Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 106, pp. 1–26.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

LAGRANGIAN STRUCTURE FOR COMPRESSIBLE FLOW IN THE HALF-SPACE WITH NAVIER BOUNDARY CONDITION

MARCELO M. SANTOS, EDSON J. TEIXEIRA

Abstract. We show the uniqueness of particle paths of a velocity field, which solves the compressible isentropic Navier-Stokes equations in the half-space $\mathbb{R}^3_+$ with the Navier boundary condition. More precisely, by energy estimates and the assumption of small energy we prove that the velocity field satisfies regularity estimates which imply the uniqueness of particle paths.

1. Introduction

This article concerns the Lagrangian structure, i.e. the uniqueness of particle paths, for the solution obtained by Hoff [16] to the Navier-Stokes system for compressible isentropic fluids, in the half-space $\mathbb{R}^3_+ = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3\,;\ x_3 > 0\}$ with the Navier boundary condition. We follow the approach of [17] but in [17] the problem is posed in the whole space $\mathbb{R}^n$ $(n = 2, 3)$, so there is no questions in [17] concerning boundary effects.

In view of the presence of the boundary, we analyze and show new estimates. For instance, to estimate the L^q norm of the second derivative of a part of the velocity field, which is denoted by $u_{F,\omega}$, we need to consider a *singular kernel* on $\partial\mathbb{R}^3_+$. For estimating this norm, we use a theorem due to the Agmon-Douglis-Nirenberg [3], i.e. Theorem 2.2 below. In fact, this part, $u_{F,\omega}$, of the velocity field satisfies a boundary value problem in the half-space (see (3.6)), to which we use the explicit formulas given by Green's functions for the half-space with Dirchlet and Neumann boundary conditions (see (2.7) and (2.8)).

The half-space has several properties that are important to our analysis, some of which we mention in Section 2 below. In addition to the aforementioned explicit formulas for Green's functions, it enjoys the *strong m-extension operator* property (see [1, Theorem 5.19]). This property implies that several classical inequalities on $\mathbb{R}^n$ holds also on $\mathbb{R}^n_+$. In particular, it is very useful the imbedding inequality (2.1) and the interpolation inequality (2.16), which we can infer from the similar inequalities on $\mathbb{R}^n$. These and some other results we shall need are explained in details in Section 2.

The crucial result in this paper, as in [17], concerning the uniqueness of particle paths, is the regularity estimates (1.14) and (1.15), stated in Theorem 1.2. To show these estimates, with the presence of the boundary (we recall that in [17] it is

---

2010 *Mathematics Subject Classification.* 35Q30, 76N10, 35Q35, 35B99.
Key words and phrases. Navier-Stokes equations; Lagrangian structure;
Navier boundary condition.

Submitted March 5, 2019. Published September 18, 2019.

considered only the initial value problem), in addition to the results mentioned in Section 2, we shall use arguments in the papers [13, 15, 18, 16, 24]. In particular, to prove Proposition 3.2 and Theorem 3.4 we use some arguments as those in [24, Lemma 3.3].

Let us now describe in more details the results we show in this paper. First, for the reader convenience, we recall the solution obtained in [16]. Consider the Navier-Stokes equations

$$\begin{gathered} \rho_t + \operatorname{div}(\rho u) = 0 \\ (\rho u^j)_t + \operatorname{div}(\rho u^j u) + P(\rho)_{x_j} = \mu \Delta u^j + \lambda \operatorname{div} u_{x_j} + \rho f^j, \quad j = 1,2,3 \end{gathered} \tag{1.1}$$

for $x = (x_1, x_2, x_3) \in \mathbb{R}^3_+$ and $t > 0$, with the Navier boundary condition

$$u(x,t) = K(x)(u^1_{x_3}(x,t), u^2_{x_3}(x,t), 0), \tag{1.2}$$

for $x = (x_1, x_2, 0) \in \partial\mathbb{R}^3_+$, $t > 0$, and with the initial condition

$$(\rho, u)\big|_{t=0} = (\rho_0, u_0). \tag{1.3}$$

Here, as usual, ρ and $u = (u^1, u^2, u^3)$ denote, respectively, the unknowns density and velocity vector field of the fluid modeled by these equations, and $P(\rho)$ is the pressure function, which is assumed to satisfy the following conditions:

$$\begin{gathered} P \in C^2([0,\bar\rho]), \quad P(0) = 0, \quad P'(\tilde\rho) > 0, \\ (\rho - \tilde\rho)[P(\rho) - P(\tilde\rho)] > 0, \quad \rho \neq \tilde\rho, \quad \rho \in [0,\bar\rho], \end{gathered} \tag{1.4}$$

for fixed numbers $\tilde\rho, \bar\rho$ such that $0 < \tilde\rho < \bar\rho$. In addition, $f = (f^1, f^2, f^3)$ is a given external force density, μ and λ are given constant viscosities, and K is a smooth and strictly positive function, also given, satisfying the following conditions:

$$\mu > 0, \quad 0 < \lambda < 5\mu/4; \tag{1.5}$$

$$K \in (W^{2,\infty} \cap W^{1,3})(\mathbb{R}^2), \quad K(x) \geq \underline{K} > 0, \tag{1.6}$$

for some constant $\underline{K} > 0$;

$$\int_{\mathbb{R}^3_+} \big[\frac{1}{2}\rho_0 |u_0|^2 + G(\rho_0)\big]\, dx \leq C_0 \tag{1.7}$$

and

$$\begin{aligned} &\sup_{t\geq 0} \|f(.,t)\|_2 + \int_0^\infty \big(\|f(.,t)\|_2 + \sigma^7 \|\nabla f(.,t)\|_4\big)\, dt \\ &+ \int_0^\infty \int_{\mathbb{R}^3_+} (|f|^2 + \sigma^5 |f_t|^2)\, dx\, dt \leq C_f, \end{aligned} \tag{1.8}$$

where

$$G(\rho) := \rho \int_{\tilde\rho}^{\rho} \frac{P(s) - P(\tilde\rho)}{s^2}\, ds,$$

C_0 and C_f are positive numbers sufficiently small and $\sigma(t) := \min\{t, 1\}$, and the quantity

$$M_q := \int_{\mathbb{R}^3_+} \rho_0 |u_0|^q + \sup_{t>0} \|f(\cdot,t)\|_q + \int_0^\infty \int_{\mathbb{R}^3_+} |f|^q\, dx\, dt \tag{1.9}$$

is finite, where $q > 6$ and satisfies

$$\frac{(q-2)^2}{4(q-1)} < \frac{\mu}{\lambda}. \tag{1.10}$$

Throughout the article, $\|\cdot\|_q$ stands for the L^q norm in $\mathbb{R}^n_+$.

Under the above conditions, Hoff [16, Theorem 1.1] established the existence of a "small energy" (i.e. for C_0, C_f sufficiently small) weak solution (ρ, u) to (1.1)-(1.3) as follows:

Given a positive number M (not necessarily small) and given $\bar\rho_1 \in (\tilde\rho, \bar\rho)$, there are positive numbers ε and $\bar C$ depending on $\tilde\rho, \bar\rho_1, \bar\rho, P, \lambda, \mu, q, M$ and on the function K, and there is a positive universal constant θ, such that, if

$$\begin{gathered} 0 \le \inf_{\mathbb{R}^3_+} \rho_0 \le \sup_{\mathbb{R}^3_+} \rho_0 \le \bar\rho_1, \\ C_0 + C_f \le \varepsilon \quad \text{and} \quad M_q \le M, \end{gathered}$$

then there is a weak solution (ρ, u) to (1.1)-(1.3) having the following (among other) properties:

The functions u, $F = (\lambda + \mu)\operatorname{div} u - P(\rho) + P(\tilde\rho)$ (the so-called *effective viscous flow*) and $\omega^{j,k} = u^j_{x_k} - u^k_{x_j}$, $j, k = 1, 2, 3$ (note that $\omega = (\omega^{j,k})$ is the *vorticity matrix*) are Hölder continuous in $\overline{\mathbb{R}^3_+} \times [\tau, \infty)$, for any $\tau > 0$;

$$C^{-1} \inf \rho_0 \le \rho \le \bar\rho \quad \text{a.e.}$$

and

$$\begin{aligned} &\sup_{t>0} \int_{\mathbb{R}^3_+} [\frac{1}{2}\rho(x,t)|u(x,t)|^2 + |\rho(x,t) - \tilde\rho|^2 + \sigma(t)|\nabla u(x,t)|^2]\, dx \\ &+ \int_0^\infty \int_{\mathbb{R}^3_+} \left[|\nabla u|^2 + \sigma^3(t)|\nabla \dot u|^2\right] dx\, dt \\ &\le \bar C (C_0 + C_f)^\theta, \end{aligned}$$

where $\dot u$ denotes the *convective derivative* of u, i.e.

$$\dot u := u_t + (\nabla u) u.$$

In addition, when $\inf_{\mathbb{R}^3_+} \rho_0 > 0$, the term $\int_0^\infty \int_{\mathbb{R}^3_+} \sigma |\dot u|^2\, dx\, dt$ can be included on the left side of (1.11).

In this article we show the following results.

Proposition 1.1. *Let assumptions* (1.4)-(1.10) *be satisfied. Then the vector field u described above (in particular, satisfying the estimate* (1.11)*) can be written as*

$$u = u_P + u_{F,\omega},$$

for some vector fields u_P, $u_{F,\omega}$ satisfying:

$$\|\nabla u_P\|_q \le C\|P - \tilde P\|_q, \tag{1.11}$$

$$\|\nabla u_{F,\omega}\|_q \le C(\|F\|_q + \|\omega\|_q + \|P - \tilde P\|_q + \|u\|_q), \tag{1.12}$$

$$\|D^2 u_{F,\omega}\|_q \le C(\|\nabla F\|_q + \|\nabla\omega\|_q + \|F\|_q + \|\omega\|_q + \|P - \tilde P\|_q + \|u\|_q), \tag{1.13}$$

for any $q \in (1, \infty)$, where C is a constant depending only on q and on arbitrary positive numbers $\underline{K}, \overline{K}$ such that $\underline{K} \le K \le \overline{K}$.

Theorem 1.2. *Let assumptions* (1.4)-(1.10) *be satisfied. Suppose that u_0 belongs to the Sobolev space $H^s(\mathbb{R}^3_+)$, for some $s \in [0, 1]$, and $\inf_{\mathbb{R}^3_+} \rho_0 > 0$. Then the solution*

(ρ, u) to problem (1.1)-(1.3)*, described above, satisfies the additional estimates:*

$$\begin{aligned} &\sup_{t>0} \sigma^{1-s} \int_{\mathbb{R}^3_+} |\nabla u|^2\,dx + \int_0^\infty \int_{\mathbb{R}^3_+} \sigma^{1-s} \rho |\dot u|^2\,dx\,dt \\ &\le C(s)(C_0 + \|u_0\|_{H^s} + C_f)^\theta, \end{aligned} \tag{1.14}$$

$$\begin{aligned} &\sup_{t>0} \sigma^{2-s} \int_{\mathbb{R}^3_+} \rho |\dot u|^2\,dx + \int_0^\infty \int_{\mathbb{R}^3_+} \sigma^{2-s} |\nabla \dot u|^2\,dx\,dt \\ &\le C(s)(C_0 + \|u_0\|_{H^s} + C_f)^\theta, \end{aligned} \tag{1.15}$$

where $C(s)$ is a constant depending only on s and on the same quantities as does C in Proposition 1.1.

Estimates (1.14) and (1.15), as in [17], imply a Lagrangian structure for the solution (ρ, u) described above to problem (1.1)-(1.3). More precisely, the following theorem, which is similar to [17, Theorem 2.5], holds for the Navier-Stokes equations (1.1) in the half-plahe $\mathbb{R}^3_+$, with the Navier boundary condition (1.2).

Theorem 1.3 (cf. [17, Theorem 2.5])**.** *Under the hypothesis in Theorem 1.2, if $s > 1/2$ then the following assertions are true:*

(a) *For each $x \in \overline{\mathbb{R}^3_+}$, there exists a unique map $X(\cdot\,, x) \in C([0, \infty)) \cap C^1((0, \infty))$ such that*
$$X(t, x) = x + \int_0^t u(X(\tau, x), \tau)\,d\tau, \quad t \in [0, \infty). \tag{1.16}$$

(b) *For each $t > 0$, the map $x \mapsto X(t, x)$ is a homeomorphism of $\overline{\mathbb{R}}^3_+$ into $\overline{\mathbb{R}}^3_+$, leaving $\partial\mathbb{R}^3_+$ invariant i.e. $X(t, \partial\mathbb{R}^3_+) \subset \partial\mathbb{R}^3_+$.*

(c) *Given $t_1, t_2 \ge 0$, the map $X(t_1, x) \mapsto X(t_2, x)$, $x \in \mathbb{R}^3_+$, is Hölder continuous, locally uniform with respect to t_1, t_2, i.e., given any $T > 0$, there exist positive numbers C, L and γ such that*
$$|X(t_2, y) - X(t_2, x)| \le C|X(t_1, y) - X(t_1, x)|^{e^{-LT^\gamma}}$$
for all $t_1, t_2 \in [0, T])$ and $x, y \in \mathbb{R}^3_+$.

(d) *Let $\mathcal{M}$ be a parametrized manifold in $\mathbb{R}^3_+$ of class C^α, for some $\alpha \in [0, 1)$, and of dimension k, where $k = 1$ or 2. Then, for each $t > 0$, $\mathcal{M}^t := X(t, \mathcal{M})$ is also a parametrized manifold of dimension k in $\mathbb{R}^3_+$, and of class C^β, where $\beta = \alpha e^{Lt^\gamma}$, being L and γ the same constants in item (c).*

We shall assume throughout the paper, without loss of generality, that the above solution (ρ, u) to (1.1)-(1.3) is smooth, since it is the limit of smooth solutions (see [16, Proposition 3.2 and §4]) and all the above estimates can be obtained by passing to the limit from corresponding uniform estimates for smooth solutions. In particular, we note that by the proof of [16, Proposition 3.2], we have that $\rho(\cdot, t), u(\cdot, t) \in H^\infty(\mathbb{R}^3_+)$ for any $t \ge 0$, if all data are smooth. Before ending this Introduction, we say some words about previous results related to this paper.

Considering the Cauchy problem, Hoff [15] established the Lagrangian structure in dimension two with the initial velocity in the Sobolev space H^s, for an arbitrary $s > 0$, while Hoff and Santos [17] proved that the velocity field was a Lipschitzian vector field, in dimension two and three, for the initial velocity in H^s, with $s > 0$ in dimension two and $s > 1/2$ in dimension three, and, as a consequence, assured the

Lagrangian structure in dimensions two and three; Zhang and Fang [26] obtained the Lagrangian structure in dimension two for the viscosity $\lambda = \lambda(\rho)$, depending only on the fluid density ρ, but with the initial velocity in $H^1(\mathbb{R}^2)$, and Maluendas [22] extended the Lagrangian structure result obtained in [17] to non isentropic fluids in dimension two.

Regarding the initial and boundary value problems, Hoff and Perepelitsa [18] proved (among other results in [18]) the Lagrangian structure in the half-plane with the initial velocity in H^1.

We end this Introduction, by describing the next sections in this paper. In Section 2 we collect several results we use in the proofs of Proposition 1.1, and theorems 1.2 and 1.3, stated above. In Section 3 we prove these three results.

2. Preliminaries

In this section we collect several results, regarding the half-space, that we shall use in the proofs of Proposition 1.1, and theorems 1.2 and 1.3, stated above. Although, the problem (1.1)-(1.3) is set in this paper in the half-space $\mathbb{R}^3_+$, some results we give in this section are stated in the half-space $\mathbb{R}^n_+$, for an arbitrary $n \geq 2$, since it does not make any relevant difference to state them only for $\mathbb{R}^3_+$.

One of the main properties of the half-space $\mathbb{R}^n_+$ is the existence of a *strong m-extension operator* $\mathcal{E}$, for any $m \in \mathbb{Z}_+$, and its explicit construction; see [1, Theorem 5.19 and its proof]. This property implies that several classical inequalities on $\mathbb{R}^n$ holds also on $\mathbb{R}^n_+$. In particular, it is very useful the inequality

$$\|u\|_{L^\infty(\mathbb{R}^n_+)} \leq C(\|u\|_{L^2(\mathbb{R}^n_+)} + \|\nabla u\|_{L^q(\mathbb{R}^n_+)}) \tag{2.1}$$

where $q > n$ is arbitrary, C is a constant depending only on n and q, and u can be any function in $C^1(\mathbb{R}^n_+)$ such that $u \in L^2(\mathbb{R}^n_+)$ and $\nabla u \in L^q(\mathbb{R}^n_+)$.

It is worth mentioning that inequality (2.1) is true with $\mathbb{R}^n_+$ replaced by any open set Ω in $\mathbb{R}^n$ that has a *strong 1-extension operator* $\mathcal{E}$ mapping $C^1(\Omega)$ into $C^1(\mathbb{R}^n)$ and a *simple (0,p)-extension operator* $\mathcal{E}_0$ such that $\nabla \circ \mathcal{E} = \mathcal{E}_0 \circ \nabla$ on $C^1(\Omega)$ (see [1, Chapter 5, §Extensions Theorems] for details on extension operators). Indeed, by the proof of Morrey's inequality [8, p. 282] it easy to see that, given a function $v \in C^1(\mathbb{R}^n)$ such that $v \in L^2(\mathbb{R}^n)$ and $\nabla v \in L^q(\mathbb{R}^n)$, where $q > n$, we have the inequality

$$\|v\|_{L^\infty(\mathbb{R}^n)} \leq C(\|v\|_{L^2(\mathbb{R}^n)} + \|\nabla v\|_{L^q(\mathbb{R}^n)}),$$

for some constant $C = C(n, q)$. Then, taking in this inequality $v = \mathcal{E}(u)$, for $u \in C^1(\Omega)$ such that $u \in L^2(\Omega)$ and $\nabla u \in L^q(\Omega)$, using the aforementioned extension operators, we obtain that

$$\begin{aligned} \|u\|_{L^\infty(\Omega)} \leq \|\mathcal{E}(u)\|_{L^\infty(\mathbb{R}^n)} &\leq C(\|\mathcal{E}(u)\|_{L^2(\mathbb{R}^n)} + \|\nabla\mathcal{E}(u)\|_{L^q(\mathbb{R}^n)}) \\ &= C(\|\mathcal{E}(u)\|_{L^2(\mathbb{R}^n)} + \|\mathcal{E}_0(\nabla u)\|_{L^q(\mathbb{R}^n)}) \\ &\leq C(\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^q(\Omega)}), \end{aligned}$$

where $q > n$ and C denote different constants depending only on n and q.

Remark 2.1. Certainly many results in this paper (in particular, the very important estimate (2.4) below) hold true if we replace the half-space $\mathbb{R}^3_+$ by any domain (i.e. an open set) Ω in $\mathbb{R}^n$ having the aforementioned extension properties, and a nice boundary – such that we can assure the existence of the Green function, with Dirichlet or Neumann boundary condition. In this regard, we believe that our main theorem in this paper, i.e. Theorem 1.2 above, and, consequently, also Theorem 1.3 above, hold true for the solution obtained by the Hoff in the paper [19] for more general 3d domains.

For convenience of the reader, we give next explicitly the Green functions for the half-space $\mathbb{R}^n_+$, and, using them, we show how to estimate solutions for some Poisson equations in $\mathbb{R}^n_+$.

The Green functions in $\mathbb{R}^n_+$, with homogeneous Dirichlet and Neumann boundary conditions, which we shall denote in this paper, respectively, by G_D and G_N, are given by (see e.g. [12, p. 121])

$$G_D(x,y) = \Gamma(x-y) - \Gamma(x-y^*) \quad \text{and} \quad G_N(x,y) = \Gamma(x-y) + \Gamma(x-y^*), \tag{2.2}$$

where $x, y \in \overline{\mathbb{R}^n_+}$, $x \neq y$, Γ is the fundamental solution of the laplacian operator in $\mathbb{R}^n$ and $y^* = (y_1^*, \cdots, y_n^*)$ is the reflection point of $y = (y_1, \cdots, y_n) \in \overline{\mathbb{R}^n_+}$ through the boundary $\partial\mathbb{R}^n_+$, i.e. $y_j^* = y_j$ for $j = 1, \cdots, n-1$ and $y_n^* = -y_n$.

Let us denote either G_D or G_N by G, for a while. A basic fact related to these Green functions we shall use is that the operator

$$g \mapsto \nabla G * g,$$

where

$$(\nabla G * g)(x) := \int_{\mathbb{R}^n_+} \nabla_x G(x,y) g(y)\, dy, \quad x \in \mathbb{R}^n_+,$$

whenever the right-hand side makes sense, maps the space $L^q(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$, for $1 \le q < n$, continuously into the space of bounded *log-lispchitzian functions* in $\mathbb{R}^n_+$, i.e. the space of continuous functions h in $\mathbb{R}^n_+$ such that

$$\|h\|_{LL} \equiv \|h\|_{LL(\mathbb{R}^n_+)} := \sup_{x \in \mathbb{R}^n_+} |h(x)| + \langle g \rangle_{LL} < \infty, \tag{2.3}$$

where

$$\langle h \rangle_{LL} := \sup_{x,y \in \mathbb{R}^n_+;\, 0<|x-y|\le 1} \frac{|h(x) - h(y)|}{|x-y|(1 - \log|x-y|)}.$$

More precisely, if $g \in L^q(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$, and $1 \le q < n$, then

$$\|\nabla G * g\|_{LL(\mathbb{R}^n_+)} \le C(\|g\|_{L^q(\mathbb{R}^n_+)} + \|g\|_{L^\infty(\mathbb{R}^n_+)}) \tag{2.4}$$

where C is a constant depending only on n and q. This follows from a similar result for $\nabla\Gamma * g$ in $\mathbb{R}^n$ and the extension (simple 0-extension) property of $\mathbb{R}^n_+$. Indeed, denoting by $\tilde{g}$ the extension of g to $\mathbb{R}^n$ by reflection through $\partial\mathbb{R}^n_+$ (i.e. $\tilde{g}(y) := g(y^*)$ when $y_n < 0$), in the case $G(x,y) = G_N(x,y) = \Gamma(x-y) + \Gamma(x-y^*)$ we have

$$\nabla G * g = \nabla\Gamma * \tilde{g},$$

where the last symbol $*$ stands for the classical convolution product in $\mathbb{R}^n$. Then

$$\begin{aligned} \|\nabla G * g\|_{LL(\mathbb{R}^n_+)} &= \|\nabla\Gamma * \tilde{g}\|_{LL(\mathbb{R}^n)} \\ &\le C(\|\tilde{g}\|_{L^q(\mathbb{R}^n)} + \|\tilde{g}\|_{L^\infty(\mathbb{R}^n)}) \\ &\le 2C(\|g\|_{L^q(\mathbb{R}^n_+)} + \|g\|_{L^\infty(\mathbb{R}^n_+)}). \end{aligned}$$

Regarding $G(x,y) = G_D(x,y) = \Gamma(x-y) - \Gamma(x-y^*)$, it is easy to see that

$$\nabla G * g = \nabla\Gamma * \tilde{g} - 2\int_{\mathbb{R}^n_+} \nabla\Gamma(x-y^*) g(y)\, dy,$$

so we obtain (2.4) similarly, since the last integral has a regular kernel.

Now, we want to give estimates to the solutions of boundary value problems for a special (for us) Poisson equation in the half-space $\mathbb{R}^n_+$ (see (2.6) and (2.14)), but let us first try to explain the importance of these estimates in this paper.

One of the ideas in the analysis of Hoff in e.g. [15] is to decompose the velocity field u, in the solutions of (1.1), as the sum of two terms, $u_{F,\omega}$ and u_P, being the term $u_{F,\omega}$ related to the distinguished quantity

$$F = (\lambda + \mu)\operatorname{div} u - P(\rho) + P(\tilde{\rho})$$

and to the vorticity matrix

$$\omega^{j,k} = u^j_{x_k} - u^k_{x_j},$$

and u_P related to the fluid pressure P. In subsection 3.1 we exhibit a similar decomposition. In [17], the vector field u_P is log-lipschitzian with respect to the spatial variable, with the log-lipschitz norm $\|u_P(\cdot,t)\|_{LL}$ (see (2.3)) locally integrable with respect to t, while $u_{F,\omega}$ is a lipschitzian vector field with respect to the spatial variable, with the Lipschitz norm

$$\|u_{F,\omega}\|_{\mathrm{Lip}} \equiv \sup_{x\in\mathbb{R}^3_+} |u_{F,\omega}(x,t)| + \sup_{x,y\in\mathbb{R}^3_+;x\neq y} |u_{F,\omega}(x,t) - u_{F,\omega}(y,t)|/|x-y| \tag{2.5}$$

also locally integrable (the hardest part to show) with respect to t. Here, this facts are also true, and we have extra difficulties to show them, in view of the presence of the boundary. For instance, to estimate the L^q norm of $D^2 u_{F,\omega}$ we need to consider a *singular kernel* on $\partial\mathbb{R}^3_+$, which we deal with the help of the following theorem due to Agmon, Douglis and Nirenberg [3, Theorem 3.3] (see also [11, Theorem II.11.6]).

Theorem 2.2. *Let $q \in (1,\infty)$ and $\kappa : \big(\overline{\mathbb{R}^n_+} \equiv \mathbb{R}^{n-1}\times[0,\infty)\big) - \{(\mathbf{0},0)\} \to \mathbb{R}$ be given by $\kappa(x,x_n) = w(\frac{(x,x_n)}{|(x,x_n)|})/|(x,x_n)|^{n-1}$, where w is a continuous function on $\overline{\mathbb{R}^n_+}\cap\mathbb{S}^{n-1}$, Hölder continuous on $\mathbb{S}^{n-1}\cap\{x_n=0\}$ and satisfies $\int_{\mathbb{S}^{n-1}} w(x,0)\,dx = 0$. Assume also that κ has continuous partial derivatives $\partial_{x_i}\kappa, i=1,2,...,n$, $\partial^2_{x_n}\kappa$ in $\mathbb{R}^n_+$ which are bounded by a constant c on $\mathbb{R}^n_+\cap\mathbb{S}^{n-1}$. Then, for any function $h\in L^q(\partial\mathbb{R}^n_+)$ that has finite seminorm*

$$\langle h\rangle_{1-1/p,p} \equiv \Big(\int_{\partial\mathbb{R}^n_+}\int_{\partial\mathbb{R}^n_+} \frac{|h(x)-h(y)|^q}{|x-y|^{n-2+q}}\,dx\,dy\Big)^{1/q},$$

the function

$$\psi(x,x_n) := \int_{\partial\mathbb{R}^n_+} \kappa(x-y,x_n)h(y)\,dy$$

belongs to $L^q(\mathbb{R}^n_+)$ and $\|\nabla\psi\|_{L^q(\mathbb{R}^n_+)} \le Cc\langle h\rangle_{1-1/q}$, where C is a constant depending only on n and q.

The coordinates functions of the vector fields $u_{F,\omega}, u_P$ in this paper, described in §3.1, satisfy boundary value problems for Poisson equations of the form

$$-\Delta v = g_{x_j} \tag{2.6}$$

in the half-space $\mathbb{R}^3_+$, for some function g, with Neumann or Dirichlet boundary condition. In this regard, we shall use the formulas

$$\begin{aligned} v(x) &= -\int_{\mathbb{R}^n_+} G_D(x,y)g(y)_{y_j}\,dy - \int_{\mathbb{R}^{n-1}} G_D(x,y)_{y_n}h(y)\,dy \\ &= \int_{\mathbb{R}^n_+} G_D(x,y)_{y_j}g(y)\,dy - \int_{\mathbb{R}^{n-1}} G_D(x,y)_{y_n}h(y)\,dy \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} v(x) &= -\int_{\mathbb{R}^n_+} G_N(x,y)g(y)_{y_j}\,dy - \int_{\mathbb{R}^{n-1}} G_N(x,y)h(y)\,dy \\ &= \int_{\mathbb{R}^n_+} G_N(x,y)_{y_j}g(y)\,dy - \int_{\mathbb{R}^{n-1}} G_N(x,y)h(y)\,dy, \end{aligned} \tag{2.8}$$

for the solutions of the the boundary value problems

$$\begin{gathered} -\Delta v = g_{x_j} \quad \text{in } \mathbb{R}^n_+ \\ v = h \quad \text{on } \mathbb{R}^{n-1} \end{gathered} \tag{2.9}$$

and

$$\begin{gathered} -\Delta v = g_{x_j} \quad \text{in } \mathbb{R}^n_+ \\ -v_{x_n} = h \quad \text{on } \mathbb{R}^{n-1}, \end{gathered} \tag{2.10}$$

respectively, for $j = 1, \cdots, n$, and $g \in H^m(\mathbb{R}^n_+), h \in H^m(\mathbb{R}^{n-1})$ with a sufficiently large m, where G_D and G_N are the Green functions in $\mathbb{R}^n_+$ with the homogeneous Dirichlet and Neumann boundary conditions, respectively (see (2.2)) and in the case $j = n$ we can assume $g|\mathbb{R}^{n-1} = 0$, without loss of generality.

We note that, extending g to a function $\tilde{g} \in H^m(\mathbb{R}^n)$ (see [1, Theorem 5.19]) we can write the integral

$$w(x) := \int_{\mathbb{R}^n_+} G(x,y)_{y_j} g(y)\,dy,$$

where $G = G_D, G_N$, in (2.7), (2.8), as

$$w(x) = \int_{\mathbb{R}^n} \Gamma(x-y)_{y_j}\tilde{g}(y)\,dy - \int_{\mathbb{R}^n_-} \Gamma(x-y)_{y_j}\tilde{g}(y)\,dy \pm \int_{\mathbb{R}^n_+} \Gamma(x-y^*)_{y_j} g(y)\,dy,$$

being the last two integrals harmonic functions in $\mathbb{R}^n_+$, since their kernels are regular, for $x \in \mathbb{R}^n_+$. The first integral satisfies the equation

$$-\Delta w = \tilde{g}_{x_j}$$

in $\mathbb{R}^n$ in the classical sense (cf. e.g. [8, §2.2, Theorem 1] where the condition of the right hand side of the Poisson equation having compact support can be replaced by the condition of being in $H^m(\mathbb{R}^n)$ for a sufficiently large m, as can be seen by checking the proof). In addition, we also can write

$$w(x) = \int_{\mathbb{R}^n_+} \Gamma(x-y)_{y_j} g(y)\,dy \pm \int_{\mathbb{R}^n_-} \Gamma(x-y)_{y_j} g(y*)\,dy = \int_{\mathbb{R}^n} \Gamma(x-y)_{y_j}[\bar{g}(y) \pm \bar{\bar{g}}(y)]dy,$$

where $\bar{g}$ and $\bar{\bar{g}}$ denote, respectively, the extensions by zero to $\mathbb{R}^n$ of g and $g(y*)$, from which, by using that the second derivative $\Gamma_{y_i y_j}$ of the fundamental solution for the laplacian in $\mathbb{R}^n$ is a *singular kernel*, we can infer the estimate

$$\|\nabla_x \int_{\mathbb{R}^n_+} G(x-y)_{y_j} g(y)\,dy\|_q \le C\|g\|_q, \tag{2.11}$$

for any $q \in (1,\infty)$, where $G = G_D, G_N$ and C is a constant depending only on n and q. On the other hand, writing

$$w(x) = -\int_{\mathbb{R}^n_+} G(x,y) g_{y_j}(y)\,dy,$$

by the same argument, we have also the estimate

$$\|D_x^2 \int_{\mathbb{R}_+^n} G(x,y)_{y_j} g(y)\,dy\|_q \le C\|\nabla g\|_q, \tag{2.12}$$

for q, G, C as in (2.11).

Regarding the boundary integrals (i.e. over $\mathbb{R}^{n-1}$) in (2.7) and(2.8), we observe that the function

$$x \mapsto \int_{\mathbb{R}^{n-1}} G_D(x,y)_{y_n} h(y)\,dy$$

defines a classical solution to (2.9), with $g = 0$, if h is continuous and bounded, as it is well known, and as for

$$\int_{\mathbb{R}^{n-1}} G_N(x,y)h(y)\,dy,$$

it defines a solution to (2.10), with also $g = 0$, if h is continuous and have a nice decay at infinity (e.g. $h \in H^m(\mathbb{R}^{n-1})$ for some large m); see [21, 4].

In addition, using the Agmon-Douglis-Nirenber (Theorem 2.2 above), we have the estimate

$$\big\|D^2 \int_{\mathbb{R}^{n-1}} G_N(\cdot,y)h(y)\,dy\big\|_{L^q(\mathbb{R}_+^n)} \le C\langle h\rangle_{1-1/q,q} \le C\|\nabla\tilde{h}\|_{L^q(\mathbb{R}_+^n)}, \tag{2.13}$$

for any $q \in (1,\infty)$, where $\tilde{h}$ is any extension to $H^1(\mathbb{R}_+^n)$ of $h \in H^1(\mathbb{R}_+^{n-1})$, C is a constant depending only on n and q, and for the last inequality we used [11, Theorem II.10.2].

It is interesting to note that the boundary value problem

$$\begin{aligned} \Delta v = 0 \quad &\text{in } \mathbb{R}_+^n \\ K v_{x_n} = v \quad &\text{on } \partial\mathbb{R}_+^n, \end{aligned} \tag{2.14}$$

which is required for the coordinates u_1 and u_2 of the vector field u in the Navier boundary condition (1.2), can be reduced to the boundary value problem (2.10) with homogeneous boundary condition (i.e. with $h = 0$ in (2.10)) through the change of variable (suggested to us by Hoff in a private communnication)

$$V = \varphi v$$

where φ is a suitable function coinciding with $e^{-K^{-1}x_n}$ on $\partial\mathbb{R}_+^n$. From this observation, using (2.11), (2.12) and that $\|G_N * v\|_q \le C\|v\|_q$, it is possible to show the estimates

$$\|\nabla v\|_q \le C\|v\|_q, \quad \|D^2 v\|_q \le C\|\nabla v\|_q \tag{2.15}$$

for the solution to problem (2.14), where $q \in (1,\infty)$ is arbitrary and C is as in (1.13).

Finally, regarding the above boundary value problems for Poisson equations, we observe that the solutions to the problems (2.9) and (2.10) given, respectively, by (2.7) and (2.8), are unique in the space $L^q(\mathbb{R}_+^n) \cap L^\infty(\mathbb{R}_+^n)$, for an arbitrary $q \in [1,\infty)$. Indeed, if v is a solution of (2.9) in $L^q(\mathbb{R}_+^n) \cap L^\infty(\mathbb{R}_+^n)$ with $g = h = 0$, extending it to $\mathbb{R}^n$ as an odd function with respect to x_n, we obtain an integrable harmonic function (in the sense of the distributions) and bounded, in $\mathbb{R}^n$, then, by Liouville's theorem, we conclude that $v = 0$. We can conclude the same result with respect to (2.10) by taking instead an even extension with respect to x_n.

Before ending this Section, we mention two facts we shall need in Section 3. The first, is a very useful inequality for us in this paper, which is the interpolation inequality

$$\|u\|_{L^q(\mathbb{R}^3_+)} \le C\|u\|_{L^2(\mathbb{R}^3_+)}^{(6-q)/2q}\|\nabla u\|_{L^2(\mathbb{R}^3_+)}^{(3q-6)/2q}, \tag{2.16}$$

which holds for any function u in the Sobolev space $H^1(\mathbb{R}^3_+)$, with $q \in [2,6]$ and C being a constant depending only on q. We note that this inequality can be obtained from the same inequality in $\mathbb{R}^3$, using the extension operators from $\mathbb{R}^3_+$ to $\mathbb{R}^3$.

To estimate the solutions of (1.1)-(1.3) in the Sobolev space H^s, $0 < s < 1$, we shall use the interpolation theory, since the space H^s is the *interpolation space* $(L^2, H^1)_{s,2}$ (see e.g. [23]). In particular, the interpolation Stein-Weiss' theorem [5, p. 115] will be very important to us.

3. Proofs

In this section, using the results presented in Section 2 and following mainly the methods in the papers [18, 15, 24] and [17], we prove Proposition 1.1 and Theorems 1.2 and 1.3.

3.1. **Proof of Proposition 1.1.** As in [18, (2.28)], we define u_P as the solution of the boundary value problem

$$\begin{gathered}(\lambda+\mu)\Delta u_P = \nabla(P-\tilde{P}), \quad \text{in } \mathbb{R}^3_+\\ u_P^3 = (u_P^2)_{x_3} = (u_P^1)_{x_3} = 0, \quad \text{on } \partial\mathbb{R}^3_+,\end{gathered} \tag{3.1}$$

i.e.

$$\begin{aligned}(\lambda+\mu)u_P^j(x) &= \int_{\mathbb{R}^3_+} G_N(x,y)_{y_j}(P-\tilde{P})(y)\,dy\\ &= \int_{\mathbb{R}^3_+} (\Gamma(x-y)+\Gamma(x-y^*))_{y_j}(P-\tilde{P})(y)\,dy,\end{aligned} \tag{3.2}$$

for $j=1,2$, $x \in \mathbb{R}^3_+$, and

$$\begin{aligned}(\lambda+\mu)u_P^3(x) &= \int_{\mathbb{R}^3_+} G_D(x,y)_{y_3}(P-\tilde{P})(y)\,dy\\ &= \int_{\mathbb{R}^3_+} (\Gamma(x-y)-\Gamma(x-y^*))_{y_3}(P-\tilde{P})(y)\,dy,\end{aligned} \tag{3.3}$$

for $x \in \mathbb{R}^3_+$; see (2.7) and (2.8). By (2.11), we have the estimate

$$\|\nabla u_P^j\|_q \le C\|P-\tilde{P}\|_q, \quad j=1,2,3, \tag{3.4}$$

for any $q \in (1,\infty)$, with C being a constant depending only on n and q.

Next we define $u_{F,\omega} = u - u_P$. Using (3.4), it follows that

$$\|\nabla u_{F,\omega}\|_q \le C(\|\nabla u\|_q + \|P-\tilde{P}\|_q) \tag{3.5}$$

for any $q \in (1,\infty)$, with C being a constant depending only on n and q.

On the other hand, by the definitions of u_P, the Navier boundary condition (1.2), and observing that the the momemtum equation (second equation in (1.1)) can be written in terms of the *effective viscous flow* F and of the vortex matrix ω as

$$(\lambda+\mu)\Delta u^j = F_{x_j} + (P-\tilde{P})_{x_j} + (\lambda+\mu)\sum_{k=1}^{3}\omega^{j,k}_{x_k},$$

we have that $u_{F,\omega}$ satisfies the boundary value problem

$$\begin{gathered}(\lambda+\mu)\Delta u_{F,\omega}=\nabla F+(\lambda+\mu)\sum_{k=1}^{3}\omega_{x_k}^{\cdot,k},\quad \text{in } \mathbb{R}^3_+\\ u^3_{F,\omega}=0,\quad (u^j_{F,\omega})_{x_3}=K^{-1}u^j,\quad j=2,3,\quad \text{on } \partial\mathbb{R}^3_+.\end{gathered}\tag{3.6}$$

Then by (2.11), (2.12) and (2.13), we have

$$\|D^2u_{F,\omega}\|_q\le C(\|\nabla F\|_q+\|\nabla\omega\|_q+\|\nabla u\|_q),\tag{3.7}$$

for q and C as in (1.13). Now, the velocity field u satisfies the boundary value problem

$$\begin{gathered}(\lambda+\mu)\Delta u=\nabla F+(\lambda+\mu)\sum_{k=1}^{3}\omega_{x_k}^{\cdot,k}+\nabla(P-\tilde P),\quad \text{in } \mathbb{R}^3_+\\ u^3=0,\quad u^j_{x_3}=K^{-1}u^j,\quad j=2,3,\quad \text{on } \partial\mathbb{R}^3_+.\end{gathered}\tag{3.8}$$

Then, by (2.11) and (2.15), we have the estimate [16, Lemma 2.3, item (b)]

$$\|\nabla u\|_q\le C(\|F\|_q+\|\omega\|_q+\|P-\tilde P\|_q+\|u\|_q)\tag{3.9}$$

where q and C are as in (1.13).

By (3.4), (3.5), (3.7) and (3.9), we conclude the proof of Proposition 1.1.

Proof of Theorem 1.2. To prove (1.14), following [15] and [18], we write $u=v+w$, where v is the solution of a linear homogeneous system with initial condition $v|_{t=0}=u_0$ and w is the solution of a linear nonhomogeneous system with initial homogeneous initial condition. More precisely, taking the differential operator $\mathcal{L}\equiv(\mathcal{L}^1,\mathcal{L}^2,\mathcal{L}^3)$ given by

$$\mathcal{L}^j(z)=\rho\dot z^j-\mu\Delta z^j-\lambda\operatorname{div} z_j,\quad j=1,2,3,\quad z=(z^1,z^2,z^3),$$

where $\dot z$ is the convective derivative of z with respect to u, i.e.

$$\dot z:=z_t+u\nabla z,$$

we define v and w as the solutions of the following initial boundary value problems

$$\begin{gathered}\mathcal{L}(v)=0,\quad \text{in } \mathbb{R}^3_+\\ (v^1,v^2,v^3)=K^{-1}(v^1_{x_3},v^2_{x_3},0),\quad \text{on } \partial\mathbb{R}^3_+\\ v(.,0)=u_0,\end{gathered}\tag{3.10}$$

and

$$\begin{gathered}\mathcal{L}(w)=-\nabla(P-\tilde P)+\rho f,\quad \text{in } \mathbb{R}^3_+\\ (w^1,w^2,w^3)=K^{-1}(w^1_{x_3},w^2_{x_3},0),\quad \text{on } \partial\mathbb{R}^3_+\\ w(.,0)=0.\end{gathered}\tag{3.11}$$

Then v and w are estimated separately. To estimate v, the interpolation theory is used, since the initial data u_0 is in H^s and H^s is the interpolation space $\left(L^2,H^1\right)_{s,2}$; see [23, p. 186 and 226]. We shall use also the Stein-Weiss' theorem for L^q spaces with weights [5, p. 115]. To estimate w, the interpolation theory is not needed, since the initial condition is null. Actually, w satisfies the estimate (1.14) with $s=0$ (equation (3.14) below).

Proposition 3.1. *If $u_0 \in H^s(\mathbb{R}^3_+)$, $0 \le s \le 1$, then for any positive number T there is a constant C independent of $(\rho, u), v, w, \rho_0, u_0$ and f such that*

$$\sup_{0\le t\le T} \sigma^{1-s}(t) \int_{\mathbb{R}^3_+} |\nabla v|^2\,dx + \int_0^T \int_{\mathbb{R}^3_+} \sigma^{1-s}(t)\rho|\dot v|^2\,dx\,dt \le C||u_0||^2_{H^s(\mathbb{R}^3_+)}. \tag{3.12}$$

Proof. We shall obtain (3.12) for $s = 1$ when $u_0 \in L^2(\mathbb{R}^3_+)$ and for $s = 0$ when $u_0 \in H^1(\mathbb{R}^3_+)$. Then (3.12) follows by interpolation.

Multiplying the equation $\rho\dot v^j = \mu\Delta v^j + \lambda(\operatorname{div} v)_j$ by v^j_t and integrating, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3_+} \rho|\dot v|^2\,dx - \int_{\mathbb{R}^3_+} \rho\dot v^j u\cdot\nabla v^j\,dx \\
&= \mu\int_{\mathbb{R}^3_+} \Delta v^j v^j_t\,dx + \lambda\int_{\mathbb{R}^3_+} (\operatorname{div} v)_j v^j_t\,dx \\
&= -\mu\int_{\mathbb{R}^3_+} \nabla v^j\cdot\nabla v^j_t\,dx + \mu\int_{\partial\mathbb{R}^3_+} v^j_t\nabla v^j.\nu\,dS_x - \lambda\int_{\mathbb{R}^3_+} (\operatorname{div} v)(\operatorname{div} v)_t\,dx \\
&\quad + \lambda\int_{\partial\mathbb{R}^3_+} (\operatorname{div} v)v^j_t\nu^j\,dS_x \\
&= -\frac{\mu}{2}\frac{d}{dt}\int_{\mathbb{R}^3_+} |\nabla v|^2\,dx - \frac{\lambda}{2}\frac{d}{dt}\int_{\mathbb{R}^3_+} |\operatorname{div} v|^2\,dx + \mu\int_{\partial\mathbb{R}^3_+} v^j_t v^j_k\nu^k\,dS_x \\
&= -\frac{1}{2}\frac{d}{dt}\Big\{\mu\int_{\mathbb{R}^3_+} |\nabla v|^2\,dx + \lambda\int_{\mathbb{R}^3_+} (\operatorname{div} v)^2\,dx + \int_{\partial\mathbb{R}^3_+} \mu K^{-1}|v|^2\,dS_x\Big\}.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\mu||\nabla v||_2^2 + \lambda||\operatorname{div} v||_2^2 + \mu\int_{\partial\mathbb{R}^3_+} K^{-1}|v|^2\,dS_x\Big) + \int_{\mathbb{R}^3_+} \rho|\dot v|^2\,dx \\
&= \int_{\mathbb{R}^3_+} \rho\dot v^j(u\cdot\nabla v^j)\,dx \\
&\le C(\bar\rho)\Big(\int_{\mathbb{R}^3_+} \rho|u|^3\,dx\Big)^{1/3}\Big(\int_{\mathbb{R}^3_+} \rho|\dot v|^2\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^3_+} |\nabla v|^6\,dx\Big)^{1/6} \\
&\le C(\bar\rho)\|\rho u\|_2^a\|\rho u\|_q^{1-a}\|\rho\dot v\|_2\|\nabla v\|_6 \\
&\le C(\bar\rho)(C_0 + C_f + M_q)^\theta\|\rho\dot v\|_2\|\nabla v\|_6,
\end{aligned}$$

for some $a \in (0,1)$, where $q > 6$ and M_q are defined in (1.10) and (1.9), θ is some universal positive constant, and we used [16, Proposition 2.1] and (1.11).

Now defining

$$\tilde F = (\lambda+\mu)\operatorname{div} v, \quad \tilde\omega^{j,k} = v^j_{x_k} - v^k_{x_j},$$

we have

$$(\lambda+\mu)\Delta v^j = \tilde F_{x_j} + (\lambda+\mu)\tilde\omega^{j,k}_{x_k}$$

and, analogously to [16, Lemma 2.3], it follows that

$$\begin{gathered}
\|\nabla v\|_q \le C(\|v\|_q + \|\tilde\omega\|_q + \|\tilde F\|_q), \\
\|\nabla\tilde F\|_q + \|\nabla\tilde\omega\|_q \le C(\|\rho\dot v\|_q + \|\nabla v\|_q + \|v\|_q),
\end{gathered}$$

for any $q \in (1, \infty)$. Thus by (2.16) and energy estimates we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3_+} \rho\dot{v}(u \cdot \nabla v^j)\, dx \\
&\le C\|\rho\dot{v}\|_2\big(\|v\|_6 + \|\tilde{w}\|_6 + \|\tilde{F}\|_6\big) \\
&\le C(C_0 + C_f)^\theta \|\rho\dot{v}\|_2 \Big(\|\nabla v\|_2 + \|\nabla\tilde{w}\|_2 + \|\nabla\tilde{F}\|_2\Big) \\
&\le C(C_0 + C_f)^\theta \|\rho\dot{v}\|_2 \left(\|\nabla v\|_2 + \|\rho\dot{v}\|_2 + \|v\|_2\right) \\
&= C(C_0 + C_f)^\theta \|\rho\dot{v}\|_2\|\nabla v\|_2 + C(C_0 + C_f)^\theta \|\rho\dot{v}\|_2^2 + C(C_0 + C_f)^\theta \|\rho\dot{v}\|_2\|v\|_2 \\
&\le C(C_0 + C_f)^\theta \|\nabla v\|_2^2 + C(C_0 + C_f)^\theta \|\rho\dot{v}\|_2^2 + C\|v\|_2^2 \\
&= C(C_0 + C_f)^\theta \int_{\mathbb{R}^3_+} |\nabla v|^2\, dx \\
&\quad + C(C_0 + C_f)^\theta \int_{\mathbb{R}^3_+} \rho|\dot{v}|2\, dx + C(C_0 + C_f)^\theta \int_{\mathbb{R}^3_+} |v|^2\, dx
\end{aligned}
$$

Therefore, if C_0, C_f are sufficiently small,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\mu\|\nabla v\|_2^2 + \lambda\|\operatorname{div} v\|_2^2 + \mu\int_{\partial\mathbb{R}^3_+} K^{-1}|v|^2\, dS_x) + \int_{\mathbb{R}^3_+} \rho|\dot{v}|^2 \\
&\le C\int_{\mathbb{R}^3_+} |\nabla v|^2\, dx + C\int_{\mathbb{R}^3_+} |v|^2\, dx,
\end{aligned}
\tag{3.13}
$$

so integrating on $(0, t)$, we obtain

$$
\begin{aligned}
&\frac{\mu}{2}\int_{\mathbb{R}^3_+} |\nabla v|^2\, dx + \frac{\lambda}{2}\int_{\mathbb{R}^3_+} |\operatorname{div} v|^2\, dx + \frac{\mu}{2}\int_{\partial\mathbb{R}^3_+} K^{-1}|v|^2\, dS_x + \int_0^T\int_{\mathbb{R}^3_+} \rho|\dot{v}|^2\, dx\, ds \\
&\le \frac{\mu}{2}\int_{\mathbb{R}^3_+} |\nabla u_0|^2\, dx + \frac{\lambda}{2}\int_{\mathbb{R}^3_+} |\operatorname{div} u_0|^2\, dx + \frac{\mu}{2}\int_{\partial\mathbb{R}^3_+} K^{-1}|u_0|^2\, dS_x \\
&\quad + C\int_0^T\int_{\mathbb{R}^3_+} |v|^2\, dx\, ds \\
&\le C\|u_0\|^2_{H^1(\mathbb{R}^3_+)},
\end{aligned}
$$

if $u_0 \in H^1(\mathbb{R}^3_+)$. On the other hand, multiplying (3.13) by $\sigma(t)$, we obtain

$$
\begin{aligned}
&-\frac{1}{2}\sigma'\Big(\mu\|\nabla v\|_2^2 + \lambda\|\operatorname{div} v\|_2^2 + \mu\int_{\partial\mathbb{R}^3_+} K^{-1}|v|^2\, dS_x\Big) \\
&\quad + \sigma\int_{\mathbb{R}^3_+} \rho|\dot{v}|^2\, dx + \frac{1}{2}\frac{d}{dt}\Big(\mu\sigma\|\nabla v\|_2^2 + \lambda\sigma\|\operatorname{div} v\|_2^2 + \mu\alpha\sigma\int_{\partial\mathbb{R}^3_+} K^{-1}|v|^2\, dS_x\Big) \\
&\le \sigma C\int_{\mathbb{R}^3_+} |\nabla v|^2\, dx + C\int_{\mathbb{R}^3_+} |v|^2\, dx,
\end{aligned}
$$

so integrating on $(0, t)$,

$$
\begin{aligned}
&\sigma\frac{\mu}{2}\|\nabla v\|_2^2 + \sigma\frac{\lambda}{2}\|\operatorname{div} v\|_2^2 + \sigma\frac{\mu}{2}\int_{\partial\mathbb{R}^3_+} K^{-1}|v|^2\, dS_x + \int_0^T\int_{\mathbb{R}^3_+} \sigma\rho|\dot{v}|^2\, dx\, ds \\
&\le \int_0^T\int_{\mathbb{R}^3_+} \sigma'|\nabla v|^2\, dx\, ds + \int_0^T\int_{\mathbb{R}^3_+} \sigma'|\operatorname{div} v|^2\, dx\, ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^T\int_{\partial\mathbb{R}^3_+}\sigma' K^{-1}|v|^2\,dx\,dS_x+\sigma\int_0^T\int_{\mathbb{R}^3_+}|\nabla v|^2\,dx+C\int_0^T\int_{\mathbb{R}^3_+}|\nabla v|^2\,dx\,ds\\
&\le C\|u_0\|_2^2\,,
\end{aligned}
$$

if $u_0\in L^2(\mathbb{R}^3_+)$. In conclusion, we have the following estimates for v:

$$
\sup_{0\le t\le T}\int_{\mathbb{R}^3_+}|\nabla v|^2\,dx+\int_0^T\int_{\mathbb{R}^3_+}\rho|\dot v|^2\,dx\,dt\le C\|u_0\|^2_{H^1(\mathbb{R}^3_+)}\,,
$$

$$
\sup_{0\le t\le T}\sigma(t)\int_{\mathbb{R}^3_+}|\nabla v|^2\,dx+\int_0^T\int_{\mathbb{R}^3_+}\sigma(t)\rho|\dot v|^2\,dx\,dt\le C\|u_0\|^2_{L^2(\mathbb{R}^3_+)}\,.
$$

In particular, for any fixed $t>0$, we have that the operator $u_0\mapsto\nabla v$ is linear continuous from $L^2(\mathbb{R}^3_+)$ into $L^2(\mathbb{R}^3_+)$ and from $H^1(\mathbb{R}^3_+)$ into $L^2(\mathbb{R}^3_+)$ with respective norms bounded by $C\sigma(t)^{-1/2}$ and C. Then by interpolation (see [23, p. 186 and 226]) we obtain

$$
\sup_{0\le t\le T}\sigma(t)^{1-s}\int_{\mathbb{R}^3_+}|\nabla v|^2\,dx\le C\|u_0\|^2_{H^s(\mathbb{R}^3_+)}\,.
$$

Also, from the above estimates, we have that the operator $u_0\mapsto\dot v$ is linear and bounded from $L^2(\mathbb{R}^3_+)$ into $L^2((0,T)\times\mathbb{R}^3_+,\sigma(t)dt\,dx)$ and from $H^1(\mathbb{R}^3_+)$ into $L^2((0,T)\times\mathbb{R}^3_+)$. Then

$$
\int_0^T\int_{\mathbb{R}^3_+}\sigma^{1-s}(t)\rho|\dot v|^2\,dx\,dt\le C\|u_0\|^2_{H^s(\mathbb{R}^3_+)}
$$

(see [5, p. 115]). □

Proposition 3.2. *For any positive number T there is a constant C independent of (ρ,u), v, w, ρ_0, u_0 and f such that*

$$
\sup_{0\le t\le T}\int_{\mathbb{R}^3_+}|\nabla w|^2\,dx+\int_0^T\int_{\mathbb{R}^3_+}\rho|\dot w|^2\,dx\,dt\le C(C_0+C_f)^\theta, \tag{3.14}
$$

for some universal positive constant θ.

Proof. Multiplying (3.11) by w_t^j, summing in j and integrating over $\mathbb{R}^3_+$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^3_+}\rho|\dot w|^2\,dx-\int_{\mathbb{R}^3_+}\rho\dot w^j u\cdot\nabla w^j\,dx\\
&=-\mu\int_{\mathbb{R}^3_+}(\nabla w^j)(\nabla w^j)_t\,dx+\ \mu\int_{\partial\mathbb{R}^3_+}w_t^j(\nabla w^j).\nu dS(x)\\
&\quad-\lambda\int_{\mathbb{R}^3_+}(\operatorname{div}w)(\operatorname{div}w)_t\,dx+\ \int_{\mathbb{R}^3_+}(P-\tilde P)(\operatorname{div}w)_t\,dx+\int_{\mathbb{R}^3_+}\rho f^j w_t^j\,dx,
\end{aligned}
$$

thence,

$$
\begin{aligned}
&\int_{\mathbb{R}^3_+}\rho|\dot w|^2\,dx+\frac{d}{dt}\Big(\frac{\mu}{2}\int_{\mathbb{R}^3_+}|\nabla w|^2\,dx+\frac{\lambda}{2}\int_{\mathbb{R}^3_+}|\operatorname{div}w|^2\,dx-\int_{\mathbb{R}^3_+}(P-\tilde P)\operatorname{div}w\Big)\\
&=\int_{\mathbb{R}^3_+}\rho\dot w^j u\cdot\nabla w^j\,dx-\int_{\mathbb{R}^3_+}P_t w_j^j\,dx+\mu\int_{\partial\mathbb{R}^3_+}w_t^j(\nabla w^j).\nu\,dS(x)+\int_{\mathbb{R}^3_+}\rho f^j w_t^j\,dx\\
&=:I_1+I_2+I_3+I_4.
\end{aligned}
$$

Let us estimate each of these integrals I_1, I_2, I_3, I_4 separately. Using estimates for w analogous to those for u in [16, Lemma 2.3] and (2.16), it is possible to show that

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^3_+} \rho \dot{w}^j u\cdot\nabla w^j\,dx \\
&\le C\Big(\int_{\mathbb{R}^3_+} \rho|\dot{w}|^2\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^3_+} \rho|u|^3\,dx\Big)^{1/3}\|\nabla w\|_6 \\
&\le C(C_0+C_f)^\theta\Big(\int_{\mathbb{R}^3_+} \rho|\dot{w}|^2\,dx\Big)^{1/2}\big(\|\rho\dot{w}\|_2+\|\nabla w\|_2+\|f\|_2+\|w\|_2+\|P-\tilde{P}\|_6\big) \\
&\le C(C_0+C_f)^\theta + C(C_0+C_f)^\theta\int_{\mathbb{R}^3_+} \rho|\dot{w}|^2\,dx + C(C_0+C_f)^\theta\int_{\mathbb{R}^3_+}|\nabla w|^2\,dx.
\end{aligned}$$

Writing the identity

$$(\lambda+\mu)\Delta w^j = \tilde{\tilde{F}}_{x_j} + (\lambda+\mu)\tilde{\tilde{\omega}}^{j,k}_{x_k} + (P-\tilde{P})_{x_j},$$

with

$$\tilde{\tilde{F}} = (\lambda+\mu)\operatorname{div} w - P(\rho) + P(\tilde{\rho})$$

and $\tilde{\tilde{\omega}}^{j,k} = w^j_{x_k} - w^k_{x_j}$, similarly to the proof of [16, Lemma 2.3], we have

$$\|\nabla\tilde{\tilde{F}}\|_q + \|\nabla\tilde{\tilde{\omega}}\|_q \le C(\|\rho\dot{w}\|_q + \|\nabla w\|_q + \|w\|_q + \|\rho f\|_q),$$

i.e.

$$\|\nabla\tilde{\tilde{F}}\|_q = \|\nabla((\lambda+\mu)\operatorname{div} w - (P-\tilde{P}))\|_q \le C(\|\rho\dot{w}\|_q + \|\rho f\|_q + \|\nabla w\|_q + \|w\|_q).$$

Thence, following [24, Lemma 3.3], we obtain

$$\begin{aligned}
I_2 &= -\int_{\mathbb{R}^3_+} P_t w^j_j\,dx \\
&= -\int_{\mathbb{R}^3_+} P'(\rho)\rho_t w^j_j\,dx \\
&= \int_{\mathbb{R}^3_+} P'(\rho)\operatorname{div}(\rho u) w^j_j\,dx \\
&= \int_{\mathbb{R}^3_+} P'(\rho)(u\cdot\nabla\rho) w^j_j\,dx + \int_{\mathbb{R}^3_+} P'(\rho)\rho\operatorname{div} u\operatorname{div} w\,dx \\
&\le \int_{\mathbb{R}^3_+} \nabla(P-\tilde{P})u\operatorname{div} w\,dx + C\int_{\mathbb{R}^3_+}|\nabla u||\nabla w|\,dx \\
&= \int_{\mathbb{R}^3_+} \operatorname{div}((P-\tilde{P})u)\operatorname{div} w\,dx - \int_{\mathbb{R}^3_+}(P-\tilde{P})(\operatorname{div} u)(\operatorname{div} w)\,dx \\
&\quad + C\int_{\mathbb{R}^3_+}|\nabla u||\nabla w|\,dx \\
&\le -\int_{\mathbb{R}^3_+}(P-\tilde{P})u\cdot\nabla(\operatorname{div} w)\,dx + C\int_{\mathbb{R}^3_+}|\nabla u||\nabla w|\,dx \\
&= -\int_{\mathbb{R}^3_+}(P-\tilde{P})u\cdot\nabla\big(\operatorname{div} w - \frac{(P-\tilde{P})}{\lambda+\mu}\big)\,dx - \int_{\mathbb{R}^3_+}(P-\tilde{P})u\cdot\nabla\big(\frac{P-\tilde{P}}{\lambda+\mu}\big)\,dx
\end{aligned}$$

$$
\begin{aligned}
&\quad + \ C\int_{\mathbb{R}^3_+} |\nabla u||\nabla w|\,dx \\
&\le C\int_{\mathbb{R}^3_+} |u||\nabla(\operatorname{div} w - \frac{(P-\tilde{P})}{\lambda+\mu})|\,dx - \frac{1}{2(\lambda+\mu)}\int_{\mathbb{R}^3_+} u\cdot\nabla\big((P-\tilde{P})^2\big)\,dx \\
&\quad + \ C\int_{\mathbb{R}^3_+} |\nabla u||\nabla w|\,dx \\
&= C\int_{\mathbb{R}^3_+} |u||\nabla(\operatorname{div} w - \frac{(P-\tilde{P})}{\lambda+\mu})|\,dx + C\int_{\mathbb{R}^3_+} \operatorname{div} u(P-\tilde{P})^2\,dx \\
&\quad + \ C\int_{\mathbb{R}^3_+} |\nabla u||\nabla w|\,dx \\
&\le C(C_0+C_f)^\theta\int_{\mathbb{R}^3_+} |\nabla(\operatorname{div} w - \frac{(P-\tilde{P})}{\lambda+\mu})|^2\,dx + C\int_{\mathbb{R}^3_+} |\nabla u|^2\,dx \\
&\quad + \ C\int_{\mathbb{R}^3_+} |\nabla w|^2\,dx \\
&\le C(C_0+C_f)^\theta + C(C_0+C_f)^\theta\int_{\mathbb{R}^3_+} \rho|\dot{w}|^2\,dx + C\int_{\mathbb{R}^3_+} |\nabla u|^2\,dx \\
&\quad + \ C\int_{\mathbb{R}^3_+} |\nabla w|^2\,dx
\end{aligned}
$$

Regarding I_3, we have

$$
\begin{aligned}
I_3 &= \int_{\mathbb{R}^3_+} \rho f^j w^j_t\,dx \\
&= \int_{\mathbb{R}^3_+} \rho f^j(\dot{w}^j - u\cdot\nabla w^j)\,dx \\
&\le C(C_0+C_f)^\theta\int_{\mathbb{R}^3_+} \rho|\dot{w}|^2\,dx + C\Big(\int_{\mathbb{R}^3_+} \rho|f|^3\,dx\Big)^{1/3}\Big(\int_{\mathbb{R}^3_+} |u|^6\,dx\Big)^{1/6} \\
&\quad \times \Big(\int_{\mathbb{R}^3_+} |\nabla w|^2\,dx\Big)^{1/2} \\
&\le C(C_0+C_f)^\theta\int_{\mathbb{R}^3_+} \rho|\dot{w}|^2\,dx \ + C(C_0+C_f)^\theta\Big(\int_{\mathbb{R}^3_+} |\nabla u|^2\,dx\Big)^{1/2} \\
&\quad \times \Big(\int_{\mathbb{R}^3_+} |\nabla w|^2\,dx\Big)^{1/2}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
I_4 &= \mu\int_{\partial\mathbb{R}^3_+} w^j_t(\nabla w^j)\cdot\nu\,dS(x) \\
&= \mu\int_{\partial\mathbb{R}^3_+} w^j_t w^j_k \nu^k\,dS(x) \\
&= -\mu\int_{\partial\mathbb{R}^3_+} w^j_t w^j_3\,dS(x)
\end{aligned}
$$

$$
\begin{aligned}
&= -\mu \int_{\partial\mathbb{R}^3_+} K^{-1} w^j_t w^j \, dS(x) \\
&= -\frac{\mu}{2} \int_{\partial\mathbb{R}^3_+} (K^{-1}|w|^2)_t \, dS(x) \\
&= -\frac{\mu}{2} \frac{d}{dt} \int_{\partial\mathbb{R}^3_+} K^{-1}|w|^2 \, dS(x)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\int_{\mathbb{R}^3_+} \rho|\dot{w}|^2 \, dx + \Big(\frac{\mu}{2} \int_{\mathbb{R}^3_+} |\nabla w|^2 \, dx + \frac{\lambda}{2} \int_{\mathbb{R}^3_+} |\operatorname{div} w|^2 \, dx \\
&- \int_{\mathbb{R}^3_+} (P - \tilde{P}) \operatorname{div} w \, dx + \frac{\mu}{2} \int_{\partial\mathbb{R}^3_+} K|w|^2 dS(x)\Big)_t \\
&\le C(C_0 + C_f)^\theta + C(C_0 + C_f)^\theta \int_{\mathbb{R}^3_+} |\nabla u|^2 \, dx + C \int_{\mathbb{R}^3_+} |\nabla u||\nabla w| \, dx \\
&\quad + C(C_0 + C_f)^\theta \int_{\mathbb{R}^3_+} \rho|\dot{w}|^2 \, dx + C \int_{\mathbb{R}^3_+} |\nabla w|^2 \, dx \\
&\quad + C(C_0 + C_f)^\theta \Big(\int_{\mathbb{R}^3_+} |\nabla u|^2 \, dx\Big)^{1/2} \Big(\int_{\mathbb{R}^3_+} |\nabla w|^2 \, dx \Big)^{1/2}.
\end{aligned}
$$

Integrating on $(0, t)$ and taking C_0, C_f sufficiently small, we obtain

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3_+} \rho|\dot{w}|^2 \, dx \, ds + \frac{\mu}{2} \int_{\mathbb{R}^3_+} |\nabla w|^2 \, dx + \frac{\lambda}{2} \int_{\mathbb{R}^3_+} |\operatorname{div} w|^2 \, dx + \frac{\mu}{2} \int_{\partial\mathbb{R}^3_+} K|w|^2 dS(x) \\
&\le C(C_0 + C_f)^\theta + \int_{\mathbb{R}^3_+} (P - \tilde{P})(\operatorname{div} w) \, dx \\
&\quad + C M_q \Big(\int_0^t \int_{\mathbb{R}^3_+} |\nabla u|^2 \, dx \, ds \Big)^{1/2} \Big(\int_0^t \int_{\mathbb{R}^3_+} |\nabla w|^2 \, dx \, ds \Big)^{1/2} \\
&\le C(C_0 + C_f)^\theta + C(C_0 + C_f)^\theta \int_{\mathbb{R}^3_+} |\nabla w|^2 \, dx,
\end{aligned}
$$

hence obtaining the result, assuming again C_0, C_f sufficiently small. □

Now, we are ready to show (1.14).

Theorem 3.3. *Let u_0 be in the Sobolev space $H^s(\mathbb{R}^3_+)$, for some $s \in [0, 1]$. Then the estimate* (1.14) *holds for the solution (ρ, u) of* (1.1)-(1.3) *obtained in* [16]*, as described here from* (1.4) *to* (1.11).

Proof. Let v and w be the solutions of (3.10) and (3.11)), respectively. Since $v|_{t=0} = u_0$, by the unicity of solution of the linear system $\mathcal{L}(z) = \nabla(P - \tilde{P}) + \rho f$, joint with the initial condition $z_{t=0} = u_0$, we have that $u = v + w$. (note that $z = v + w$ and $z = u$ are both solutions of this problem.) Thus, by (3.12) and (3.14), we obtain (1.14). □

Next we shall use (1.14) to show the estimate (1.15).

Theorem 3.4. *Let be $u_0 \in H^s(\mathbb{R}^3_+)$, for some $s \in (1/2, 1]$ and (ρ, u) as in Theorem 3.3. Then we have the estimate* (1.15).

Proof. Writing the momentum equation as

$$\rho\dot{u}^j + P_j = \mu\Delta u^j + \lambda \operatorname{div} u_j + \rho f^j,$$

and applying the operator $\sigma^m \dot{u}^j\big(\partial_t(\cdot) + \operatorname{div}(.u)\big)$, $m \geq 1$, as in [15] and [24], we have

$$\begin{aligned}
&\sigma^m \rho\dot{u}^j_t\dot{u}^j + \sigma^m\rho u\cdot\nabla\dot{u}^j\dot{u}^j + \sigma^m\dot{u}^jP_{jt} + \sigma^m\dot{u}^j\operatorname{div}(P_ju)\\
&= \mu\sigma^m\dot{u}^j(\Delta u^j_t + \operatorname{div}(u\Delta u^j)) + \lambda\sigma^m\dot{u}^j(\partial_t\partial_j\operatorname{div} u + \operatorname{div}(u\partial_j \operatorname{div} u))\\
&\quad + \sigma^m\rho\dot{u}^jf^j_t + \sigma^m\rho u^kf^j_k.
\end{aligned}$$

Note that

$$\begin{aligned}
&\sigma^m \rho\dot{u}^j_t\dot{u}^j + \sigma^m\rho u\cdot\nabla\dot{u}^j\dot{u}^j\\
&= \frac{\sigma^m}{2}\big(\rho\partial_t(|\dot{u}|^2) + \rho u\cdot\nabla(|\dot{u}|^2)\big)\\
&= \partial_t\big(\frac{\sigma^m}{2}\rho|\dot{u}|^2\big) - \frac{m}{2}\sigma^{m-1}\sigma'\rho|\dot{u}|^2 - \frac{\sigma^m}{2}\rho_t|\dot{u}|^2 + \frac{\sigma^m}{2}\rho u\cdot\nabla(|\dot{u}|^2).
\end{aligned}$$

Integrating on $\mathbb{R}^3_+$, it follows that

$$\begin{aligned}
&\Big(\frac{\sigma^m}{2}\int_{\mathbb{R}^3_+}\rho|\dot{u}|^2\,dx\Big)_t - \frac{m}{2}\sigma'\sigma^{m-1}\int_{\mathbb{R}^3_+}\rho|\dot{u}|^2\,dx\\
&= -\sigma^m\int_{\mathbb{R}^3_+}\dot{u}^j\left(P_{jt} + \operatorname{div}(P_ju)\right)\,dx + \mu\sigma^m\int_{\mathbb{R}^3_+}\dot{u}^j(\Delta u^j_t + \operatorname{div}(u\Delta u^j))\,dx\\
&\quad + \lambda\sigma^m\int_{\mathbb{R}^3_+}\dot{u}^j(\partial_t\partial_j\operatorname{div} u + \operatorname{div}(u\partial_j\operatorname{div} u))\,dx + \sigma^m\int_{\mathbb{R}^3_+}(\rho\dot{u}^jf^j_t + \rho u^kf^j_k)\,dx\\
&=: N_1 + N_2 + N_3 + N_4\,.
\end{aligned}$$

Let us estimate each of these terms separately. Integrating by parts, we have

$$\begin{aligned}
N_1 &= -\int_{\mathbb{R}^3_+}\sigma^m\dot{u}^j\left(\partial_tP_j + \operatorname{div}(P_ju)\right)\,dx\\
&= \sigma^m\int_{\mathbb{R}^3_+}\dot{u}^j_jP'\rho_t\,dx - \int_{\partial\mathbb{R}^3_+}\sigma^m\dot{u}^j\nu^jP_t\,dS_x + \sigma^m\int_{\mathbb{R}^3_+}\dot{u}^j_kP_ju^k\\
&\quad - \int_{\partial\mathbb{R}^3_+}\sigma^m\dot{u}^jP_ju.\nu\,dS_x\\
&= \sigma^m\int_{\mathbb{R}^3_+}\dot{u}^j_jP'(-\rho\operatorname{div} u - u\cdot\nabla\rho)\,dx + \sigma^m\int_{\mathbb{R}^3_+}\dot{u}^j_kP_ju^k\,dx\\
&= -\sigma^m\int_{\mathbb{R}^3_+}P'\rho\dot{u}^j_j\operatorname{div} u\,dx - \sigma^m\int_{\mathbb{R}^3_+}\dot{u}^j_ju\cdot\nabla P\,dx\\
&\quad - \sigma^m\int_{\mathbb{R}^3_+}(P-\tilde{P})(\dot{u}^j_{jk}u^k + \dot{u}^j_ku^k_j)\,dx + \sigma^m\int_{\partial\mathbb{R}^3_+}(P-\tilde{P})\dot{u}^j_ku^k\nu^j\,dS_x\\
&= -\sigma^m\int_{\mathbb{R}^3_+}P'\rho\dot{u}^j_j\operatorname{div} u\,dx + \sigma^m\int_{\mathbb{R}^3_+}(P-\tilde{P})(\dot{u}^j_{jk}u^k + \dot{u}^j_ju^k_k)\,dx\\
&\quad - \sigma^m\int_{\partial\mathbb{R}^3_+}(P-\tilde{P})(\dot{u}^j_ju.\nu)\,dS_x - \sigma^m\int_{\mathbb{R}^3_+}(P-\tilde{P})(\dot{u}^j_{jk}u^k + \dot{u}^j_ku^k_j)\,dx
\end{aligned}$$

$$
\begin{aligned}
&= -\sigma^m \int_{\mathbb{R}^3_+} \Big(P'\rho \dot u^j_j \operatorname{div} u - P\dot u^j_j u^k_k + P\dot u^j_k u^k_j \Big)\, dx \\
&\le C(\bar\rho)\sigma^m \|\nabla u\|_2 \|\nabla \dot u\|_2 \\
&\le C(\bar\rho)C(\varepsilon)\sigma^m \|\nabla u\|_2^2 + C(\bar\rho)\varepsilon\sigma^m \|\nabla \dot u\|_2^2.
\end{aligned}
$$

$$
\begin{aligned}
N_2 &= \mu\sigma^m \int_{\mathbb{R}^3_+} \dot u^j (\Delta u^j_t + \operatorname{div}(u\Delta u^j))\, dx \\
&= \mu\sigma^m \int_{\mathbb{R}^3_+} (\dot u^j u^j_{kkt} + \dot u^j (u^k u^j_{ll})_k)\, dx \\
&= -\mu\sigma^m \{ \int_{\mathbb{R}^3_+} (\dot u^j_k u^j_{kt})\, dx - \int_{\partial\mathbb{R}^3_+} \dot u^j u^j_{kt} \nu^k \, dS_x \} \\
&\quad - \mu\sigma^m \{ \int_{\mathbb{R}^3_+} \dot u^j_k u^k u^j_{ll}\, dx - \int_{\partial\mathbb{R}^3_+} \dot u^j u^k u^j_{ll} \nu^k \, dS_x \} \\
&= -\mu\sigma^m \{ \int_{\mathbb{R}^3_+} \dot u^j_k (\dot u^j_k - (u\cdot\nabla u^j)_k)\, dx - \int_{\partial\mathbb{R}^3_+} \dot u^j u^j_{kt} \nu^k \, dS_x \} \\
&\quad + \mu\sigma^m \{ \int_{\mathbb{R}^3_+} (\dot u^j_{kl} u^k u^j_l + \dot u^j_k u^k_l u^j_l)\, dx - \int_{\partial\mathbb{R}^3_+} \dot u^j_k u^k u^j_l \nu^l \, dS_x \} \\
&= -\mu\sigma^m \{ \int_{\mathbb{R}^3_+} |\nabla \dot u|^2\, dx - \int_{\mathbb{R}^3_+} \dot u^j_k u^l_k u^j_l \, dx - \int_{\mathbb{R}^3_+} \dot u^j_k u^l u^j_{kl}\, dx \\
&\quad - \int_{\partial\mathbb{R}^3_+} \dot u^j u^j_{kt} \nu^k \, dS_x \} + \mu\sigma^m \{ \int_{\mathbb{R}^3_+} (\dot u^j_{kl} u^k u^j_l + \dot u^j_k u^k_l u^j_l)\, dx \\
&\quad - \int_{\partial\mathbb{R}^3_+} \dot u^j_k u^k u^j_l \nu^l \, dS_x \} \\
&= -\mu\sigma^m \{ \int_{\mathbb{R}^3_+} |\nabla \dot u|^2\, dx - \int_{\mathbb{R}^3_+} \dot u^j_k u^l_k u^j_l \, dx + \int_{\mathbb{R}^3_+} (\dot u^j_{kl} u^l u^j_k + \dot u^j_k u^l_l u^j_k)\, dx \\
&\quad - \int_{\partial\mathbb{R}^3_+} \dot u^j u^j_{kt} \nu^k \, dS_x \} + \mu\sigma^m \Big\{ \int_{\mathbb{R}^3_+} (\dot u^j_{kl} u^k u^j_l + \dot u^j_k u^k_l u^j_l)\, dx \\
&\quad - \int_{\partial\mathbb{R}^3_+} \dot u^j_k u^k u^j_l \nu^l \, dS_x \Big\} \\
&= -\mu\sigma^m \Big\{ \int_{\mathbb{R}^3_+} |\nabla \dot u|^2\, dx - \int_{\mathbb{R}^3_+} (\dot u^j_k u^l_k u^j_l - \dot u^j_k u^l_l u^j_k + \dot u^j_k u^k_l u^j_l)\, dx \Big\} \\
&\quad + \mu\sigma^m \int_{\partial\mathbb{R}^3_+} (\dot u^j u^j_{kt} \nu^k - \dot u^j_k u^k u^j_l \nu^l)\, dS_x;
\end{aligned}
$$

to estimate the boundary term above, we write

$$
\mu\sigma^m \int_{\partial\mathbb{R}^3_+} (\dot u^j u^j_{kt} \nu^k - \dot u^j_k u^k u^j_l \nu^l)\, dS_x =: N_{21} + N_{22},
$$

using that if $h \in (C^1 \cap W^{1,1})(\overline{\mathbb{R}^3_+})$, then

$$
\int_{\partial\mathbb{R}^3_+} h(x)\, dS_x = \int_{\{0\le x_3\le 1\}} [h(x) + (x_3 - 1) h_{x_3}(x)]\, dx.
$$

Observe that we can assume $j \neq 3$ in N_{21} and $k \neq 3$ in N_{22} without loss of generality, since $u^3 = 0$ on $\partial\mathbb{R}^3_+$. Let us show how to estimate the term N_{21} above. The term N_{22} can be estimate similarly.

$$\begin{aligned}
N_{21} &= -\mu\sigma^m \int_{\partial\mathbb{R}^3_+} \dot{u}^j u^j_{3t}\, dS_x \\
&= -\mu\sigma^m \int_{\partial\mathbb{R}^3_+} K^{-1}\dot{u}^j u^j_t\, dS_x \\
&= -\mu\sigma^m \int_{\partial\mathbb{R}^3_+} K^{-1}\dot{u}^j(\dot{u}^j - u^k u^j_k)\, dS_x \\
&= -\mu\sigma^m \int_{\partial\mathbb{R}^3_+} K^{-1}|\dot{u}|^2\, dS_x + \mu\sigma^m \int_{\partial\mathbb{R}^3_+} K^{-1}\dot{u}^j u^k u^j_k\, dS_x \\
&\le \mu\sigma^m \int_{\partial\mathbb{R}^3_+} K^{-1}\dot{u}^j u^k u^j_k\, dS_x \\
&= \mu\sigma^m \int_{\{0\le x_3\le 1\}} K^{-1}(\dot{u}^j u^k u^j_k + (x_3-1)[\dot{u}^j_3 u^k u^j_k + \dot{u}^j u^k_3 u^j_k + \dot{u}^j u^k u^j_{k3}])\, dx \\
&\le C\mu\sigma^m \int_{\mathbb{R}^3_+} (|\nabla\dot{u}||u||\nabla u| + |\dot{u}||\nabla u||u| + |\dot{u}||\nabla u|^2)\, dx \\
&\quad - \mu\sigma^m \int_{\{0\le x_3\le 1\}} (x_3-1)(K^{-1}\dot{u}^j_k u^k u^j_3 + K^{-1}\dot{u}^j u^k_k u^j_3 + (K^{-1})_k \dot{u}^j u^k u^j_3)\, dx \\
&\quad + \mu\sigma^m \int_{\{x_3=0\}\cup\{x_3=1\}} K^{-1}(x_3-1)\dot{u}^j u^k u^j_3 \nu^k\, dS_x.
\end{aligned}$$

Note that the above boundary term is null, since for $x_3 = 0$ we have $u^k\nu^k = 0$ and for $x_3 = 1$ the term $(x_3 - 1)$ vanishes the integrand. Thus,

$$N_{21} \le C\sigma^m \int_{\mathbb{R}^3_+} (|\nabla\dot{u}||u||\nabla u| + |\dot{u}||\nabla u||u| + |\dot{u}||\nabla u|^2)\, dx.$$

Regarding N_3, setting $D = \operatorname{div} u$, we have

$$\begin{aligned}
N_3 &= \lambda\sigma^m \int_{\mathbb{R}^3_+} \dot{u}^j\, (\partial_t\partial_j \operatorname{div} u + \operatorname{div}(u\partial_j \operatorname{div} u))\, dx \\
&= -\lambda\sigma^m \int_{\mathbb{R}^3_+} (\dot{u}^j_j D_t)\, dx + \lambda\sigma^m \int_{\partial\mathbb{R}^3_+} \dot{u}^j D_t \nu^j\, dS_x \\
&\quad + \lambda\sigma^m \int_{\mathbb{R}^3_+} \dot{u}^j (DD_j + u^k D_{jk})\, dx \\
&= -\lambda\sigma^m \int_{\mathbb{R}^3_+} (\dot{u}^j_j D_t)\, dx + \lambda\sigma^m \int_{\mathbb{R}^3_+} \dot{u}^j DD_j\, dx + \int_{\mathbb{R}^3_+} \dot{u}^j u^k D_{jk}\, dx \\
&=: N_{31} + N_{32} + N_{33}.
\end{aligned}$$

Note that $N_{31} = -\lambda\sigma^m \int_{\mathbb{R}^3_+} \dot{u}^j_j D_t\, dx = -\lambda\sigma^m \int_{\mathbb{R}^3_+} \dot{u}^j_j \dot{D}\, dx + \lambda\sigma^m \int_{\mathbb{R}^3_+} \dot{u}^j_j u\cdot\nabla D\, dx$. For N_{32}, we have

$$N_{32} = \frac{\lambda}{2}\sigma^m \int_{\mathbb{R}^3_+} \dot{u}^j (|D|^2)_j\, dx$$

$$
\begin{aligned}
&= -\frac{\lambda}{2}\sigma^m \int_{\mathbb{R}^3_+} \dot{u}^j_j |D|^2 \,dx + \frac{\lambda}{2}\sigma^m \int_{\partial\mathbb{R}^3_+} \dot{u}^j \nu^j |D|^2 \,dS_x \\
&\le C\sigma^m \int_{\mathbb{R}^3_+} |\nabla \dot{u}||\nabla u|^2 \,dx \\
&\le C\varepsilon\sigma^m \int_{\mathbb{R}^3_+} |\nabla \dot{u}|^2 \,dx + C\sigma^m \int_{\mathbb{R}^3_+} |\nabla u|^4 \,dx.
\end{aligned}
$$

$$
\begin{aligned}
N_{33} &= \lambda\sigma^m \int_{\mathbb{R}^3_+} u^k \dot{u}^j D_{jk} \,dx \\
&= -\lambda\sigma^m \int_{\mathbb{R}^3_+} (\dot{u}^j_j u^k D_k + \dot{u}^j u^k_j D_k) \,dx + \lambda\sigma^m \int_{\partial\mathbb{R}^3_+} D_k u^k \dot{u}^j \nu^j \,dS_x \\
&= -\lambda\sigma^m \int_{\mathbb{R}^3_+} \dot{u}^j_j u \cdot \nabla D \,dx + \lambda\sigma^m \int_{\mathbb{R}^3_+} (\dot{u}^j_k u^k_j D + \dot{u}^j u^k_{kj} D) \,dx \\
&\quad - \lambda\sigma^m \int_{\partial\mathbb{R}^3_+} \dot{u}^j u^k_j u^l_l \nu^k \,dS_x \\
&= -\lambda\sigma^m \int_{\mathbb{R}^3_+} \dot{u}^j_j u \cdot \nabla D \,dx + \lambda\sigma^m \int_{\mathbb{R}^3_+} \dot{u}^j_k u^k_j D \,dx - \frac{\lambda}{2}\sigma^m \int_{\mathbb{R}^3_+} \dot{u}^j_j |D|^2 \,dx \\
&\quad + \frac{\lambda}{2}\sigma^m \int_{\partial\mathbb{R}^3_+} \dot{u}^j \nu^j |D|^2 \,dS_x.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&N_{31} + N_{33} \\
&= -\lambda\sigma^m \int_{\mathbb{R}^3_+} \dot{u}^j_j \dot{D} \,dx + \lambda\sigma^m \int_{\mathbb{R}^3_+} \dot{u}^j_k u^k_j D \,dx - \frac{\lambda}{2}\sigma^m \int_{\mathbb{R}^3_+} \dot{u}^j_j |D|^2 \,dx \\
&\le -\lambda\sigma^m \int_{\mathbb{R}^3_+} \dot{u}^j_j \dot{D} \,dx + \varepsilon\sigma^m \int_{\mathbb{R}^3_+} |\nabla \dot{u}|^2 \,dx + C\sigma^m \int_{\mathbb{R}^3_+} |\nabla u|^4 \,dx \\
&= -\lambda\sigma^m \int_{\mathbb{R}^3_+} \dot{D}(u^j_t + u \cdot \nabla u^j)_j \,dx + \varepsilon\sigma^m \int_{\mathbb{R}^3_+} |\nabla \dot{u}|^2 \,dx \\
&\quad + C\sigma^m \int_{\mathbb{R}^3_+} |\nabla u|^4 \,dx \\
&= -\lambda\sigma^m \int_{\mathbb{R}^3_+} \dot{D}(D_t + u \cdot \nabla D + u^k_j u^j_k) \,dx + \varepsilon\sigma^m \int_{\mathbb{R}^3_+} |\nabla \dot{u}|^2 \,dx \\
&\quad + C\sigma^m \int_{\mathbb{R}^3_+} |\nabla u|^4 \,dx \\
&= -\lambda\sigma^m \int_{\mathbb{R}^3_+} |\dot{D}|^2 \,dx - \lambda\sigma^m \int_{\mathbb{R}^3_+} \dot{D} u^k_j u^j_k \,dx + \varepsilon\sigma^m \int_{\mathbb{R}^3_+} |\nabla \dot{u}|^2 \,dx \\
&\quad + C\sigma^m \int_{\mathbb{R}^3_+} |\nabla u|^4 \,dx \\
&\le -\lambda\sigma^m \int_{\mathbb{R}^3_+} |\dot{D}|^2 \,dx + \varepsilon\sigma^m \int_{\mathbb{R}^3_+} |\nabla \dot{u}|^2 \,dx + C\sigma^m \int_{\mathbb{R}^3_+} |\nabla u|^4 \,dx.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
N_4 &= \sigma^m \int_{\mathbb{R}^3_+} (\rho \dot{u}^j f^j_t + \rho \dot{u}^j u^k f^j_k)\,dx \\
&\le \varepsilon \sigma^{m-1} \int_{\mathbb{R}^3_+} \rho |\dot{u}|^2\,dx + C\sigma^{m+1} \int_{\mathbb{R}^3_+} |f_t|^2\,dx \\
&\quad + \varepsilon \sigma^m \int_{\mathbb{R}^3_+} \rho |\dot{u}|^2\,dx + C\sigma^{m+1} \int_{\mathbb{R}^3_+} |\nabla f|^2 |u|^2\,dx \\
&\le \varepsilon \sigma^{m-1} \int_{\mathbb{R}^3_+} \rho |\dot{u}|^2\,dx + C\sigma^{m+1} \int_{\mathbb{R}^3_+} |f_t|^2\,dx \\
&\quad + C(\sigma^{(3-3s)/2} \int_{\mathbb{R}^3_+} |u|^4\,dx)^{1/2} (\sigma^{(4m+1+3s)/2} \int_{\mathbb{R}^3_+} |\nabla f|^4\,dx)^{1/2} \\
&\le \varepsilon \sigma^{m-1} \int_{\mathbb{R}^3_+} \rho |\dot{u}|^2\,dx + C\sigma^{m+1} \int_{\mathbb{R}^3_+} |f_t|^2\,dx \\
&\quad + C(C_0 + C_f)^\theta (\sigma^{(4m+1+3s)/2} \int_{\mathbb{R}^3_+} |\nabla f|^4\,dx)^{1/2},
\end{aligned}
$$

since

$$
\begin{aligned}
\sigma^{(3-3s)/2} \|u\|_4^4 &\le C\sigma^{(3-3s)/2} \|u\|_2 \|\nabla u\|_2^3 \\
&\le C(C_0 + C_f)^\theta \sigma^{(3-3s)/2} \|\nabla u\|_2^3 \\
&= C(C_0 + C_f)^\theta (\int_{\mathbb{R}^3_+} \sigma^{1-s} |\nabla u|^2\,dx)^{3/2} \le C(C_0 + C_f)^\theta.
\end{aligned}
$$

With these estimates, we arrive at

$$
\begin{aligned}
&\Big(\frac{\sigma^m}{2} \int_{\mathbb{R}^3_+} \rho |\dot{u}|^2\,dx\Big)_t - \frac{m}{2} \sigma' \sigma^{m-1} \int_{\mathbb{R}^3_+} \rho |\dot{u}|^2\,dx \\
&\le C(\bar{\rho}) C(\varepsilon) \sigma^m \int_{\mathbb{R}^3_+} |\nabla u|^2\,dx + C(\bar{\rho}) \varepsilon \sigma^m \int_{\mathbb{R}^3_+} |\nabla \dot{u}|^2\,dx \\
&\quad - \mu \sigma^m \int_{\mathbb{R}^3_+} |\nabla \dot{u}|^2\,dx + C\sigma^m \int_{\mathbb{R}^3_+} |\nabla u|^4\,dx \\
&\quad + C\varepsilon \sigma^m \int_{\mathbb{R}^3_+} |\nabla \dot{u}|^2\,dx + C\sigma^m \int_{\mathbb{R}^3_+} |\nabla u|^4\,dx - \lambda \sigma^m \int_{\mathbb{R}^3_+} |\dot{D}|^2\,dx \\
&\quad + C\sigma^m \int_{\mathbb{R}^3_+} |\dot{u}|^2\,dx + C\sigma^m \int_{\mathbb{R}^3_+} |u|^4\,dx \\
&\quad + \varepsilon \sigma^{m-1} \int_{\mathbb{R}^3_+} \rho |\dot{u}|^2\,dx + C\sigma^{m+1} \int_{\mathbb{R}^3_+} |f_t|^2\,dx \\
&\quad + C(C_0 + C_f)^\theta \Big(\sigma^{(4m+1+3s)/2} \int_{\mathbb{R}^3_+} |\nabla f|^4\,dx\Big)^{1/2}.
\end{aligned}
$$

Integrating on $(0,T)$, taking $m = 2 - s$ and using (1.14), we obtain

$$
\sigma^m \int_{\mathbb{R}^3_+} \rho |\dot{u}|^2\,dx + \int_0^T \int_{\mathbb{R}^3_+} \sigma^m |\nabla \dot{u}|^2\,dx\,ds
$$

$$
\begin{aligned}
&\le C(C_0+C_f)^\theta + C\int_0^T\int_{\mathbb{R}^3_+}\sigma^m(|\nabla u|^4+|u|^4)\,dx\,ds\\
&\quad+\int_0^T\sigma^{3-s}\int_{\mathbb{R}^3_+}|f_t|^2\,dx\,dt+\int_0^T\sigma^{(9-s)/4}\Big(\int_{\mathbb{R}^3_+}|\nabla f|^4\,dx\Big)^{1/2}dt\\
&\le C(C_0+C_f)^\theta + C\int_0^T\int_{\mathbb{R}^3_+}\sigma^m(|\nabla u|^4+|u|^4)\,dx\,dt.
\end{aligned}
$$

To conclude the result, we must estimate the term $\int_0^T\sigma^{2-s}\int_{\mathbb{R}^3_+}(|\nabla u|^4+|u|^4)\,dx\,d\tau$. Using (2.16), we have

$$
\begin{aligned}
\int_0^T\sigma^{2-s}\|u\|_4^4\,d\tau &\le \int_0^T\sigma^{2-s}\|u\|_2\|\nabla u\|_2^3\,d\tau\\
&=\int_0^T\sigma^{2-s}\Big(\int_{\mathbb{R}^3_+}|u|^2\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^3_+}|\nabla u|^2\,dx\Big)^{3/2}d\tau\\
&\le C(C_0+C_f)^\theta\int_0^T\sigma^{2-s}\Big(\int_{\mathbb{R}^3_+}|\nabla u|^2\,dx\Big)^{3/2}d\tau\\
&\le C(C_0+C_f)^\theta\int_0^T\sigma^{\frac{1+s}{2}}\Big(\sigma^{1-s}\int_{\mathbb{R}^3_+}|\nabla u|^2\,dx\Big)^{3/2}d\tau\\
&\le C(C_0+C_f)^\theta.
\end{aligned}
$$

On the other hand, following [24, Lemma 3.3], and using energy estimates and (1.14), we estimate

$$
\begin{aligned}
&\int_0^T\sigma^{2-s}\int_{\mathbb{R}^3_+}|\nabla u|^4\,dx\,d\tau\\
&=\int_0^T\sigma^{2-s}\|\nabla u\|_4^4\,d\tau\\
&\le C\int_0^T\sigma^{2-s}\|\nabla u\|_2\Big(\|\rho\dot u\|_2+\|\nabla u\|_2+\|u\|_2+\|f\|_2+\|P-\tilde P\|_6\Big)^3\,d\tau\\
&\le C\int_0^T\sigma^{2-s}\|\nabla u\|_2\Big(\|\rho\dot u\|_2^3+\|\nabla u\|_2^3+\|u\|_2^3+\|f\|_2^3+\|P-\tilde P\|_6^3\Big)\,d\tau\\
&\le C(C_0+C_f)^\theta + C\int_0^T\sigma^{2-s}\|\nabla u\|_2\|\rho\dot u\|_2^3\,d\tau\\
&\le C(C_0+C_f)^\theta + C\int_0^T\sigma^{2-s}\Big(\int_{\mathbb{R}^3_+}|\nabla u|^2\,dx\Big)^{1/2}\\
&\quad\times\Big(\int_{\mathbb{R}^3_+}\rho|\dot u|^2\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^3_+}\rho|\dot u|^2\,dx\Big)\,d\tau\\
&\le C(C_0+C_f)^\theta + C\int_0^T\sigma^{\frac{2s-1}{2}}\Big(\sigma^{1-s}\int_{\mathbb{R}^3_+}|\nabla u|^2\,dx\Big)^{1/2}\\
&\quad\times\Big(\sigma^{2-s}\int_{\mathbb{R}^3_+}\rho|\dot u|^2\,dx\Big)^{1/2}\Big(\sigma^{1-s}\int_{\mathbb{R}^3_+}\rho|\dot u|^2\,dx\Big)\,d\tau
\end{aligned}
$$

$$\leq C(C_0+C_f)^\theta + C(C_0+C_f)^\theta \sup_{0\leq t\leq T}\Big(\sigma^{2-s}\int_{\mathbb{R}^3_+}\rho|\dot{u}|^2\,dx\Big)^{1/2}.$$

Therefore, using once more that C_0, C_f are sufficiently small, we obtain (1.15). □

Proof of Theorem 1.3. The proof of Theorem 1.3 using Theorem 1.2 is similar to the proof of [17, Theorem 2.5]. Thus, here we just give an overview of this proof, showing some details which may be peculiar to our case.

The proof of the existence of the particle paths $X(t,x)$ satisfying (1.16) is obtained through the following estimate, uniformly with respect to smooth solutions:

$$|X(t_1,x)-X(t_2,x)| \leq \int_{t_1}^{t_2} ||u(t,\cdot)||_\infty dt \leq C\int_{t_1}^{t_2}(\|u(t,\cdot)\|_2 + \|\nabla u(t,\cdot)\|_q)dt,$$

where $q>3$, and for the last inequality we used (2.1). Indeed, from this estimate it is possible to show, after several other estimates, that $X(t,x)$ is Hölder continuous in t, uniformly with respect to smooth solutions.

The uniqueness (of particle paths) follows from the estimate

$$\int_0^T \langle u(.,t)\rangle_{LL} dt \leq CT^\gamma,$$

for a fixed and arbitrary $T>0$, where C and γ are positive constants, uniform with respect to u, cf. [17, lemmas 3.1 and 3.2].

For the proof of item (b) of Theorem 1.3, first we observe that the injectivity and openness of the map $x\mapsto X(t,x)$ can be shown exactly as in [17]. To show the surjectivity, we use the particles paths starting at $t_0>0$, i.e. the map

$$X(\cdot,x_0;t_0)\in C([0,+\infty),\overline{\mathbb{R}^3_+})\cap C^1((0,+\infty);\overline{\mathbb{R}^3_+})$$

such that

$$X(t,x_0;t_0) = x_0 + \int_{t_0}^t u(X(\tau,x_0),\tau)\,d\tau$$

(see [17, Corollary 2.3]): given $y\in\overline{\mathbb{R}^3_+}$, let $Y(s)=X(s;y,t)$, $s\in[0,t]$. Since the curves $Y(s)$ e $X(s,Y(0))$ satisfy $Z'(s)=u(Z(s),s)$, $Z(0)=Y(0)$, we have $Y(s)=X(s;Y(0))$, $s\in[0,t]$, so $y=Y(t)=X(t;Y(0))$, which shows the surjectivity of the map $X(t,\cdot):\overline{\mathbb{R}^3_+}\to\overline{\mathbb{R}^3_+}$. The continuity is a direct consequence of item (c).

To show the invariance of the boundary $\partial\mathbb{R}^3_+$ by the flux, let $x=(x_1,x_2,0)\in\partial\mathbb{R}^3_+$. Defining $X^i(\cdot,x)$, for $i=1,2$, by

$$X^i(t,x) = x_i + \int_0^t u^i(X^i(\tau,x),\tau)\,d\tau,$$

we have that $Y(t,x) := (X^1(t,x),X^2(t,x),0)$ is a path which lies in $\partial\mathbb{R}^3_+$ and satisfies

$$dY(t,x)/dt = u(Y(t,x),t),\quad t>0,\quad Y(0)=x,$$

since $u^3=0$ on $\partial\mathbb{R}^3_+$, so, by uniqueness (item (a)) we have $Y(t,x)=X(t,x)$, for all $t\geq 0$, and thus we conclude the invariance of the boundary by the flux $X(t,\cdot)$.

The proofs of items (c) and (d) can be done exactly as the proofs of [17, Theorem 2.5 (c),(d)]. Actually, the proof of item (d) is a direct consequence of item (c) and the definition of a parametrized manifold of class C^α, which is the image of a map $\psi: U\to\mathbb{R}^3_+$ of class C^α, where U is an open set of $\mathbb{R}^k$, $k=1$ or 2.

Acknowledgments. E. J. Teixeira wants to thank CAPES-Coordenação de Aperfeiçoamento de Pessoal de Nível Superior, Brazil, for the financial support.

References

[1] Adams, Robert A.; Fournier, John J. F.; *Sobolev spaces.* Second edition. Pure and Applied Mathematics (Amsterdam), **140**. Elsevier/Academic Press, Amsterdam, 2003.

[2] Agarwal, R. P.; Lakshmikantham, V.; *Uniqueness and nonuniqueness criteria for ordinary differential equations.* Series in Real Analysis, **6**. World Scientific Publishing Co., Inc., River Edge, NJ, 1993.

[3] Agmon, S.; Douglis, A.; Nirenberg, L.; *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I.* Comm. Pure Appl. Math., **12** (1959), 623-727.

[4] Armitage, D. H.; *The Neumann problem for a function harmonic in* $\mathbb{R}^n \times (0,\infty)$. Arch. Rational Mech. Anal., **63** (1976), no. 1, 89-105.

[5] Bergh, Jöran; Löfström, Jörgen; *Interpolation spaces. An introduction.*Grundlehren der Mathematischen Wissenschaften, **223**. Springer-Verlag, Berlin-New York, 1976.

[6] Calderón, A. P.; Zygmund, A.; *On singular integrals.* Amer. J. Math. **78** (1956), 289-309.

[7] Chemin, Jean-Yves; *Perfect incompressible fluids.* Oxford Lecture Series in Mathematics and its Applications, **14**. The Clarendon Press, Oxford University Press, New York, 1998.

[8] Evans, Lawrence C.; *Partial differential equations.* Second edition. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010.

[9] Feireisl, Eduard; *Dynamics of viscous compressible fluids.* Oxford Lecture Series in Mathematics and its Applications, **26**. Oxford University Press, Oxford, 2004.

[10] Folland, Gerald B.; *Real analysis. Modern techniques and their applications.* Second edition. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999.

[11] Galdi, G. P.; *An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems.* Second edition. Springer Monographs in Mathematics. Springer, New York, 2011.

[12] Gilbarg, David; Trudinger, Neil S.; *Elliptic partial differential equations of second order.* Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[13] Hoff, David; *Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data.* J. Differential Equations, **120** (1995), no. 1, 215-254.

[14] Hoff, David; *Discontinuous solutions of the Navier-Stokes equations for multidimensional flows of heat-conducting fluids.* Arch. Rational Mech. Anal. **139** (1997), no. 4, 303-354.

[15] Hoff, David; *Dynamics of singularity surfaces for compressible, viscous flows in two space dimensions.* Comm. Pure Appl. Math., **55** (2002), no. 11, 1365-1407.

[16] Hoff, David; *Compressible flow in a half-space with Navier boundary conditions.* J. Math. Fluid Mech., **7** (2005), no. 3, 315-338.

[17] Hoff, David; Santos, Marcelo M.; *Lagrangean structure and propagation of singularities in multidimensional compressible flow.* Arch. Ration. Mech. Anal., **188** (2008), no. 3, 509-543.

[18] Hoff, David; Perepelitsa, Misha; *Boundary tangency for density interfaces in compressible viscous fluid flows.* J. Differential Equations, **253** (2012), no. 12, 3543-3567.

[19] Hoff, David; *Local solutions of a compressible flow problem with Navier boundary conditions in general three-dimensional domains.* SIAM J. Math. Anal., **44** (2012), no. 2, 633-650.

[20] Duoandikoetxea, Javier; *Fourier analysis.* Graduate Studies in Mathematics, **29**. American Mathematical Society, Providence, RI, 2001.

[21] Love, E. R.; *The Neumann problem for a function harmonic in a half space.* Arch. Rational Mech. Anal., **53** (1973/74), 187-202.

[22] Maluendas, Pedro; Santos, Marcelo M.; *Lagrangian structure for two dimensional non-barotropic compressible fluids.* J. Math. Anal. Appl., **473** (2019), no. 2, 934-951.

[23] Triebel, Hans; *Interpolation theory, function spaces, differential operators.* North-Holland Mathematical Library, **18**. North-Holland Publishing Co., Amsterdam-New York, 1978.

[24] Huang, Xiangdi; Li, Jing; Xin, Zhouping; *Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations.* Comm. Pure Appl. Math. **65** (2012), no. 4, 549-585.

[25] Ziemer, William P.; *Weakly differentiable functions. Sobolev spaces and functions of bounded variation.* Graduate Texts in Mathematics, **120**. Springer-Verlag, New York, 1989.

[26] Zhang, Ting; Fang, Daoyuan; *Compressible flows with a density-dependent viscosity coefficient.* SIAM J. Math. Anal., **41** (2009/10), no. 6, 2453-2488.

Marcelo M. Santos
Departamento de Matemática, IMECC-UNICAMP, Instituto de Matemática, Estatística e Computação Científica, Universidade Estadual de Campinas, Rua Sérgio Buarque de Holanda, 651, 13083-859 - Campinas - SP, Brazil
Email address: `msantos@ime.unicamp.br`

Edson J. Teixeira
Departamento de Matemática, UFV (Universidade Federal de Viçosa), Av. PH. Rolfs, s/n, 36570-900 - Viçosa - MG, Brazil
Email address: `edson.teixeira@ufv.br`