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SHARP TRUDINGER-MOSER INEQUALITIES WITH HOMOGENEOUS WEIGHTS

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Abstract. We investigate sharp Trudinger-Moser type inequalities with the homogeneous weight satisfying a natural curvature-dimension bound condition. Also we study the optimal versions of these inequalities with best constants on both finite and infinite volume domains on Euclidean spaces.

1. Introduction

The main motivation for this article is the Trudinger-Moser inequality studied by Moser in [38].

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Then for each $f \in C_0^\infty(\Omega)$ with $\int_\Omega |\nabla f|^N dx \le 1$, we have*

$$\frac{1}{|\Omega|}\int_\Omega \exp\left\{N\omega_{N-1}^{\frac{1}{N-1}}|f|^{\frac{N}{N-1}}\right\}dx \le c_1 \tag{1.1}$$

where c_1 is a constant depending only on N. Moreover, the value $N\omega_{N-1}^{\frac{1}{N-1}}$ is optimal.

The above result is a sharp version, with the explicit optimal constant of the embedding $W_0^{1,N}(\Omega) \subset L_{\varphi_N}(\Omega)$, where $L_{\varphi_N}(\Omega)$ is the Orlicz space associated with the Young function $\varphi_N(t) = \exp(\beta|t|^{N/(N-1)}) - 1$ for some $\beta > 0$, Ithat was studied by Pohozaev [40], Trudinger [43] and Yudovich [44]. This is widely considered as a replacement of the well-known Sobolev embeddings in the border-line cases.

Because of their important roles in the literature, see [12, 13], Trudinger-Moser type inequalities have been investigated intensively in many different settings. See [5, 11, 14, 15, 16, 17, 18, 19, 21, 22, 26, 27, 28, 30, 31, 32, 35, 36, 41], to name just a few. When the volume of Ω is infinite, the Trudinger-Moser inequality (1.1) becomes trivial. In this aspect, we have the following versions of the Trudinger-Moser type inequalities that can be found in [1, 33, 37].

Theorem 1.2. *Let $0 \le \beta < \beta_N = N\omega_{N-1}^{\frac{1}{N-1}}$. Then one has*

$$\sup_{f\in W^{1,N}(\mathbb{R}^N):\|\nabla f\|_N\le 1}\frac{1}{\|f\|_N^N}\int_{\mathbb{R}^N}\phi_N(\beta|f|^{\frac{N}{N-1}})\,dx < \infty. \tag{1.2}$$

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$$\sup_{f\in W^{1,N}(\mathbb{R}^N):\|\nabla f\|_N^N+\|f\|_N^N\le 1}\int_{\mathbb{R}^N}\phi_N(\beta_N|f|^{\frac{N}{N-1}})\,dx<\infty. \tag{1.3}$$

$$\sup_{f\in W^{1,N}(\mathbb{R}^N):\|\nabla f\|_N\le 1}\frac{1}{\|f\|_N^N}\int_{\mathbb{R}^N}\frac{\phi_N(\beta_N|f|^{\frac{N}{N-1}})}{(1+|f|^{\frac{N}{N-1}})}\,dx<\infty. \tag{1.4}$$

Here

$$\phi_N(t)=e^t-\sum_{j=0}^{N-2}\frac{t^j}{j!}.$$

The constant β_N is sharp. Moreover, the last supremum is infinite when the power $\frac{N}{N-1}$ in the denominator of (1.4) *is replaced by $p<\frac{N}{N-1}$.*

Another interesting problem is to study the Trudinger-Moser type inequalities with the presence of weights $\omega(x)$. For instance, when $\omega(x)=\frac{1}{|x|^\beta}$, the singular Trudinger-Moser type inequalities, which are the interpolations of the Hardy inequality and Trudinger-Moser type inequalities, were set up in [3, 4, 20, 39]. We also mention that, motivated by an open question raised by Haïm Brezis [6, 7], Cabré and Ros-Oton studied in [8] the problem of the regularity of stable solutions to reaction-diffusion problems of double revolution and then established in [9] the Sobolev, Morrey, Trudinger and isoperimetric inequalities with monomial weight x^A. The optimal versions with best constant of the Trudinger-Moser type inequalities with monomial densities were also investigated by Lam [24].

Motivated by the sharp Trudinger-Moser type inequalities with monomial weights in [24], and the functional and geometric inequalities with homogeneous weights (see [10, 25]), we will study the sharp Trudinger-Moser type inequalities with homogeneous weight $\omega(x)$.

Let $\Sigma\subset\mathbb{R}^N$ be an open convex cone with vertex at the origin. Let ω be a continuous function in $\overline{\Sigma}$, positive in Σ, $\omega=0$ on $\partial\Sigma$, and positively homogeneous of degree $A\ge 0$. Let

$$N_A=N+A$$

and denote

$$m_\omega(E)=\int_E\omega(x)N_Ax.$$

We also assume that $\omega^{1/A}$ is concave in Σ if $A>0$. We note that this condition is equivalent to the nonnegativity of the Bakry-Émery Ricci tensor in dimension N_A. In other words, Σ with the reference measure $\omega(x)\,dx$ satisfies $CD(0,N_A)$. We note that many interesting examples of the density ω were provided in [10, 25].

We also use the following notation:

$$\Omega^\Sigma=\Omega\cap\Sigma,\quad w_{N-1,\omega}=\int_{\partial B_1^\Sigma}\omega(x)d\sigma.$$

We note that

$$\int_{B_1^\Sigma}\omega(x)\,dx=\int_0^1\int_{\partial B_r^\Sigma}\omega(x)d\sigma dr=\int_0^1 r^{N_A-1}w_{N-1,\omega}dr=\frac{1}{N_A}w_{N-1,\omega}.$$

Our first main result in this paper is to prove the following sharp Trudinger-Moser inequality with homogeneous density on domains of finite volume.

Theorem 1.3. *With the above notation we have*

$$\sup_{\int_\Sigma |\nabla f|^{N_A}\omega(x)\,dx\le 1} \frac{1}{m_\omega(\operatorname{supp}(f))}\int_\Sigma \big[\exp\big(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}}\big)-1\big]\omega(x)\,dx<\infty$$

where

$$\beta_{N_A,\omega}=N_A\Big(\int_{\partial B_1^\Sigma}\omega(x)\,d\sigma\Big)^{\frac{1}{N_A-1}}$$

is the best constant. Here the supremum is taken over f that is Lipschitz continuous function in Σ, $m_\omega\{x\in\Sigma : |f(x)>t|\}$ is finite for every positive t, and $m_\omega(\operatorname{supp}(f))<\infty$.

When $\omega=1$ and $\Sigma=\mathbb{R}^N$, we recover the classical Trudinger-Moser inequality on bounded domains in [38]. When $\omega(x)=x^A$ is the monomial weight, we obtain the optimal Trudinger-Moser inequality in [24].

Our next goal of this article is to establish the sharp Trudinger-Moser inequalities with homogeneous density on domain of infinite volume. Let

$$\phi_{N_A}(t)=\sum_{k\in\mathbb{N}:k\ge N_A-1}\frac{t^k}{k!}.$$

Then we have the following versions of the Trudinger-Moser inequalities on the whole domain Σ in the spirit of [1, 29, 30, 33, 37]:

Theorem 1.4. *Let $K>1$ and $\beta<\beta_{N_A,\omega}$. There exist constants $C(N_A,\omega)>0$ and $C(N_A,\omega,K)>0$ such that*

(i) for all $f\in C_c^\infty(\overline{\Sigma})$ for which $\int_\Sigma|\nabla f|^{N_A}\omega(x)\,dx\le 1$, we have

$$\int_\Sigma \frac{\phi_{N_A}\big(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}}\big)}{\big(1+|f|^{\frac{N_A}{N_A-1}}\big)}\omega(x)\,dx\le C(N_A,\omega)\int_\Sigma |f|^{N_A}\omega(x)\,dx. \tag{1.5}$$

The constant $\beta_{N_A,\omega}$ is optimal. Moreover, the inequality does not hold when the power $\frac{N_A}{N_A-1}$ is replaced by $p<\frac{N_A}{N_A-1}$.

(ii) for all $f\in C_c^\infty(\overline{\Sigma})$ for which $\int_\Sigma|\nabla f|^{N_A}\omega(x)\,dx\le 1$, we have

$$\int_\Sigma \phi_{N_A}\big(\beta|f|^{\frac{N_A}{N_A-1}}\big)\omega(x)\,dx\le \frac{C(N_A,\omega)}{\beta_{N_A,\omega}-\beta}\int_\Sigma |f|^{N_A}\omega(x)\,dx. \tag{1.6}$$

(iii) for all $f\in C_c^\infty(\overline{\Sigma})$ for which $\int_\Sigma|\nabla f|^{N_A}\omega(x)\,dx<1$, we have

$$\begin{aligned}&\int_\Sigma \phi_{N_A}\Big(\frac{K^{\frac{1}{N_A-1}}\beta_{N_A,\omega}}{\big(K-1+\int_\Sigma|\nabla f|^{N_A}\omega(x)\,dx\big)^{\frac{1}{N_A-1}}}|f|^{\frac{N_A}{N_A-1}}\Big)\omega(x)\,dx\\ &\le C(N_A,\omega,K)\frac{\int_\Sigma|f|^{N_A}\omega(x)\,dx}{1-\int_\Sigma|\nabla f|^{N_A}\omega(x)\,dx}\end{aligned} \tag{1.7}$$

(iv) for all $f\in C_c^\infty(\overline{\Sigma})$ for which $\int_\Sigma|\nabla f|^{N_A}\omega(x)\,dx+\int_\Sigma|f|^{N_A}\omega(x)\,dx\le 1$, we have

$$\int_\Sigma \phi_{N_A}(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}})\omega(x)\,dx\le C(N_A,\omega). \tag{1.8}$$

The constant $\beta_{N_A,\omega}$ is sharp.

Let us denote

$$\mathrm{STM}(\beta) = \sup_{f\in C_c^\infty(\overline{\Sigma}):\int_\Sigma |\nabla f|^{N_A}\omega(x)\,dx\le 1} \frac{1}{\int_\Sigma |f|^{N_A}\omega(x)\,dx}\int_\Sigma \phi_{N_A}\big(\beta|f|^{\frac{N_A}{N_A-1}}\big)\omega(x)\,dx$$

$$\mathrm{TM} = \sup_{f\in C_c^\infty(\overline{\Sigma}):\int_\Sigma |\nabla f|^{N_A}\omega(x)\,dx+\int_\Sigma |f|^{N_A}\omega(x)\,dx\le 1}\int_\Sigma \phi_{N_A}\big(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}}\big)\omega(x)\,dx,$$

$$\begin{aligned}\mathrm{ITM}_K = &\sup_{f\in C_c^\infty(\overline{\Sigma}):\int_\Sigma |\nabla f|^{N_A}\omega(x)\,dx\le 1}\frac{1-\int_\Sigma |\nabla f|^{N_A}\omega(x)\,dx}{\int_\Sigma |f|^{N_A}\omega(x)\,dx}\\ &\times\int_\Sigma \phi_{N_A}\Big(\frac{K^{\frac{1}{N_A-1}}\beta_{N_A,\omega}}{\big(K-1+\int_\Sigma |\nabla f|^{N_A}\omega(x)\,dx\big)^{\frac{1}{N_A-1}}}|f|^{\frac{N_A}{N_A-1}}\Big)\omega(x)\,dx.\end{aligned}$$

Then we show the following relations:

Theorem 1.5. *For $K>1$, we have*

$$\mathrm{ITM}_K = \frac{K}{K-1}\,\mathrm{TM} = \frac{K}{K-1}\sup_{\beta\in(0,\beta_{N_A,\omega})}\Big[\frac{1-(\frac{\beta}{\beta_{N_A,\omega}})^{N_A-1}}{(\frac{\beta}{\beta_{N_A,\omega}})^{N_A-1}}\Big]\,\mathrm{STM}(\beta).$$

See [23] for similar results for the non-weighted case and [24] for the ones with monomial weights.

2. Sharp Trudinger-Moser inequality with homogeneous weight on a bounded domain

We first recall the following result by Adams [2]:

Lemma 2.1. *Let $1<p<\infty$ and $a(s,t)$ be a non-negative measurable function on $[0,\infty)\times[0,\infty)$ such that (a.e.)*

$$a(s,t)\le 1,\quad \text{when } 0<s<t, \tag{2.1}$$

$$\sup_{t>0}\Big(\int_t^\infty a(s,t)^{p'}ds\Big)^{1/p'} = b<\infty. \tag{2.2}$$

Then there is a constant $c_0=c_0(p,b)$ such that if for $\psi\ge 0$,

$$\int_0^\infty \psi(s)^p ds\le 1\,. \tag{2.3}$$

Then

$$\int_0^\infty e^{-F(t)}dt\le c_0\,, \tag{2.4}$$

where

$$F(t) = t-\Big(\int_0^\infty a(s,t)\psi(s)ds\Big)^{p'}. \tag{2.5}$$

Our approach to prove Theorem 1.3 relies on the classical symmetrization argument. However, since we have to deal with the nonradial densities, we now need to set up a weighted version of the rearrangement. Actually, we have the following version of the rearrangement argument:

Lemma 2.2. *For any Lipschitz continuous function f in Σ such that $m_\omega\{x\in\Sigma:|f(x)>t|\}$ is finite for every positive t, there exists a radial rearrangement $f^{\#}$ of f such that*

(a) $f^{\#}$ *is radially decreasing,*
(b) $m_\omega(\{|f|>t\}) = m_\omega(\{f^{\#}>t\})$ *for all* t,
(c) *for every Young function* Φ *(that is,* Φ *maps* $[0,\infty)$ *into* $[0,\infty)$*, vanishes at* 0*, and is convex and increasing):*

$$\int_\Sigma \Phi(|\nabla f^{\#}|)\omega(x)\,dx \le \int_\Sigma \Phi(|\nabla f|)\omega(x)\,dx\,,$$

(d) $\int_\Sigma \Psi(f)\omega(x)\,dx = \int_\Sigma \Psi(f^{\#})\omega(x)\,dx$ *for a nondecreasing function* $\Psi:\mathbb{R}^+\to\mathbb{R}^+$.

Indeed, by [42], we have that whenever balls minimize the isoperimetric quotient with a weight ω, there exists a radial rearrangement which preserves $\int \Psi(f)\omega(x)\,dx$ and decreases $\int |\nabla f|^p\omega(x)\,dx$. Hence, by combining this fact with the results about the isoperimetric inequalities with homogeneous weights (see [10, 25] for example) and the layer cake representation (see [34]), we obtain Lemma 2.2.

Proof of Theorem 1.3. By applying Lemma 2.2, we assume that f is radially non-increasing with $\overline{\operatorname{supp}(f)} = \overline{B_R^\Sigma}$. Hence, using polar coordinates, we obtain

$$\int_\Sigma |\nabla f|^{N_A}\omega(x)\,dx = w_{N-1,\omega}\int_0^R |f'|^{N_A} r^{N_A-1}dr$$

and

$$\begin{aligned}
&\int_\Sigma \Big[\exp\big(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}}\big)-1\Big]\omega(x)\,dx\\
&= \int_0^R \Big(\int_{\partial B_r^\Sigma}\Big[\exp(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}})-1\Big]\omega(x)d\sigma\Big)dr\\
&= w_{N-1,\omega}\int_0^R \Big[\exp\big(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}}\big)-1\Big]r^{N_A-1}dr.
\end{aligned}$$

Now, define a function v:

$$v(t) = Pf(Re^{-t/N_A})$$

where

$$P = N_A\Big(\frac{1}{N_A}w_{N-1,\omega}\Big)^{1/N_A}.$$

Then, by direct computations

$$\begin{aligned}
\int_0^R |f'(r)|^{N_A}r^{N_A-1}dr &= \int_0^R |f'\big(Re^{-t/N_A}\big)|^{N_A}\big(Re^{-t/N_A}\big)^{N_A-1}R\frac{1}{N_A}e^{-t/N_A}dt\\
&= (\frac{N_A}{P})^{N_A}\frac{1}{N_A}\int_0^R |v'(t)|^{N_A}dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^R \Big[\exp\big(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}}\big)-1\Big]r^{N_A-1}dr\\
&= R^{N_A}\frac{1}{N_A}\int_0^\infty \Big[\exp\Big(\frac{\beta_{N_A,\omega}}{P^{\frac{N_A}{N_A-1}}}|v(t)|^{\frac{N_A}{N_A-1}}\Big)-1\Big]e^{-t}dt.
\end{aligned}$$

Noting that

$$(\frac{N_A}{P})^{N_A}\frac{1}{N_A}w_{N-1,\omega} = 1,$$

we obtain $\int_0^R |v'(t)|^{N_A} dt \le 1$. Now we can apply Lemma 2.1 with

$$\psi = v', \quad a(s,t) = \begin{cases} 1 & 0 \le s \le t \\ 0 & t < s \end{cases}$$

to find a constant $C_0 = C_0(N_A)$ such that

$$\begin{aligned} &\frac{1}{N_A}\int_0^\infty \Big[\exp\Big(\frac{\beta_{N_A,\omega}}{P^{\frac{N_A}{N_A-1}}}|v(t)|^{\frac{N_A}{N_A-1}}\Big) - 1\Big]e^{-t}dt \\ &= \frac{1}{N_A}\int_0^\infty \Big[\exp\big(|v(t)|^{\frac{N_A}{N_A-1}}\big) - 1\Big]e^{-t}dt \le C_0. \end{aligned}$$

Hence

$$\begin{aligned} \int_\Sigma \Big[\exp\big(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}}\big) - 1\Big]\omega(x)\,dx &\le C_0(N_A)R^{N_A}w_{N-1,\omega} \\ &= C_0(N_A)\frac{w_{N-1,\omega}}{\int_{B_1^\Sigma}\omega(x)\,dx} m_\omega(B_R^\Sigma). \end{aligned}$$

In other words,

$$\int_\Sigma \Big[\exp\big(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}}\big) - 1\Big]\omega(x)\,dx \le C_1(N_A)m_\omega(\mathrm{supp}(f)).$$

Now, we introduce the following Moser type sequence:

$$f_k(x) = \Big(\frac{1}{w_{N-1,\omega}}\Big)^{1/N_A} \begin{cases} \big(\frac{k}{N_A}\big)^{\frac{N_A-1}{N_A}} & 0 \le |x| \le e^{-k/N_A} \\ \big(\frac{N_A}{k}\big)^{1/N_A}\log\big(\frac{1}{|x|}\big) & e^{-k/N_A} < |x| < 1 \\ 0 & |x| \ge 1 \end{cases} \tag{2.6}$$

to prove that the constant $\beta_{N_A,\omega}$ is optimal. We have that

$$\begin{aligned} \int_\Sigma |\nabla f_k|^{N_A}\omega(x)\,dx &= w_{N-1,\omega}\int_0^1 |f_k'(r)|^{N_A} r^{N_A-1}\,dr \\ &= w_{N-1,\omega}\int_{e^{-k/N_A}}^1 \frac{N_A}{kw_{N-1,\omega}}\frac{1}{r}\,dr = 1. \end{aligned}$$

Hence, for all $\beta > \beta_{N_A,\omega}$,

$$\begin{aligned} &\int_\Sigma \Big[\exp(\beta|f_k|^{\frac{N_A}{N_A-1}}) - 1\Big]\omega(x)\,dx \\ &\gtrsim \int_0^{e^{-k/N_A}} \exp\Big(\beta\Big|\big(\frac{1}{w_{N-1,\omega}}\big)^{1/N_A}\big(\frac{k}{N_A}\big)^{\frac{N_A-1}{N_A}}\Big|^{\frac{N_A}{N_A-1}}\Big)r^{N_A-1}dr \\ &\gtrsim \exp\big[\frac{\beta}{\beta_{N_A,\omega}}k\big]e^{-k} \to \infty \quad \text{as } k \to \infty. \end{aligned}$$

□

3. Sharp Trudinger-Moser type inequalities with homogeneous density on Σ

Using [24, Theorem 2.1] and a simple scaling argument, we can deduce the following Radial Sobolev inequality in the spirit of Ibrahim-Masmoudi-Nakanishi [21].

Theorem 3.1. *There exists a constant $C>0$ such that for any radially nonnegative nonincreasing function φ satisfying $f(R)>1$ and*

$$w_{N-1,\omega}\int_R^\infty |\varphi'(t)|^{N_A}t^{N_A-1}dt\le K$$

for some $R,K>0$, then we have

$$\frac{\exp\big[\frac{\beta_{N_A,\omega}}{K^{\frac{1}{N_A-1}}}\varphi^{\frac{N_A}{N_A-1}}(R)\big]}{\varphi^{\frac{N_A}{N_A-1}}(R)}R^{N_A}\le C\frac{\int_R^\infty |\varphi(t)|^{N_A}t^{N_A-1}N_At}{K^{\frac{N_A}{N_A-1}}}.$$

Also, by arguments as in [24], we obtain the following lemma.

Lemma 3.2. *We have*

$$\mathrm{STM}(\beta)=\sup_{f\in C_c^\infty(\overline{\Sigma}):\int_\Sigma|\nabla f|^{N_A}\omega(x)\,dx=1=\int_\Sigma|f|^{N_A}\omega(x)\,dx}\int_\Sigma\phi_{N_A}\big(\beta|f|^{\frac{N_A}{N_A-1}}\big)\omega(x)\,dx.$$

Proof of Theorem 1.4. We will first prove (i). By Lemma 2.2, we just need to consider smooth, nonnegative and radially nonincreasing function f. Choose $R_1=R_1(f)$ such that

$$\begin{gathered}\int_{B_{R_1}^\Sigma}|\nabla f|^{N_A}\omega(x)\,dx=w_{N-1,\omega}\int_0^{R_1}|f_r|^{N_A}r^{N_A-1}dr\le 1-\delta_0,\\ \int_{\Sigma\setminus B_{R_1}^\Sigma}|\nabla f|^{N_A}\omega(x)\,dx=w_{N-1,\omega}\int_{R_1}^\infty|f_r|^{N_A}r^{N_A-1}dr\le\delta_0.\end{gathered}$$

Here $\delta_0\in(0,1)$ is fixed and does not depend on f.

Applying the Holder's inequality, for $0<r_1\le r_2\le R_1$, we obtain

$$\begin{aligned}f(r_1)-f(r_2)&\le\int_{r_1}^{r_2}-f_r dr\\ &\le\Big(\int_{r_1}^{r_2}|f_r|^{N_A}r^{N_A-1}dr\Big)^{1/N_A}\Big(\ln\frac{r_2}{r_1}\Big)^{\frac{N_A-1}{N_A}}\\ &\le\Big(\frac{1-\delta_0}{w_{N-1,\omega}}\Big)^{1/N_A}\Big(\ln\frac{r_2}{r_1}\Big)^{\frac{N_A-1}{N_A}}.\end{aligned}\tag{3.1}$$

Similarly, for $R_1\le r_1\le r_2$, we obtain

$$f(r_1)-f(r_2)\le\Big(\frac{\delta_0}{w_{N-1,\omega}}\Big)^{1/N_A}\Big(\ln\frac{r_2}{r_1}\Big)^{\frac{N_A-1}{N_A}}.\tag{3.2}$$

We now set $R_0:=\inf\{r>0:f(r)\le1\}\in[0,\infty)$. Obviously $f(s)\le1$ when $s\ge R_0$. Moreover, we just need to consider the case $R_0>0$.

Now, we write

$$\int_\Sigma\frac{\phi_{N_A}(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}})}{1+|f|^{\frac{N_A}{N_A-1}}}\omega(x)\,dx=\int_{B_{R_0}^\Sigma}+\int_{\Sigma\setminus B_{R_0}^\Sigma}\frac{\phi_{N_A}(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}})}{1+|f|^{\frac{N_A}{N_A-1}}}\omega(x)\,dx.$$

Note that

$$\begin{aligned}\int_{\Sigma\setminus B^{\Sigma}_{R_0}} \frac{\phi_{N_A}\big(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}}\big)}{1+|f|^{\frac{N_A}{N_A-1}}}\omega(x)\,dx &\le C\int_{\{f\le 1\}}|f|^{N_A}\omega(x)\,dx \\ &\le C\int_{\Sigma}|f|^{N_A}\omega(x)\,dx\end{aligned} \tag{3.3}$$

since $f\le 1$ on $\Sigma\setminus B^{\Sigma}_{R_0}$. Hence, we just need to estimate

$$I := \int_{B^{\Sigma}_{R_0}} \frac{\phi_{N_A}\big(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}}\big)}{1+|f|^{\frac{N_A}{N_A-1}}}\omega(x)\,dx.$$

Case 1: $0<R_0\le R_1$. Using (3.1), for $0<r\le R_0$, we have

$$|f(r)|\le 1+\Big(\frac{1-\delta_0}{w_{N-1,\omega}}\Big)^{1/N_A}\Big(\ln\frac{R_0}{r}\Big)^{\frac{N_A-1}{N_A}}.$$

By using the elementary inequality

$$(a+b)^{\frac{N_A}{N_A-1}}\le(1+\delta)a^{\frac{N_A}{N_A-1}}+A(\delta)b^{\frac{N_A}{N_A-1}},$$

where

$$A(\delta)=\Big(1-\frac{1}{(1+\delta)^{N_A-1}}\Big)^{\frac{1}{1-N_A}},$$

we obtain

$$|f|^{\frac{N_A}{N_A-1}}(r)\le(1+\delta_1)(\frac{1-\delta_0}{w_{N-1,\omega}})^{1/(N_A-1)}\ln\frac{R_0}{r}+A(\delta_1).$$

Thus, with $\delta_1=(1+\delta_0)^{1/(N_A-1)}-1$, we have

$$\begin{aligned} I&=\int_{B^{\Sigma}_{R_0}} \frac{\phi_{N_A}\big(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}}\big)}{\big(1+|f|^{\frac{N_A}{N_A-1}}\big)}\omega(x)\,dx\\ &\le\int_{B^{\Sigma}_{R_0}}\exp\Big(\beta_{N_A,\omega}(1+\delta_1)(\frac{1-\delta_0}{w_{N-1,\omega}}big)^{1/(N_A-1)}\ln\frac{R_0}{r}+\beta A(\delta_1)\Big)\omega(x)\,dx\\ &\le CR_0^{\beta_{N_A,\omega}(1+\delta_1)((1-\delta_0)/w_{N-1,\omega})^{1/(N_A-1)}}\\ &\quad\times w_{N-1,\omega}\int_0^{R_0} r^{N_A-1-\beta_{N_A,\omega}(1+\delta_1)((1-\delta_0)/w_{N-1,\omega})^{1/(N_A-1)}}dr\\ &\le CR_0^{N_A}w_{N-1,\omega}\\ &\le C\int_{\Sigma}|f|^{N_A}\omega(x)\,dx.\end{aligned} \tag{3.4}$$

Case 2: $0<R_1<R_0$. We write

$$I=\int_{B^{\Sigma}_{R_1}}+\int_{B^{\Sigma}_{R_0}\setminus B^{\Sigma}_{R_1}}\frac{\phi_{N_A}\big(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}}\big)}{1+|f|^{\frac{N_A}{N_A-1}}}\omega(x)\,dx=:I_1+I_2.$$

By (3.2), for $r\ge R_1$, we obtain

$$f(r)-f(R_0)\le\Big(\frac{\delta_0}{w_{N-1,\omega}}\Big)^{1/N_A}\Big(\ln\frac{R_0}{r}\Big)^{\frac{N_A-1}{N_A}}.$$

That is,

$$|f(r)| \le 1 + \Big(\frac{\delta_0}{w_{N-1,\omega}}\Big)^{1/N_A}\Big(\ln\frac{R_0}{r}\Big)^{\frac{N_A-1}{N_A}}.$$

Hence,

$$|f|^{\frac{N_A}{N_A-1}}(r) \le (1+\delta_2)\Big(\frac{\delta_0}{w_{N-1,\omega}}\Big)^{\frac{1}{N_A-1}}\ln\frac{R_0}{r} + A(\delta_2).$$

Then by choosing $\delta_2 > 0$ such that $N_A - \beta_{N_A,\omega}(1+\delta_2)(\delta_0/w_{N-1,\omega})^{\frac{1}{N_A-1}} > 0$, we have the estimate

$$\begin{aligned}
I_2 &= \int_{B^\Sigma_{R_0}\setminus B^\Sigma_{R_1}} \frac{\phi_{N_A}\big(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}}\big)}{1+|f|^{\frac{N_A}{N_A-1}}}\omega(x)\,dx\\
&\le C w_{N-1,\omega}\int_{R_1}^{R_0}\exp\Big(\beta_{N_A,\omega}(1+\delta_2)\big(\frac{\delta_0}{w_{N-1,\omega}}\big)^{\frac{1}{N_A-1}}\ln\frac{R_0}{r} + \beta A(\delta_2)\Big)r^{N_A-1}dr\\
&\le C w_{N-1,\omega} R_0^{\beta_{N_A,\omega}(1+\delta_2)(\delta_0/w_{N-1,\omega})^{\frac{1}{N_A-1}}}\\
&\quad\times \frac{R_0^{N_A-\beta_{N_A,\omega}(1+\delta_2)(\delta_0/w_{N-1,\omega})^{\frac{1}{N_A-1}}} - R_1^{N_A-\beta_{N_A,\omega}(1+\delta_2)(\delta_0/w_{N-1,\omega})^{\frac{1}{N_A-1}}}}{N_A-\beta_{N_A,\omega}(1+\delta)(\delta_0/w_{N-1,\omega})^{\frac{1}{N_A-1}}}\\
&\le \frac{C w_{N-1,\omega}}{N_A-\beta_{N_A,\omega}(1+\delta_2)(\delta_0/w_{N-1,\omega})^{\frac{1}{N_A-1}}}(R_0^{N_A}-R_1^{N_A})\\
&\le C\int_{B^\Sigma_{R_0}\setminus B^\Sigma_{R_1}}\omega(x)\,dx\\
&\le C\int_\Sigma |f|^{N_A}\omega(x)\,dx.
\end{aligned}$$

Now to estimate $I_1 = \int_{B^\Sigma_{R_1}}\frac{\phi_{N_A}(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}})}{1+|f|^{\frac{N_A}{N_A-1}}}\omega(x)\,dx$ with $f(R_1) > 1$, we first set

$$v(r) = f(r) - f(R_1) \quad \text{on } 0 \le r \le R_1.$$

Then

$$\int_{B^\Sigma_{R_1}}|\nabla v|^N\omega(x)\,dx = \int_{B^\Sigma_{R_1}}|\nabla f|^N\omega(x)\,dx \le 1-\delta_0.$$

Also,

$$|f|^{\frac{N_A}{N_A-1}}(r) \le (1+\delta)v^{\frac{N_A}{N_A-1}}(r) + A(\delta)f^{\frac{N_A}{N_A-1}}(R_1) \text{ for } 0 \le r \le R_1.$$

With

$$0 < \delta = \Big(\frac{1}{1-\delta_0}\Big)^{\frac{1}{N_A-1}} - 1$$

and

$$A(\delta) = \Big(1 - \frac{1}{(1+\delta)^{N_A-1}}\Big)^{\frac{1}{1-N_A}} = \delta_0^{\frac{1}{1-N_A}},$$

we obtain

$$I_1 = \int_{B^\Sigma_{R_1}}\frac{\phi_{N_A}(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}})}{1+|f|^{\frac{N_A}{N_A-1}}}\omega(x)\,dx \tag{3.5}$$

$$\le \frac{e^{\beta_{N_A,\omega}A(\delta)f^{\frac{N_A}{N_A-1}}(R_1)}}{|f|^{\frac{N_A}{N_A-1}}(R_1)}\int_{B^{\Sigma}_{R_1}} e^{\beta_{N_A,\omega}(1+\delta)v^{\frac{N_A}{N_A-1}}(r)}\omega(x)\,dx \tag{3.6}$$

$$= \frac{e^{\beta_{N_A,\omega}A(\delta)f^{\frac{N_A}{N_A-1}}(R_1)}}{|f|^{\frac{N_A}{N_A-1}}(R_1)}\int_{B^{\Sigma}_{R_1}} e^{\beta_{N_A,\omega}w^{\frac{N_A}{N_A-1}}(r)}\omega(x)\,dx \tag{3.7}$$

where $w = (1+\delta)^{\frac{N_A-1}{N_A}}v$.

Note that $\operatorname{supp}(w) \subset B^{\Sigma}_{R_1}$ and

$$\int_{B^{\Sigma}_{R_1}} |\nabla w|^{N_A}\omega(x)\,dx = (1+\delta)^{N_A-1}\int_{B^{\Sigma}_{R_1}} |\nabla v|^{N_A}\omega(x)\,dx \le (1+\delta)^{N_A-1}(1-\delta_0) = 1.$$

Hence, using Theorem 1.3, we deduce that

$$\int_{B^{\Sigma}_{R_1}} e^{\beta_{N_A,\omega}w^{\frac{N_A}{N_A-1}}(r)}\omega(x)\,dx \le C\int_{B^{\Sigma}_{R_1}}\omega(x)\,dx \le CR_1^{N_A}\int_{B^{\Sigma}_1}\omega(x)\,dx. \tag{3.8}$$

Also, applying Theorem 3.1, we obtain

$$\begin{aligned}
&\frac{\exp\big(\beta_{N_A,\omega}A(\delta)f^{\frac{N_A}{N_A-1}}(R_1)\big)}{|f|^{\frac{N_A}{N_A-1}}(R_1)}R_1^{N_A}\int_{B^{\Sigma}_1}\omega(x)\,dx\\
&= \frac{\exp\big(\frac{\beta_{N_A,\omega}}{\delta_0^{\frac{1}{N_A-1}}}f^{\frac{N_A}{N_A-1}}(R_1)\big)}{|f|^{\frac{N_A}{N_A-1}}(R_1)}R_1^{N_A}\int_{B^{\Sigma}_1}\omega(x)\,dx\\
&\le C\delta_0^{\frac{N_A}{N_A-1}}\int_{\Sigma\setminus B^{\Sigma}_{R_1}}|f|^{N_A}\omega(x)\,dx\\
&\le C\int_{\Sigma}|f|^{N_A}\omega(x)\,dx.
\end{aligned}\tag{3.9}$$

Combining (3.5), (3.8) and (3.9), we obtain the desired result.

To show that $\beta_{N_A,\omega}$ is sharp we can use the Moser type sequence as in the proof of Theorem 1.3. The fact that the power $\frac{N_A}{N_A-1}$ in the denominator of (1.5) is also optimal can be proved as follows:

$$\begin{aligned}
\int_{\Sigma}|f_k|^{N_A}\omega(x)\,dx &= w_{N-1,\omega}\int_0^1 |f_k(r)|^{N_A}r^{N_A-1}dr\\
&= w_{N-1,\omega}\int_0^{e^{-k/N_A}}\Big(\frac{1}{w_{N-1,\omega}}\Big)\Big(\frac{k}{N_A}\Big)^{N_A-1}r^{N_A-1}\,dr\\
&\quad + w_{N-1,\omega}\int_{e^{-k/N_A}}^1\Big(\frac{N_A}{kw_{N-1,\omega}}\Big)|\log(\frac{1}{r})|^{N_A}r^{N_A-1}\,dr\\
&\lesssim e^{-k}k^{N_A-1}+\frac{1}{k}\lesssim\frac{1}{k}
\end{aligned}$$

and

$$\int_{\Sigma}\frac{\phi_{N_A}\big(\beta_{N_A,\omega}|f_k|^{\frac{N_A}{N_A-1}}\big)}{(1+|f_k|^p)}\omega(x)\,dx$$

$$
\begin{aligned}
&\gtrsim \int_0^{e^{-k/N_A}} \frac{\phi_{N_A}\big(\beta_{N_A,\omega}(\frac{1}{w_{N-1,\omega}})^{\frac{1}{N_A-1}}\frac{k}{N_A}\big)}{\big(1+|(\frac{1}{w_{N-1,\omega}})^{1/N_A}(\frac{k}{N_A})^{\frac{N_A-1}{N_A}}|^p\big)} r^{N_A-1}\,dr \\
&\gtrsim \int_0^{e^{-\frac{k}{N_A}}} \frac{\phi_{N_A}(k)}{k^{\frac{p(N_A-1)}{N_A}}} r^{N_A-1}\,dr \\
&\gtrsim \frac{\phi_{N_A}(k)e^{-k}}{k^{\frac{p(N_A-1)}{N_A}}} \gtrsim \frac{1}{k^{\frac{p(N_A-1)}{N_A}}}
\end{aligned}
$$

Hence, since we need

$$
\frac{1}{k^{\frac{p(N_A-1)}{N_A}}} \lesssim \frac{1}{k},
$$

we deduce that

$$
p \geq \frac{N_A}{N_A-1}.
$$

It is obvious that we can use (i) to deduce the first part of (ii) (that is, (ii) without the exact asymptotic behavior $\frac{1}{\beta_{N_A,\omega}-\beta}$). To study the asymptotic behavior part, we let $0<\beta \lneqq \beta_{N_A,\omega}$ and $f \in C_c^\infty(\overline{\Sigma})$: $\int_\Sigma |\nabla f|^{N_A}\omega(x)\,dx \leq 1$. Then we have

$$
\int_\Sigma \phi_{N_A}\big(\beta|f|^{\frac{N_A}{N_A-1}}\big)\omega(x)\,dx = \int_{|f|\leq 1} + \int_{|f|>1} \phi_{N_A}\big(\beta|f|^{\frac{N_A}{N_A-1}}\big)\omega(x)\,dx.
$$

The first integral can be estimated as follows:

$$
\begin{aligned}
\int_{|f|\leq 1} \phi_{N_A}\Big(\beta|f|^{\frac{N_A}{N_A-1}}\Big)\omega(x)\,dx &= \int_{|f|\leq 1} \sum_{k=N_A-1}^{\infty} \frac{\beta^k}{k!}|f|^{k\frac{N_A}{N_A-1}}\omega(x)\,dx \\
&\leq \sum_{k=N_A-1}^{\infty} \frac{\beta^k}{k!} \int_\Sigma |f|^{N_A}\omega(x)\,dx \\
&\leq \frac{C_0(N_A,\omega)}{\beta_{N_A,\omega}-\beta} \int_\Sigma |f|^{N_A}\omega(x)\,dx.
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\int_{|f|>1} \phi_{N_A}\Big(\beta|f|^{\frac{N_A}{N_A-1}}\Big)\omega(x)\,dx \\
&\leq \int_{|f|>1} \exp\Big(\beta|f|^{\frac{N_A}{N_A-1}}\Big)\omega(x)\,dx \\
&\leq \int_{|f|>1} \exp\Big(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}}\Big)\exp\Big([\beta-\beta_{N_A,\omega}]|f|^{\frac{N_A}{N_A-1}}\Big)\omega(x)\,dx \\
&\leq C_0(N_A,\omega)\int_{|f|>1} \frac{\exp\big(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}}\big)}{(\beta_{N_A,\omega}-\beta)|f|^{\frac{N_A}{N_A-1}}}\omega(x)\,dx \\
&\leq \frac{C_1(N_A,\omega)}{\beta_{N_A,\omega}-\beta}\int_{|f|>1} \frac{\phi_{N_A}\big(\beta_{N_A,\omega}|f|^{\frac{N_A}{N_A-1}}\big)}{1+|f|^{\frac{N_A}{N_A-1}}}\omega(x)\,dx \\
&\leq \frac{C_2(N_A,\omega)}{\beta_{N_A,\omega}-\beta}.
\end{aligned}
$$

We now prove (iii). Let $f \in C_c^\infty(\overline{\Sigma})$ be such that $\int_\Sigma |\nabla f|^{N_A}\omega(x)\,dx < 1$. Then if

$$\int_\Sigma |\nabla f|^{N_A}\omega(x)\,dx \le \big(\frac{K-1}{K}\big)^{N_A},$$

we can set $v = \frac{K}{K-1} f$ and obtain $\int_\Sigma |\nabla v|^{N_A}\omega(x)\,dx \le 1$. Moreover,

$$\begin{aligned}
&\int_\Sigma \phi_{N_A}\Big(\frac{K^{\frac{1}{N_A-1}}\beta_{N_A,\omega}}{\big(K-1+\int_\Sigma |\nabla f|^{N_A}\omega(x)\,dx\big)^{\frac{1}{N_A-1}}}|f|^{\frac{N_A}{N_A-1}}\Big)\omega(x)\,dx \\
&= \int_\Sigma \phi_{N_A}\Big(\frac{K^{\frac{1}{N_A-1}}\beta_{N_A,\omega}}{\big(K-1+\int_\Sigma |\nabla f|^{N_A}\omega(x)\,dx\big)^{\frac{1}{N_A-1}}}\big(\frac{K-1}{K}\big)^{\frac{N_A}{N_A-1}}|v|^{\frac{N_A}{N_A-1}}\Big)\omega(x)\,dx \\
&\le \int_\Sigma \phi_{N_A}\Big(\big(\frac{K-1}{K}\big)\beta_{N_A,\omega}|v|^{\frac{N_A}{N_A-1}}\Big)\omega(x)\,dx \\
&\le C_0(K,N_A,\omega)\int_\Sigma |v|^{N_A}\omega(x)\,dx \\
&\le C_1(K,N_A,\omega)\int_\Sigma |f|^{N_A}\omega(x)\,dx.
\end{aligned}$$

On the other hand, if

$$1 > \int_\Sigma |\nabla f|^{N_A}\omega(x)\,dx > \big(\frac{K-1}{K}\big)^{N_A},$$

then we can set

$$w = \frac{f}{\big[\int_\Sigma |\nabla f|^{N_A}\omega(x)\,dx\big]^{1/N_A}},$$

$$\beta = \frac{K^{\frac{1}{N_A-1}}}{\big(K-1+\int_\Sigma |\nabla f|^{N_A}\omega(x)\,dx\big)^{\frac{1}{N_A-1}}}\Big[\int_\Sigma |\nabla f|^{N_A}\omega(x)\,dx\Big]^{\frac{1}{N_A-1}}\beta_{N_A,\omega}.$$

We note that

$$\beta_{N_A,\omega} > \beta \ge \frac{K^{\frac{1}{N_A-1}}}{\big(K-1+(\frac{K-1}{K})^{N_A}\big)^{\frac{1}{N_A-1}}}\Big[\big(\frac{K-1}{K}\big)^{N_A}\Big]^{\frac{1}{N_A-1}}\beta_{N_A,\omega},$$

$$\frac{1}{\beta_{N_A,\omega}-\beta} \le C(K,N_A,\omega)\frac{1}{1-\int_\Sigma |\nabla f|^{N_A}\omega(x)\,dx}.$$

Hence, by Statement (ii), we obtain

$$\begin{aligned}
&\int_\Sigma \phi_{N_A}\Big(\frac{K^{\frac{1}{N_A-1}}\beta_{N_A,\omega}}{\big(K-1+\int_\Sigma |\nabla f|^{N_A}\omega(x)\,dx\big)^{\frac{1}{N_A-1}}}|f|^{\frac{N_A}{N_A-1}}\Big)\omega(x)\,dx \\
&\le C_0(N_A,\omega)\frac{1}{\beta_{N_A,\omega}-\beta}\frac{\int_\Sigma |f|^{N_A}\omega(x)\,dx}{\int_\Sigma |\nabla f|^{N_A}\omega(x)\,dx} \\
&\le C_1(K,N_A,\omega)\frac{\int_\Sigma |f|^{N_A}\omega(x)\,dx}{1-\int_\Sigma |\nabla f|^{N_A}\omega(x)\,dx}.
\end{aligned}$$

Lastly, (iv) is a direct consequence of (iii). □

We now use the scaling technique to prove the equivalence of the above the Trudinger-Moser inequalities.

Proof of Theorem 1.5. We first claim that

$$\mathrm{ITM}_K = \frac{K}{K-1} \sup_{\beta\in(0,\beta_{N_A,\omega})} \Big[\frac{1-\big(\frac{\beta}{\beta_{N_A,\omega}}\big)^{N_A-1}}{\big(\frac{\beta}{\beta_{N_A,\omega}}\big)^{N_A-1}}\Big] \mathrm{STM}(\beta).$$

Indeed, for any $f \in C_c^\infty(\overline{\Sigma}) \setminus \{0\}$ for which

$$\int_\Sigma |\nabla f|^{N_A} \omega(x)\,dx = 1 = \int_\Sigma |f|^{N_A}\omega(x)\,dx,$$

we define $v(x) = \mu f(\lambda x)$ and obtain

$$\int_\Sigma |\nabla v|^{N_A}\omega(x)\,dx = \mu^{N_A}, \quad \int_\Sigma |v|^{N_A}\omega(x)\,dx = \frac{\mu^{N_A}}{\lambda^{N_A}}.$$

We also obtain

$$\int_\Sigma \phi_{N_A}\Big(\beta|f|^{\frac{N_A}{N_A-1}}\Big)\omega(x)\,dx = \lambda^{N_A}\int_\Sigma \phi_{N_A}\Big(\frac{\beta}{\mu^{\frac{N_A}{N_A-1}}}|v|^{\frac{N_A}{N_A-1}}\Big)\omega(x)\,dx.$$

Hence, by choosing μ and λ such that

$$\frac{\beta}{\mu^{\frac{N_A}{N_A-1}}} = \frac{K^{\frac{1}{N_A-1}}\beta_{N_A,\omega}}{\big(K-1+\int_\Sigma |\nabla v|^{N_A}\omega(x)\,dx\big)^{\frac{1}{N_A-1}}};$$

that is,

$$\mu^{N_A} = \frac{K-1}{K\big(\frac{\beta_{N_A,\omega}}{\beta}\big)^{N_A-1}-1} = \big(\frac{\beta}{\beta_{N_A,\omega}}\big)^{N_A-1}\frac{K-1+\mu^{N_A}}{K},$$

and $\frac{\mu^{N_A}}{\lambda^{N_A}} = 1-\mu^{N_A}$, we deduce

$$\begin{aligned}
&\int_\Sigma \phi_{N_A}\Big(\beta|f|^{\frac{N_A}{N_A-1}}\Big)\omega(x)\,dx \\
&= \frac{\mu^{N_A}}{1-\mu^{N_A}}\int_\Sigma \phi_{N_A}\Big(\frac{K^{\frac{1}{N_A-1}}\beta_{N_A,\omega}}{\big(K-1+\int_\Sigma|\nabla v|^{N_A}\omega(x)\,dx\big)^{\frac{1}{N_A-1}}}|v|^{\frac{N_A}{N_A-1}}\Big)\omega(x)\,dx \\
&\le \frac{K-1}{K}.\frac{\big(\frac{\beta}{\beta_{N_A,\omega}}\big)^{N_A-1}}{1-\big(\frac{\beta}{\beta_{N_A,\omega}}\big)^{N_A-1}}\,\mathrm{ITM}_K\,.
\end{aligned}$$

By Lemma 3.2, we have

$$\mathrm{STM}(\beta) \le \frac{K-1}{K}.\frac{\big(\frac{\beta}{\beta_{N_A,\omega}}\big)^{N_A-1}}{1-\big(\frac{\beta}{\beta_{N_A,\omega}}\big)^{N_A-1}}\,\mathrm{ITM}_K\,.$$

Also, noting that the above process can be reversed, we obtain

$$\mathrm{ITM}_K = \frac{K}{K-1} \sup_{\beta\in(0,\beta_{N_A,\omega})} \Big[\frac{1-\big(\frac{\beta}{\beta_{N_A,\omega}}\big)^{N_A-1}}{\big(\frac{\beta}{\beta_{N_A,\omega}}\big)^{N_A-1}}\Big] \mathrm{STM}(\beta).$$

Similarly, we can argue as above and as in [30] to deduce

$$\mathrm{TM} = \sup_{\beta\in(0,\beta_{N_A,\omega})} \Big[\frac{1-\big(\frac{\beta}{\beta_{N_A,\omega}}\big)^{N_A-1}}{\big(\frac{\beta}{\beta_{N_A,\omega}}\big)^{N_A-1}}\Big] \mathrm{STM}(\beta).$$

□

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