $\{t_i\}$ converging to zero such that $\{C_{j,i}\} := \{C_{j,t_i}\}$ converges to cones $\{Q_j\}$ which are a V-minimizer of $\mathcal{F}_S$ in $\mathbb{R}^n$. Moreover $\{Q_j\}$ are cylinders with axes through 0 and x_0.

Remark 8.2 (Existence of isolated triple points)**.** It is not clear that we need to assume that a point such as x_0 exists in dimension 3. Indeed, in dimension 3, we conjecture that if there is a triple point in a minimal configuration of cones, then there will be a full line of these triple points.

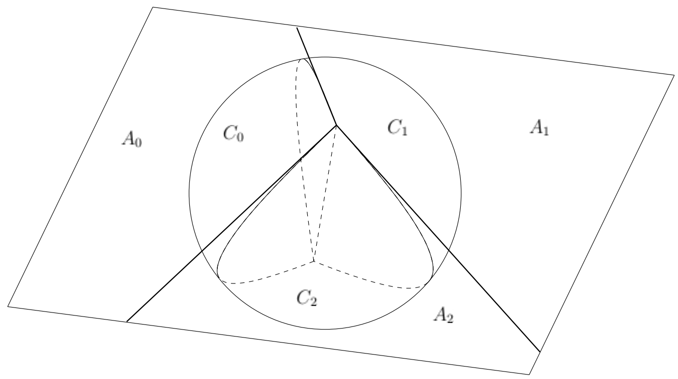


FIGURE 5. Cones in the tangent plane to the blow up sphere.

Proof. We may assume $x_0 = (0, 0, \dots, 0, a), a \neq 0$. We have

$$\chi_{C_{j,t}}(x) = \chi_{C_j}(x_0 + t(x - x_0)) \tag{8.2}$$

and so

$$\mathcal{F}_S(\{C_{j,t}\}, B(x_0, \rho)) = t^{1-n}\mathcal{F}_S(\{C_j\}, B(x_0, \rho t)) = \rho^{n-1}\mathcal{F}_S(\{C_j\}, B(x_0, 1)) \tag{8.3}$$

The argument in the proof of Theorem 7.4 implies the existence of a sequence $\{t_i\}$ converging to 0 such that $\{C_{j,i}\}$ converges to cones $\{Q_j\}$, each with a vertex at x_0, and that V-minimize $\mathcal{F}_S$ over $\mathbb{R}^n$.

It remains to prove that $\{Q_j\}$ are cylinders with axes through 0 and x_0. This is equivalent to the existence of sets $\{A_j\} \subseteq \mathbb{R}^{n-1}$ such that $\{Q_j\} = \{A_j\} \times \mathbb{R}$. Because the $\{C_j\}$ are all cones with vertex at 0, we have $\langle x, D\chi_{C_j}\rangle = 0$ and hence

$$aD_n\chi_{C_j} = -\langle x - x_0, D\chi_{C_j}\rangle. \tag{8.4}$$

Thus

$$|D_n\chi_{C_j}| \le \frac{|x - x_0|}{|x_0|}|D\chi_{C_j}| \tag{8.5}$$

and then

$$\begin{aligned}
\sum_{j=0}^{2}\alpha_j\int_{B(x_0,\rho)}|D_n\chi_{C_{j,t}}| &= t^{1-n}\sum_{j=0}^{2}\alpha_j\int_{B(x_0,\rho t)}|D_n\chi_{C_j}| \\
&\le \frac{t^{2-n}\rho}{|x_0|}\sum_{j=0}^{2}\alpha_j\int_{B(x_0,\rho t)}|D\chi_{C_j}| \\
&= \frac{t^{2-n}\rho}{|x_0|}\mathcal{F}_S\left(\{C_j\}, B(x_0, \rho t)\right) \\
&\le C\frac{\rho^n t}{|x_0|}\,.
\end{aligned}$$

Thus

$$D_n\chi_{Q_j} = \lim_{i\to\infty} D_n\chi_{C_{j,t_i}} = 0, \quad j = 0, 1, 2. \tag{8.6}$$

However, for almost all $s < t$, by Theorem 2.6,

$$\int_{\mathcal{B}_R}|\chi_{Q_{j,s}} - \chi_{Q_{j,t}}|\,d\mathcal{H}^{n-1} \le \int_{\mathcal{B}_R\times(s,t)}|D_n\chi_{Q_j}| = 0 \tag{8.7}$$

where $\chi_{Q_{j,r}}(y) = \chi_{Q_j}(y,r)$. This implies the existence of sets $A_j \subseteq \mathbb{R}^{n-1}$ such that for almost all r and s we have

$$\chi_{Q_j}(y,s) = \chi_{Q_j}(y,r) = \chi_{A_j}(y) \tag{8.8}$$

for $j = 0,1,2$ and almost all $y \in \mathbb{R}^{n-1}$. Thus

$$Q_j = A_j \times \mathbb{R}, \quad j = 0,1,2. \tag{8.9}$$

□

Since $\{Q_j\}$'s are cones, for each $t > 0$, $(y,s) \in \mathbb{R}^{n-1} \times \mathbb{R}$ we have

$$\chi_{A_j}(ty) = \chi_{Q_j}(ty,ts) = \chi_{Q_j}(y,s) = \chi_{A_j}(y), \quad j = 0,1,2, \tag{8.10}$$

which implies $\{A_j\}$ are also cones. We consider the case where $x_0 \in \partial B_1$. Then we get a blow up limit in the tangent plane at that point. We now turn to the task of classifying the behavior in this tangent plane.

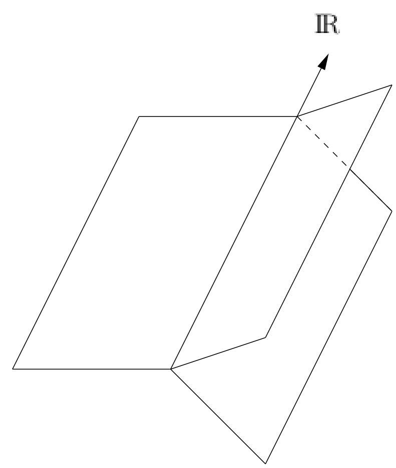


Figure 6. Cylinders in the second blow up limit.

Theorem 8.3. *Suppose $\{Q_j\} = \{A_j\} \times \mathbb{R}$ are V-permissible cylinders in $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$. If $\{Q_j\}$ is a V-minimizer of $\mathcal{F}_S$ in $\mathbb{R}^n$ then $\{A_j\}$ is a V-minimizer of $\mathcal{F}_S$ in $\mathbb{R}^{n-1}$. If we remove all of the volume constraints in the previous statements then the result still holds.*

See Figure 6.

Proof. Without the volume constraints the proof only becomes simpler, so it suffices to prove the statements where we include the volume restrictions. Suppose $\{Q_j\}$ is a V-minimizer of $\mathcal{F}_S$ in $\mathbb{R}^n$. If $\{A_j\}$ is not a V-minimizer of $\mathcal{F}_S$ in $\mathbb{R}^{n-1}$, then there exists $\epsilon > 0$, $R > 0$, and sets $\{E_j\}$ coinciding with $\{A_j\}$ outside some compact set $H \subseteq \tilde{\mathcal{B}}_R$ such that

$$\mathcal{F}_S(\{E_j\}, \tilde{\mathcal{B}}_R) \le \mathcal{F}_S(\{A_j\}, \tilde{\mathcal{B}}_R) - \epsilon. \tag{8.11}$$

Let $T > 0$ and set

$$M_j = \begin{cases} E_j \times (-T,T) & \text{in } |x_n| < T \\ Q_j & \text{outside } |x_n| < T \end{cases} \tag{8.12}$$

for $j = 0, 1, 2$, giving $\{M_j\} = \{Q_j\}$ outside $H \times [-T, T]$. Hence

$$\mathcal{F}_S\Big(\{Q_j\}, \tilde{\mathcal{B}}_R \times [-T, T]\Big) \leq \mathcal{F}_S\Big(\{M_j\}, \tilde{\mathcal{B}}_R \times [-T, T]\Big). \tag{8.13}$$

However, we have

$$\mathcal{F}_S\Big(\{Q_j\}, \tilde{\mathcal{B}}_R \times [-T, T]\Big) = 2T\mathcal{F}_S\Big(\{A_j\}, \tilde{\mathcal{B}}_R\Big) \tag{8.14}$$

and

$$\begin{aligned} \mathcal{F}_S\Big(\{M_j\}, \tilde{\mathcal{B}}_R \times [-T, T]\Big) &\leq 2T\mathcal{F}_S\Big(\{E_j\}, \tilde{\mathcal{B}}_R\Big) + 2\omega_{n-1}R^{n-1}\sum_{j=0}^{2}\alpha_j \\ &\leq 2T\mathcal{F}_S\Big(\{A_j\}, \tilde{\mathcal{B}}_R\Big) - 2T\epsilon + 2\omega_{n-1}R^{n-1}\sum_{j=0}^{2}\alpha_j \end{aligned}$$

This contradicts (8.13) for sufficiently large T, say, $T > \frac{\omega_{n-1}}{\epsilon}R^{n-1}\sum_{j=0}^{2}\alpha_j$. □

Note that at this point we have proven everything in Theorem1.1 except the angle condition.

If we weaken our definition of minimality by abandoning the volume constraint again, then we are able to prove the converse. We expect it is true with the volume constraint, but Figure 7 illustrates the difficulty in generalizing the following proof. Namely, the volume constraint could be satisfied globally, while individual slices did not preserve the induced $(n-1)$-dimensional volume constraints.

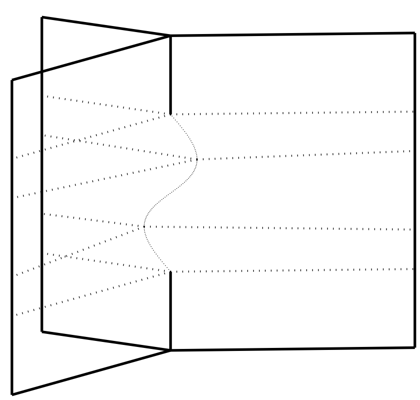


FIGURE 7. A visualization of the difficulty in generalizing Theorem 8.4. Perhaps each perpendicular slice is a minimizer in $\mathbb{R}^{n-1}$.

Theorem 8.4. *Suppose $\{Q_j\} = \{A_j\} \times \mathbb{R}$ are permissible cylinders in $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ without the volume constraint condition. Then $\{A_j\}$ is minimal in $\mathbb{R}^{n-1}$ implies $\{Q_j\}$ is minimal in $\mathbb{R}^n$.*

Proof. Suppose $\{A_j\}$ is minimal in $\mathbb{R}^{n-1}$ and let $\{M_j\}$ be permissible Caccioppoli sets in $\mathbb{R}^n$ coinciding with $\{Q_j\}$ outside some compact set K. Recall that $\tilde{\mathcal{B}}_R$ denotes the ball in $\mathbb{R}^{n-1}$ centered at 0 with radius R, and choose $T > 0$ such that

$$K \subseteq \tilde{\mathcal{B}}_T \times (-T, T). \tag{8.15}$$

Let $\{M_{j,t}\} \subseteq \mathbb{R}^{n-1}$ be defined by

$$\chi_{M_{j,t}}(y) = \chi_{M_j}(y, t), \quad j = 0, 1, 2. \tag{8.16}$$

Then [10, Lemma 9.8] gives

$$\int_{A_j} |D\chi_{M_j}| \le \int_{-T}^{T} dt \int_{\tilde{\mathcal{B}}_T} |D\chi_{M_{j,t}}|. \tag{8.17}$$

Note $M_{j,t} = A_j$ outside compact sets $H_{j,t} \subseteq \tilde{\mathcal{B}}_T$ for $j = 0, 1, 2$, and $M_{j,t}$ are permissible. Hence

$$\mathcal{F}_S\Big(\{A_j\}, \tilde{\mathcal{B}}_T\Big) \le \mathcal{F}_S\Big(\{M_{j,t}\}, \tilde{\mathcal{B}}_T\Big). \tag{8.18}$$

Therefore

$$\begin{aligned} \mathcal{F}_S\Big(\{M_j\}, \tilde{\mathcal{B}}_T \times (-T,T)\Big) &\ge \sum_{j=0}^{2} \alpha_j \int_{-T}^{T} dt \int_{\tilde{\mathcal{B}}_T} |D\chi_{A_j}| \\ &= \mathcal{F}_S\Big(\{Q_j\}, \tilde{\mathcal{B}}_T \times (-T,T)\Big) \end{aligned}$$

which implies $\{Q_j\}$ is minimal. □

9. Classification

We turn to classifying the possible minimal configurations in $\mathbb{R}^2$. First we point out that using the tools of mass-minimizing integral currents, Morgan [22, Theorem 4.3] showed that the triple junction points are isolated in $\mathbb{R}^2$. Under assumption of sufficient regularity Elcrat, Neel and Siegel [6] showed that the following Neumann angle condition holds

$$\frac{\sin \gamma_{01}}{\sigma_{01}} = \frac{\sin \gamma_{02}}{\sigma_{02}} = \frac{\sin \gamma_{12}}{\sigma_{12}}, \tag{9.1}$$

and their proof carries over to $\mathbb{R}^2$ directly. Here γ_{12} is the angle at the triple point measured within E_0, γ_{02} is the angle at the triple point measured within E_1, and γ_{01} is the angle at the triple point measured within E_2. We are able to prove the following theorem.

Theorem 9.1 (Angle condition result)**.** *Let $\{A_j\}$ be D-minimal or V-minimal cones in $\mathbb{R}^2$ with vertices at the origin. Then each A_j is formed of precisely one connected component, and the angle condition* (9.1) *is satisfied.*

Corollary 9.2 (Volume constraints of blowups)**.** *No matter what volume constraints we impose on minimization problem, the angles at the triple points for blowup limits are independent of everything except the constants which come from the surface tensions.*

Note that the preceding two results wrap up the proof of Theorem 1.1. We will start by dealing with the case without volume constraints, and we will proceed by contradiction, but first we will need some preliminary propositions. Before the first proposition, we record here a basic lemma which can be proven with no more than high school trigonometry.

Lemma 9.3 (Basic trigonometry lemma). *Given σ_{01}, σ_{02}, σ_{12} satisfying the strict triangle inequality given in* (3.2)*, there exists a unique triple of real numbers Γ_{01}, Γ_{02}, Γ_{12} in $(0,\pi)$ which satisfy both*

$$\Gamma_{01} + \Gamma_{02} + \Gamma_{12} = 2\pi$$

and (9.1).

Proof. We provide a sketch: First, because of the strict triangle inequality, there is a unique triangle (up to reflection and congruence, of course) with sides with lengths given by σ_{01}, σ_{02}, and σ_{12}. Now let θ_{ij} be the angle opposite σ_{ij}. Then the law of sines gives us:

$$\frac{\sin\theta_{01}}{\sigma_{01}} = \frac{\sin\theta_{02}}{\sigma_{02}} = \frac{\sin\theta_{12}}{\sigma_{12}}\,.$$

Of course, the angles just given sum to π and not 2π, but their supplementary angles sum to 2π and have the same value when plugged into the sine function. Define $\Gamma_{ij} := \pi - \theta_{ij}$ and everything is satisfied. □

Now, most of the calculus that we need to do has to be done on a suitable triangle, so we start with a definition of a "good triangle" and then give the calculus proposition which will be the main engine in the rest of our proofs in this section.

Definition 9.4 (Good Triangles). Given a blowup limit to our minimization problem, we define a good triangle to be a pair $(T, \tilde{P})$ consisting of a triangle T (whose vertices we label as P_0, P_1, and P_2), and a point $\tilde{P}$ which is in the interior of the triangle such that the following hold:

1. For $i, j \in \{0, 1, 2\}$ with $i \neq j$ we have that the angle between the vector from $\tilde{P}$ to P_i and the vector from $\tilde{P}$ to P_j is exactly Γ_{ij}.
2. If $\{i, j, k\}$ is a permutation of $\{0, 1, 2\}$, then the open segment from P_i to P_j has the k^{th} fluid as data.

To simplify the exposition, we can assume without loss of generality that the ordering of the vertices is counter-clockwise with respect to the triangle T. See Figure 8.

Definition 9.5 (Basic cost function). Given any good triangle $(T, \tilde{P})$ with the vertices of T labeled as $P_j := (a_j, b_j)$, and where for the sake of simplifying notation we let

$$\zeta_0 := \sigma_{12}, \quad \zeta_1 := \sigma_{02}, \quad \zeta_2 := \sigma_{01}\,,$$

we define the basic cost function

$$C(x,y) := \sum_{j=0}^{2} \zeta_j \sqrt{(x - a_j)^2 + (y - b_j)^2}\,. \tag{9.2}$$

The cost function $C(x, y)$ is continuous on the closed bounded triangle, T, and so it must attain a minimum there.

Proposition 9.6 (Minimization on good triangles). *The unique $D-$minimizer on a good triangle $(T, \tilde{P})$ with $\tilde{P} = (\tilde{x}, \tilde{y})$ is the configuration formed by letting E_i be the triangular region with P_j, P_k, and $\tilde{P}$ as vertices, where we let (i, j, k) run through the three permutations: $\{(0, 1, 2), (1, 2, 0), (2, 0, 1)\}$. Furthermore, the basic cost function has the following properties:*

(A) *The Hessian D^2C is positive definite in the interior of T.*

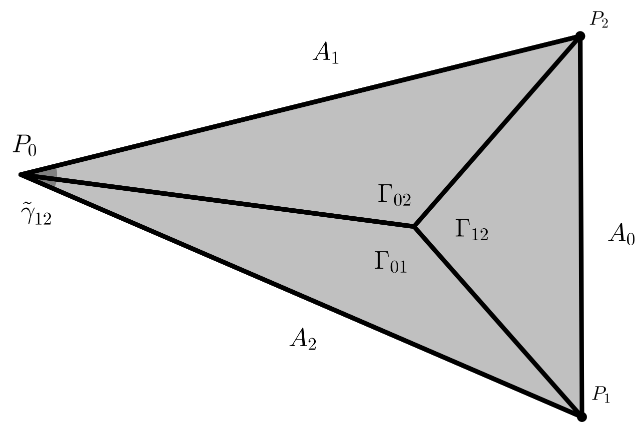


FIGURE 8. The triangle T.

(B) *$\nabla C(x, y)$ is zero if and only if $(x, y) = (\tilde{x}, \tilde{y})$.*
(C) *The cost C has a unique minimum at $(\tilde{x}, \tilde{y})$.*
(D) *If we let $\tilde{v}_j$ denote the vector from $(\tilde{x}, \tilde{y})$ to (a_j, b_j), then the angles*

$$\gamma_{ij} := \arccos \frac{\tilde{v}_i \cdot \tilde{v}_j}{|\tilde{v}_i| \cdot |\tilde{v}_j|}$$

satisfy $\gamma_{ij} \equiv \Gamma_{ij}$ and therefore automatically obey the relation

$$\frac{\sin \gamma_{01}}{\sigma_{01}} = \frac{\sin \gamma_{12}}{\sigma_{12}} = \frac{\sin \gamma_{02}}{\sigma_{02}}, \tag{9.3}$$

which is derived by using the Calculus of Variations in [6].

Proof. Our statements about the minimizer follow from our statements about the cost function, so we will skip immediately to proving those facts. In order to simplify our computations, we start by defining the following notation: For $j = 0$, 1, or 2, we define:

$$X_j := x - a_j\,, \quad Y_j := y - b_j\,,$$
$$\beta_j := (X_j^2 + Y_j^2)^{1/2}\,, \quad W_j := \zeta_j/\beta_j\,, \quad Z_j := \zeta_j/\beta_j^3\,.$$

All sums are assumed to be sums from $j = 0$ to 2. With our new notation, we easily compute:

$$C(x, y) = \sum \zeta_j \beta_j\,, \quad C_x(x, y) = \sum W_j X_j\,,$$
$$C_y(x, y) = \sum W_j Y_j\,, \quad C_{xx}(x, y) = \sum Z_j Y_j^2\,,$$
$$C_{xy}(x, y) = \sum Z_j X_j Y_j\,, \quad C_{yy}(x, y) = \sum Z_j X_j^2\,.$$

The trace of the Hessian D^2C is obviously strictly positive. The determinant of the Hessian D^2C is equal to

$$\Big(\sum Z_j X_j^2\Big)\Big(\sum Z_j Y_j^2\Big) - \Big(\sum Z_j X_j Y_j\Big)^2$$

and by using the Schwarz inequality on a set of three points with delta measures on each one weighted by Z_j we easily see that this determinant is nonnegative. On the other hand, by noting that equality in the Schwarz inequality only happens when one function is a multiple of the other, and that $(X_0, X_1, X_2) = \alpha(Y_0, Y_1, Y_2)$ would mean that the slopes of the vectors from $\tilde{P}$ to each vertex are the same, we can easily rule out equality, and so we conclude that D^2C is positive definite, thereby proving (A).

At this point we note that (D) follows immediately by definition of what a good triangle is, and (B) implies (C), now that we have our statement about the Hessian. In fact, it will suffice to show that the gradient vanishes at $(\tilde{x}, \tilde{y})$, as the uniqueness of the critical point of the cost function follows from positive definiteness of the Hessian. Thus, the gradient condition that we now need to show is equivalent to showing that

$$0 = \sum W_j X_j = \sum W_j Y_j \tag{9.4}$$

holds when $(x, y) = (\tilde{x}, \tilde{y})$.

We compute the sines of the angles, γ_{ij}, by giving them a zero z-component and then taking cross products (while carefully following the right-hand rule and recalling our convention about the counter-clockwise orientation of (P_0, P_1, P_2)):

$$\sin\gamma_{01} = \frac{(v_0 \times v_1)\cdot \hat{k}}{|v_0|\cdot|v_1|} = \frac{X_0Y_1 - X_1Y_0}{\beta_0\beta_1}\,, \tag{9.5}$$

$$\sin\gamma_{12} = \frac{(v_1 \times v_2)\cdot \hat{k}}{|v_1|\cdot|v_2|} = \frac{X_1Y_2 - X_2Y_1}{\beta_1\beta_2}\,, \tag{9.6}$$

$$\sin\gamma_{02} = \frac{(v_2 \times v_0)\cdot \hat{k}}{|v_2|\cdot|v_0|} = \frac{X_2Y_0 - X_0Y_2}{\beta_2\beta_0}\,, \tag{9.7}$$

where $\hat{k}$ is, as usual, the unit vector in the positive z direction.

Observe that

$$\frac{\sin\gamma_{01}}{\sigma_{01}} = \frac{X_0Y_1 - X_1Y_0}{\zeta_2\beta_0\beta_1} \quad\text{and}\quad \frac{\sin\gamma_{12}}{\sigma_{12}} = \frac{X_1Y_2 - X_2Y_1}{\zeta_0\beta_1\beta_2}\,. \tag{9.8}$$

Because we are assuming that we are at the point $(\tilde{x}, \tilde{y})$, we know that

$$\frac{X_0Y_1 - X_1Y_0}{\zeta_2\beta_0\beta_1} = \frac{X_1Y_2 - X_2Y_1}{\zeta_0\beta_1\beta_2}\,. \tag{9.9}$$

By cross multiplication and some cancellation of the β_1 we see that we have

$$\begin{aligned} W_0(X_0Y_1 - X_1Y_0) &= \frac{\zeta_0(X_0Y_1 - X_1Y_0)}{\beta_0} \\ &= \frac{\zeta_2(X_1Y_2 - X_2Y_1)}{\beta_2} = W_2(X_1Y_2 - X_2Y_1)\,. \end{aligned} \tag{9.10}$$

Now we note that

$$\begin{aligned} W_2(X_1Y_2 - X_2Y_1) &= W_0(X_0Y_1 - X_1Y_0) \\ &= W_0X_0Y_1 - W_0X_1Y_0 + W_1X_1Y_1 - W_1X_1Y_1 \end{aligned}$$

$$\begin{aligned} &= Y_1(W_0X_0 + W_1X_1) - X_1(W_0Y_0 + W_1Y_1) \\ &= Y_1(W_0X_0 + W_1X_1 + W_2X_2) - W_2(X_2Y_1) \\ &\quad - X_1(W_0Y_0 + W_1Y_1 + W_2Y_2) + W_2(X_1Y_2) \end{aligned}$$

and so we have

$$0 = Y_1(W_0X_0 + W_1X_1 + W_2X_2) - X_1(W_0Y_0 + W_1Y_1 + W_2Y_2)\,. \tag{9.11}$$

Arguing in the exact same way with each of the other combinations of angles, we see that

$$0 = Y_0(W_0X_0 + W_1X_1 + W_2X_2) - X_0(W_0Y_0 + W_1Y_1 + W_2Y_2), \tag{9.12}$$

$$0 = Y_2(W_0X_0 + W_1X_1 + W_2X_2) - X_2(W_0Y_0 + W_1Y_1 + W_2Y_2)\,. \tag{9.13}$$

Here again, if both $W_0X_0 + W_1X_1 + W_2X_2$ and $W_0Y_0 + W_1Y_1 + W_2Y_2$ did not vanish, then we would come to a contradiction by having the slopes of v_0, v_1, and v_2 all equal. Thus we have the nontrivial direction of (B), and (C) follows. □

Now we turn to a task which is essentially Euclidean geometry which will allow us to produce a good triangle.

Proposition 9.7 (Existence of Good Triangles). *Let $\{A_j\}$ be a permissible configuration of cones in $\mathbb{R}^2$ with vertices at the origin, and assume that as we move through a counterclockwise rotation, we have a sector which we will call A_2 which is a subset of E_2, followed by a sector which we will call A_0 which is a subset of E_0, followed by a sector which we will call A_1 which is a subset of E_1. Furthermore, assume that the angle of the opening for A_0 is strictly less than the real number Γ_{12}. Then letting P_0 be the origin, there exists a point $\tilde{P}$ within the infinite sector A_0, such that we can find a point P_1 on the ray between A_0 and A_2 and a point P_2 on the ray between A_0 and A_1 such that the triangle formed with vertices given by the P_j together with the point $\tilde{P}$ forms a good triangle. See Figure 9.*

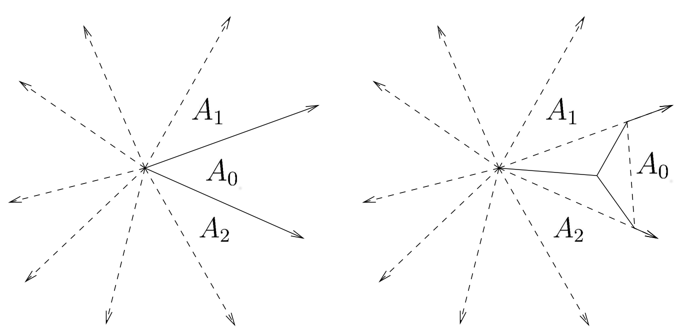


FIGURE 9. The basic setting for proposition 9.7.

Remark 9.8 (Key Assumption). We note that after relabeling things and/or reflecting things we see that the only real assumption we make is that the angle of the opening for A_0 is strictly smaller than Γ_{12}.

Proof. We consider the set of points with distance one from the origin which intersects the solid sector A_0, and we plan to make one of these points the point $\tilde{P}$ in our good triangle. At each point on that set we extend three rays with the following two properties:

(1) One of the rays passes through the origin.
(2) Going counter-clockwise from the rays passing through the origin, the angles between the rays are Γ_{01} followed by Γ_{12} followed by Γ_{02}.

Going counter-clockwise starting with the ray that passes through the origin, we will refer to these rays as the "zeroth ray," the "first ray," and the "second ray," respectively.

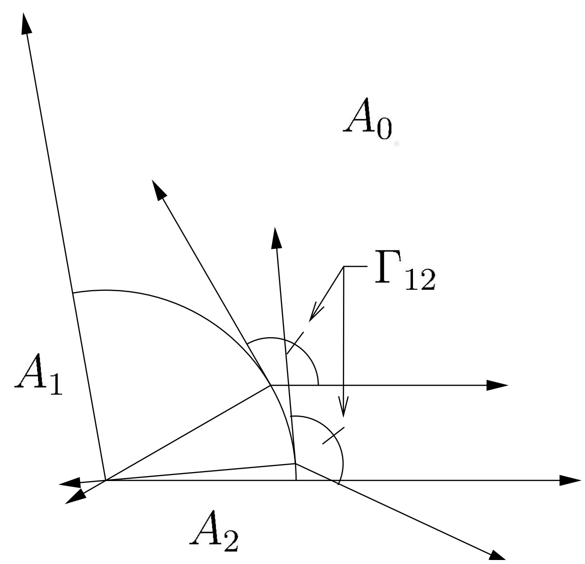


FIGURE 10. The basic picture.

In Figure 10 we have placed these three rays for two of the points with distance one from the origin and we have labeled the angle which has measure Γ_{12}. We can choose our coordinate system so that the border between A_0 and A_2 is the positive x-axis, and then owing to the fact that all of the $\Gamma_{ij} < \pi$, we see that with $\tilde{P}$ having $\theta > 0$ but sufficiently small, we must have an intersection of the first ray with $\partial A_0 \cap \partial A_2$. In Figure 10 we have chosen one of our potential $\tilde{P}$'s to have θ equal to five degrees. Now if the second ray has a nonempty intersection with $\partial A_0 \cap \partial A_1$, then we are done by letting P_1 and P_2 be the two points of intersection that we have found already. On the other hand, it is not necessarily the case that the second ray will intersect ∂A_1 if θ is sufficiently small. Assuming that there is no intersection we consider what happens as we increase θ while recalling that the main hypothesis guarantees that Γ_{12} is larger than the angle between the rays on either side of A_0. In particular, this hypothesis guarantees that the second ray will be parallel to $\partial A_1 \cap \partial A_0$ at a value of θ which we can call θ_1 which is strictly less than the value of θ which we call θ_2 where the first ray is parallel to $\partial A_2 \cap \partial A_0$. Then by taking θ strictly between θ_1 and θ_2 we are guaranteed a frame of three vectors with all of the desired intersections. (In Figure 10 we have chosen θ_2 as another value of θ where we plotted the three relevant rays. Decreasing θ from that value very slightly gives us what we need.) $\square$

The next proposition shows that at blow up limits we have a distinct sector for each fluid and not multiple sectors for any of the fluids.

Proposition 9.9 (One sector per fluid at blowups)**.** *Let* $\{A_j\}$ *be D-minimal cones in* $\mathbb{R}^2$ *with vertices at the origin. Then each* A_j *is formed of precisely one connected component.*

Proof. The first observation we need is that if we don't have all three fluids in any three consecutive sectors, then the triangle inequality guarantees an improvement by "filling in" near the triple point. See Figure 11 where we have only A_0 and A_1 in three consecutive sectors on the left hand side, and where we have an immediate improvement on the right hand side.

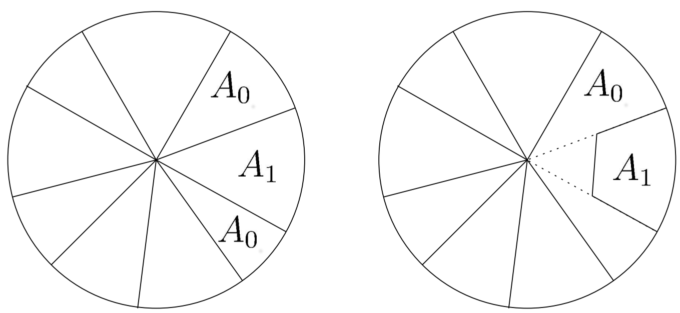


FIGURE 11. An improvement when three consecutive sectors have only two fluids.

Thus, it follows that if we have more than three sectors, then we must have at least six sectors.

Now by renaming and/or relabeling we can assume without loss of generality that we have the situation depicted on the left hand side of Figure 12. Furthermore, using the fact that we have at least six sectors now, we can assume that the angle of the sector for A_0 on the left hand side of the figure is less than or equal to $\Gamma_{12}/2$ which is strictly less than Γ_{12}. Now of course we can apply Proposition 9.7 to obtain the existence of a good triangle, followed by Proposition 9.6 to come to a contradiction. □

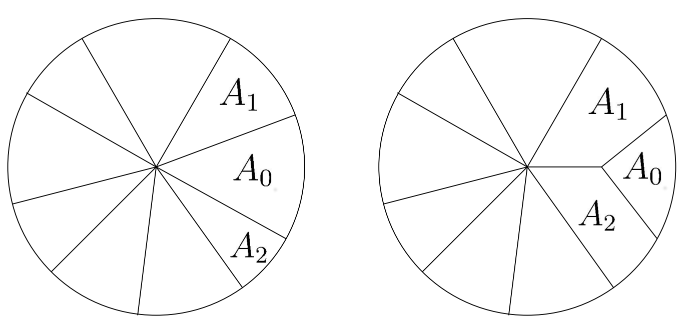


FIGURE 12. An improvement when three consecutive sectors have only two fluids.

Proof of Theorem 9.1. In fact, at this point, the D-minimal situation is essentially complete. The final observation needed is that if the angles are not exactly what they are supposed to be, then one of the angles is smaller than the corresponding Γ_{ij} and then (after renaming the indices if necessary) we can invoke Proposition 9.7 followed by Proposition 9.6 to achieve the desired result. Thus, we turn immediately to the V-minimal case.

The key observation in the V-minimal case is that we can actually improve Almgren's Volume Adjustment Lemma by removing any lack of uniformity when our configuration consists solely of cones. Indeed, we suppose toward a contradiction that we have a V-minimal configuration of cones which does not satisfy the angle condition. In this case, it follows from the D-minimal proof that we can lower the energy by some amount within B_1 if we temporarily ignore the volume constraint. On the other hand, by considering our sectors on a large enough disk, we can restore the volume constraint by adding or subtracting rectangles along the boundaries of the sectors at a cost which is bounded by twice the width of the rectangle times the largest σ_{ij}. See Figure 13. Of course, since we can choose our disk to be as large as we like, our rectangles can have arbitrarily small width, and therefore we can fix the volume constraint with a loss to our energy which is as small as we like. The arbitrarily small width that we can have for these rectangles also guarantees that even if one of our sectors is very thin, by shrinking the width of the rectangle if necessary, we do not have to worry about having an intersection with more than one of the rays bounding our sector. Thus, the original configuration could not possibly have been the V-minimizer on our large disk and that gives us the desired contradiction. □

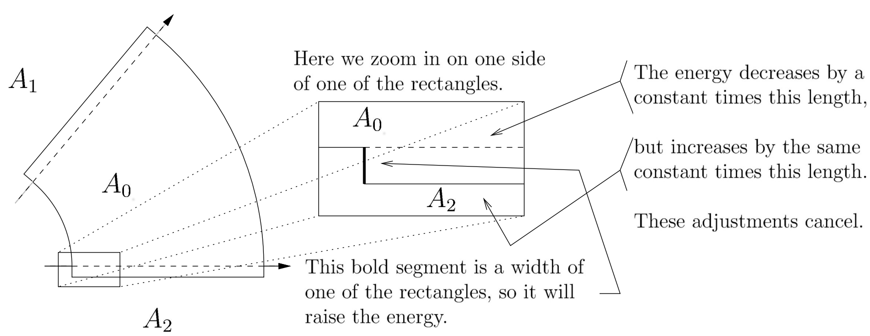


Figure 13. An example of using rectangles to adjust the volumes.

Remark 9.10 (Rectangles are not optimal, but very convenient)**.** The competing variations built by rectangles could be immediately improved by using smoother connections to the old boundary, however, we prefer the explicit construction presented in the proof.

We note that Futer, Gnepp, McMath, Munson, Ng, Pahk, and Yoder [9] studied planar cones that are minimizing, which is similar to some of the results above, however they proved their results using a calibration argument as in [13]. Lawlor and Morgan give a criteria for a configuration of immiscible fluids to be energy minimizing (see [13, equation 1, section 1.2]) in the case where the interfaces are pieces of planes, and presumably, this equation is equivalent to the Neumann angle condition in dimension 2 or 3, although they make no direct claims of this fact. Using Morgan's result that there are only finitely many triple points in the tangent plane [22], we may use our results to conclude that there are only finitely many triple points on the blow up sphere ∂B_1. Classifying this finite number of free boundary points remains an open problem.

10. Concluding comments

It is with great sadness that we must report that our collaborator Alan Elcrat passed away suddenly on December 20th, 2013. He was an energetic and hard-working mathematician, and a good friend and mentor. It is without doubt that the current work would not have been completed without him, and that future works will be more difficult without his insight.

Finally, to close with some cheer, we wish to thank Luis Silvestre and especially Frank Morgan for useful conversations. Silvestre helped us with certain aspects of the coarea formula, and Morgan assisted us greatly in understanding Allard's work. We also wish to thank the referees for their expertise with geometric measure theory and for their very constructive criticisms of earlier drafts of this work. Finally, the third author was a postdoc at Kansas State University when this project began, and he was also partially supported by an REP grant from Texas State University in 2012 for work on this project.

References

[1] W. K. Allard; On the first variation of a varifold, *Ann. Math.,* 95 (1972), 417–491.
[2] F. J. Almgren Jr.; Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints, *Mem. AMS,* no. 165 (1976).
[3] F. J. Almgren Jr.; Almgren's big regularity paper. Q-valued functions minimizing Dirichlet's integral and the regularity of area-minimizing rectifiable currents up to codimension 2. World Scientific Monograph Series in Mathematics, 1. World Scientific Publishing Co., Inc., 2000.
[4] L. Ambrosio, N. Fusco, D. Pallara; Functions of Bounded Variation and Free Discontinuity Problems, Oxford, 2000.
[5] I. Blank; Sharp results for the regularity and stability of the free boundary in the obstacle problem, *Indiana Univ. Math. J.,* 50 (2001), 1077–1112.
[6] A. Elcrat, R. Neel, D. Siegel; *Equilibrium configurations for a floating drop*, *J. Math. Fluid Mech.,* no. 4, 6(2004), 405–429.
[7] L. C. Evans, R. Gariepy; *Measure Theory and Fine Properties of Functions*, CRC Press, 1992.
[8] H. Federer; *Geometric measure theory, Die Grundlehren der mathematischen Wissenschaften*, Band 153 Springer-Verlag New York Inc., 1969
[9] D. Futer, A. Gnepp, D. McMath, B. Munson, T. Ng, S.H. Pahk, C. Yoder; Cost-minimizing networks among immiscible fluids in $\mathbb{R}^2$, *Pacific J. Math.* 196 (2000), no. 2, 395–414.
[10] E. Giusti; *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, 1984.
[11] M. Hutchings, F. Morgan, M. Ritoré, A. Ros; Proof of the Double Bubble Conjecture, *Ann. Math.,* **155** (2002), 459–489.
[12] M. de La Place; *Celestial mechanics. Vols. I–IV*; translated from the French, with commentary, by Nathaniel Bowditch, Chelsea Publishing Co. 1966
[13] G. Lawlor, F. Morgan; Paired calibrations applied to soap films, immiscible fluids, and surfaces or networks minimizing other norms, *Pacific J. Math.,* 166 (1994), no. 1, 55—83.
[14] G. P. Leonardi; Blow-up of oriented boundaries. *Rend. Sem. Mat. Univ. Padova,* 103 (2000), 211–232.
[15] G. P. Leonardi; Infiltrations in immiscible fluids systems, *Proc. Roy. Soc. Edinburgh Sect. A,* no. 2, 131 (2001), 425–436.

[16] G. P. Leonardi; Partitions with prescribed mean curvatures, *Manuscripta Math.,* 107 (2002), no. 1, 111–133.
[17] F. Maggi; Sets of finite perimeter and geometric variational problems, Cambridge University Press, 2012.
[18] U. Massari; The parametric problem of capillarity: the case of two and three fluids, *Astérisque,* 118 (1984), 197–203.
[19] U. Massari, M. Miranda; Minimal surfaces of codimension one. North-Holland Mathematics Studies, vol. 91. North-Holland Publishing Co., Amsterdam (1984).
[20] U. Massari, I. Tamanini; Regularity properties of optimal segmentations, *J. Reine Angew. Math.*, 420 (1991), 61–84.
[21] F. Morgan; Soap bubbles in $\mathbb{R}^2$ and in surfaces. *Pac. J. Math.,* 165 (1994), 347–361.
[22] F. Morgan; Immiscible fluid clusters in $\mathbb{R}^2$ and $\mathbb{R}^3$, *Mich. Math. J.,* 45 (1998), 441–450.
[23] F. Morgan; Clusters with multiplicities in $\mathbb{R}^2$. *Pacific J. Math.,* 221 (2005), no. 1, 123–146.
[24] F. Morgan; *Geometric Measure Theory: a Beginner's Guide*, Elsevier/Academic Press, fifth edition, 2016.
[25] J. Taylor; The structure of singularities in soap-bubble and soap-film-like minimal surfaces, *Ann. Math.,* 103 (1976), 489–539.
[26] B. White; Tangent cones to two-dimensional area-minimizing integral currents are unique. *Duke Math. J.*, 50 (1983), no. 1, 143–160.
[27] B. White; Existence of least-energy configurations of immiscible fluids. *J. Geom. Anal.*, 6 (1996), no. 1, 151–161.

Ivan Blank
Department of Mathematics, Kansas State University, Manhattan, KS 66506, USA
Email address: blanki@math.ksu.edu

Alan Elcrat (deceased)
Department of Mathematics, Wichita State University, Wichita, KS 67260, USA

Raymond Treinen
Department of Mathematics, Texas State University, 601 University drive, San Marcos, TX 78666, USA
Email address: rt30@txstate.edu