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LIMIT CYCLES IN PIECEWISE SMOOTH PERTURBATIONS OF A QUARTIC ISOCHRONOUS CENTER

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Abstract. This article concerns the bifurcation of limit cycles from a quartic integrable and non-Hamiltonian system. By using the first order averaging method and some mathematical technique on estimating the number of the zeros, we show that under a class of piecewise smooth quartic perturbations, seven is a lower and twelve an upper bound for the maximum number of limit cycles bifurcating from the unperturbed quartic isochronous center.

1. Introduction and statement of main results

Non-smooth phenomena exists in mechanics, electrical engineering and the theory of automatic control, etc. [2, 4, 13, 16]. Motivated by the practical problems, a great interest in the limit cycles of piecewise smooth differential systems, which belong to non-smooth systems, has emerged in the recent years, see for instance [18, 19, 20, 21, 22, 23, 24, 25]. Many innovative methods and theoretical results have been established since the first studies on the piecewise linear differential systems appeared in the book [1]. For example, the conjecture that a class of piecewise liénard equations with $n+1$ intervals has up to $2n$ limit cycles was proved in [30]. By developing new methods for computing the Lyapunov exponents, Hopf bifurcation of non-smooth systems was discussed [8, 9, 14]. The Melnikov method for Hopf and homoclinic bifurcations was extended to non-smooth systems [3, 12, 15, 20, 21]. In addition, the first order Melnikov functions for planar piecewise smooth Hamiltonian systems were applied to study Poincaré bifurcation [19], while the averaging theory for discontinuous dynamical systems was developed to detect limit cycles of piecewise continuous dynamical systems [23]. By using the averaging theory, an estimate was presented on the number of limit cycles bifurcating from the period annulus around the linear center with the singular line parallel to the switching line [18]. More results on this topic can be found in [5, 7, 10, 11, 13, 16, 22, 24, 25] and the references therein.

However, non-smooth property leads to some new bifurcation phenomena which are unique to non-smooth systems (see the review book [5]), such as the border collision bifurcation including grazing bifurcation, corner-collision bifurcation and sliding bifurcation. Besides, non-smooth systems possess some properties which

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are impossible for smooth systems. For example, the study in [15, 24] shows that piecewise linear system can have two limit cycles surrounding the origin. Moreover, three limit cycles surrounding a unique equilibrium may exists in piecewise linear systems [7]. In [8], it is even proved that piecewise quadratic system can have nine small amplitude limit cycles. Hence the study on the limit cycles of non-smooth system is very interesting and challenging. So far it seems that there are few papers in the literature studying limit cycle bifurcations inside the class of piecewise smooth polynomial differential systems of higher degree. In the present paper we choose the following quartic integrable and non-Hamiltonian system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}-y + x^3y + xy^3\\ x + x^2y^2 + y^4\end{pmatrix} \tag{1.1}$$

to study the limit cycles bifurcation under a class of piecewise smooth quartic perturbations as follows

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{cases}\begin{pmatrix}-y + x^3y + xy^3 + \varepsilon P_1(x,y)\\ x + x^2y^2 + y^4 + \varepsilon Q_1(x,y)\end{pmatrix}, & x > 0\\ \begin{pmatrix}-y + x^3y + xy^3 + \varepsilon P_2(x,y)\\ x + x^2y^2 + y^4 + \varepsilon Q_2(x,y)\end{pmatrix}, & x < 0\end{cases} \tag{1.2}$$

where $0 < |\varepsilon| \ll 1$, $P_i(x,y)$, $Q_i(x,y)$, $i = 1,2$ are respectively the quartic polynomials in the variables x and y, given by

$$\begin{aligned} P_1(x,y) &= \sum_{i+j=1,4} a_{ij}x^iy^j, \quad Q_1(x,y) = \sum_{i+j=1,4} b_{ij}x^iy^j,\\ P_2(x,y) &= \sum_{i+j=1,4} c_{ij}x^iy^j, \quad Q_2(x,y) = \sum_{i+j=1,4} d_{ij}x^iy^j. \end{aligned} \tag{1.3}$$

Obviously, system (1.1) has

$$H(x,y) = \frac{1}{3(x^2+y^2)^{3/2}} - \frac{x}{(x^2+y^2)^{1/2}} = h$$

as its first integral with the integrating factor $\mu = (x^2+y^2)^{-5/2}$, and has the unique finite singularity $(0,0)$ as its isochronous center. The period annulus denoted by

$$\{(x,y) : H(x,y) = h,\ h \in (1,+\infty)\}$$

starts at the center $(0,0)$ and terminates with the separatrix passing the infinite degenerate singularity on the equator. The phase portrait of system (1.1) is shown in Figure 1.

The objective of this article is to give lower and upper bounds for the maximum number of limit cycles of system (1.2) bifurcating from the periodic orbits of quartic isochronous center (1.1). It is challenging to estimate the number of limit cycles in non-smooth perturbations of a polynomial differential system of high degree, which is closely related to the simple zeros of the corresponding averaged function. The techniques we use mainly include the first order averaging method and some effective results on extended Chebyshev systems with positive accuracy. Our efforts in the present paper focus on simplifying the averaged function and determining the number of simple zeros of Wronskian determinants of $W_7(t)$ and $W_8(t)$ (see Section 5 for more details), whose expressions are the collection of the arc tangent function. After making some appropriate transformations, qualitative analysis and

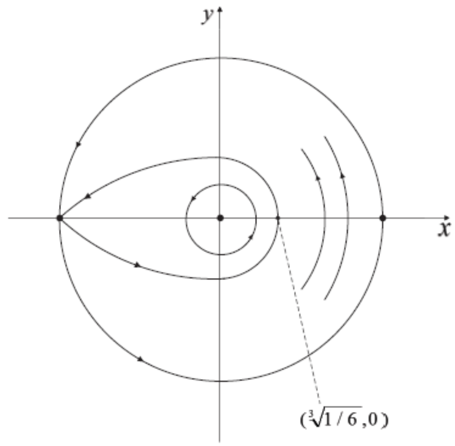

FIGURE 1. Phase portrait of system (1.1) in the Poincaré disk

algebraic calculation, we obtained the main results on the limit cycles bifurcation from the isochronous center (1.1).

Theorem 1.1. *For system* (1.2) *with* $|\varepsilon| \neq 0$ *sufficiently small, we have the following:*

- (a) *there exists a system* (1.2) *which has at least seven limit cycles bifurcating from the periodic orbits of the quartic isochronous center* (1.1)*,*
- (b) *at most twelve limit cycles bifurcate from the periodic orbits of the quartic isochronous center* (1.1)*.*

With a similar as for Theorem 1.1, we have the following theorem.

Theorem 1.2. *For any sufficiently small* $|\varepsilon| \neq 0$*, results (a) and (b) in Theorem 1.1 are also true for the piecewise smooth system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -y + x^3y + xy^3 + \varepsilon A_1(x,y) \\ x + x^2y^2 + y^4 + \varepsilon B_1(x,y) \end{pmatrix}, & x > 0, \\ \begin{pmatrix} -y + x^3y + xy^3 + \varepsilon A_2(x,y) \\ x + x^2y^2 + y^4 + \varepsilon B_2(x,y) \end{pmatrix}, & x < 0, \end{cases} \tag{1.4}$$

where

$$A_1(x,y) = \sum_{i+j=1,4} a_{ij}x^iy^j + a_{11}xy + a_{21}x^2y + a_{03}y^3,$$
$$B_1(x,y) = \sum_{i+j=1,4} b_{ij}x^iy^j + b_{20}x^2 + b_{02}y^2 + b_{12}xy^2 + b_{30}x^3,$$
$$A_2(x,y) = \sum_{i+j=1,4} c_{ij}x^iy^j + c_{11}xy + c_{21}x^2y + c_{03}y^3,$$
$$B_2(x,y) = \sum_{i+j=1,4} d_{ij}x^iy^j + d_{20}x^2 + d_{02}y^2 + d_{12}xy^2 + d_{30}x^3.$$

Based on Theorems 1.1-1.2, we obtain the following result.

Corollary 1.3. *labelc1 For systems* (1.2) *and* (1.4)*, the following two results hold.*

(1) *The lower bound of the maximum number of limit cycles of systems* (1.2) *and* (1.4) *bifurcating from the periodic orbits of the corresponding unperturbed quartic isochronous centers* $(1.2)|_{\varepsilon=0}$ *and* $(1.4)|_{\varepsilon=0}$ *is respectively seven.*
(2) *The upper bound of the maximum number of systems* (1.2) *and* (1.4) *bifurcating from the periodic orbits of the corresponding unperturbed quartic isochronous centers* $(1.2)|_{\varepsilon=0}$ *and* $(1.4)|_{\varepsilon=0}$ *is respectively twelve.*

Remark 1.4. Note that system (1.1) has been studied in [27, 28] under two classes of continuous perturbations, which turns out that at most two and three limit cycles can bifurcate from the unperturbed system using the first order averaging method. Comparing this result with Theorem 1.1, it shows that for system (1.1), more limit cycles can be produced under discontinuous perturbations than continuous perturbations by the first order averaging method.

The rest of this paper is organized as follows. In Section 2, we briefly present the averaging theory for discontinuous differential systems, Sturm's Theorem and some useful results. Section 3 is dedicated to derive the averaged function of system (1.2). Theorem 1.1 is proved in Sections 4 and 5.

2. Preliminary results

In this section, we summarize the first order averaging theory for discontinuous differential systems, introduce the transformation lemma, and present Sturm's Theorem and some important results on ECT-systems, which will be used in the proof of the main results. See the book [26] for a more general introduction to averaging methods, [29] for the details about ECT-systems.

2.1. **Averaging theory.**

Lemma 2.1 ([23]). *Consider the discontinuous differential system*

$$\frac{dr}{d\theta} = \varepsilon F(\theta, r) + \varepsilon^2 R(\theta, r, \varepsilon), \tag{2.1}$$

with

$$F(\theta, r) = F_1(\theta, r) + \operatorname{sign}(h(\theta, r))F_2(\theta, r),$$
$$R(\theta, r, \varepsilon) = R_1(\theta, r, \varepsilon) + \operatorname{sign}(h(\theta, r))R_2(\theta, r, \varepsilon),$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $R_1, R_2 : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ *and* $h : \mathbb{R} \times D \to \mathbb{R}$ *are continuous functions,* T*- periodic in the first variable* θ *and* D *is an open subset of* $\mathbb{R}^n$. *We also suppose that* h *is a* C^1 *function having zero as a regular value, and the sign function* $\operatorname{sign}(u)$ *is given by*

$$\operatorname{sign}(u) = \begin{cases} 1 & u > 0, \\ 0 & u = 0, \\ -1 & u < 0. \end{cases}$$

Define the averaged function $f : D \to \mathbb{R}^n$ *as*

$$f(r) = \int_0^T F(\theta, r)\, d\theta. \tag{2.2}$$

Assume that the following hypotheses (i), (ii) *and* (iii) *hold.*

(i) F_1, F_2, R_1, R_2 *and* h *are locally Lipschitz with respect to* r.
(ii) *There exists an open bounded subset* $C \subset D$ *such that for the sufficiently small* $|\varepsilon| > 0$, *every orbit starting in* $\overline{C}$ $(C \cup \partial C)$ *reaches the set of discontinuity only at its crossing regions.*
(iii) *For* $a \in C$ *with* $f(a) = 0$, *there exists a neighborhood* V *of* a *such that* $f(z) \neq 0$ *for all* $z \in \overline{V} \backslash \{a\}$ *and the Brouwer degree function* $d_B(f, V, a) \neq 0$.

Then, for sufficiently small $|\varepsilon| > 0$ *there exists a T-periodic solution* $r(\theta, \varepsilon)$ *of system* (2.1) *such that* $r(0, \varepsilon) \to a$ *as* $\varepsilon \to 0$.

Remark 2.2. If f is a C^1 function and the Jacobian determinant $J_f(a) \neq 0$, then the hypothesis (iii) in Lemma 2.1 holds, see [6].

2.2. **Transformation lemma.** Consider a planar differential system

$$\begin{aligned} \dot{x} &= P(x, y) + \varepsilon p(x, y), \\ \dot{y} &= Q(x, y) + \varepsilon q(x, y), \end{aligned} \tag{2.3}$$

where the functions $P(x, y), Q(x, y), p(x, y), q(x, y) : \mathbb{R}^2 \to \mathbb{R}$ are continuous, and ε is a small parameter. Suppose that system (2.3) with $\varepsilon = 0$ has a continuous family of periodic orbits

$$\{\gamma_h\} \subset \{(x, y) | H(x, y) = h, \quad h \in (h_c, h_s)\}$$

around the center $(0, 0)$, where $H(x, y)$ is a first integral of system $(2.3)|_{\varepsilon=0}$ and h_c and h_s correspond to the center and the separatrix polycycle, respectively.

Lemma 2.3 ([6])**.** *Consider system* (2.3) *with* $\varepsilon = 0$ *and its first integral* $H = H(x, y)$. *Assume that* $xQ(x, y) - yP(x, y) \neq 0$ *for all* (x, y) *in the period annulus. Let* $\rho : (\sqrt{h_c}, \sqrt{h_s}) \times [0, 2\pi) \to [0, +\infty)$ *be a continuous function such that*

$$H(\rho(r, \theta) \cos\theta, \rho(r, \theta) \sin\theta) = r^2, \tag{2.4}$$

for all $r \in (\sqrt{h_c}, \sqrt{h_s})$ *and all* $\theta \in [0, 2\pi)$. *Then the differential equation which describes the dependence between the square root of energy* $r = \sqrt{h}$ *and the angle* θ *for system* (2.3) *is*

$$\frac{dr}{d\theta} = \varepsilon \frac{\mu(x^2 + y^2)(Qp - Pq)}{2r(Qx - Py) + 2r\varepsilon(qx - py)}, \tag{2.5}$$

where $\mu = \mu(x, y)$ *is the integral factor of system* (2.3) *with* $\varepsilon = 0$ *corresponding to the first integral* H, *and* $x = \rho(r, \theta) \cos\theta$, $y = \rho(r, \theta) sin\theta$, *and* P, Q, p *and* q *are defined as before.*

Remark 2.4. For the integrable and non-Hamiltonian systems, it is generally difficult to find the suitable transformations as described in Lemma 2.3.

2.3. **Sturm's theorem and ECT-systems.**

Lemma 2.5 (Sturm's theorem)**.** *Assume that a univariate polynomial* $p(x)$ *with square-free factor has the definition in* $(a, b]$. *Its Sturm sequence is given as follows*

$$\begin{gathered} p_0(x) := p(x), \quad p_1(x) := p'(x), \\ p_2(x) := -\operatorname{rem}(p_0(x), p_1(x)), \quad p_3(x) := -\operatorname{rem}(p_1(x), p_2(x)), \dots, \\ p_{i+1}(x) := -\operatorname{rem}(p_{i-1}(x), p_i(x)), \dots, 0 = \operatorname{rem}(p_{m-1}(x), p_m(x)), \end{gathered}$$

where rem$(p_{i-1}(x), p_i(x))$ *stands for the reminder of* $p_{i-1}(x)$ *divided by* $p_i(x)$. *Let* $\sigma(\xi)$ *denote the number of sign variation of the Sturm sequence at the point* ξ.

Then the number of distinct real roots of $p(x)$ in the half-open interval $(a, b]$ is $\sigma(a) - \sigma(b)$.

To prove the main results, some definitions of ECT-systems and useful results in [29] are needed.

Let $h_1, h_2, \dots, h_n$ be analytic functions on an open interval L of $\mathbb{R}$. An ordered set $[h_1, h_2, \dots, h_n]$ is an extended complete Chebyshev system (in short, ECT-system) on L if, for all $i = 1, 2, \dots, n$, any nontrivial linear combination

$$\lambda_1 h_1(x) + \lambda_2 h_2(x) + \dots + \lambda_i h_i(x) \tag{2.6}$$

has at most $i-1$ isolated zeros on L counted with multiplicities. For more details, see the book[17].

Sometimes the standard results on ECT-systems can not be directly applied to bound the number of zeros of $h(x) = \lambda_1 h_1(x) + \lambda_2 h_2(x) + \dots + \lambda_n h_n(x)$. In order to study the maximum number of simple zeros of the function $h(x)$, we quote the following result from [29] which provides a very effective estimation for the number of simple zeros.

Lemma 2.6. *Let $[h_1, h_2, \dots, h_n]$ be an ordered set of analytic functions on the open interval L, and $W_k(x), k = 1, 2, \dots, n$ be the Wronskian determinant for the functions $h_1, h_2, \dots, h_k$ depending on x:*

$$W_k(x) = \begin{vmatrix} h_1(x) & h_2(x) & \dots & h_k(x) \\ h_1'(x) & h_2'(x) & \dots & h_k'(x) \\ \vdots & \vdots & \ddots & \vdots \\ h_1^{(k-1)}(x) & h_2^{(k-1)}(x) & \dots & h_k^{(k-1)}(x) \end{vmatrix}. \tag{2.7}$$

Assume that all the zeros ν_i of W_i are simple for $i = 1, 2, \dots, n$. Then the number of isolated zeros for every linear combination (2.6) does not exceed

$$n - 1 + \nu_n + \nu_{n-1} + 2(\nu_{n-2} + \dots + \nu_1) + \lambda_{n-1} + \dots + \lambda_4,$$

where $\lambda_i = \min(2\nu_i, \nu_{i-3} + \dots + \nu_1)$ for $i = 4, \dots, n-1$.

3. Properties of the averaged function of system (1.2)

For

$$H(x, y) = \frac{1}{3(x^2 + y^2)^{3/2}} - \frac{x}{(x^2 + y^2)^{1/2}},$$

we choose the function $\rho = \rho(r, \theta)$ as follows in a very technical way

$$\rho(r, \theta) = \frac{r}{(r^2 + 3\cos\theta)^{1/3}}, \tag{3.1}$$

such that

$$H\left(\rho(r, \theta)\cos\theta, \rho(r, \theta)\sin\theta\right) = \frac{r^2}{3}, \quad r \in (\sqrt{3}, +\infty).$$

According to Lemma 2.3, in the coordinates

$$x = \frac{r\cos\theta}{(r^2 + 3\cos\theta)^{1/3}}, \quad y = \frac{r sin\theta}{(r^2 + 3\cos\theta)^{1/3}}, \quad r \in (\sqrt{3}, +\infty), \tag{3.2}$$

system (1.2) is transformed to

$$\frac{dr}{d\theta} = \begin{cases} \varepsilon X_1(\theta, r) + \varepsilon^2 Y_1(\theta, r, \varepsilon), & \text{if } \cos\theta > 0 \\ \varepsilon X_2(\theta, r) + \varepsilon^2 Y_2(\theta, r, \varepsilon), & \text{if } \cos\theta < 0, \end{cases} \tag{3.3}$$

where

$$
\begin{aligned}
&X_i(\theta,r)\\
&=\frac{3\Big(r^2+3\cos\theta\Big)^{1/3}}{2r}\Big[(r^2\cos\theta+2\cos^2\theta+1)P_i(\theta,r)+(r^2+2\cos\theta\sin\theta)Q_i(\theta,r)\Big],\\
&Y_i(\theta,r,\varepsilon)\\
&=\frac{-X_i(\theta,r)\big[\cos\theta(r^2+3\cos\theta)Q_i(\theta,r)-\sin\theta(r^2+3\cos\theta)P_i(\theta,r)\big]}{\big(r^2+3\cos\theta\big)^{2/3}+\varepsilon\big[\cos\theta(r^2+3\cos\theta)Q_i(\theta,r)-\sin\theta(r^2+3\cos\theta)P_i(\theta,r)\big]},
\end{aligned}
$$

and $P_i(\theta,r)$, $Q_i(\theta,r)$ are respectively derived from $P_i(x,y)$ and $Q_i(x,y)$ given in (1.3) and the variable changes (3.2) for $i=1,2$.

Let

$$
\begin{gathered}
F_i(\theta,r)=\frac{1}{2}\Big[X_1(\theta,r)-(-1)^iX_2(\theta,r)\Big],\\
R_i(\theta,r,\varepsilon)=\frac{1}{2}\Big[Y_1(\theta,r,\varepsilon)-(-1)^iY_2(\theta,r,\varepsilon)\Big],\quad i=1,2\,.
\end{gathered}
$$

Then system (3.3) takes the form

$$
\frac{dr}{d\theta}=\varepsilon F(\theta,r)+\varepsilon^2R(\theta,r,\varepsilon), \tag{3.4}
$$

where

$$
\begin{gathered}
F(\theta,r)=F_1(\theta,r)+\operatorname{sign}(\cos\theta)F_2(\theta,r),\\
R(\theta,r,\varepsilon)=R_1(\theta,r,\varepsilon)+\operatorname{sign}(\cos\theta)R_2(\theta,r,\varepsilon).
\end{gathered}
$$

Based on Lemma 2.1, the averaged function of system (3.4) is

$$
f(r)=\int_0^{2\pi}F(\theta,r)d\theta=\int_{-\pi/2}^{\pi/2}X_1(\theta,r)d\theta+\int_{\pi/2}^{3\pi/2}X_2(\theta,r)d\theta. \tag{3.5}
$$

After substitution of $X_1(\theta,r)$ and $X_2(\theta,r)$ into the averaged function (3.5), we have

$$
\begin{aligned}
&f(r)\\
&=\frac{3}{2r}\int_{-\pi/2}^{\pi/2}\big(r^2+3\cos\theta\big)^{1/3}\Big[(r^2\cos\theta+2\cos^2\theta+1)\sum_{i+j=1,4}a_{ij}\frac{\cos^i\theta\sin^j\theta}{\big(r^2+3\cos\theta\big)^{\frac{i+j}{3}}}\\
&\quad+(r^2+2\cos\theta sin\theta)\sum_{i+j=1,4}b_{ij}\frac{\cos^i\theta\sin^j\theta}{(r^2+3\cos\theta)^{\frac{i+j}{3}}}\Big]d\theta\\
&\quad+\frac{3}{2r}\int_{\pi/2}^{3\pi/2}\big(r^2+3\cos\theta\big)^{1/3}\Big[(r^2\cos\theta+2\cos^2\theta+1)\sum_{i+j=1,4}c_{ij}\frac{\cos^i\theta\sin^j\theta}{(r^2+3\cos\theta)^{\frac{i+j}{3}}}\\
&\quad+(r^2+2\cos\theta\sin\theta)\sum_{i+j=1,4}d_{ij}\frac{\cos^i\theta\sin^j\theta}{\big(r^2+3\cos\theta\big)^{\frac{i+j}{3}}}\Big]d\theta\\
=&\frac{3}{2r}\int_{-\pi/2}^{\pi/2}\Big\{a_{10}(r^2\cos^2\theta+2\cos^3\theta+\cos\theta)+b_{01}(r^2sin\theta+2\cos\theta\sin^2\theta)\\
&\quad+\frac{1}{r^2+3\cos\theta}\Big[a_{04}+(a_{04}+b_{13})r^2\cos\theta+(a_{22}+2b_{13})\cos^2\theta\\
&\quad+(-2a_{04}+a_{22}+b_{31}-2b_{13})r^2\cos^3\theta
\end{aligned}
$$

$$
\begin{aligned}
&+ (-3a_{04} + a_{40} + a_{22} - 4b_{13} + 2b_{31}) \cos^4 \theta \\
&+ (a_{04} + a_{40} - a_{22} + b_{13} - b_{31}) r^2 \cos^5 \theta \\
&+ 2(a_{04} + a_{40} - a_{22} + b_{13} - b_{31}) \cos^6 \theta \Big] \Big\} d\theta \\
&+ \frac{3}{2r} \int_{\pi/2}^{3\pi/2} \Big\{ c_{10}(r^2 \cos^2 \theta + 2\cos^3 \theta + \cos\theta) + d_{01}(r^2 sin\theta + 2\cos\theta \sin^2 \theta) \\
&+ \frac{1}{r^2 + 3\cos\theta} \Big[c_{04} + (c_{04} + d_{13}) r^2 \cos\theta + (c_{22} + 2d_{13}) \cos^2 \theta \\
&+ (-2c_{04} + c_{22} + d_{31} - 2d_{13}) r^2 \cos^3 \theta \\
&+ (-3c_{04} + c_{40} + c_{22} - 4d_{13} + 2d_{31}) \cos^4 \theta \\
&+ (c_{04} + c_{40} - c_{22} + d_{13} - d_{31}) r^2 \cos^5 \theta \\
&+ 2(c_{04} + c_{40} - c_{22} + d_{13} - d_{31}) \cos^6 \theta \Big] \Big\} d\theta.
\end{aligned}
$$

To simplify the averaged function $f(r)$, we list some useful results on the integrals which can be obtained by a straightforward calculation.

Proposition 3.1. *The following equalities hold:*

$$
\begin{gathered}
\int_{-\pi/2}^{\pi/2} \frac{1}{r^2 + 3\cos\theta} d\theta = \frac{4}{\sqrt{r^4 - 9}} \arctan \sqrt{\frac{r^2 - 3}{r^2 + 3}}, \\
\int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{r^2 + 3\cos\theta} d\theta = \frac{\pi}{3} - \frac{4r^2}{3\sqrt{r^4 - 9}} \arctan \sqrt{\frac{r^2 - 3}{r^2 + 3}}, \\
\int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta}{r^2 + 3\cos\theta} d\theta = \frac{2}{3} - \frac{\pi r^2}{9} + \frac{4r^4}{9\sqrt{r^4 - 9}} \arctan \sqrt{\frac{r^2 - 3}{r^2 + 3}}, \\
\int_{-\pi/2}^{\pi/2} \frac{\cos^3\theta}{r^2 + 3\cos\theta} d\theta = \frac{\pi}{6} - \frac{2r^2}{9} + \frac{\pi r^4}{27} - \frac{4r^6}{27\sqrt{r^4 - 9}} \arctan \sqrt{\frac{r^2 - 3}{r^2 + 3}}, \\
\int_{-\pi/2}^{\pi/2} \frac{\cos^4\theta}{r^2 + 3\cos\theta} d\theta = \frac{4}{9} - \frac{\pi r^2}{18} + \frac{2\pi r^4}{27} - \frac{\pi r^6}{81} + \frac{4r^8}{81\sqrt{r^4 - 9}} \arctan \sqrt{\frac{r^2 - 3}{r^2 + 3}}, \\
\begin{aligned}
\int_{-\pi/2}^{\pi/2} \frac{\cos^5\theta}{r^2 + 3\cos\theta} d\theta &= \frac{\pi}{8} - \frac{4r^2}{27} + \frac{\pi r^4}{54} - \frac{2r^6}{81} + \frac{\pi r^8}{243} \\
&\quad - \frac{4r^{10}}{243\sqrt{r^4 - 9}} \arctan \sqrt{\frac{r^2 - 3}{r^2 + 3}},
\end{aligned} \\
\begin{aligned}
\int_{-\pi/2}^{\pi/2} \frac{\cos^6\theta}{r^2 + 3\cos\theta} d\theta &= \frac{16}{45} - \frac{\pi r^2}{24} + \frac{4r^4}{81} - \frac{\pi r^6}{162} + \frac{2r^8}{243} - \frac{\pi r^{10}}{729} \\
&\quad + \frac{4r^{12}}{729\sqrt{r^4 - 9}} \arctan \sqrt{\frac{r^2 - 3}{r^2 + 3}},
\end{aligned} \\
\int_{\pi/2}^{3\pi/2} \frac{1}{r^2 + 3\cos\theta} d\theta = \frac{2\pi}{\sqrt{r^4 - 9}} - \frac{4}{\sqrt{r^4 - 9}} \arctan \sqrt{\frac{r^2 - 3}{r^2 + 3}}, \\
\int_{\pi/2}^{3\pi/2} \frac{\cos\theta}{r^2 + 3\cos\theta} d\theta = \frac{\pi}{3} - \frac{2\pi r^2}{3\sqrt{r^4 - 9}} + \frac{4r^2}{3\sqrt{r^4 - 9}} \arctan \sqrt{\frac{r^2 - 3}{r^2 + 3}},
\end{gathered}
$$

$$\int_{\pi/2}^{3\pi/2}\frac{\cos^2\theta}{r^2+3\cos\theta}d\theta=-\frac{2}{3}-\frac{\pi r^2}{9}+\frac{2\pi r^4}{9\sqrt{r^4-9}}-\frac{4r^4}{9\sqrt{r^4-9}}\arctan\sqrt{\frac{r^2-3}{r^2+3}},$$

$$\begin{aligned}\int_{\pi/2}^{3\pi/2}\frac{\cos^3\theta}{r^2+3\cos\theta}d\theta&=\frac{\pi}{6}+\frac{2r^2}{9}+\frac{\pi r^4}{27}-\frac{2\pi r^6}{27\sqrt{r^4-9}}\\&\quad+\frac{4r^6}{27\sqrt{r^4-9}}\arctan\sqrt{\frac{r^2-3}{r^2+3}},\\
\int_{\pi/2}^{3\pi/2}\frac{\cos^4\theta}{r^2+3\cos\theta}d\theta&=-\frac{4}{9}-\frac{\pi r^2}{18}-\frac{2\pi r^4}{27}-\frac{\pi r^6}{81}+\frac{2\pi r^8}{81\sqrt{r^4-9}}\\&\quad-\frac{4r^8}{81\sqrt{r^4-9}}\arctan\sqrt{\frac{r^2-3}{r^2+3}},\\
\int_{\pi/2}^{3\pi/2}\frac{\cos^5\theta}{r^2+3\cos\theta}d\theta&=\frac{\pi}{8}+\frac{4r^2}{27}+\frac{\pi r^4}{54}+\frac{2r^6}{81}+\frac{\pi r^8}{243}-\frac{2\pi r^{10}}{243\sqrt{r^4-9}}\\&\quad+\frac{4r^{10}}{243\sqrt{r^4-9}}\arctan\sqrt{\frac{r^2-3}{r^2+3}},\\
\int_{\pi/2}^{3\pi/2}\frac{\cos^6\theta}{r^2+3\cos\theta}d\theta&=-\frac{16}{45}-\frac{\pi r^2}{24}-\frac{4r^4}{81}-\frac{\pi r^6}{162}-\frac{2r^8}{243}-\frac{\pi r^{10}}{729}+\frac{2\pi r^{12}}{729\sqrt{r^4-9}}\\&\quad-\frac{4r^{12}}{729\sqrt{r^4-9}}\arctan\sqrt{\frac{r^2-3}{r^2+3}}.\end{aligned}$$

Using Proposition 3.1, we obtain the averaged function as stated in Proposition 3.2.

Proposition 3.2. *The averaged function $f(r)$ can be rewritten as*

$$\begin{aligned}f(r)=&\frac{1}{r}[k_1f_1(r)+k_2f_2(r)+k_3f_3(r)+k_4f_4(r)+k_5f_5(r)\\&+k_6f_6(r)+k_7f_7(r)+k_8f_8(r)],\quad r\in(\sqrt{3},+\infty),\end{aligned}\tag{3.6}$$

where

$$f_1(r)=1,\quad f_2(r)=r^2,$$
$$f_3(r)=2r^8\sqrt{r^4-9}\arctan\sqrt{\frac{r^2-3}{r^2+3}}-\frac{\pi}{2}r^{10}+3r^8+\frac{9\pi}{4}r^6-9r^4,$$
$$f_4(r)=2r^4\sqrt{r^4-9}\arctan\sqrt{\frac{r^2-3}{r^2+3}}-\frac{\pi}{2}r^6+3r^4,$$
$$f_5(r)=\sqrt{r^4-9}\arctan\sqrt{\frac{r^2-3}{r^2+3}},\quad f_6(r)=r^8\sqrt{r^4-9}-2r^{10}+\frac{9}{2}r^6,$$
$$f_7(r)=r^4\sqrt{r^4-9}-r^6,\quad f_8(r)=\sqrt{r^4-9},$$

and

$$\begin{aligned}k_1=&\frac{1}{15}(105a_{10}+30b_{01}+26a_{40}+9a_{22}-14a_{04}+4b_{31}+6b_{13}-105c_{10}-30d_{01}\\&-26c_{40}-9c_{22}+14c_{04}-4d_{31}-6d_{13}),\\
k_2=&\frac{\pi}{48}(36a_{10}+36b_{01}-a_{40}-3a_{22}+15a_{04}+b_{31}+3b_{13}-36c_{10}+36d_{01}-c_{40}\\&-3c_{22}+15c_{04}+d_{31}+3d_{13}),\end{aligned}$$

$$\begin{gathered} k_3=\frac{1}{243}(-a_{40}+a_{22}-a_{04}+b_{31}-b_{13}+c_{40}-c_{22}+c_{04}-d_{31}+d_{13}),\\ k_4=\frac{1}{27}(-a_{22}+2a_{04}+b_{13}+c_{22}-2c_{04}-d_{13}),\\ k_5=\frac{2}{3}(-a_{04}+c_{04}),\quad k_6=\frac{\pi}{243}(-c_{40}+c_{22}-c_{04}+d_{31}-d_{13}),\\ k_7=\frac{\pi}{27}(-c_{22}+2c_{04}+d_{13}),\quad k_8=-\frac{\pi}{3}c_{04}. \end{gathered}$$

Moreover, the coefficients $k_1,k_2,\dots,k_8$ can be chosen arbitrarily.

Proof. After substitution of Proposition 3.1 and by a straightforward calculation, we obtain the formula (3.6) of $f(r)$.

The second result follows from

$$\Big|\frac{\partial(k_1,k_2,k_3,k_4,k_5,k_6,k_7,k_8)}{\partial(a_{10},a_{04},a_{22},b_{31},c_{04},c_{40},c_{22},d_{13})}\Big|=-\frac{\pi^4}{258280326}\neq 0, \tag{3.7}$$

which implies that $k_1,k_2,\dots,k_8$ are independent. So the coefficients of the function $f_i(r), i=1,2,\dots,8$ in (3.6) can be chosen arbitrarily. $\square$

For convenience, we consider $F(r)=rf(r)$ which has the same zeros as $f(r)$ in $r\in(\sqrt{3},+\infty)$. Since the expression of $F(r)$ includes the square root terms as $\sqrt{r^4-9}$ and $\sqrt{\frac{r^2-3}{r^2+3}}$, the first and the most important step is to eliminate these ones. As a result of $r\in(\sqrt{3},+\infty)$, let

$$r^2=\frac{3(1+t^2)}{1-t^2},\quad t\in(0,1), \tag{3.8}$$

then

$$\sqrt{r^4-9}=\frac{6t}{1-t^2},\quad \sqrt{\frac{r^2-3}{r^2+3}}=t. \tag{3.9}$$

Substituting (3.9) into the formula $F(r)$, we obtain

$$\begin{aligned} G(t):=&F(r)\Big|_{r=\sqrt{\frac{3(1+t^2)}{1-t^2}}}\\ =&\frac{1}{4(t^2-1)^5}[m_1g_1(t)+m_2g_2(t)+m_3g_3(t)+m_4g_4(t)+m_5g_5(t)\\ &+m_6g_6(t)+m_7g_7(t)+m_8g_8(t)]. \end{aligned} \tag{3.10}$$

Let

$$\begin{aligned} g(t)=&m_1g_1(t)+m_2g_2(t)+m_3g_3(t)+m_4g_4(t)+m_5g_5(t)\\ &+m_6g_6(t)+m_7g_7(t)+m_8g_8(t), \end{aligned} \tag{3.11}$$

where

$$\begin{gathered} g_1(t)=(t^2-1)^5,\quad g_2(t)=(t^2+1)(t^2-1)^4,\\ g_3(t)=t(t^2-1)^4,\quad g_4(t)=(t^2-1)^2(t^2+1)^2(t-1)^2,\\ g_5(t)=(t^2+1)^3(3-4t+10t^2-4t^3+3t^4),\\ g_6(t)=t(t^2-1)^4\arctan(t),\\ g_7(t)=(t^2-1)^2(t^2+1)^2[\pi-2+(\pi+2)t^2-8t\arctan(t)],\\ \begin{aligned} g_8(t)=&(t^2+1)^2[3\pi-8+(21\pi-24)t^2+(21\pi+24)t^4+(3\pi+8)t^6\\ &-(48t+96t^3+48t^5)\arctan(t)], \end{aligned} \end{gathered} \tag{3.12}$$

and

$$\begin{aligned} &m_1 = 4k_1, \quad m_2 = -12k_2, \quad m_3 = -24k_8, \quad m_4 = 108k_7, \\ &m_5 = 486k_6, \quad m_6 = -24k_5, \quad m_7 = 54k_4, \quad m_8 = 81k_3. \end{aligned} \tag{3.13}$$

In summary, we have the following proposition.

Proposition 3.3. *In $r \in (\sqrt{3}, +\infty)$, the number of zeros of $f(r)$ is the same as that of $F(r)$, which also equals to the number of zeros of $g(t)$ defined by* (3.11) *for $t \in (0,1)$.*

4. Proof of Theorem 1.1(a)

We start by studying the properties of the function $g(t)$ in $t \in (0,1)$.

Proposition 4.1. *The generating functions $g_1(t), g_2(t), \dots, g_8(t)$ of $g(t)$ defined by* (3.11) *are linearly independent. And the coefficients $m_1, m_2, \dots, m_8$ can be chosen arbitrarily.*

Proof. First we prove that $g_1(t), g_2(t), \dots, g_8(t)$ are linearly independent functions. Suppose that

$$e_1 g_1(t) + e_2 g_2(t) + \dots + e_8 g_8(t) \equiv 0, \quad t \in (0,1).$$

We need to show $e_i = 0$ for all $i \in \{1, 2, \dots, 8\}$. To this end let

$$h(t) = e_1 g_1(t) + e_2 g_2(t) + \dots + e_8 g_8(t).$$

Then we can get Taylor expansions of the function $h(t)$ near $t = 0$:

$$h(t) = s_0 + s_1 t + s_2 t^2 + s_3 t^3 + s_4 t^4 + s_5 t^5 + s_6 t^6 + s_7 t^7 + s_8 t^8 + \mathcal{O}(t^9),$$

where

$$\begin{gathered} s_0 = -e_1 + e_2 + e_4 + 3e_5 + (\pi - 2)e_7 + (3\pi - 8)e_8, \quad s_1 = e_3 - 2e_4 - 4e_5, \\ s_2 = 5e_1 - 3e_2 + e_4 + 19e_5 + e_6 + (\pi - 6)e_7 + (27\pi - 88)e_8, \\ s_3 = -4e_3 - 16e_5, \\ s_4 = -10e_1 + 2e_2 - 2e_4 + 42e_5 - \frac{13}{3}e_6 + \big(-2\pi + \frac{20}{3}\big)e_7 + (66\pi - 208)e_8, \\ s_5 = 6e_3 + 4e_4 - 24e_5, \\ s_6 = 10e_1 + 2e_2 - 2e_4 + 42e_5 + \frac{113}{15}e_6 + \big(-2\pi + \frac{52}{5}\big)e_7 + \big(66\pi - \frac{1008}{5}\big)e_8, \\ s_7 = -4e_3 - 16e_5, \\ s_8 = -5e_1 - 3e_2 + e_4 + 19e_5 - \frac{243}{35}e_6 + \big(\pi - \frac{130}{21}\big)e_7 + \big(27\pi - \frac{3064}{35}\big)e_8. \end{gathered}$$

Noting that $s_3 = s_7$ and that the Jacobian determinant

$$\Big|\frac{\partial(s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_8)}{\partial(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8)}\Big| = -\frac{398032896}{7}\pi + \frac{55312384}{7}\pi^2 + \frac{158527913984}{1575} \neq 0,$$

we have that $s_0, s_1, \dots, s_6, s_8$ are independent. Thus $h(t) \equiv 0$ implies that $s_0 = s_1 = \dots = s_6 = s_8 = 0$. Consequently, the functions $g_1(t), g_2(t), \dots, g_8(t)$ defined by (3.12) are linearly independent.

It follows from

$$\Big|\frac{\partial(m_1, m_2, \dots, m_8)}{\partial(k_1, k_2, \dots, k_8)}\Big| = 6347497291776 \neq 0,$$

$$|\frac{\partial(k_1,k_2,k_3,k_4,k_5,k_6,k_7,k_8)}{\partial(a_{10},a_{04},a_{22},b_{31},c_{04},c_{40},c_{22},d_{13})}| = -\frac{\pi^4}{258280326} \neq 0$$

that $m_1, m_2, \dots, m_8$ are independent. Hence the coefficients $m_1, m_2, \dots, m_8$ in (3.11) can be chosen arbitrarily. □

Lemma 4.2 ([10])**.** *Consider n linearly independent analytical functions $f_i(x) : D \to \mathbb{R}$, $i = 1, 2, \dots, n$, where $D \subset \mathbb{R}$ is an interval. Suppose that there exists $k \in \{1, 2, \dots, n\}$ such that $f_k(x)$ has constant sign. Then there exist n constants $c_i, i = 1, 2, \dots, n$, such that $\sum_{i=1}^{n} c_i f_i(x)$ has at least $n-1$ simple zeros in D.*

Proof of Theorem 1.1 (a). By Lemma 4.2 and Proposition 4.1, there exists a linear combination of $g_1(t), g_2(t), \dots, g_8(t)$, namely $\widetilde{g}(t) = \lambda_1 g_1(t) + \lambda_2 g_2(t) + \dots + \lambda_8 g_8(t)$, such that the function $\widetilde{g}(t)$ has at least seven simple zeros. This means that we can get the averaged function $\widetilde{f}(r)$ corresponding to $\widetilde{g}(t)$ having at least seven simple zeros in $r \in (\sqrt{3}, +\infty)$. Following this fact and Lemma 2.1, we obtain the result in Theorem 1.1(a). □

5. Proof of Theorem 1.1(b)

This section is devoted to explore the upper bound of the maximum number of zeros of the averaged function $f(r)$ given by (3.6). Proposition 3.3 in Section 3 has shown that the number of zeros of $f(r)$ in $r \in (\sqrt{3}, +\infty)$ is the same as that of zeros of $g(t)$ in $t \in (0, 1)$. Hence, we investigate the zeros of $g(t)$ instead of $f(r)$ in the following.

For $g_1(t), g_2(t), \dots, g_8(t)$, after a straightforward calculation, we obtain

$$\begin{aligned}
W_1(t) &= (t-1)^5(t+1)^5, \quad W_2(t) = -4t(t-1)^8(t+1)^8, \\
W_3(t) &= 4(t-1)^{12}(t+1)^{12}, \quad W_4(t) = 192(t+1)^{11}(t-1)^{17}, \\
W_5(t) &= 92160(t+1)^7(t-1)^{13}(5t^{10} + 14t^9 + 177t^8 + 72t^7 + 714t^6 + 84t^5 \\
&\quad + 714t^4 + 72t^3 + 177t^2 + 14t + 5), \\
W_6(t) &= -\frac{2949120}{(t^2+1)^5}(t-1)^{15}(t+1)^9(4 - 141t - 574t^2 - 2051t^3 - 8564t^4 \\
&\quad + 4823t^5 - 33090t^6 + 36793t^7 - 51744t^8 + 36793t^9 - 33090t^{10} \\
&\quad + 4823t^{11} - 8564t^{12} - 2051t^{13} - 574t^{14} - 141t^{15} + 4t^{16}), \\
W_7(t) &= \frac{135(t-1)^{11}(t+1)^{11}}{8388608(t^2+1)^9}\left(W_{71}(t) + W_{72}(t)\arctan(t)\right), \\
W_8(t) &= \frac{1215(t-1)^{13}(t+1)^{13}}{4611686018427387904}\Big(W_{81}(t) + W_{82}(t)\arctan(t) \\
&\quad + W_{83}(t)\arctan^2(t)\Big),
\end{aligned} \tag{5.1}$$

where $W_{71}(t)$, $W_{72}(t)$, $W_{81}(t)$, $W_{82}(t)$ and $W_{83}(t)$ are cumbersome polynomials give by (A.1) in the appendix.

Based on (5.1) and Sturm's Theorem, a straightforward calculation leads to the following result.

Proposition 5.1. (1) *The Wronskian determinants $W_1(t), W_2(t), W_3(t), W_4(t)$ and $W_5(t)$ do not vanish for $t \in (0, 1)$;*

(2) *$W_6(t)$ has only one simple zero in the open interval $(0, 1)$.*

For $W_7(t)$ and $W_8(t)$, we have Propositions 5.2 and 5.3.

Proposition 5.2. *$W_7(t)$ has a unique zero in $t \in (0,1)$ which is simple.*

Proof. Let
$$\overline{W}_7(t) = W_{71}(t) + W_{72}(t)\arctan(t),$$
then $\overline{W}_7(t)$ has the same zeros as $W_7(t)$ in $t \in (0,1)$.

Using Sturm's Theorem, we can know that $W_{71}(t) < 0, W_{72}(t) > 0$ for all $t \in (0,1)$. Let
$$T(t) = \frac{W_{71}(t)}{W_{72}(t)} + \arctan(t),$$
then we have $\overline{W}_7(t) = W_{72}(t)T(t)$, and
$$T'(t) = \frac{27021597764222976t(t^2+1)^2 Z(t)}{W_{72}^2(t)},$$
where $Z(t)$ is a polynomial of degree 40 given by (6.2) in the appendix. Using Sturm's Theorem, we can obtain that $Z(t)$ has exactly two zeros, denoted by $\bar{t}_1$ and $\bar{t}_2$, in $t \in (0,1)$. Using Maple software, $\bar{t}_1$ and $\bar{t}_2$ can be easily isolated as $\bar{t}_1 \in [\frac{53401}{2097152}, \frac{106803}{4194304}]$ and $\bar{t}_2 \in [\frac{39169}{262144}, \frac{78339}{524288}]$. A straightforward calculation shows that $Z(0) > 0$ and $Z(1) > 0$, thus the function $T(t)$ is monotonically increasing in the intervals $(0, \bar{t}_1)$ and $(\bar{t}_2, 1)$, and monotonically decreasing in the interval $(\bar{t}_1, \bar{t}_2)$.

Using Maple software, we obtain $T(0) < 0$, $T(1) > 0$, $T(\bar{t}_1) < 0$ and $T(\bar{t}_2) < 0$. Therefore, the function $T(t)$ has a unique zero in $(\bar{t}_2, 1)$ which is simple. This completes the proof. □

Proposition 5.3. *$W_8(t)$ has exactly two zeros in $t \in (0,1)$ and both zeros are simple.*

Proof. Let
$$\overline{W}_8(t) = W_{81}(t) + W_{82}(t)\arctan(t) + W_{83}(t)\arctan^2(t),$$
which has the same zeros as $W_8(t)$ in $t \in (0,1)$. We will study $\overline{W}_8(t)$ instead of $W_8(t)$.

Using Sturm's Theorem, we know that $W_{81}(t) < 0$ and $W_{82}(t) > 0$, $W_{83}(t) < 0$ for all $t \in (0,1)$. Moreover, we have
$$\begin{aligned}\Delta(t) &= W_{82}^2(t) - 4W_{81}(t)W_{83}(t)\\ &= 2852213850513516153367582212096 00(t^2+1)^2 D(t),\end{aligned} \tag{5.2}$$
where $D(t)$ is a polynomial of degree 36 without multi-factor given in Appendix 6.3. By Sturm's Theorem, it shows that $D(t) > 0$ for all $t \in (0,1)$.

Let
$$\mu_1(t) = \frac{-W_{82}(t) + \sqrt{\Delta(t)}}{2W_{83}(t)}, \quad \mu_2(t) = \frac{-W_{82}(t) - \sqrt{\Delta(t)}}{2W_{83}(t)}, \quad \mu(t) = \arctan(t),$$
then
$$\overline{W}_8(t) = W_{83}(t)(\mu(t) - \mu_1(t))(\mu(t) - \mu_2(t)).$$
This means that the abscissae of the intersection points of $\mu(t)$ with $\mu_1(t)$ and $\mu_2(t)$ coincide with the zeros of the function $\overline{W}_8(t)$. Hence we only need to count the number of zeros of $\nu_1(t) := \mu_1(t) - \mu(t)$ and $\nu_2(t) := \mu_2(t) - \mu(t)$.

Firstly, we investigate the number of zeros of the function $\nu_1(t)$. According to the definition of $\nu_1(t)$, we have

$$\frac{d\nu_1}{dt} = \frac{-t\big(M_1(t) + N_1(t)\sqrt{\overline{\Delta}(t)}\big)}{1979120929996800(t^2+1)M_2(t)N_2(t)\sqrt{\Delta(t)}},$$

where

$$\sqrt{\overline{\Delta}(t)} = \frac{\sqrt{\Delta(t)}}{1152921504606846976},$$

the polynomials $M_1(t)$, $N_1(t)$, $M_2(t)$ and $N_2(t)$ are given in Appendix 6.3. Moreover, applying Sturm's Theorem, we have $M_2(t)N_2(t) > 0$ for all $t \in (0,1)$. Therefore, it is enough to study the zeros of the function $M_1(t) + N_1(t)\sqrt{\overline{\Delta}(t)}$ instead of $d\nu_1/dt$, which are also the zeros of the following function

$$M_1^2(t) - N_1^2(t)\overline{\Delta}(t) = 3865470566400(t-1)^2(t+1)^2(t^2+1)^5 M(t),$$

where $M(t)$ is a polynomial of degree 82 given in appendix 6.3. Using Sturm's Theorem and Maple software, we obtain that $M(t)$ has $t_1 \in [\frac{765}{4096}, \frac{6121}{32768}], t_2 \in [\frac{4449}{16384}, \frac{2225}{8192}]$ and $t_3 \in [\frac{1013}{1024}, \frac{8105}{8192}]$ as its zeros in $t \in (0,1)$. By analyzing the sign of $M_1(t)$ and $N_1(t)$ at the points t_1, t_2 and t_3, we obtain that only t_2 and t_3 are the zeros of $d\nu_1/dt$.

After direct calculation and the analysis, we obtain that $\nu_1(0) > 0$, $\nu_1(t_2) < 0$, $\nu_1(t_3) > 0$ and $\nu_1(1) > 0$. Therefore, the function $\nu_1(t)$ has exactly two simple zeros in $t \in (0,1)$.

Secondly, the analysis about the function $\nu_2(t)$ is similar to the above case. Consequently, we obtain that the function $\nu_2(t)$ has no zero in the interval $(0,1)$.

In summary, the function $\overline{W}_8(t)$ has two simple zeros in $t \in (0,1)$, so the Wronskian determinant $W_8(t)$ has two simple zeros in $t \in (0,1)$. This completes the proof of Proposition 5.3. □

Proof of Theorem 1.1 (b). Using Propositions 5.1-5.3, we obtain that $W_1(t)$, $W_2(t)$, $W_3(t)$, $W_4(t)$ and $W_5(t)$ do not vanish in $t \in (0,1)$, $W_6(t)$ and $W_7(t)$ have a unique simple zero in $t \in (0,1)$ respectively, and $W_8(t)$ has exactly two simple zeros in $t \in (0,1)$. Thus using Lemma 2.6, we obtain that the number of isolated zeros of the function $g(t)$ given in (3.11) does not exceed

$$8 - 1 + 2 + 1 + 2 \times 1 = 12,$$

which implies that the averaged function $f(r)$ has at most twelve zeros in $r \in (\sqrt{3}, +\infty)$. By virtue of Lemma 2.1, we obtain Theorem 1.1 (b). □

6. Appendix

6.1. **A.1.**

$$\begin{aligned}
W_{71}(t) = &- 901342619428862\,72t^{24} + 574208952489738\,24t^{23} \\
&+ 6168366923881833472t^{22} - 9336138005069240729\,6t^{21} \\
&+ 23167356180217973964\,8t^{20} - 104444906955739614412\,8t^{19} \\
&+ 151901069884658339020\,8t^{18} - 358819416234912907264\,0t^{17} \\
&+ 371378018314314789683\,2t^{16} - 534431884601111294771\,2t^{15} \\
&+ 387460104831422064230\,4t^{14} - 348423203088070239846\,4t^{13}
\end{aligned}$$

$$
\begin{aligned}
&+ 8851675358414345338888t^{12} - 2721882154463992872296t^{11}\\
&- 1780046586218136207360t^{10} + 9731190420817320345600t^{9}\\
&- 173123389593436736716 8t^{8} + 5698149922156506644448t^{7}\\
&- 5772527981077972254722t^{6} + 11005545031847161036 8t^{5}\\
&- 568090159286647521 28t^{4} + 380153011001235763 2t^{3}\\
&+ 7442633638994796t^{2} - 57420895248973824\, t - 62269395476345.
\end{aligned}
$$

$$
\begin{aligned}
W_{72}(t) =&3377699720527872\,(t^2+1)^3(17t^{18} - 1215t^{16} + 16880t^{15} - 29364t^{14}\\
&+ 125472t^{13} - 85780t^{12} + 141840t^{11} - 27018t^{10} + 5056t^{9} - 27018t^{8}\\
&+ 141840t^{7} - 85780t^{6} + 125472t^{5} - 29364t^{4}\\
&+ 16880t^{3} - 1215t^{2} + 17).
\end{aligned}
$$

$$
\begin{aligned}
W_{81}(t) =&118589152849670871101771361107769841 1t^{20}\\
&- 2266004888243560189771265096527380480t^{19}\\
&+ 3835522024174547959633389010613268920t^{18}\\
&- 4714044509535283562829585653516206080t^{17}\\
&+ 3881022138916230431364589772180153865t^{16}\\
&- 2148878442618162181729270487675043840t^{15}\\
&- 1683587088444303269065255220992513200t^{14}\\
&- 5244580781673026505805039792910499840t^{13}\\
&- 6504111571087226756841681728185558450t^{12}\\
&- 21659231139657798060138444726982410240t^{11}\\
&- 716839526339247403929574155957043 2t^{10}\\
&- 2663998246602653876614717122528935936 0t^{9}\\
&+ 6520837826701808329738108022055495090t^{8}\\
&- 11678566756143409397689586845860495360t^{7}\\
&+ 2243677705024017984489213850889196720t^{6}\\
&- 1058739491241092199782953625554255872t^{5}\\
&- 1658102767738292126602207791694603785t^{4}\\
&+ 510033906986216282204618273639628 80t^{3}\\
&- 962907095624491580737423254276895160t^{2}\\
&- 279221262008854551511106177728512 0\, t\\
&- 54228499683958932407596982 83.
\end{aligned}
$$

$$
W_{82}(t) = - 16888498602639360\,(t^2+1)(1341744427115282783 43t^{18}
$$

$$
\begin{aligned}
&- 114031142565020958720t^{17} + 189677878383987820241t^{16}\\
&- 120336182043339653120t^{15} + 75259129312831332460t^{14}\\
&+ 120912642795643076608t^{13} - 3503509027040144092t^{12}\\
&+ 9655717601082343424t^{11} - 597756385533301886398t^{10}\\
&- 308910905640597061632t^{9} - 1154215927107858985538t^{8}\\
&+ 9655717601082343424t^{7} - 647825083481790469412t^{6}\\
&+ 120912642795643076608t^{5} - 64810778177331781740t^{4}\\
&- 120336182043339653120t^{3} + 3130228862997854511t^{2}\\
&- 114031142565020958720\,t - 165332199491799367).
\end{aligned}
$$

$$
\begin{aligned}
W_{83}(t) =&4563542160821625845388131539353600\,(t-1)(t+1)(211t^{8} - 340t^{6}\\
&+ 818t^{4} - 340t^{2} + 211)(t^{2}+1)^{5}.
\end{aligned}
$$

6.2. **A.2.**

$$
\begin{aligned}
Z(t) =&27021597764222976t(13510798882112896t^{40} - 557320453887148800t^{39}\\
&+ 702561541869938048t^{38} + 14568018894637086976t^{37}\\
&- 26400101015648370944t^{36} - 594309643526618057472t^{35}\\
&+ 8369831821077324280832t^{34} - 31926223798113055981312t^{33}\\
&+ 111167461187962183024384t^{32} - 11762947363129579100416t^{31}\\
&- 1758032932760516097727744t^{30} + 10919540674355160671449856t^{29}\\
&- 43640914019377804089929216t^{28} + 132773796854708013459128064\,t^{27}\\
&- 329600636163665084127721472t^{26} + 690790880313384305445008128t^{25}\\
&- 1243529438957662468719622272t^{24} + 1945832602586207995294138368t^{23}\\
&- 2667488186096564230379160704t^{22} + 3217510671604529979408769664\,t^{21}\\
&- 3423771052004058459647616249t^{20} + 3217510671604529924739735808t^{19}\\
&- 2667488186096564185492657483t^{18} + 1945832602586207939629673864t^{17}\\
&- 1243529438957662457746067088t^{16} + 690790880313384269397499822\,t^{15}\\
&- 329600636163665074867438968t^{14} + 132773796854707981703554310t^{13}\\
&- 43640914019377781658542970t^{12} + 10919540674355152051925870t^{11}\\
&- 1758032932760503352465558t^{10} - 11762947363128648582330t^{9}\\
&+ 111167461187962661085448t^{8} - 31926223798112199097958t^{7}\\
&+ 8369831821076973826368t^{6} - 594309643526604041918t^{5}\\
&- 26400101015644673797t^{4} + 14568018894636651594t^{3}\\
&+ 702561541869769241t^{2} - 557320453887099798\,t + 13510798882111488).
\end{aligned}
$$

$$
D(t) =1988501876129629500474705489760443483505t^{36}
$$

$$
\begin{aligned}
&+ 5884500572433515234379892676130793963118t^{34}\\
&+ 605609151411869753708915666107244087369t^{32}\\
&+ 43403895762289312289381530865074236817408t^{31}\\
&- 83326798894808862552273652737549113512368t^{30}\\
&+ 281632057350203689330390282952609649131520t^{29}\\
&- 339528795975034095415480066886365024048652t^{28}\\
&+ 678809005652106319968345619462656062652416t^{27}\\
&- 555930300928416176019762380766425394374776t^{26}\\
&+ 806289180568398165762752168272129307967488t^{25}\\
&- 338568141236418399632409945569940635178492t^{24}\\
&+ 768996116339222076327406729182906345848832t^{23}\\
&+ 19346189464397818679236507699066293500464t^{22}\\
&+ 1245902406756596634662873385740421520424960t^{21}\\
&+ 168530043036960113627313245814994103787086t^{20}\\
&+ 1377590181459632184339865975836731859009536t^{19}\\
&+ 886245187942348322666541942973828390552340t^{18}\\
&- 59974653752892282730714799986761443311616t^{17}\\
&+ 1982197348295878629950986704693415027236430t^{16}\\
&- 1462561668541303465068912603777582902542336t^{15}\\
&+ 1877456207255899142320772683203168908655152t^{14}\\
&- 1182759951478895156946986977153738623418368t^{13}\\
&+ 789113167879454688412117734071632371329540t^{12}\\
&- 340239082216778844038353935241933198721024t^{11}\\
&+ 120883842434726856318939067238891247441800t^{10}\\
&- 23896876747383011358282864079587474669568t^{9}\\
&+ 550656586770623643578827126624169535988t^{8}\\
&+ 1127054208895437603409452578172708061184t^{7}\\
&- 193844721745407853396522073439738810800t^{6}\\
&- 66660031965017298888426839047824998400t^{5}\\
&+ 31017349107330149006541280157099231305t^{4}\\
&- 1031028333780566802157428913880580498t^{2}\\
&+ 27261506022822924725379464415987057.
\end{aligned}
$$

6.3. **A.3.**

$$
\begin{aligned}
M_1(t) =&4096673085353190194511879500402655232t^{48}\\
&+ 112783926943216041984000t^{47}
\end{aligned}
$$

$$
\begin{aligned}
& - 176749032758087775067883548852928970752t^{46} \\
& + 122821328122299003308402253182502174720 0t^{45} \\
& - 350881093875752697111662418991088371302 4t^{44} \\
& + 1027009195452474790468319572464079208448 0t^{43} \\
& - 1553072647484713993710263875503952016179 2t^{42} \\
& + 3017671632543544269362814594974157373440 0t^{41} \\
& - 2675225051622893657469643699136699919564 8t^{40} \\
& + 3678248948273431156267509267407541633024 0t^{39} \\
& - 527599692257381817424637974584489481830 4t^{38} \\
& + 1874058288789466481845615446166528327680 0t^{37} \\
& + 3943003703865515492233566631576190627020 8t^{36} \\
& + 5625714361398962817202352791854187020288 0t^{35} \\
& + 3759936936989666045766764271771407806464 0t^{34} \\
& + 18161216395392800679436208119316037500928 0t^{33} \\
& + 1310817773236262749541250324827724133171 2t^{32} \\
& + 24140234783996394499450601232150772580352 0t^{31} \\
& + 6218291469951402488323198867409252232268 8t^{30} \\
& + 18884492129530643318574548577060919443456 0t^{29} \\
& + 15876444560546744967173879301725144660019 2t^{28} \\
& + 16542487394536389293397129285961871523840 0t^{27} \\
& + 25390482918558914102329385429361182221849 6t^{26} \\
& + 14397924032738680438875885116135332380672 0t^{25} \\
& + 35417086745241374707008079584271352058374 4t^{24} \\
& + 3612238736194159111129495812031018696704 0t^{23} \\
& + 38153457052599939146207996068954043508787 2t^{22} \\
& - 1414512731509481039284242004381236461568 0t^{21} \\
& + 26550647667813564536401362373812007215264 0t^{20} \\
& + 7964395759720586983867450443342916091904 0t^{19} \\
& + 13154285071692427490381815429880182555417 6t^{18} \\
& + 11033198233016289002538130237517274808320 0t^{17} \\
& + 13510154164753879691438364399009276527635 2t^{16} \\
& - 2661567122805709358372351615565815611392 0t^{15} \\
& + 19164761613683722171430702698804647958809 6t^{14} \\
& - 12702003788001790287485820624044795363328 0t^{13} \\
& + 14730302174418614283949898690625172118758 4t^{12}
\end{aligned}
$$

$$
\begin{aligned}
&- 8513443053061088790436363800197008982016 0t^{11}\\
&+ 5248807313803039951742896645136690199144 7t^{10}\\
&- 2087881439484670147800968312773561810944 0t^{9}\\
&+ 682588109395799469125835690926563829801 6t^{8}\\
&- 123545086428863706574143636821489221632 0t^{7}\\
&+ 28706402053952889048372787969647560730t^{6}\\
&+ 46012062126586609978860499737237258240t^{5}\\
&- 7284327511482175992305996261232662188t^{4}\\
&- 1886299262964271953864180210912460800t^{3}\\
&+ 729611530820898442630207636388052583t^{2}\\
&+ 1145461758017037926400\,t - 125271081245742098196107432624005 40.
\end{aligned}
$$

$$
\begin{aligned}
N_1(t) =&5192289730544231936t^{28} - 2101702402744986882 56t^{26}\\
&+ 810690154173195878400t^{25} - 2728769321208895530304t^{24}\\
&+ 1933353098783647334400t^{23} - 4289869168445257579776t^{22}\\
&- 1687574778619436728320t^{21} - 1370094094634001316800t^{20}\\
&- 2511311297963171512320t^{19} - 950648044803963584256t^{18}\\
&- 836031346326456238080t^{17} - 4380301932747977724608t^{16}\\
&- 1946112359477941370880t^{15} - 6046571210673199090432t^{14}\\
&- 1946112359477941370880t^{13} - 4380301932747980630144t^{12}\\
&- 836031346326456238080t^{11} - 950648044803960622685t^{10}\\
&- 2511311297963171512320t^{9} - 1370094094634002837216t^{8}\\
&- 1687574778619436728320t^{7} - 4289869168445257120094t^{6}\\
&+ 1933353098783647334400t^{5} - 2728769321208895456412t^{4}\\
&+ 810690154173195878400t^{3} - 210170240274498844621t^{2}\\
&+ 5192289730544271028.
\end{aligned}
$$

$$
\begin{aligned}
M_2(t) &= 211t^{10} - 551t^{8} + 1158t^{6} - 1158t^{4} + 551t^{2} - 211.\\
N_2(t) =&211t^{20} + 504t^{18} + 513t^{16} + 1232t^{14} + 1886t^{12} - 1886t^{8} - 1232t^{6} - 513t^{4}\\
&- 504t^{2} - 211.
\end{aligned}
$$

$$
\begin{aligned}
M(t) =&\Big(211t^{8} - 340t^{6} + 818t^{4} - 340t^{2} + 211\Big)^{2}\\
&\times \Big(306768716292446258373807219716099333925072520491428741 12t^{66}\\
&+ 53695913464341334517506106987677682656870 40t^{65}\\
&- 33283039894015533115478292402746232502108683008086324019 20t^{64}
\end{aligned}
$$

$$
\begin{aligned}
&+ 376016615324939168143311781438765967659416202357160689008 64t^{63}\\
&+ 777354965623947277501198048139694037055794098811606925312t^{62}\\
&- 1335399741081820531706641405430829656028318416508891846148096t^{61}\\
&+ 1072318094264336851486955856849615367304107125312812817684 8896t^{60}\\
&- 5317730213494594986267787539679958730633284279764337136395 8784t^{59}\\
&+ 1979358385389455373556172148417119056696504654718114820427 61216t^{58}\\
&- 5976057595362657756866032732289917474216871519262254119906 18112t^{57}\\
&+ 1525783074161600867430969784000207780338108533389732553877 880832t^{56}\\
&- 3376858688280205576142813752934988636398375449728950474295 476224t^{55}\\
&+ 6587102631237989580875362429492511063151935917814943981437 452288t^{54}\\
&- 1145596053046038242064391265740922354724030419110835432115 6145152t^{53}\\
&+ 1789764316105298154040965351987586520137743825054944016978 8833792t^{52}\\
&- 2524527751278355353677086353916711863321996640694301227014 2128128t^{51}\\
&+ 3225449944369885117442058600892546885566202492556140766017 0936320t^{50}\\
&- 3741767872729447968594354495839340696337822653878907335610 8857344t^{49}\\
&+ 3959008580464834558645049410429357653679728642133335891578 8724224t^{48}\\
&- 3868320154951593369376435583964542759098719366839288241628 9579008t^{47}\\
&+ 3608716053484432378919169296539122317319817909574727506744 1673216t^{46}\\
&- 3449994613201898687595496660097758185086879051546807605786 7845632t^{45}\\
&+ 3699635472118710053515593293506944616521484656117545776228 3550464t^{44}\\
&- 4564115478916566549964772789927689549228353106587246721185 4020608t^{43}\\
&+ 6021117600755258026245306648394859257338184160437680067745 1691776t^{42}\\
&- 7767862813669348856068946570862070283703324808543071539932 0633344t^{41}\\
&+ 9298439747688315134736950330341793661545427429290321623663 8158848t^{40}\\
&- 1009286835751364425453935211435411638661377182762985387220 23964672t^{39}\\
&+ 9846007997851655450295892595597380925250086617767370722659 4975424t^{38}\\
&- 8633494088807908857495486095665899457560644048769772283039 3180160t^{37}\\
&+ 6911733606367532737541030932647124973945475021882537587848 3985088t^{36}\\
&- 5337457472685392600963190418440767608781724467521724207660 5317120t^{35}\\
&+ 4482189171801462584353133730225809086815118502982628027911 4133568t^{34}\\
&- 4574111487890049491932397932633863295956588248282605516932 0419328t^{33}\\
&+ 5408213239549714869683708655972299244090721181398710907231 3855808t^{32}\\
&- 6465338627169093565718196864963366427872044776753943555250 9480448t^{31}\\
&+ 7168792171707415152587894117076803760605058361527778382325 6895980t^{30}\\
&- 7143853646449728164762982022266805721136426609380640911909 0343936t^{29}
\end{aligned}
$$

$$
\begin{aligned}
&+ 634551373609715076090763467694106025687201361694272497196386573 00t^{28}\\
&- 50147607966744507918204958175674079859234312919069319424575012864t^{27}\\
&+ 35208665212721782248514721807635270183515823060006914310784332400t^{26}\\
&- 21882945547657565562905129327955436178288985354211105136028876800t^{25}\\
&+ 11954092927965701609277582135070077780172869692368653334094663651t^{24}\\
&- 5670262858758178119978437161547820024142210116010929214124589056t^{23}\\
&+ 2286803652644538481185600281622000818915320224254855702671704487t^{22}\\
&- 754829976075168666057850204877299439166265785614654918931185664t^{21}\\
&+ 188258928876760239471287985747967712862372128643293067403664758t^{20}\\
&- 27482109511582887740269423190297150878290506816801802588258304t^{19}\\
&- 1457731460174043377674274933026952892397833263471456808659826t^{18}\\
&+ 1637629127692112544148906971787044581645163705608508158705664t^{17}\\
&- 244144081343365225181759489703971345903109218359380820577777t^{16}\\
&- 2916813939231717794400093422738384814633769798807419009105 92t^{15}\\
&+ 1410769823812540805884803220594796757880218652456723173978 71t^{14}\\
&- 358166164236905289258762721647258564556244290332580573511 68t^{13}\\
&+ 329868053306063210847820910957200504973735269326390901474 3t^{12}\\
&+ 74917083015684082805934881418531610914402666328336162304 0t^{11}\\
&- 20376283817772217022316865914054552158126201082097920919 7t^{10}\\
&- 54416850940847772011552007231437190025115624301046595584t^{9}\\
&+ 3034903814398173765398618841431814555996036107928720256 2t^{8}\\
&- 1014789942075700630070917465090579799425427186589349888t^{7}\\
&- 2472541394067730453828179532344256940766023087915882826t^{6}\\
&+ 696094060873805246779586521006276650359339490575568896t^{5}\\
&- 159128256451594753085916183360407423930476725199291 93t^{4}\\
&- 115451108582182947188213033911875630042659874182318 08t^{3}\\
&+ 7910552368924030136004673284717160688142549915596 39t^{2}\\
&- 1667608633543318811186582143090759884 80\,t\\
&- 4522589232023915633658478185240030199129129090316\Big).
\end{aligned}
$$

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References

[1] A. Andronov, A. Vitt, S. Khaikin; *Theory of oscillations*, Pergamon Press, Oxford, 1966.
[2] S. Banerjee, G. Verghese; *Nonlinear phenomena in power electronics: attractors, bifurcations, chaos, and nonlinear control*, Wiley-IEEE Press, New York, 2001.

[3] F. Battelli, M. Fečkan; *Bifurcation and chaos near sliding homoclinics*, J. Differential Equations, 248 (2010), 2227-2262.
[4] M. Bernardo, C. Budd, A. R. Champneys, P. Kowalczyk; *Piecewise-smooth dynamical systems: theory and applications*, Springer, New York, 2008.
[5] M. di Bernardo, C. Budd, A. R. Champneys, P. Kowalczyk, A. B. Nordmark, G. Olivar, P. T. Piiroinen; *Bifurcations in nonsmooth dynamical systems*, SIAM Rev., 50 (2008), 629-701.
[6] A. Buică, J. Llibre; *Averaging methods for finding periodic orbits via Brouwer degree*, Bull. Sci. Math., 128 (2004), 7-22.
[7] C. Buzzi, C. Pessoa, J. Torregrosa; *Piecewise linear perturbations of a linear center*, Discrete Contin. Dyn. Syst., 33 (2013), 3915-3936.
[8] X. Chen, Z. Du; *Limit cycles bifurcate from centers of discontinuous quadratic systems*, Comput. Math. Appl., 59 (2010), 3836-3848.
[9] B. Coll, A. Gasull, R. Prohens; *Degerate Hopf bifurcations in discontinuous planar systems*, J. Math. Anal. Appl., 253 (2001), 671-690.
[10] B. Coll, A. Gasull, R. Prohens; *Bifurcation of limit cycles from two families of centers*, Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal., 12 (2005), 275-287.
[11] Z. Du, Y. Li, W. Zhang, *Bifurcations of periodic orbits in a class of planar Filippov systems*, Nonlinear Anal., 69(2008), 3610-3628.
[12] Z. Du, W. Zhang; *Melnikov method for homoclinic bifurcation in nonlinear impact oscillations*, Comput. Math. Appl., 50 (2005), 445-458.
[13] A. F. Filippov; *Differential equation with discontinuous right-hand sides*, Kluwer Academic, Amsterdam, 1988.
[14] A. Gasull, J. Torregrosa; *Center-focus problem for discontinuous planar differnetial equations*, Internat. J. Bifur. Chaos, 13 (2003), 1755-1865.
[15] M. Han, W. Zhang; *On Hopf bifurcations in nonsmooth planar systems*, J. Differential Equations, 248 (2010), 2399-2416.
[16] M. Kukucka; *Non-smooth dynamical systems*, Springer-Verlag, Berlin, Heidelberg, 2000.
[17] J. Karlin, W. Studden; *T-systems: with application in analysis and statistics*, Pure Appl. Math., Interscience Publishers, New York, 1966.
[18] S. Li, C. Liu; *A linear estimate of the number of limit cycles for some planar piecewise quadratic differential systems*, J. Math. Anal. Appl., 428 (2015), 1354-1367.
[19] X. Liu, M. Han; *Bifurcations of limit cycles by perturbing piecewise Hamiltonian systems*, Internat. J. Bifur. Chaos, 20 (2010), 1379-1390.
[20] F. Liang, M. Han, R. G. Romanovski; *Bifurcation of limit cycles by perturbing a piecewise linear Hamiltonian system with a homoclinic loop*, Nonlinear Anal., 75 (2012), 4355-4374.
[21] F. Liang, M. Han, X. Zhang; *Bifurcation of limit cycles from generalized homoclinic loops in planar piecewise smooth systems*, J. Differential Equations, 255 (2013), 4403-4436.
[22] J. Llibre, A. C. Mereu; *Limit cycles for discontinuous quadratic differential systems with two zones*, J. Math. Anal. Appl., 413 (2014), 763-775.
[23] J. Llibre, D. Novaes, M. Teixeira; *On the birth of limit cycles for non-smooth dynamical systems*, Bull. Sci. Math., 139 (2015), 229-244.
[24] J. Llibre, M. A. Teixeira, J. Torregrosa; *Lower bounds for the maximal number of limit cycles of discontinuous piecewise linear differential systems with a straight line of separation*, Internat. J. Bifur. Chaos, 23 (2013), 1350066 (10pages).
[25] R. M. Martins, A. C. Mereu; *Limit cycles in discontinuous classical Liéard equations*, Nonlinear Anal. Real. World Appl., 20 (2014), 67-73.
[26] J. Sanders, F. Verbulst; *Averaging methods in nonlinear dynamic systems*, Spring-Verlag, New York, 1985.
[27] L. Peng, Z. Feng; *Bifurcation of limit cycles from quartic isochronous systems*, Electron. J. Differential Equations, 2014, no. 95 (2014), 1-14.
[28] L. Peng, Z. Feng; *Limit cycles from a cubic reversible system via the third-order averaging method,* Electron. J. Differential Equations, 2015, no. 111 (2015), 1-27.
[29] D. Novaes, J. Torregrosa; *On extended Chebyshev systems with positive accuracy*, J. Math. Anal. Appl., 448 (2017), 171-186.
[30] A. Tonnelier; *On the number of limit cycles in piecewise-linear liénard systems*, Internat. J. Bifur. Chaos, 15 (2005), 1417-1422.

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