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NULL CONTROLLABILITY OF A COUPLED SYSTEM OF DEGENERATE PARABOLIC EQUATIONS WITH LOWER ORDER TERMS

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Abstract. This article concerns the null controllability of a control system governed by coupled degenerate parabolic equations with lower order terms. For these equations, the convection terms cannot be controlled by the diffusion terms. We establish a Carleman estimate and an observability inequality by using Carleman estimates for single degenerate parabolic equations with lower order term and some energy estimates. Then we prove that the system with two controls is null controllable. Finally, we show the null controllability of the system with one control, by constructing suitable controls.

1. Introduction

In this article, we study the null controllability of the following control system governed by coupled degenerate parabolic equations with lower order terms

$$u_t - (x^{\lambda_1} u_x)_x + b_1(x,t)u_x + c_1(x,t)u + c_2(x,t)v = h(x,t)\chi_\omega, \quad (x,t) \in Q_T\,, \tag{1.1}$$

$$v_t - (x^{\lambda_2} v_x)_x + b_2(x,t)v_x + c_3(x,t)u + c_4(x,t)v = 0, \quad (x,t) \in Q_T\,, \tag{1.2}$$

$$u(0,t) = v(0,t) = 0, \quad u(1,t) = v(1,t) = 0, \quad t \in (0,T), \tag{1.3}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in (0,1), \tag{1.4}$$

where $0 < \lambda_1, \lambda_2 < 1$, $Q_T = (0,1) \times (0,T)$, $b_1, b_2 \in W^{2,1}_\infty(Q_T)$, $c_1, c_2, c_3, c_4 \in L^\infty(Q_T)$, h is a control function, χ_ω is the characteristic function of $\omega = (x_0, x_1)$ with $0 < x_0 < x_1 < 1$, $u_0, v_0 \in L^2(0,1)$. Note that, the equations (1.1) and (1.2) are degenerate at the boundary $x = 0$. The coupled equations (1.1) and (1.2) are the linear version of some models in mathematical biology and physics, such as the Keller-Segel model [8] and the Lotka-Volterra model [24].

Controllability theory for nondegenerate parabolic equations has been widely investigated over the previous forty years and has been almost completed (see, e.g. [6]). Recently, the controllability theory for degenerate parabolic equations has been studied and some results have been known ([3, 7, 9, 10, 12, 14, 16, 17, 23, 26, 27, 28, 25] and the references therein). Among these, the null controllability of the following degenerate parabolic system has been extensively studied,

$$u_t - (x^{\lambda} u_x)_x + c(x,t)u = h(x,t)\chi_\omega, \quad (x,t) \in Q_T\,, \tag{1.5}$$

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$$\left.\begin{aligned} u(0,t)=u(1,t)=0, &\quad \text{if } 0<\lambda<1,\ t\in(0,T)\\ (x^\lambda u_x)(0,t)=u(1,t)=0, &\quad \text{if } \lambda\ge 1,\ t\in(0,T), \end{aligned}\right\} \tag{1.6}$$

$$u(x,0)=u_0(x),\quad x\in(0,1)\,. \tag{1.7}$$

System (1.5)–(1.7) was proved to be null controllable if $0<\lambda<2$ in [3, 10, 23], while not if $\lambda\ge 2$ in [9]. Although system (1.5)–(1.7) is not null controllable in the case $\lambda\ge 2$, it was shown to be regional null controllable and approximate controllable in $L^2(0,1)$ for each $\lambda>0$ (see e.g. [9, 26]). Flores and Teresa [16] studied the linear degenerate convection-diffusion equation

$$u_t-(x^\lambda u_x)_x+x^{\lambda/2}b(x,t)u_x+c(x,t)u=h(x,t)\chi_\omega,\quad (x,t)\in Q_T \tag{1.8}$$

with $b\in L^\infty(Q_T)$, and they proved the null controllability of system (1.8), (1.6), and (1.7) if $0<\lambda<2$. It is noted that the convection term can be controlled by the diffusion term in (1.8). Wang and Du [27] studied the linear degenerate convection-diffusion equation

$$u_t-(x^\lambda u_x)_x+(b(x,t)u)_x+c(x,t)u=h(x,t)\chi_\omega,\quad (x,t)\in Q_T \tag{1.9}$$

with $b\in L^\infty(Q_T)$, and they showed the null controllability of (1.9), (1.6) and (1.7) if $0<\lambda<1/2$. Here, the restriction $0<\lambda<1/2$ is optimal when one establishes the Carleman estimate in the same way as in [27]. Furthermore, in [28], the authors investigated the linear system

$$u_t-(x^\lambda u_x)_x+b(x,t)u_x+c(x,t)u=h(x,t)\chi_\omega,\quad (x,t)\in Q_T$$

with (1.6) and (1.7) and they proved that the system is null controllable if $b\in W^{2,1}_\infty(Q_T)$ and $0<\lambda<1$. But the other case ($1\le\lambda<2$) is still unknown. Note that, in [27, 28], the convection term cannot be controlled by the diffusion term. For the controllability theory of the nondegenerate coupled systems, we refer to [4, 5, 18, 19, 20, 21]. As to the degenerate parabolic system (1.1)–(1.4), [1, 2, 22] considered the special case that $b_1=b_2=0$ in Q_T, [11, 13] studied the special case that $b_1=b_2=0$ in Q_T and $\lambda_1=\lambda_2$. Du and Xu [15] proved the null controllability of the system

$$\begin{gathered} u_t-(x^\lambda u_x)_x+b_1(x,t)u_x+b_2(x,t)v_x+c_{11}(x,t)u+c_{12}(x,t)v=h(x,t)\chi_w,\\ (x,t)\in Q_T\,, \end{gathered} \tag{1.10}$$

$$v_t-(x^\lambda v_x)_x+b_3(x,t)v_x+c_{21}(x,t)u+c_{22}(x,t)v=0,\quad (x,t)\in Q_T\,, \tag{1.11}$$

$$u(0,t)=v(0,t)=0\quad \text{if } 0<\lambda<1,\quad t\in(0,T), \tag{1.12}$$

$$(x^\lambda u_x)(0,t)=(x^\lambda v_x)(0,t)=0\quad \text{if } 1\le\lambda<2,\quad t\in(0,T), \tag{1.13}$$

$$u(1,t)=v(1,t)=0,\quad t\in(0,T), \tag{1.14}$$

$$u(x,0)=u_0(x),\quad v(x,0)=v_0(x),\quad x\in(0,1), \tag{1.15}$$

where $b_i\in L^\infty((0,T;W^{1,\infty}(0,1)))$ with

$$|b_i(x,t)|\le Kx^{\lambda/2},\quad (x,t)\in Q_T\,,\quad i=1,2,3. \tag{1.16}$$

They proved that system (1.10)–(1.15) is null controllable if $0<\lambda<2$. In equations (1.10) and (1.11), the convection terms can be controlled by the diffusion terms owing to (1.16). Moreover, [29] considered the following semilinear degenerate parabolic cascade system with general convection terms

$$u_t-(x^\lambda u_x)_x+(P_1(x,t,u))_x+F_1(x,t,u)=h(x,t)\chi_\omega,\quad (x,t)\in Q_T\,, \tag{1.17}$$

$$v_t - (x^\lambda v_x)_x + (P_2(x,t,v))_x + F_2(x,t,u,v) = 0, \quad (x,t) \in Q_T\,, \tag{1.18}$$

$$u(0,t) = v(0,t) = 0, \quad u(1,t) = v(1,t) = 0, \quad t \in (0,T), \tag{1.19}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in (0,1), \tag{1.20}$$

and proved that system (1.17)–(1.20) is null controllable if $0 < \lambda < 1/2$. It is noted that the convection terms cannot be controlled by the diffusion terms in the equations (1.17) and (1.18).

In this article, we study the null controllability of the degenerate parabolic system (1.1)–(1.4), where the control acts on only one equation. In particular, the convection terms cannot be controlled by the diffusion terms. By using a Carleman estimate for the case of a single degenerate parabolic equation with lower order term [28] and some energy estimates, we establish a Carleman estimate and the observability inequality for solutions to the conjugate problem. Then we can prove that the system with two controls is null controllable by the observability inequality. By means of this null controllability result, we can construct suitable controls for the system (1.1)–(1.4).

This article is organized as follows. In section 2, we prove the null controllability of the system with two controls by establishing the energy estimates, the Carleman estimate and the observability inequality. Subsequently, the null controllability of the system (1.1)–(1.4) is proved in section 3.

2. Carleman estimate and null controllability of the system with two controls

In this section, we prove the null controllability of the following system with two controls

$$u_t - (x^{\lambda_1} u_x)_x + b_1(x,t)u_x + c_1(x,t)u + c_2(x,t)v = h_1(x,t)\chi_{\tilde\omega}, \quad (x,t) \in Q_T\,, \tag{2.1}$$

$$v_t - (x^{\lambda_2} v_x)_x + b_2(x,t)v_x + c_3(x,t)u + c_4(x,t)v = h_2(x,t)\chi_{\tilde\omega}, \quad (x,t) \in Q_T\,, \tag{2.2}$$

subject to conditions (1.3) and (1.4), where $\tilde\omega \Subset \omega$ is an open interval such that

$$\operatorname{supp} c_2 \subset \tilde\omega \times [0,T]. \tag{2.3}$$

Equations (2.1) and (2.2) are degenerate at the boundary $x = 0$. We first consider the regularized problem

$$u^\eta_t - ((x+\eta)^{\lambda_1} u^\eta_x)_x + b_1(x,t)u^\eta_x + c_1(x,t)u^\eta + c_2(x,t)v^\eta = f_1(x,t), \quad (x,t) \in Q_T\,, \tag{2.4}$$

$$v^\eta_t - ((x+\eta)^{\lambda_2} v^\eta_x)_x + b_2(x,t)v^\eta_x + c_3(x,t)u^\eta + c_4(x,t)v^\eta = f_2(x,t), \quad (x,t) \in Q_T\,, \tag{2.5}$$

$$u^\eta(0,t) = v^\eta(0,t) = 0, \quad u^\eta(1,t) = v^\eta(1,t) = 0, \quad t \in (0,T), \tag{2.6}$$

$$u^\eta(x,0) = u_0(x), \quad v^\eta(x,0) = v_0(x), \quad x \in (0,1), \tag{2.7}$$

where $0 < \lambda_1, \lambda_2 < 1$, $0 < \eta < 1$, $b_i \in W^{2,1}_\infty(Q_T)$ for $i = 1,2$, $c_j \in L^\infty(Q_T)$ for $1 \le j \le 4$, $f_1, f_2 \in L^2(Q_T)$, and $u_0, v_0 \in L^2(0,1)$. Thanks to the classical theory on parabolic equations, there exists a unique solution (u^η, v^η) with $u^\eta, v^\eta \in L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;H^1(0,1))$ to the problem (2.4)–(2.7). Furthermore, the solution (u^η, v^η) satisfies the following a priori estimates.

Lemma 2.1. *Assume that* $0 < \lambda_1, \lambda_2 < 1$, $0 < \eta < 1$, $b_i \in W^{2,1}_\infty(Q_T)$ *with* $\|b_i\|_{L^\infty(Q_T)} \le K$ *and* $\|(b_i)_x\|_{L^\infty(Q_T)} \le K$ $(i = 1, 2)$, $c_j \in L^\infty(Q_T)$ *with* $\|c_j\|_{L^\infty(Q_T)} \le K$ $(1 \le j \le 4)$, $f_1, f_2 \in L^2(Q_T)$, *and* $u_0, v_0 \in L^2(0, 1)$. *Then, the solution* (u^η, v^η) *of problem* (2.4)–(2.7) *satisfies*

$$
\begin{aligned}
&\|u^\eta\|_{L^\infty(0,T;L^2(0,1))} + \|(x+\eta)^{\lambda_1/2}u^\eta_x\|_{L^2(Q_T)} + \|v^\eta\|_{L^\infty(0,T;L^2(0,1))}\\
&+ \|(x+\eta)^{\lambda_2/2}v^\eta_x\|_{L^2(Q_T)}\\
&\le M(\|f_1\|_{L^2(Q_T)} + \|f_2\|_{L^2(Q_T)} + \|u_0\|_{L^2(0,1)} + \|v_0\|_{L^2(0,1)}),
\end{aligned} \tag{2.8}
$$

$$
\begin{aligned}
&\Big|\int_0^1 (u^\eta(x,t_2) - u^\eta(x,t_1))\xi(x)\,\mathrm{d}x\Big| + \Big|\int_0^1 (v^\eta(x,t_2) - v^\eta(x,t_1))\xi(x)\,\mathrm{d}x\Big|\\
&\le M(t_2-t_1)^{1/2}(\|f_1\|_{L^2(Q_T)} + \|f_2\|_{L^2(Q_T)} + \|u_0\|_{L^2(0,1)}\\
&\quad + \|v_0\|_{L^2(0,1)})\|\xi\|_{H^1(0,1)}, \quad 0 \le t_1 < t_2 \le T, \quad \xi \in H^1(0,1),
\end{aligned} \tag{2.9}
$$

$$
\begin{aligned}
&\int_0^{T-\delta}\int_0^1 (u^\eta(x,\tau+\delta) - u^\eta(x,\tau))^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&+ \int_0^{T-\delta}\int_0^1 (v^\eta(x,\tau+\delta) - v^\eta(x,\tau))^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&\le M\delta^{1/2}(\|f_1\|^2_{L^2(Q_T)} + \|f_2\|^2_{L^2(Q_T)} + \|u_0\|^2_{L^2(0,1)} + \|v_0\|^2_{L^2(0,1)}),\\
&\quad 0 < \delta < T,
\end{aligned} \tag{2.10}
$$

where $M > 0$ *is a constant depending only on* K, T, λ_1, *and* λ_2.

The above lemma is similar to [29, Lemma 2.1], where the special case that $\lambda_1 = \lambda_2$ was considered. Here we omit the proof.

By using the a priori estimates (2.8)–(2.10), one can prove the well-posedness of the problem (2.1), (2.2), (1.3), and (1.4) in a standard way (it is referred to [27, 28] for the case of a single equation). That is to say, one has

Lemma 2.2. *For any* $h_1, h_2 \in L^2(Q_T)$ *and* $u_0, v_0 \in L^2(0,1)$, *problem* (2.1), (2.2), (1.3), *and* (1.4) *admits a unique solution* (u, v) *with* u, v, $x^{\lambda_1/2}u_x$, $x^{\lambda_2/2}v_x \in L^2(Q_T)$. *Furthermore,* $u, v \in L^\infty(0,T;L^2(0,1)) \cap C_w([0,T];L^2(0,1))$. *Here, a function* $\zeta \in C_w([0,T];L^2(0,1))$ *means that* $\int_0^1 \zeta(x,t)\gamma(x)\,\mathrm{d}x \in C([0,T])$ *for each* $\gamma \in L^2(0,1)$.

To show the null controllability of system (2.1), (2.2), (1.3), and (1.4), we establish a Carleman estimate and an observability inequality for solutions to its conjugate problem

$$-y_t - (x^{\lambda_1}y_x)_x - (b_1(x,t)y)_x + c_1(x,t)y + c_3(x,t)z = 0, \quad (x,t) \in Q_T\,, \tag{2.11}$$
$$-z_t - (x^{\lambda_2}z_x)_x - (b_2(x,t)z)_x + c_2(x,t)y + c_4(x,t)z = 0, \quad (x,t) \in Q_T\,, \tag{2.12}$$
$$y(0,t) = z(0,t) = 0, \quad y(1,t) = z(1,t) = 0, \quad t \in (0,T), \tag{2.13}$$
$$y(x,T) = y_T(x), \quad z(x,T) = z_T(x), \quad x \in (0,1). \tag{2.14}$$

Theorem 2.3 (Carleman Estimate)**.** *Assume that* $b_1, b_2 \in W^{2,1}_\infty(Q_T)$, c_1, c_2, c_3, $c_4 \in L^\infty(Q_T)$, *and* (2.3) *holds. There exist two constants* $s_0 > 0$ *and* $M_0 > 0$ *depending only on* $\|b_i\|_{W^{2,1}_\infty(Q_T)}$ $(i = 1,2)$, $\|c_i\|_{L^\infty(Q_T)}$ $(1 \le i \le 4)$, $\tilde\omega$, $\operatorname{supp} c_2$, κ_1, κ_2, T, λ_1 *and* λ_2, *such that for each* $y_T, z_T \in L^2(0,1)$ *and each* $s \ge s_0$, *the solution*

(y,z) to (2.11)–(2.14) satisfies

$$\int_0^T\int_0^1 \Big((s\theta x^{\lambda_1}y_x^2+s^3\theta^3x^{2-\lambda_1}y^2)\mathrm{e}^{2s\varphi_1}+(s\theta x^{\lambda_2}z_x^2+s^3\theta^3x^{2-\lambda_2}z^2)\mathrm{e}^{2s\varphi_2}\Big)\,\mathrm{d}x\,\mathrm{d}t \\ \le M_0\int_0^T\int_{\tilde\omega}(y^2+z^2)\,\mathrm{d}x\,\mathrm{d}t,$$

where

$$\varphi_i(x,t)=\theta(t)g_i(x,t),\quad (x,t)\in Q_T\,,\quad i=1,2,$$
$$\theta(t)=\frac{1}{(t(T-t))^4},\quad t\in(0,T),$$
$$g_i(x)=\frac{\kappa_i(x^{2-\lambda_i}-2)}{2-\lambda_i},\quad x\in(0,1),\quad i=1,2,$$

while $\kappa_1,\kappa_2>0$ are constants such that $g_1\le g_2$ in $(0,1)$.

Proof. By a regularization process and some a prior estimates (see, e.g., [27, 28]), one can assume that $y,z\in C^2(\overline{Q}_T)$. In the proof, M_i $(1\le i\le 7)$ and s_i $(i=1,2,3)$ are generic positive constants depending only on $\|b_i\|_{W^{2,1}_\infty(Q_T)}$ $(i=1,2)$, $\|c_i\|_{L^\infty(Q_T)}$ $(1\le i\le 4)$, $\tilde\omega$, $\operatorname{supp} c_2$, κ_1, κ_2, T, λ_1 and λ_2.

Choose open intervals ω_1 and ω_2 such that $\operatorname{supp} c_2\subset\omega_1\times[0,T]$ and $\omega_1\Subset\omega_2\Subset\tilde\omega$. Let $\psi,\xi\in C^\infty([0,1])$ satisfy

$$\psi\begin{cases}=1, & x\in[0,\inf\omega_1]\,,\\ \in[0,1], & x\in\omega_1\,,\\ =0, & x\in[\sup\omega_1,1]\,,\end{cases}\qquad \xi\begin{cases}=1, & x\in\omega_1\,,\\ \in[0,1], & x\in\omega_2\setminus\omega_1,\\ =0, & x\in[0,1]\setminus\omega_2\,.\end{cases}$$

Set

$$w(x,t)=\psi(x)y(x,t),\quad (x,t)\in\overline{Q}_T\,,$$
$$W(x,t)=\psi(x)z(x,t),\quad (x,t)\in\overline{Q}_T\,.$$

Then (w,W) solves

$$w_t+(x^{\lambda_1}w_x)_x+(b_1w)_x-c_1w=\rho_1,\quad (x,t)\in Q_T\,, \tag{2.15}$$
$$W_t+(x^{\lambda_2}W_x)_x+(b_2W)_x-c_4W=\rho_2,\quad (x,t)\in Q_T\,, \tag{2.16}$$

where

$$\rho_1(x,t)=(x^{\lambda_1}\psi'(x)y(x,t))_x+x^{\lambda_1}\psi'(x)y_x(x,t)+b_1(x,t)\psi'(x)y(x,t) \\ +c_3(x,t)\psi(x)z(x,t),\quad (x,t)\in Q_T\,,$$
$$\rho_2(x,t)=(x^{\lambda_2}\psi'(x)z(x,t))_x+x^{\lambda_2}\psi'(x)z_x(x,t)+b_2(x,t)\psi'(x)z(x,t) \\ +c_2(x,t)\psi(x)y(x,t),\quad (x,t)\in Q_T\,.$$

Using the Carleman estimate established in [28, Theorem 3.1] for (2.15) and (2.16), one obtains $M_1>0$ and $s_1>0$ such that

$$\int_0^T\int_0^1((sx^{\lambda_1}\theta w_x^2+s^3x^{2-\lambda_1}\theta^3w^2)\mathrm{e}^{2s\varphi_1}+(sx^{\lambda_2}\theta W_x^2+s^3x^{2-\lambda_2}\theta^3W^2)\mathrm{e}^{2s\varphi_2})\,\mathrm{d}x\,\mathrm{d}t \\ \le M_1\Big(\int_0^T\int_0^1\rho_1^2\mathrm{e}^{2s\varphi_1}\,\mathrm{d}x\,\mathrm{d}t+\int_0^T s\theta(t)\mathrm{e}^{2s\varphi_1(1,t)}w_x^2(1,t)dt$$

$$
\begin{aligned}
&+\int_0^T\int_0^1 \rho_2^2 \mathrm{e}^{2s\varphi_2}\,\mathrm{d}x\,\mathrm{d}t+\int_0^T s\theta(t)\mathrm{e}^{2s\varphi_2(1,t)}W_x^2(1,t)dt\Big)\\
&= M_1\int_0^T\int_0^1(\rho_1^2\mathrm{e}^{2s\varphi_1}+\rho_2^2\mathrm{e}^{2s\varphi_2})\,\mathrm{d}x\,\mathrm{d}t,\quad s\ge s_1.
\end{aligned}
$$

The definitions of ρ_1, ρ_2 and (2.3) yield

$$
\begin{aligned}
&\int_0^T\int_0^1((sx^{\lambda_1}\theta w_x^2+s^3x^{2-\lambda_1}\theta^3w^2)\mathrm{e}^{2s\varphi_1}\\
&+(sx^{\lambda_2}\theta W_x^2+s^3x^{2-\lambda_2}\theta^3W^2)\mathrm{e}^{2s\varphi_2})\,\mathrm{d}x\,\mathrm{d}t\\
&\le M_2\int_0^T\int_{\omega_1}(\mathrm{e}^{2s\varphi_1}(y^2+y_x^2)+\mathrm{e}^{2s\varphi_2}(z^2+z_x^2+y^2))\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+M_2\int_0^T\int_0^1 c_3^2\psi^2z^2\mathrm{e}^{2s\varphi_1}\,\mathrm{d}x\,\mathrm{d}t\\
&\le 2M_2\int_0^T\int_{\omega_1}\mathrm{e}^{2s\varphi_2}(y^2+y_x^2+z^2+z_x^2)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+M_2\int_0^T\int_0^1 c_3^2W^2\mathrm{e}^{2s\varphi_1}\,\mathrm{d}x\,\mathrm{d}t,\quad s\ge s_1.
\end{aligned}
\tag{2.17}
$$

Hardy's inequality gives

$$
\begin{aligned}
&\int_0^T\int_0^1 c_3^2W^2\mathrm{e}^{2s\varphi_1}\,\mathrm{d}x\,\mathrm{d}t\\
&\le\frac{4}{(1-\lambda_2)^2}\|c_3\|^2_{L^\infty(Q_T)}\int_0^T\int_0^1 x^{\lambda_2}((W\mathrm{e}^{s\varphi_2})_x)^2\,\mathrm{d}x\,\mathrm{d}t\\
&\le\frac{8}{(1-\lambda_2)^2}\|c_3\|^2_{L^\infty(Q_T)}\int_0^T\int_0^1(x^{\lambda_2}W_x^2\mathrm{e}^{2s\varphi_2}+s^2\kappa_2^2x^{2-\lambda_2}\theta^2W^2\mathrm{e}^{2s\varphi_2})\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

which, together with (2.17), leads to that there exist $M_3>0$ and $s_2>0$ such that

$$
\begin{aligned}
&\int_0^T\int_0^1((sx^{\lambda_1}\theta w_x^2+s^3x^{2-\lambda_1}\theta^3w^2)\mathrm{e}^{2s\varphi_1}\\
&+(sx^{\lambda_2}\theta W_x^2+s^3x^{2-\lambda_2}\theta^3W^2)\mathrm{e}^{2s\varphi_2})\,\mathrm{d}x\,\mathrm{d}t\\
&\le M_3\int_0^T\int_{\omega_1}\mathrm{e}^{2s\varphi_2}(y^2+y_x^2+z^2+z_x^2)\,\mathrm{d}x\,\mathrm{d}t,\quad s\ge s_2.
\end{aligned}
\tag{2.18}
$$

From (2.11)–(2.13), we obtain

$$
\begin{aligned}
0&=\int_0^T\frac{d}{dt}\int_0^1\xi^2\mathrm{e}^{2s\varphi_2}(y^2+z^2)\,\mathrm{d}x\,\mathrm{d}t\\
&=2s\int_0^T\int_0^1\xi^2(\varphi_2)_t\mathrm{e}^{2s\varphi_2}(y^2+z^2)\,\mathrm{d}x\,\mathrm{d}t+2\int_0^T\int_0^1\xi^2\mathrm{e}^{2s\varphi_2}(yy_t+zz_t)\,\mathrm{d}x\,\mathrm{d}t\\
&=2s\int_0^T\int_0^1\xi^2(\varphi_2)_t\mathrm{e}^{2s\varphi_2}(y^2+z^2)\,\mathrm{d}x\,\mathrm{d}t+2\int_0^T\int_0^1\xi^2\mathrm{e}^{2s\varphi_2}(x^{\lambda_1}y_x^2+x^{\lambda_2}z_x^2)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+4\int_0^T\int_0^1\xi\xi_x\mathrm{e}^{2s\varphi_2}(yx^{\lambda_1}y_x+zx^{\lambda_2}z_x)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+ 4s\int_0^T\int_0^1 \xi^2(\varphi_2)_x e^{2s\varphi_2}(yx^{\lambda_1}y_x + zx^{\lambda_2}z_x)\,\mathrm{d}x\,\mathrm{d}t\\
&- 2\int_0^T\int_0^1 \xi^2 e^{2s\varphi_2}(y(b_1y)_x + z(b_2z)_x)\,\mathrm{d}x\,\mathrm{d}t\\
&+ 2\int_0^T\int_0^1 \xi^2 e^{2s\varphi_2}(c_1y^2 + c_4z^2)\,\mathrm{d}x\,\mathrm{d}t\\
&+ 2\int_0^T\int_0^1 \xi^2 e^{2s\varphi_2}(c_2yz + c_3yz)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\int_0^T\int_0^1 \xi^2 e^{2s\varphi_2}(x^{\lambda_1}y_x^2 + x^{\lambda_2}z_x^2)\,\mathrm{d}x\,\mathrm{d}t\\
&\le \frac{1}{2}\int_0^T\int_0^1 \xi^2 e^{2s\varphi_2}(x^{\lambda_1}y_x^2 + x^{\lambda_2}z_x^2)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + M_4(1+s^2)\int_0^T\int_{\omega_2} \theta^2 e^{2s\varphi_2}(y^2+z^2)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_0^T\int_{\omega_1} e^{2s\varphi_2}y_x^2\,\mathrm{d}x\,\mathrm{d}t &\le \frac{1}{x_0^{\lambda_1}}\int_0^T\int_0^1 \xi^2 e^{2s\varphi_2}x^{\lambda_1}y_x^2\,\mathrm{d}x\,\mathrm{d}t\\
&\le \frac{2M_4}{x_0^{\lambda_1}}(1+s^2)\int_0^T\int_{\omega_2}\theta^2 e^{2s\varphi_2}y^2\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{2.19}
$$

$$
\begin{aligned}
\int_0^T\int_{\omega_1} e^{2s\varphi_2}z_x^2\,\mathrm{d}x\,\mathrm{d}t &\le \frac{1}{x_0^{\lambda_2}}\int_0^T\int_0^1 \xi^2 e^{2s\varphi_2}x^{\lambda_2}z_x^2\,\mathrm{d}x\,\mathrm{d}t\\
&\le \frac{2M_4}{x_0^{\lambda_2}}(1+s^2)\int_0^T\int_{\omega_2}\theta^2 e^{2s\varphi_2}z^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.20}
$$

Note that

$$
0 < e^{2s\varphi_2(x,t)} < 1,\quad 0 < (1+s^2)\theta^2(t)e^{2s\varphi_2(x,t)} \le M_5,\quad s \ge s_2,\ (x,t)\in Q_T\,. \tag{2.21}
$$

It follows from (2.18)–(2.21) that

$$
\begin{aligned}
&\int_0^T\int_0^1 ((sx^{\lambda_1}\theta w_x^2 + s^3x^{2-\lambda_1}\theta^3w^2)e^{2s\varphi_1} + (sx^{\lambda_2}\theta W_x^2 + s^3x^{2-\lambda_2}\theta^3W^2)e^{2s\varphi_2})\,\mathrm{d}x\,\mathrm{d}t\\
&\le M_6\int_0^T\int_{\tilde\omega} e^{2s\varphi_2}(y^2+z^2)\,\mathrm{d}x\,\mathrm{d}t,\quad s\ge s_2.
\end{aligned} \tag{2.22}
$$

Set

$$
\begin{aligned}
U(x,t) &= y(x,t) - w(x,t),\quad (x,t)\in \overline{Q}_T\,,\\
V(x,t) &= z(x,t) - W(x,t),\quad (x,t)\in \overline{Q}_T\,.
\end{aligned}
$$

By using the classical Carleman estimate, we can prove by the similar process as in [28] that there exist $M_7 > 0$ and $s_3 > 0$ such that

$$
\int_0^T\int_0^1 ((sx^{\lambda_1}\theta U_x^2 + s^3x^{2-\lambda_1}\theta^3U^2)e^{2s\varphi_1} + (sx^{\lambda_2}\theta V_x^2 + s^3x^{2-\lambda_2}\theta^3V^2)e^{2s\varphi_2})\,\mathrm{d}x\,\mathrm{d}t
$$

$$\le M_7 \int_0^T\int_{\tilde\omega}(y^2+z^2)\,\mathrm{d}x\,\mathrm{d}t, \quad s\ge s_3,$$

which, together with (2.22), completes the proof. □

We remark that in the proof of Theorem 2.3, (2.3) is needed generally. However, (2.3) is not needed for the special case $\lambda_1=\lambda_2$.

Theorem 2.4 (Observability Inequality)**.** *Assume that $b_1, b_2 \in W^{2,1}_\infty(Q_T)$, c_1, c_2, c_3, c_4 belong to $L^\infty(Q_T)$, and (2.3) holds. There exists $M>0$ depending only on $\|b_i\|_{W^{2,1}_\infty(Q_T)}$ $(i=1,2)$, $\|c_i\|_{L^\infty(Q_T)}$ $(1\le i\le 4)$, $\tilde\omega$, $\operatorname{supp} c_2$, κ_1, κ_2, T, λ_1 and λ_2, such that for each $y_T, z_T\in L^2(0,1)$, the solution (y,z) to problem (2.11)–(2.14) satisfies*

$$\int_0^1 (y^2(x,0)+z^2(x,0))dx \le M\int_0^T\int_{\tilde\omega}(y^2+z^2)\,\mathrm{d}x\,\mathrm{d}t.$$

Proof. It is assumed that $y,z\in C^2(\overline{Q}_T)$ as in Theorem 2.3. Multiplying (2.11) and (2.12) by y and z, respectively, and then integrating over $(0,1)$ with respect to x, one gets

$$-\frac12\frac{d}{dt}\int_0^1 y^2\,\mathrm{d}x+\int_0^1 x^{\lambda_1}y_x^2\mathrm{d}x+\int_0^1 b_1(x,t)yy_x\,\mathrm{d}x+\int_0^1 c_1y^2\mathrm{d}x+\int_0^1 c_3yz\mathrm{d}x=0,$$
$$-\frac12\frac{d}{dt}\int_0^1 z^2\,\mathrm{d}x+\int_0^1 x^{\lambda_2}z_x^2\mathrm{d}x+\int_0^1 b_2(x,t)zz_x\,\mathrm{d}x+\int_0^1 c_2yz\,\mathrm{d}x+\int_0^1 c_4z^2\mathrm{d}x=0$$

for $t\in(0,T)$. Hölder's inequality and Hardy's inequality yield

$$-\frac{d}{dt}\int_0^1(y^2+z^2)\,\mathrm{d}x\le \tilde M\int_0^1(y^2+z^2)\,\mathrm{d}x, \quad t\in(0,T),$$

where $\tilde M>0$ depends only on $\|b_i\|_{L^\infty(Q_T)}$ $(i=1,2)$, $\|c_i\|_{L^\infty(Q_T)}$ $(1\le i\le 4)$, λ_1 and λ_2. Thus

$$\int_0^1((y(x,0))^2+(z(x,0))^2)\,\mathrm{d}x\le \mathrm{e}^{\tilde Mt}\int_0^1(y^2(x,t)+z^2(x,t))\,\mathrm{d}x, \tag{2.23}$$

for $t\in(0,T)$. Integrating (2.23) over $[T/4,3T/4]$ leads to

$$\frac T2\int_0^1((y(x,0))^2+(z(x,0))^2)\,\mathrm{d}x\le \mathrm{e}^{3\tilde MT/4}\int_{T/4}^{3T/4}\int_0^1(y^2+z^2)\,\mathrm{d}x\,\mathrm{d}t. \tag{2.24}$$

The theorem can be proved from (2.24), the Hardy inequality and Theorem 2.3. □

We remark that in Theorem 2.4, (2.3) is not needed for the special case $\lambda_1=\lambda_2$.

The null controllability of the system (2.1), (2.2), (1.3), and (1.4) follows from the observability inequality (Theorem 2.4). The proof is standard and it is omitted. That is to say, one has

Proposition 2.5. *Assume that $b_1,b_2\in W^{2,1}_\infty(Q_T)$, $c_1,c_2,c_3,c_4\in L^\infty(Q_T)$, and (2.3) holds. For each $u_0,v_0\in L^2(0,1)$, there exist $h_1,h_2\in L^2(Q_T)$, such that the solution (u,v) to the problem (2.1), (2.2), (1.3), and (1.4) satisfies $u(\cdot,T)=v(\cdot,T)=0$ in $(0,1)$.*

3. Null controllability of the system with one control

In this section, we study the null controllability of system (1.1)–(1.4). As the nondegenerate case [19], it is assumed that there exists an open interval $\tilde{\omega} \Subset \hat{\omega} \Subset \omega$ such that

$$\inf_{\hat{\omega}\times[0,T]} |c_3| > 0, \quad (c_1)_x,\, (c_2)_x,\, (c_3)_x,\, (c_4)_x,\, (c_3)_t, (c_3)_{xx} \in L^\infty(\hat{\omega}\times(0,T)). \tag{3.1}$$

Theorem 3.1. *Assume that $b_1, b_2 \in W^{2,1}_\infty(Q_T)$, $c_1, c_2, c_3, c_4 \in L^\infty(Q_T)$, and (2.3) and (3.1) hold. For each $u_0, v_0 \in L^2(0,1)$, there exists $h \in L^2(Q_T)$, such that the solution (u,v) to the problem (1.1)–(1.4) satisfies $u(\cdot,T) = v(\cdot,T) = 0$ in $(0,1)$.*

Proof. Choose two open intervals ω_1, ω_2 such that $\tilde{\omega} \Subset \omega_1 \Subset \omega_2 \Subset \hat{\omega}$. Let $\eta \in C^\infty([0,T])$ and $\rho \in C_0^\infty([0,1])$ such that

$$\begin{gathered} 0 \le \eta(t) \le 1, \quad 0 \le t \le T, \\ \eta = 1 \text{ in } (0,T/3), \quad \eta = 0 \text{ in } (2T/3,T), \\ 0 \le \rho(x) \le 1, \quad 0 \le x \le 1, \\ \rho = 1 \text{ in } \omega_1, \quad \rho = 0 \text{ in } (0,1)\setminus\omega_2. \end{gathered}$$

For $u_0, v_0 \in L^2(0,1)$, it follows from Proposition 2.5 that there exist $h_1, h_2 \in L^2(Q_T)$, such that the solution $(\hat{u}, \hat{v})$ to the problem (2.1), (2.2), (1.3), and (1.4) satisfies $\hat{u}(\cdot,T) = \hat{v}(\cdot,T) = 0$ in $(0,1)$. Denote $(\check{u}, \check{v})$ to be the solution to problem (2.1), (2.2), (1.3), and (1.4) with null controls. Thanks to the classical L^2 theory for the equations of $\check{u}, \check{v}$ in $(\hat{\omega}\setminus\tilde{\omega})\times(0,T)$, together with (3.1), one gets

$$\check{u}_t\,,\, \check{u}_x,\, \check{u}_{xx}\,,\, \check{u}_{xxx}\,,\, \check{u}_{xt},\, \check{v}_t\,,\, \check{v}_x,\, \check{v}_{xx}\,,\, \check{v}_{xxx}\,,\, \check{v}_{xt} \in L^2(0,T; L^2_{loc}(\hat{\omega}\setminus\tilde{\omega})). \tag{3.2}$$

Set

$$\begin{aligned} \hat{U}(x,t) &= \hat{u}(x,t) - \eta(t)\check{u}(x,t), \quad (x,t) \in \overline{Q}_T\,. \\ \hat{V}(x,t) &= \hat{v}(x,t) - \eta(t)\check{v}(x,t), \quad (x,t) \in \overline{Q}_T\,. \end{aligned}$$

Then $(\hat{U}, \hat{V})$ solves

$$\begin{gathered} \hat{U}_t - (x^{\lambda_1}\hat{U}_x)_x + b_1(x,t)\hat{U}_x + c_1(x,t)\hat{U} + c_2(x,t)\hat{V} = -\eta'(t)\check{u}(x,t) + h_1(x,t)\chi_{\tilde{\omega}}, \\ (x,t) \in Q_T\,, \\ \hat{V}_t - (x^{\lambda_2}\hat{V}_x)_x + b_2(x,t)\hat{V}_x + c_3(x,t)\hat{U} + c_4(x,t)\hat{V} = -\eta'(t)\check{v}(x,t) + h_2(x,t)\chi_{\tilde{\omega}}, \\ (x,t) \in Q_T\,, \\ \hat{U}(0,t) = \hat{V}(0,t) = 0, \quad \hat{U}(1,t) = \hat{V}(1,t) = 0, \quad t \in (0,T), \\ \hat{U}(x,0) = 0, \quad \hat{V}(x,0) = 0, \quad x \in (0,1), \end{gathered}$$

and satisfies $\hat{U}(\cdot,T) = \hat{V}(\cdot,T) = 0$ in $(0,1)$. Furthermore, by using the classical L^2 theory for the equations of $\hat{U}, \hat{V}$ in $(\hat{\omega}\setminus\tilde{\omega})\times(0,T)$, we can get from (3.1) and (3.2) that

$$\hat{U}_t\,, \hat{U}_x\,, \hat{U}_{xx}\,, m\hat{U}_{xxx}\,, \hat{U}_{xt}, \hat{V}_t\,, \hat{V}_x\,, \hat{V}_{xx}\,, \hat{V}_{xxx}\,, \hat{V}_{xt} \in L^2((\omega_2\setminus\omega_1)\times(0,T)). \tag{3.3}$$

Define

$$\begin{aligned} U(x,t) &= (1-\rho(x))\hat{U}(x,t) + Z(x,t), \quad (x,t) \in \overline{Q}_T\,, \\ V(x,t) &= (1-\rho(x))\hat{V}(x,t), \quad (x,t) \in \overline{Q}_T\,, \end{aligned}$$

and

$$\begin{aligned} h(x,t) = &\rho(x)\eta'(t)\check{u}(x,t) + 2x^{\lambda_1}\hat{U}_x(x,t)\rho'(x) + (x^{\lambda_1}\rho'(x))'\hat{U}(x,t) \\ &+ Z_t(x,t) - (x^{\lambda_1}Z_x(x,t))_x + b_1(x,t)Z_x(x,t) \\ &- b_1(x,t)\rho'(x)\hat{U}(x,t) + c_1(x,t)Z(x,t), \quad (x,t)\in Q_T\,, \end{aligned} \tag{3.4}$$

where

$$Z(x,t) = \begin{cases} \frac{-\rho(x)\eta'(t)\check{v}(x,t) - 2\rho'(x)x^{\lambda_2}\hat{V}_x(x,t) - (x^{\lambda_2}\rho'(x))'\hat{V}(x,t)}{c_3(x,t)} \\ +\frac{b_2(x,t)\rho'(x)\hat{V}(x,t)}{c_3(x,t)}, \qquad \text{if } (x,t)\in\hat{\omega}\times(0,T), \\ 0, \qquad \text{if } (x,t)\in\big((0,1)\setminus\hat{\omega}\big)\times(0,T). \end{cases}$$

Note that $\operatorname{supp}\rho \subset \omega_2 \Subset \hat{\omega}$. It follows from (3.1)–(3.3) that $h \in L^2(Q_T)$. Then, we can verify that (U,V) is the solution to the problem

$$\begin{gathered} U_t - (x^{\lambda_1}U_x)_x + b_1(x,t)U_x + c_1(x,t)U + c_2(x,t)V \\ = -\eta'(t)\check{u}(x,t) + h(x,t)\chi_\omega, \quad (x,t)\in Q_T\,, \\ V_t - (x^{\lambda_2}V_x)_x + b_2(x,t)V_x + c_3(x,t)U + c_4(x,t)V = -\eta'(t)\check{v}(x,t), \quad (x,t)\in Q_T\,, \\ U(0,t) = V(0,t) = 0, \quad U(1,t) = V(1,t) = 0, \quad t\in(0,T), \\ U(x,0) = 0, \quad V(x,0) = 0, \quad x\in(0,1), \end{gathered}$$

and satisfies $U(\cdot,T) = V(\cdot,T) = 0$ in $(0,1)$. Set

$$\begin{aligned} u(x,t) &= U(x,t) + \eta(t)\check{u}(x,t), \quad (x,t)\in Q_T\,, \\ v(x,t) &= V(x,t) + \eta(t)\check{v}(x,t), \quad (x,t)\in Q_T\,. \end{aligned}$$

Then, (u,v) solves problem (1.1)–(1.4) with h given by (3.4), and satisfies $u(\cdot,T) = v(\cdot,T) = 0$ in $(0,1)$. □

We remark that in Theorem 3.1, conidtion (2.3) is not needed for the special case $\lambda_1 = \lambda_2$.

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