Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 111, pp. 1–24.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF POSITIVE SOLUTIONS FOR SYSTEMS OF NONLINEAR STURM-LIOUVILLE DIFFERENTIAL EQUATIONS WITH WEIGHT FUNCTIONS

JINFANG HE, LU YANG

Abstract. This article concerns the existence of component-wise positive solutions for systems of nonlinear Sturm-Liouville differential equations with weight functions, in which one nonlinear term is uniformly superlinear or uniformly sublinear, and the other is locally uniformly superlinear or locally uniformly sublinear. As applications, we consider the existence of global component-wise positive solutions to the corresponding nonlinear eigenvalue problem with respect to positive parameters. The discussion is based on the product formula of fixed point index on product cone, the fixed point index theory in cones and Leray-Schauder degree theory.

1. Introduction and statement of main results

In this article, we consider the existence of component-wise positive solutions for the following system of nonlinear Sturm-Liouville differential equations with weight functions

$$\begin{gathered}
-(p(t)u')' + q(t)u = w_1(t)f_1(t,u,v), \quad t \in (0,1),\\
-(p(t)v')' + q(t)v = w_2(t)f_2(t,u,v), \quad t \in (0,1),\\
au(0) - bp(0)u'(0) = cu(1) + dp(1)u'(1) = 0,\\
av(0) - bp(0)v'(0) = cv(1) + dp(1)v'(1) = 0,
\end{gathered} \tag{1.1}$$

and the existence of global component-wise positive solutions to the corresponding nonlinear eigenvalue problem

$$\begin{gathered}
-(p(t)u')' + q(t)u = \lambda w_1(t)f_1(t,u,v), \quad t \in (0,1),\\
-(p(t)v')' + q(t)v = \mu w_2(t)f_2(t,u,v), \quad t \in (0,1),\\
au(0) - bp(0)u'(0) = cu(1) + dp(1)u'(1) = 0,\\
av(0) - bp(0)v'(0) = cv(1) + dp(1)v'(1) = 0,
\end{gathered} \tag{1.2}$$

with respect to positive parameters λ and μ, here functions p, q, w_i, f_i $(i = 1, 2)$ and constants a, b, c, d satisfy the following conditions:

(H1) $p \in C^1([0,1], \mathbb{R}_0^+), q \in C([0,1], \mathbb{R}^+)$ and $\{w_1, w_2\} \subset C([0,1], \mathbb{R}^+)\backslash\{0\}$, here $\mathbb{R}_0^+ = (0,\infty)$, and $\mathbb{R}^+ = [0,\infty)$;

2010 *Mathematics Subject Classification.* 34B18, 47H11, 47N20.
Key words and phrases. Positive solution; nonlinear eigenvalue problem; fixed point; strict lower and upper solutions; Leray-Schauder degree.

Submitted March 24, 2019. Published September 26, 2019.

(H2) $a, b, c, d \geqslant 0$ and $(a+b)(c+d) > 0$, in addition, $q(t) \not\equiv 0$ if $a = c = 0$;
(H3) $\{f_1, f_2\} \subset C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$.

The elliptic boundary value problem arises in many areas of applied mathematics and physics, and only its positive solution is significant in practice (see [1, 2, 4, 14, 19, 22, 29]). First, we give the definitions of positive solutions for systems of nonlinear Sturm-Liouville differential equations.

Definition 1.1. Let $C^+[0,1] = \{u \in C[0,1] | \, u(t) \geqslant 0, \ \forall t \in [0,1]\}$. We say that (u, v) is one positive solution to system (1.1) (resp. (1.2)), if $(u, v) \in C^+[0,1] \times C^+[0,1] \backslash \{(0,0)\}$ satisfies (1.1) (resp. (1.2)); we say that (u, v) is one component-wise positive solution to system (1.1) (resp. (1.2)), if $(u, v) \in [C^+[0,1] \backslash \{0\}] \times [C^+[0,1] \backslash \{0\}]$ satisfies (1.1) (resp. (1.2)).

In recent years, the study of positive solutions for ordinary elliptic systems and the study of positive radial solutions for elliptic systems in annular domains have received considerable attention, see [6, 13, 15, 16, 25, 26, 31, 32, 35] and the references therein. These references discussed mainly (1.1) or (1.2) for the special case $p(t) = w_1(t) = w_2(t) = 1$ and $q(t) = b = d = 0$, and established some interesting results by introducing some new types of sublinear or superlinear conditions except for the classical sublinear or superlinear conditions (i.e., $\lim_{|(u,v)| \to 0} \frac{f(u,v)}{u+v} = \infty$ and $\lim_{|(u,v)| \to \infty} \frac{f(u,v)}{u+v} = 0$, or $\lim_{|(u,v)| \to 0} \frac{f(u,v)}{u+v} = 0$ and $\lim_{|(u,v)| \to \infty} \frac{f(u,v)}{u+v} = \infty$) and applying the fixed point theorems of cone expansion/compression type, the lower and upper solutions method and the fixed point index theory in cones, and especially extended the relevant results on the scalar second-order ordinary differential equation. For instance, Ubilla and co-authors [31, 32] have developed new notions of local superlinearity and superlinearity, where they have considered the existence, nonexistence and multiplicity of positive solutions for elliptic systems and ordinary elliptic systems involving parameters, and obtained some crucial results by the fixed point theorem of cone expansion/compression type, the lower and upper solutions method and topological degree theory. Subsequently, Cheng [6] introduced two classes of nonlinear terms (i.e. uniformly superlinear and uniformly sublinear terms) and obtained some results on the existence of component-wise positive solutions under the so-called uniformly superlinear and uniformly sublinear conditions by use of the fixed point index formula on product cone and fixed point index theory in cones.

Recently, there have been also many extensive attentions (see [3, 5, 8, 17, 18, 23, 24, 27, 28, 30, 33, 34] and references therein) for the Sturm-Liouville boundary value problem

$$\begin{gathered} -(p(t)u')' + q(t)u = f(t, u), \quad t \in (0,1), \\ au(0) - bp(0)u'(0) = cu(1) + dp(1)u'(1) = 0. \end{gathered} \tag{1.3}$$

Usually, the nonlinearity f has some conditions involving the limiting behavior of $f(t,u)/u$ at zero and at infinity, for example, the classical sublinear condition (i.e., $\lim_{u \to 0} f(t,u)/u = \infty$ and $\lim_{u \to \infty} f(t,u)/u = 0$) or the classical superlinear condition (i.e., $\lim_{u \to 0} f(t,u)/u = 0$ and $\lim_{u \to \infty} f(t,u)/u = \infty$). In addition, Li [28] and Sun and Zhang [33, 34] considered (1.3) under the general sublinear condition (i.e. $\lim_{u \to 0} f(t,u)/u > \delta_1 > \lim_{u \to \infty} f(t,u)/u$) or general superlinear condition (i.e. $\lim_{u \to 0} f(t,u)/u < \delta_1 < \lim_{u \to \infty} f(t,u)/u$) involving the principal eigenvalue δ_1 of Sturm-Liouville operator, in some sense their conditions are optimal. As to the classical sub-superlinear case (i.e. $\lim_{u \to 0} f(t,u)/u = \infty$ and $\lim_{u \to \infty} f(t,u)/u = \infty$), in [8] Cheng and Dai have discussed the following Sturm-Liouville nonlinear eigenvalue problem

$$\begin{gathered} -(p(t)u')' + q(t)u = \lambda f(t,u), \quad t \in (0,1), \\ au(0) - bp(0)u'(0) = cu(1) + dp(1)u'(1) = 0, \end{gathered} \tag{1.4}$$

and showed that there exists a positive real number λ_* such that problem (1.4) has at least two solutions, at least one or no positive solutions according to $\lambda \in (0,\lambda_*)$, $\lambda = \lambda_*$ or $\lambda > \lambda_*$.

In this paper, we intend to improve and extend the relevant results in [6, 8, 28] to systems of nonlinear Sturm-Liouville differential equations with weight functions (i.e. systems (1.1) and (1.2)). In concrete, we try to weaken the uniformly superlinear or uniformly sublinear conditions used in [6] to some extent such that our results include the relevant results in [6, 8, 28] at the same time, in particular, our results are also new even for the scalar Sturm-Liouville differential equation. For convenience and simplicity, denote by $\lambda_{i,1}$ the first eigenvalue of linear eigenvalue problem with a weight function

$$\begin{gathered} -(p(t)u')' + q(t)u = \lambda w_i(t)u, \quad t \in (0,1), \\ au(0) - bp(0)u'(0) = cu(1) + dp(1)u'(1) = 0. \end{gathered} \tag{1.5}$$

It is well-known that $\lambda_{i,1} > 0$ is a simple eigenvalue with a positive eigenfunction $e_{i,1}(t)$ such that $e_{i,1}(t) > 0$ for $t \in (0,1)$ and $\|e_{i,1}\| = \max_{t\in[0,1]} |e_{i,1}(t)| = 1$ $(i = 1,2)$.

Our main results are the following. First, we present some results (see Theorems 1.2–1.6) about the existence of component-wise positive solutions to system (1.1). In the case that the nonlinear terms are so called "super-sublinear", we have the following result.

Theorem 1.2. *Assume that f_1 and f_2 respectively satisfy the following hypotheses:*

(H4)
$$\limsup_{u\to 0^+} \max_{t\in[0,1]} \frac{f_1(t,u,v)}{u} < \lambda_{1,1} < \liminf_{u\to +\infty} \min_{t\in[0,1]} \frac{f_1(t,u,v)}{u}$$
uniformly with respect to $v \in \mathbb{R}^+$;

(H5)
$$\liminf_{v\to 0^+} \min_{t\in[0,1]} \frac{f_2(t,u,v)}{v} > \lambda_{2,1} > \limsup_{v\to +\infty} \max_{t\in[0,1]} \frac{f_2(t,u,v)}{v}$$
uniformly with respect to $u \in [0,M]$,

where $M \in \mathbb{R}^+$ is arbitrary. Then system (1.1) *has at least one component-wise positive solution.*

Remark 1.3. In particular, if $w_1(t) = w_2(t) = 1$, f_1 and f_2 is respectively independent of v and u, then Theorem 1.2 implies the results on nonlinear Sturm-Liouville boundary value problem (1.3) (see [28]). In addition, if $p(t) = w_1(t) = w_2(t) = 1$ and $q(t) = b = d = 0$, then Theorem 1.2 implies the main result in [6] and the locally uniformly sublinear condition (H5) is weaker than the uniformly sublinear condition used in [6].

The "sub-superlinear" case is different from the "super-sublinear" case, since the uniformly sublinear term f_1 need to be controlled at the infinity for a priori estimates of solution component u. On this purpose, we introduce condition $(H7)$ and get the following result.

Theorem 1.4. *Assume that f_1 and f_2 satisfy the following conditions:*

(H6)
$$\liminf_{u\to 0^+}\min_{t\in[0,1]}\frac{f_1(t,u,v)}{u} > \lambda_{1,1} > \limsup_{u\to+\infty}\max_{t\in[0,1]}\frac{f_1(t,u,v)}{u}$$
uniformly with respect to $v\in\mathbb{R}^+$;

(H7) $\limsup_{v\to+\infty}\max_{t\in[0,1]} f_1(t,u,v) = g(u)$ *uniformly with respect to $u\in[0,M]$, here g is locally bounded;*

(H8)
$$\limsup_{v\to 0^+}\max_{t\in[0,1]} f_2(t,u,v)/v < \lambda_{2,1} < \liminf_{v\to+\infty}\min_{t\in[0,1]}\frac{f_2(t,u,v}{v}$$
uniformly with respect to $u\in[0,M]$,

where $M\in\mathbb{R}^+$ is arbitrary. Then system (1.1) *has at least one component-wise positive solution.*

From the proofs of Theorems 1.2 and 1.4, it is not difficult to obtain the following results on the "super-superlinear" and "sub-sublinear" cases.

Theorem 1.5. *Assume* (H4) *and* (H8) *are satisfied. Then system* (1.1) *has at least one component-wise positive solution.*

Theorem 1.6. *Assume* (H5)–(H7) *are satisfied. Then system* (1.1) *has at least one component-wise positive solution.*

As a direct conclusion of *Theorems* 1.2, 1.4, 1.5 and 1.6, we have the following corollary.

Corollary 1.7. *Assume that f_1 satisfies one of the following conditions:*

(H4*) $\limsup_{u\to 0^+}\max_{t\in[0,1]} f_1(t,u,v)/u = 0$ *and*
$$\liminf_{u\to+\infty}\min_{t\in[0,1]}\frac{f_1(t,u,v)}{u} = +\infty,$$
uniformly with respect to $v\in\mathbb{R}^+$;

(H6*) $\liminf_{u\to 0^+}\min_{t\in[0,1]} f_1(t,u,v)/u = +\infty$ *and*
$$\limsup_{u\to+\infty}\max_{t\in[0,1]}\frac{f_1(t,u,v)}{u} = 0,$$
uniformly with respect to $v\in\mathbb{R}^+$, and condition (H7) *is valid;*

and assume that f_2 satisfies one of the following conditions:

(H5*) $\liminf_{v\to 0^+}\min_{t\in[0,1]} f_2(t,u,v)/v = +\infty$ *and*
$$\limsup_{v\to+\infty}\max_{t\in[0,1]}\frac{f_2(t,u,v)}{v} = 0$$
uniformly with respect to $u\in[0,M]$;

(H8*) $\limsup_{v\to 0^+}\max_{t\in[0,1]} f_2(t,u,v)/v = 0$ *and*
$$\liminf_{v\to+\infty}\min_{t\in[0,1]}\frac{f_2(t,u,v)}{v} = +\infty$$
uniformly with respect to $u\in[0,M]$,

where $M\in\mathbb{R}^+$ is arbitrary. Then problem (1.2) *has at least one component-wise positive solution for all $(\lambda,\mu)\in\mathbb{R}_0^+\times\mathbb{R}_0^+$.*

Secondly, we give a result (see Theorem 1.10) about the multiplicity of positive solutions to system (1.1). On this purpose, we need a concept of a strict upper solution. Let us consider the following more general system of nonlinear Sturm-Liouville differential equations

$$\begin{gathered}
-(p(t)u')' + q(t)u = \mathcal{F}(t,u,v), \quad t \in (0,1),\\
-(p(t)v')' + q(t)v = \mathcal{G}(t,u,v), \quad t \in (0,1),\\
au(0) - bp(0)u'(0) = cu(1) + dp(1)u'(1) = 0,\\
av(0) - bp(0)v'(0) = cv(1) + dp(1)v'(1) = 0,
\end{gathered} \tag{1.6}$$

where $\mathcal{F}, \mathcal{G} : \mathscr{D} \to \mathbb{R}$ is a continuous function and $\mathscr{D} \subset [0,1] \times \mathbb{R}^2$.

Definition 1.8. For $\alpha_u, \alpha_v \in C^2(0,1) \cap C^1[0,1]$, (α_u, α_v) is said to be a lower (strict lower) solution of (1.6) if $(t, \alpha_u(t), \alpha_v(t)) \in \mathscr{D}$ for all $t \in (0,1)$ and

$$\begin{gathered}
-(p(t)\alpha_u')' + q(t)\alpha_u - \mathcal{F}(t,\alpha_u,\alpha_v) \leqslant 0 \ (<0), \quad t \in (0,1),\\
-(p(t)\alpha_v')' + q(t)\alpha_v - \mathcal{G}(t,\alpha_u,\alpha_v) \leqslant 0 \ (<0), \quad t \in (0,1),\\
a\alpha_u(0) - bp(0)\alpha_u'(0) \leqslant 0 \ (<0),\ c\alpha_u(1) + dp(1)\alpha_u'(1) \leqslant 0 \ (<0),\\
a\alpha_v(0) - bp(0)\alpha_v'(0) \leqslant 0 \ (<0),\ c\alpha_v(1) + dp(1)\alpha_v'(1) \leqslant 0 \ (<0).
\end{gathered}$$

An upper (strict upper) solution $(\beta_u, \beta_v) \in (C^2(0,1) \cap C^1[0,1]) \times (C^2(0,1) \cap C^1[0,1])$ can also be defined if it satisfies the reverse of the above inequalities.

Definition 1.9. For a function $\mathcal{F} : \mathscr{D} \to \mathbb{R}$, $\mathcal{F}(t,u,v)$ is said to be quasi-monotone nondecreasing with respect to v (or u) if for fixed t,

$$\begin{gathered}
\mathcal{F}(t,u,v_1) \leqslant \mathcal{F}(t,u,v_2) \quad \text{as } v_1 \leqslant v_2,\\
(\text{or } \mathcal{F}(t,u_1,v) \leqslant \mathcal{F}(t,u_2,v) \quad \text{as } u_1 \leqslant u_2).
\end{gathered}$$

Theorem 1.10. *Suppose that $\{w_1, w_2\} \subset C([0,1], \mathbb{R}_0^+)$, and that f_1 and f_2 satisfy the following assumptions:*

- (H9) *f_1 (resp. f_2) is quasi-monotone nondecreasing with respect to v (resp. u);*
- (H10) $\liminf_{u\to 0} \min_{t\in[0,1]} f_1(t,u,0)/u > \lambda_{1,1}$ *and*
$$\liminf_{v\to 0} \min_{t\in[0,1]} \frac{f_2(t,0,v)}{v} > \lambda_{2,1};$$
- (H11) $\liminf_{u\to \infty} \min_{t\in[0,1]} f_1(t,u,0)/u > \lambda_{1,1}$ *and*
$$\liminf_{v\to \infty} \min_{t\in[0,1]} \frac{f_2(t,0,v)}{v} > \lambda_{2,1}.$$

In addition, if system (1.1) *has a strict upper solution (β_u, β_v), then system* (1.1) *has at least two positive solutions.*

Finally, we provide the following result about the existence of a global component-wise positive solutions to problem (1.2).

Theorem 1.11. *Assume that $\{w_1, w_2\} \subset C([0,1], \mathbb{R}_0^+)$, and that f_1 and f_2 satisfy (H9) and the following conditions:*

- (H10*) $f_i \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}_0^+)$ *$(i = 1,2)$;*
- (H11*) $\liminf_{u\to \infty} \min_{t\in[0,1]} f_1(t,u,0)/u = \infty$ *and*
$$\liminf_{v\to \infty} \min_{t\in[0,1]} \frac{f_2(t,0,v)}{v} = \infty.$$

Then there exist a simple arc $\Gamma_0 \subset \mathbb{R}_0^+ \times \mathbb{R}_0^+$ excluding both end points $(\lambda_*, 0)$ and $(0, \mu_*)$, which separates $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ into two disjoint subsets $\mathscr{O}_1$ and $\mathscr{O}_2$ such that problem (1.2) has at least one or at least two component-wise positive solutions according to (λ, μ) in Γ_0 or $\mathscr{O}_1$ respectively, and has no solutions for $(\lambda, \mu) \in \mathscr{O}_2$.

In plane, a curve is said to be a simple arc if it is homeomorphic with a straight line segment (see [36] for more details).

Remark 1.12. It is remarkable that our conditions (H9) and (H11*) are weaker than the nondecreasing conditions and the classical superlinear conditions used in [16, 26]. In particular, if $\lambda = \mu$, $w_1(t) = w_2(t) = 1$, $f_1(t, u, v) = f(t, u)$ and $f_2(t, u, v) = f(t, v)$, then Theorem 1.11 covers the result on Sturm-Liouville nonlinear eigenvalue problem (1.4) (see [8]).

On the existence results of component-wise positive solutions for system (1.1), the difficulty in proving is how to construct a proper cone and open sets in the cone. In order that the features of weight functions and nonlinear terms can be exploited better, we choose a product cone $K \times K$ with $K \subset C^+[0, 1]$ being a proper sub-cone and construct a proper open set $O_1 \times O_2 \subset [K\backslash\{0\}] \times [K\backslash\{0\}]$, and then seek solutions to system (1.1) in $O_1 \times O_2$, here the main idea derives from [10, 13, 7, 12]. Motivated by the idea in [9, 11], we analyze the structure and properties of the parameters set $\mathscr{S}$ (defined in Section 4) and obtain the existence and nonexistence of component-wise positive solutions to problem (1.2). In addition, for dealing with the multiplicity of component-wise positive solutions to problem (1.2), we consider a general system of Sturm-Liouville differential equations with quasi-monotone nondecreasing nonlinear terms and establish the relation between Leray-Schauder degree and a pair of strict lower and upper solutions for this system (see Theorem 2.7 in Section 2), here the idea involved partly comes from [2, 20]. Based on Theorem 2.7 and the construction of a pair of strict lower and upper solutions, we can show the multiplicity of positive solutions for systems (1.1) and (1.2).

This article is organized as follows: in Section 2, we state some preliminaries. In Section 3, we prove our main results on problem (1.1). In Section 4, as applications, we prove the global existence of component-wise positive solutions to the nonlinear eigenvalue problem (1.2).

2. Preliminaries

In this section, we shall establish some functional analytic framework and change problems (1.1) and (1.2) into the equivalent fixed point problems. At the same time, we will give some useful preliminary results.

2.1. **Equivalent fixed point problems.**

First, for a given $h \in C[0, 1]$, we recall the construction and properties of Green's function of the linear boundary value problem

$$\begin{gathered} -(p(t)u')' + q(t)u = h(t), \qquad t \in (0, 1), \\ au(0) - bp(0)u'(0) = cu(1) + dp(1)u'(1) = 0. \end{gathered} \tag{2.1}$$

For this matter, let $\varphi(t) \in C^2[0,1]$ be the unique solution of the linear boundary value problem

$$\begin{gathered}-(p(t)\varphi')' + q(t)\varphi = 0, \quad t \in (0,1),\\ a\varphi(0) - bp(0)\varphi'(0) = 0,\\ c\varphi(1) + dp(1)\varphi'(1) = 1,\end{gathered} \tag{2.2}$$

and $\psi(t) \in C^2[0,1]$ be the unique solution of the linear boundary value problem

$$\begin{gathered}-(p(t)\psi')' + q(t)\psi = 0, \quad t \in (0,1),\\ a\psi(0) - bp(0)\psi'(0) = 1,\\ c\psi(1) + dp(1)\psi'(1) = 0.\end{gathered} \tag{2.3}$$

Then, by the maximum principles $\varphi, \psi \geqslant 0$, moreover $\varphi(t), \psi(t) > 0$ for all $t \in (0,1)$.

Lemma 2.1 ([28]). *$\varphi'(t) > 0$, $\forall t \in (0,1]$, $\psi'(t) < 0$, $\forall t \in [0,1)$, and*

$$p(t)(\varphi'(t)\psi(t) - \varphi(t)\psi'(t)) \equiv \rho > 0, \quad t \in [0,1], \tag{2.4}$$

where ρ is a positive constant.

Let $h \in C[0,1]$, then problem (2.1) has a unique solution

$$u(t) = \int_0^1 G(t,s)h(s)\,\mathrm{d}s, \quad t \in [0,1], \tag{2.5}$$

where the Green's function $G : [0,1] \times [0,1] \to \mathbb{R}^+$ is

$$G(t,s) = \begin{cases} \frac{1}{\rho}\varphi(t)\psi(s), & 0 \leqslant t \leqslant s \leqslant 1,\\ \frac{1}{\rho}\varphi(s)\psi(t), & 0 \leqslant s \leqslant t \leqslant 1,\end{cases} \tag{2.6}$$

and $G(t,s)$ has the following properties:

(i) $G(t,s) = G(s,t)$ for all $t,s \in [0,1]$;
(ii) $G(t,s) > 0$ for all $t,s \in (0,1)$;
(iii) $G(t,s) \leqslant G(s,s)$ for all $t,s \in [0,1]$;
(iv) $G(t,s) \geqslant \delta G(t,t)G(s,s)$ for all $t,s \in [0,1]$, where $\delta > 0$ is a constant.

It is well known that $C[0,1]$ is a Banach space with the maximum norm $\|u\| = \max_{t\in[0,1]} |u(t)|$, and $C^+[0,1] := \{u \in C[0,1] : u(t) \geqslant 0,\ \forall t \in [0,1]\}$ is a total cone of $C[0,1]$.

Lemma 2.2 ([28]). *Let $h \in C^+[0,1]$, then the solution of (2.1) satisfies*

$$u(t) \geqslant \delta G(t,t)\|u\|, \quad \forall t \in [0,1].$$

Noticing that $\{w_1, w_2\} \subset C([0,1], \mathbb{R}^+)\backslash\{0\}$, there exists $\{t_1, t_2\} \subset (0,1)$ such that $w_1(t_1) > 0$ and $w_2(t_2) > 0$. Choose $\delta_0 \in (0, 1/2)$ such that $\{t_1, t_2\} \subset (\delta_0, 1-\delta_0)$. Now we construct a sub-cone K of $C^+[0,1]$ as follows

$$K = \{u \in C^+[0,1] : u(t) \geqslant \sigma\|u\|, \forall t \in [\delta_0, 1-\delta_0]\},$$

here $\sigma = \delta\varphi(\delta_0)\psi(1-\delta_0)/\rho > 0$. For convenience, we also introduce some subsets in cone K,

$$K_r = \{u \in K : \|u\| < r\}, \quad \partial K_r = \{u \in K : \|u\| = r\},$$
$$\overline{K_r} = \{u \in K : \|u\| \leqslant r\}, \quad \text{for } r > 0.$$

For $\tau \in [0,1]$, define the mappings $T_{\tau,1}$, $T_{\tau,2} : C^+[0,1] \times C^+[0,1] \to C^+[0,1]$ and T_τ, $T^\tau_{\lambda,\mu} : C^+[0,1] \times C^+[0,1] \to C^+[0,1] \times C^+[0,1]$ by

$$\begin{gathered}
T_{\tau,1}(u,v)(t) = \int_0^1 G(t,s)w_1(s)[\tau f_1(s,u(s),v(s)) + (1-\tau)f_1(s,u(s),0)]\,\mathrm{d}s,\\
T_{\tau,2}(u,v)(t) = \int_0^1 G(t,s)w_2(s)[\tau f_2(s,u(s),v(s)) + (1-\tau)f_2(s,0,v(s))]\,\mathrm{d}s,\\
T_\tau(u,v)(t) = (T_{\tau,1}(u,v)(t), T_{\tau,2}(u,v)(t)),\\
T^\tau_{\lambda,\mu}(u,v)(t) = (\lambda T_{\tau,1}(u,v)(t), \mu T_{\tau,2}(u,v)(t)) \equiv (A^\tau_\lambda(u,v)(t), B^\tau_\mu(u,v)(t)).
\end{gathered} \tag{2.7}$$

It is clear that the existence of component-wise positive solutions of system (1.1) (resp. (1.2)) is equivalent to the existence of fixed points of T_1 (resp. $T^1_{\lambda,\mu}$) in $[K\backslash\{0\}] \times [K\backslash\{0\}]$.

Lemma 2.3. *$T_\tau(K \times K) \subset K \times K$ and $T_\tau : K \times K \to K \times K$ is completely continuous.*

Proof. For $(u,v) \in K \times K$, we show that $T_\tau(u,v) \in K \times K$, i.e., $T_{\tau,1}(u,v) \in K$ and $T_{\tau,2}(u,v) \in K$. Let

$$h(t) = w_1(t)[\tau f_1(t,u(t),v(t)) + (1-\tau)f_1(t,u(t),0)],$$

then $T_{\tau,1}(u,v)(t)$ is the solution of (2.1). By (2.6), Lemmas 2.1, 2.2, we have

$$\begin{aligned}
T_{\tau,1}(u,v)(t) &\geqslant \delta G(t,t)\|T_{\tau,1}(u,v)\|\\
&\geqslant \frac{\delta}{\rho}\varphi(\delta_0)\psi(1-\delta_0)\|T_{\tau,1}(u,v)\| = \sigma\|T_{\tau,1}(u,v)\|, \quad t \in [\delta_0, 1-\delta_0].
\end{aligned}$$

Similarly,

$$T_{\tau,2}(u,v)(t) \geqslant \sigma\|T_{\tau,2}(u,v)\|, \quad t \in [\delta_0, 1-\delta_0].$$

Consequently $T_{\tau,1}(u,v) \in K$ and $T_{\tau,2}(u,v) \in K$, thus $T_\tau(K \times K) \subset K \times K$. By Ascoli-Arzelà theorem, we known that $T_\tau : K \times K \to K \times K$ is completely continuous. □

Remark 2.4. Let $T(\tau,u,v)(t) = T_\tau(u,v)(t)$ and $T_{\lambda,\mu}(\tau,u,v)(t) = T^\tau_{\lambda,\mu}(u,v)(t)$, then by the Ascoli-Arzelà theorem, T, $T_{\lambda,\mu} : [0,1] \times K \times K \to K \times K$ are both compactly continuous.

2.2. **Fixed point index and its properties.** Now we recall some concepts and results about the fixed point index (see [9, 14, 22]), which will be used in the proofs of our theorems. Let X be a Banach space and let $P \subset X$ be a closed convex cone in X. Assume that W is a bounded open subset of X with boundary ∂W, and let $A : P \cap \overline{W} \to P$ be a completely continuous operator. If $Au \neq u$ for $u \in P \cap \partial W$, then the fixed point index $i(A, P \cap W, P)$ is defined. One important fact is that if $i(A, P \cap W, P) \neq 0$ then A has a fixed point in $P \cap W$. The following results are useful in our proofs.

Lemma 2.5 ([22, 37]). *Let E be a Banach space and let $P \subset E$ be a closed convex cone in E. For $r > 0$, denote $P_r = \{u \in P : \|u\| < r\}$, $\partial P_r = \{u \in P : \|u\| = r\}$. Let $A : P \to P$ be completely continuous. Then the following conclusions are valid:*

(i) *if $\mu Au \neq u$ for every $u \in \partial P_r$ and $u \in (0,1]$, then $i(A, P_r, P) = 1$;*
(ii) *If mapping A satisfies the following two conditions:*
  (a) *$\inf_{u \in \partial P_r} \|Au\| > 0$;*

(b) $\mu Au \neq u$ *for every* $u \in \partial P_r$ *and* $\mu \geq 1$,

then $i(A, P_r, P) = 0$.

Lemma 2.6 ([9]). *Let* X *be a real Banach space,* $P_i \subset X$ *be a closed convex cone,* W_i *be a bounded open subset of* X *with boundary* ∂W_i $(i = 1, 2)$ *and* $P = P_1 \times P_2$, $W = W_1 \times W_2$. *Assume that* $T : P \cap \overline{W} \to P$ *is completely continuous and that there exist compactly continuous mappings* $A_i : P_i \cap \overline{W_i} \to P_i$ *and* $H : (P \cap \overline{W}) \times [0, 1] \to P$ *such that*

(a) $H(\cdot, 1) = T, H(\cdot, 0) = A$, *where* $A(u, v) := (A_1 u, A_2 v)$ *for all* $(u, v) \in P \cap \overline{W}$;

(b) $A_i u_i \neq u_i$ *for all* $u_i \in P_i \cap \partial W_i$;

(c) $H(w, \tau) \neq w$ *for all* $(w, \tau) \in (P \cap \partial W) \times (0, 1]$.

Then

$$i(T, P \cap W, P) = i(A_1, P_1 \cap W_1, P_1) \cdot i(A_2, P_2 \cap W_2, P_2).$$

2.3. **Sub and super solutions and Leray-Schauder degree.** Now we consider the fixed point operator associated with (1.6), i.e., the compact operator $T : C[0, 1] \times C[0, 1] \to C[0, 1] \times C[0, 1]$ defined by $T(\eta, \xi) := (u, v)$ if

$$\begin{gathered} -(p(t)u')' + q(t)u = \mathcal{F}(t, \eta, \xi), \quad t \in (0, 1), \\ -(p(t)v')' + q(t)v = \mathcal{G}(t, \eta, \xi), \quad t \in (0, 1), \\ au(0) - bp(0)u'(0) = cu(1) + dp(1)u'(1) = 0, \\ av(0) - bp(0)v'(0) = cv(1) + dp(1)v'(1) = 0. \end{gathered} \tag{2.8}$$

In fact, by (2.5), the operator T above can be expressed as

$$T(\eta, \xi)(t) = \Big(\int_0^1 G(t, s)\mathcal{F}(s, \eta(s), \xi(s))\, \mathrm{d}s, \int_0^1 G(t, s)\mathcal{G}(s, \eta(s), \xi(s))\, \mathrm{d}s \Big). \tag{2.9}$$

The following theorem provides the relation between Leray-Schauder degree of compactly continuous field $id - T$ and a pair of strict lower and upper solutions for (1.6).

Theorem 2.7. *Let* (α_u, α_v) *and* (β_u, β_v) *be a strict lower solution and a strict upper solution of* (1.6) *respectively such that*

(i) $\alpha_u(t) < \beta_u(t)$, *and* $\alpha_v(t) < \beta_v(t)$, *for all* $t \in [0, 1]$;

(ii) $\mathscr{E}_\alpha^\beta := \{(t, u, v) \in [0, 1] \times \mathbb{R}^2 : \alpha_u(t) < u < \beta_u(t)$ *and* $\alpha_v(t) < v < \beta_v(t)\} \subset \mathscr{D}$;

(iii) $\mathcal{F}(t, u, v)$ *is quasi-monotone nondecreasing with respect to* v *and* $\mathcal{G}(t, u, v)$ *is quasi-monotone nondecreasing with respect to* u.

Denote $\Omega = \{(u, v) \in C[0, 1] \times C[0, 1] : \alpha_u < u < \beta_u$ *and* $\alpha_v < v < \beta_v$ on $[0, 1]\}$, *then*

$$\deg(id - T, \Omega, (\theta, \theta)) = 1,$$

in particular, problem (1.6) *has at least one solution* (u, v) *such that*

$$\alpha_u(t) < u(t) < \beta_u(t), \quad \text{and} \quad \alpha_v(t) < v(t) < \beta_v(t), \quad \text{for all } t \in [0, 1].$$

Proof. Define the modified functions

$$\mathcal{F}^*(t, u, v) = \mathcal{F}(t, p_1(t, u, v), p_2(t, u, v)), \quad \mathcal{G}^*(t, u, v) = \mathcal{G}(t, p_1(t, u, v), p_2(t, u, v)),$$

where p_i are given by

$$p_1(t, u, v) = \max\{\alpha_u(t), \min\{u, \beta_u(t)\}\}, \quad p_2(t, u, v) = \max\{\alpha_v(t), \min\{v, \beta_v(t)\}\}.$$

Then $\mathcal{F}^*, \mathcal{G}^* : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous and bounded.

Let us consider a modified problem

$$\begin{gathered} -(p(t)u')' + q(t)u = \mathcal{F}^*(t,u,v), \quad t \in (0,1), \\ -(p(t)v')' + q(t)v = \mathcal{G}^*(t,u,v), \quad t \in (0,1), \\ au(0) - bp(0)u'(0) = cu(1) + dp(1)u'(1) = 0, \\ av(0) - bp(0)v'(0) = cv(1) + dp(1)v'(1) = 0, \end{gathered} \tag{2.10}$$

and corresponding compact operator $T^* : C[0,1]\times C[0,1] \to C[0,1]\times C[0,1]; (\eta,\xi) \mapsto (u,v)$ if

$$\begin{gathered} -(p(t)u')' + q(t)u = \mathcal{F}^*(t,\eta,\xi), \quad t \in (0,1), \\ -(p(t)v')' + q(t)v = \mathcal{G}^*(t,\eta,\xi), \quad t \in (0,1), \\ au(0) - bp(0)u'(0) = cu(1) + dp(1)u'(1) = 0, \\ av(0) - bp(0)v'(0) = cv(1) + dp(1)v'(1) = 0. \end{gathered} \tag{2.11}$$

Now we have the following claims.

Claim 1. $T^*(\eta,\xi) = T(\eta,\xi)$ for all $(\eta,\xi) \in \Omega$. It follows from the definitions of modified functions and operators T^* and T.

Claim 2. If (u,v) is a solution of (2.10), then $(u,v) \in \Omega$. We only prove that $\alpha_u < u$ on $[0,1]$, the remainder is similar and omitted here.

First, we show that $\alpha_u \leqslant u$ on $[0,1]$. By contradiction, assume that there exists a $t_0 \in (0,1)$ such that $u(t_0) < \alpha_u(t_0)$. Then there is an interval $[t_1,t_2] \subset [0,1]$ such that one of the following cases is valid:

(i) $t_1, t_2 \in [0,1]$ and $u(t) < \alpha_u(t)$ for all $t \in (t_1,t_2)$, $u(t_i) - \alpha_u(t_i) = 0$ for $i = 1,2$;

(ii) $t_1 = 0$, $t_2 \in (0,1)$ and $u(t) < \alpha_u(t)$ for all $t \in [t_1,t_2)$, $u(t_2) - \alpha_u(t_2) = 0$;

(iii) $t_1 \in (0,1)$, $t_2 = 1$ and $u(t) < \alpha_u(t)$ for all $t \in (t_1,t_2]$, $u(t_1) - \alpha_u(t_1) = 0$;

(iv) $t_1 = 0$, $t_2 = 1$ and $u(t) < \alpha_u(t)$ for all $t \in [t_1,t_2]$.

Thus, $\alpha_u - u$ has a positive maximum on $[t_1,t_2]$. On the other hand,

$$\begin{aligned} &-(p(t)(\alpha_u(t) - u(t))')' + q(t)(\alpha_u(t) - u(t)) \\ &< \mathcal{F}(t,\alpha_u(t),\alpha_v(t)) - \mathcal{F}^*(t,u(t),v(t)) \\ &= \mathcal{F}(t,\alpha_u(t),\alpha_v(t)) - \mathcal{F}(t,\alpha_u(t),p_2(t,u(t),v(t))) \leqslant 0 \end{aligned}$$

for all $t \in (t_1,t_2)$. Hence, $\max_{t\in[t_1,t_2]}(\alpha_u - u)(t) \leqslant \max\{(\alpha_u - u)^+(t_1), (\alpha_u - u)^+(t_2)\}$ by the maximum principle (see [21]). Moreover, combining with the boundary conditions of α_u and u, the cases (ii), (iii) and (iv) above can not happen. Therefore, $\alpha_u - u \leqslant 0$ on $[t_1,t_2]$ from the case (i), which contradicts that $\alpha_u - u$ has a positive maximum on $[t_1,t_2]$.

Next, we prove that $\alpha_u < u$ on $[0,1]$. Obviously, $\alpha_u < u$ on $(0,1)$. In fact, by contradiction, assume that there exists a $t^* \in (0,1)$ such that $\alpha_u(t^*) = u(t^*)$. Then

$$\begin{gathered} u'(t^*) = \alpha_u'(t^*), \quad (\alpha_u - u)''(t^*) \leqslant 0, \\ -(p(\alpha_u - u)')'(t^*) + q(t^*)(\alpha_u - u)(t^*) = -p(t^*)(\alpha_u - u)''(t^*) \geqslant 0. \end{gathered}$$

However,

$$\begin{aligned} &-(p(\alpha_u - u)')'(t^*) + q(t^*)(\alpha_u - u)(t^*) \\ &< \ \mathcal{F}(t^*,\alpha_u(t^*),\alpha_v(t^*)) - \mathcal{F}^*(t^*,u(t^*),v(t^*)) \end{aligned}$$

$$= \mathcal{F}(t^*, \alpha_u(t^*), \alpha_v(t^*)) - \mathcal{F}(t^*, \alpha_u(t^*), p_2(t^*, u(t^*), v(t^*))) \leqslant 0,$$

which is a contradiction. Furthermore, combining the proved fact that $\alpha_u < u$ on $(0,1)$ with the boundary conditions of α_u and u, it is easy to prove that $\alpha_u(t) < u(t)$ also for $t = 0, 1$.

Claim 3. There exists an open ball

$$B_r = \{(u,v) : \max\{\max_{t\in[0,1]} |u(t)|, \max_{t\in[0,1]} |v(t)|\} < r\} \subset C[0,1] \times C[0,1],$$

such that $T^*(C[0,1] \times C[0,1]) \subset B_r$ and $\Omega \subset B_r$. It is sufficient to notice that $\mathcal{F}^*$ and $\mathcal{G}^*$ are both continuous and bounded.

Combining the claims with the excision and homotopy invariance of Leray-Schauder degree, we have

$$\begin{aligned}
\deg(\mathrm{id} - T, \Omega, (\theta,\theta)) &= \deg(\mathrm{id} - T^*, \Omega, (\theta,\theta)) \\
&= \deg(\mathrm{id} - T^*, B_r, (\theta,\theta)) \\
&= \deg(\mathrm{id}, B_r, (\theta,\theta)) = 1.
\end{aligned}$$

The proof is complete. □

3. Proofs of Theorems 1.2, 1.4 and 1.10

As for Theorems 1.2-1.4, our main idea of proofs is to choose a bounded open set $D = (K_{R_1}\backslash\overline{K_{r_1}}) \times (K_{R_2}\backslash\overline{K_{r_2}})$ in product cone $K \times K$ (here $R_j > r_j > 0$ are to be determined for $j = 1, 2$), such that the family of operators $\{T_\tau\}_{\tau\in[0,1]}$ satisfy the sufficient conditions for the homotopy invariance of fixed point index on ∂D. Furthermore, Lemmas 2.5-2.6 can be applied to T_1.

Proof of Theorem 1.2. In turn, we will determine r_1, R_1, r_2 and R_2 according to the following steps.

Step 1. From the uniformly superlinear assumption of f_1 at $u = 0$, there are $\varepsilon \in (0, \lambda_{1,1})$ and $r_1 > 0$ such that

$$\tau f_1(t,u,v) + (1-\tau) f_1(t,u,0) \leqslant (\lambda_{1,1} - \varepsilon) u, \quad \forall t \in [0,1],\ (u,v) \in [0,r_1] \times \mathbb{R}^+. \tag{3.1}$$

We claim that

$$\mu T_{\tau,1}(u,v) \neq u, \quad \forall \mu \in (0,1] \text{ and } (u,v) \in \partial K_{r_1} \times K. \tag{3.2}$$

In fact, if it were not true, then there exist $\mu_0 \in (0,1]$ and $(u_0,v_0) \in \partial K_{r_1} \times K$, such that $\mu_0 T_{\tau,1}(u_0,v_0) = u_0$, that is, (u_0,v_0) satisfies the following differential equation

$$\begin{gathered}
-(p(t)u_0')' + q(t)u_0 = \mu_0 w_1(t)[\tau f_1(t,u_0,v_0) + (1-\tau) f_1(t,u_0,0)], \\
a u_0(0) - b p(0) u_0'(0) = c u_0(1) + d p(1) u_0'(1) = 0.
\end{gathered} \tag{3.3}$$

In combination with (3.1), it follows that

$$-(p(t)u_0')' + q(t)u_0 \leqslant w_1(t)[\tau f_1(t,u_0,v_0) + (1-\tau) f_1(t,u_0,0)] \leqslant (\lambda_{1,1} - \varepsilon) w_1(t) u_0.$$

Multiplying the both sides of this inequality by $e_{1,1}(t)$ and integrating on $[0,1]$, we obtain that

$$\lambda_{1,1} \int_0^1 w_1(t) u_0(t) e_{1,1}(t)\,\mathrm{d}t \leqslant (\lambda_{1,1} - \varepsilon) \int_0^1 w_1(t) u_0(t) e_{1,1}(t)\,\mathrm{d}t.$$

Noticing that $\int_0^1 w_1(t) u_0(t) e_{1,1}(t)\,\mathrm{d}t > 0$, hence $\lambda_{1,1} \leqslant \lambda_{1,1} - \varepsilon$, which is a contradiction!

Step 2. From the uniformly superlinear hypothesis of f_1 at $u = +\infty$, there exist $\varepsilon > 0$ and $m > 0$ such that

$$\tau f_1(t, u, v) + (1 - \tau) f_1(t, u, 0) \geqslant (\lambda_{1,1} + \varepsilon) u, \tag{3.4}$$

for all $t \in [0, 1]$, $(u, v) \in [m, +\infty) \times \mathbb{R}^+$; thus

$$\tau f_1(t, u, v) + (1 - \tau) f_1(t, u, 0) \geqslant (\lambda_{1,1} + \varepsilon) u - C_{1,1}, \tag{3.5}$$

for all $t \in [0, 1]$, $(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$, where $C_{1,1} = (\lambda_{1,1} + \varepsilon) m$.

Now we can prove that there exists a $R_1 > r_1$ such that

$$\mu T_{\tau,1}(u, v) \neq u, \quad \inf_{u \in \partial K_{R_1}} \|T_{\tau,1}(u, v)\| > 0, \quad \forall \mu \geqslant 1, (u, v) \in \partial K_{R_1} \times K. \tag{3.6}$$

First, if there are $(u_0, v_0) \in K \times K$ and $\mu_0 \geqslant 1$ such that $u_0 = \mu_0 T_{\tau,1}(u_0, v_0)$, together with (3.5), we obtain

$$\begin{aligned} -(p(t)u_0')' + q(t)u_0 &\geqslant w_1(t)[\tau f_1(t, u_0, v_0) + (1 - \tau) f_1(t, u_0, 0)] \\ &\geqslant w_1(t)[(\lambda_{1,1} + \varepsilon) u_0 - C_{1,1}]. \end{aligned}$$

It follows that

$$\begin{aligned} &\lambda_{1,1} \int_0^1 w_1(t) u_0(t) e_{1,1}(t) \,\mathrm{d}t \\ &\geqslant (\lambda_{1,1} + \varepsilon) \int_0^1 w_1(t) u_0(t) e_{1,1}(t) \,\mathrm{d}t - C_{1,1} \int_0^1 w_1(t) e_{1,1}(t) \,\mathrm{d}t, \end{aligned}$$

which yields

$$\int_0^1 w_1(t) u_0(t) e_{1,1}(t) \,\mathrm{d}t \leqslant \frac{C_{1,1}}{\varepsilon} \int_0^1 w_1(t) e_{1,1}(t) \,\mathrm{d}t.$$

Furthermore, in view of the definition of cone K, we know that

$$\sigma \int_{\delta_0}^{1-\delta_0} w_1(t) e_{1,1}(t) \,\mathrm{d}t \|u_0\| \leqslant \frac{C_{1,1}}{\varepsilon} \int_0^1 w_1(t) e_{1,1}(t) \,\mathrm{d}t;$$

that is,

$$\|u_0\| \leqslant \frac{C_{1,1}}{\sigma\varepsilon} \int_0^1 w_1(t) e_{1,1}(t) \,\mathrm{d}t \Big(\int_{\delta_0}^{1-\delta_0} w_1(t) e_{1,1}(t) \,\mathrm{d}t \Big)^{-1} =: R^*. \tag{3.7}$$

Therefore, as $R > R^*, u \neq \mu T_{\lambda,1}(u, v)$ for all $(u, v) \in \partial K_R \times K$ and $\mu \geqslant 1$. In addition, if $R > m/\sigma$, then by use of (3.4) we know that for all $(u, v) \in \partial K_R \times K$,

$$\begin{aligned} &\|T_{\tau,1}(u, v)\| \\ &\geqslant T_{\tau,1}(u, v)(\frac{1}{2}) \\ &= \int_0^1 G(\frac{1}{2}, s) w_1(s)[\tau f_1(s, u(s), v(s)) + (1 - \tau) f_1(s, u(s), 0)] \,\mathrm{d}s \\ &\geqslant \delta G(\frac{1}{2}, \frac{1}{2}) \int_0^1 G(s, s) w_1(s)[\tau f_1(s, u(s), v(s)) + (1 - \tau) f_1(s, u(s), 0)] \,\mathrm{d}s \\ &\geqslant \delta G(\frac{1}{2}, \frac{1}{2}) \int_{\delta_0}^{1-\delta_0} G(s, s) w_1(s)[\tau f_1(s, u(s), v(s)) + (1 - \tau) f_1(s, u(s), 0)] \,\mathrm{d}s \\ &\geqslant \frac{\delta}{\rho} \varphi(\delta_0) \psi(1 - \delta_0) G(\frac{1}{2}, \frac{1}{2}) \int_{\delta_0}^{1-\delta_0} w_1(s)[\tau f_1(s, u(s), v(s)) + (1 - \tau) f_1(s, u(s), 0)] \,\mathrm{d}s \end{aligned}$$

$$\geqslant (\lambda_{1,1}+\varepsilon)\sigma^2 G(\frac{1}{2},\frac{1}{2})\int_{\delta_0}^{1-\delta_0} w_1(s)\,\mathrm{d}s\cdot\|u\|\,.$$

It follows that $\inf_{(u,v)\in\partial K_R\times K}\|T_{\tau,1}(u,v)\|>0$. We choose $R_1>\max\{r_1,R^*,m/\sigma\}$.

Step 3. In view of the locally uniformly sublinear assumption of f_2 at $v=0$, there exist $\varepsilon>0$ and $r_2>0$ such that

$$\tau f_2(t,u,v)+(1-\tau)f_2(t,0,v)\geqslant(\lambda_{2,1}+\varepsilon)v, \tag{3.8}$$

for all $t\in[0,1]$, $(u,v)\in[0,R_1]\times[0,r_2]$. By (3.8) and the proof similar to steps 1 and 2, we can deduce that

$$\mu T_{\tau,2}(u,v)\neq v,\quad \inf_{v\in\partial K_{r_2}}\|T_{\tau,2}(u,v)\|>0,\quad \forall\mu\geqslant1,\ (u,v)\in\overline{K_{R_1}}\times\partial K_{r_2}. \tag{3.9}$$

Step 4. From the locally uniformly sublinear hypothesis of f_2 at $v=+\infty$, there exist $\varepsilon\in(0,\lambda_{2,1})$, $n>0$ and $C_2>0$ such that

$$\tau f_2(t,u,v)+(1-\tau)f_2(t,0,v)\leqslant(\lambda_{2,1}-\varepsilon)v, \tag{3.10}$$

for all $t\in[0,1]$, $(u,v)\in[0,R_1]\times[n,+\infty)$, and

$$\tau f_2(t,u,v)+(1-\tau)f_2(t,0,v)\leqslant(\lambda_{2,1}-\varepsilon)v+C_2, \tag{3.11}$$

for all $t\in[0,1]$, $(u,v)\in[0,R_1]\times\mathbb{R}^+$. From (3.11) and the similar argument used in step 2, it can be proved that if $v_0=\mu_0T_{\tau,2}(u_0,v_0)$ for $(u_0,v_0)\in\overline{K_{R_1}}\times K$ and $\mu_0\in(0,1]$, then

$$\|v_0\|\leqslant\frac{C_2}{\sigma\varepsilon}\int_0^1 w_2(t)e_{2,1}(t)\,\mathrm{d}t\Big(\int_{\delta_0}^{1-\delta_0} w_2(t)e_{2,1}(t)\,\mathrm{d}t\Big)^{-1}:=R'. \tag{3.12}$$

Hence, we choose $R_2>\max\{r_2,R'\}$, then

$$\mu T_{\tau,2}(u,v)\neq v,\quad \forall\mu\in(0,1]\text{ and }(u,v)\in\overline{K_{R_1}}\times\partial K_{R_2}. \tag{3.13}$$

Now we choose an open set $D=(K_{R_1}\backslash\overline{K_{r_1}})\times(K_{R_2}\backslash\overline{K_{r_2}})$. Based on the expressions (3.2), (3.6), (3.9) and (3.13), it is easy to verify that $\{T_\tau\}_{\tau\in[0,1]}$ satisfy the sufficient conditions for the homotopy invariance of fixed point index on ∂D. Hence, by Lemmas 2.5 and 2.6, we obtain

$$\begin{aligned} i(T_1,D,K\times K)&=\prod_{j=1}^2 i(T_{0,j},K_{R_j}\backslash\overline{K_{r_j}},K)\\ &=\prod_{j=1}^2[i(T_{0,j},K_{R_j},K)-i(T_{0,j},K_{r_j},K)]=-1. \end{aligned}$$

Therefore, system (1.1) has at least one component-wise positive solution. □

Proof of Theorem 1.4. Similar to the arguments used in the proof of Theorem 1.2, we will determine r_1, R_1, r_2 and R_2 in turn.

Step 1. In view of the uniformly sublinear assumption of f_1 at $u=0$, there are $\varepsilon>0$ and $r_1>0$ such that

$$\tau f_1(t,u,v)+(1-\tau)f_1(t,u,0)\geqslant(\lambda_{1,1}+\varepsilon)u, \tag{3.14}$$

for all $t\in[0,1]$, $(u,v)\in[0,r_1]\times\mathbb{R}^+$. Now we can deduce that

$$\mu T_{\tau,1}(u,v)\neq u,\quad \inf_{u\in\partial K_{r_1}}\|T_{\tau,1}(u,v)\|>0,\quad \forall\mu\geqslant1,\ (u,v)\in\partial K_{r_1}\times K. \tag{3.15}$$

First, by contradiction suppose that there are $\mu_0 \geqslant 1$ and $(u_0, v_0) \in \partial K_{r_1} \times K$ such that $u_0 = \mu_0 T_{\tau,1}(u_0, v_0)$, together with (3.14), we have that

$$-(p(t)u_0')' + q(t)u_0 \geqslant w_1(t)[\tau f_1(t, u_0, v_0) + (1-\tau)f_1(t, u_0, 0)] \geqslant (\lambda_{1,1} + \varepsilon)w_1(t)u_0.$$

It follows that

$$\lambda_{1,1} \int_0^1 w_1(t)u_0(t)e_{1,1}(t)\,\mathrm{d}t \geqslant (\lambda_{1,1} + \varepsilon)\int_0^1 w_1(t)u_0(t)e_{1,1}(t)\,\mathrm{d}t,$$

which yields a contradiction $\lambda_{1,1} \geqslant \lambda_{1,1} + \varepsilon$.

In addition, by (3.14) we know that for all $(u, v) \in \partial K_{r_1} \times K$,

$$\begin{aligned}
\|T_{\tau,1}(u, v)\| &\geqslant T_{\tau,1}(u, v)(\frac{1}{2}) \\
&= \int_0^1 G(\frac{1}{2}, s)w_1(s)[\tau f_1(s, u(s), v(s)) + (1-\tau)f_1(s, u(s), 0)]\,\mathrm{d}s \\
&\geqslant (\lambda_{1,1} + \varepsilon)\int_0^1 G(\frac{1}{2}, s)w_1(s)u(s)\,\mathrm{d}s \\
&\geqslant (\lambda_{1,1} + \varepsilon)\delta^2 G(\frac{1}{2}, \frac{1}{2})\int_0^1 w_1(s)G^2(s, s)\,\mathrm{d}s \cdot \|u\|,
\end{aligned} \tag{3.16}$$

which implies that $\inf_{(u,v)\in\partial K_{r_1}\times K} \|T_{\tau,1}(u, v)\| > 0$.

Step 2. From the uniformly sublinear hypothesis of f_1 at $u = +\infty$ and condition (H7), there exist $\varepsilon \in (0, \lambda_{1,1})$, $n > 0$ and $C_2 > 0$ such that

$$\tau f_1(t, u, v) + (1-\tau)f_1(t, u, 0) \leqslant (\lambda_{1,1} - \varepsilon)u, \tag{3.17}$$

for all $t \in [0, 1]$, $(u, v) \in [n, +\infty) \times \mathbb{R}^+$, and

$$\tau f_1(t, u, v) + (1-\tau)f_1(t, u, 0) \leqslant (\lambda_{1,1} - \varepsilon)u + C_2, \tag{3.18}$$

for all $t \in [0, 1]$, $(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$.

From (3.18) and the similar argument used in step 2 of proof of Theorem 1.2, it can be proved that if $u_0 = \mu_0 T_{\tau,1}(u_0, v_0)$ for $(u_0, v_0) \in K \times K$ and $\mu_0 \in (0, 1]$, then $\|u_0\| \leqslant R'$ (defined in (3.12) with $w_2(t)$ and $e_{2,1}(t)$ replaced by $w_1(t)$ and $e_{1,1}(t)$ respectively). Hence, we choose $R_1 > \max\{r_1, R'\}$, then

$$\mu T_{\tau,1}(u, v) \neq u, \quad \forall \mu \in (0, 1],\ (u, v) \in \overline{K_{R_1}} \times K. \tag{3.19}$$

Step 3. From the locally uniformly superlinear assumption of f_2 at $v = 0$, there are $\varepsilon \in (0, \lambda_{2,1})$ and $r_2 > 0$ such that

$$\tau f_2(t, u, v) + (1-\tau)f_2(t, 0, v) \leqslant (\lambda_{2,1} - \varepsilon)v, \tag{3.20}$$

for all $t \in [0, 1]$, $(u, v) \in [0, R_1] \times [0, r_2]$. We claim that

$$\mu T_{\tau,2}(u, v) \neq v,\ \forall \mu \in (0, 1] \quad \text{and} \quad (u, v) \in \overline{K_{R_1}} \times \partial K_{r_2}. \tag{3.21}$$

In fact, if this were not true, then there exist $\mu_0 \in (0, 1]$ and $(u_0, v_0) \in \overline{K_{R_1}} \times \partial K_{r_2}$, such that $\mu_0 T_{\tau,2}(u_0, v_0) = v_0$. In combination with (3.20), it follows that

$$-(p(t)v_0')' + q(t)v_0 \leqslant w_2(t)[\tau f_2(t, u_0, v_0) + (1-\tau)f_2(t, 0, v_0)] \leqslant (\lambda_{2,1} - \varepsilon)w_2(t)v_0.$$

Multiplying the both sides of this inequality by $e_{2,1}(t)$ and integrating on $[0, 1]$, we obtain that

$$\lambda_{2,1} \int_0^1 w_2(t)v_0(t)e_{2,1}(t)\,\mathrm{d}t \leqslant (\lambda_{2,1} - \varepsilon)\int_0^1 w_2(t)v_0(t)e_{2,1}(t)\,\mathrm{d}t,$$

which yields that $\lambda_{2,1} \leqslant \lambda_{2,1} - \varepsilon$, a contradiction!

Step 4. From the locally uniformly superlinear hypothesis of f_2 at $v = +\infty$, there exist $\varepsilon > 0$ and $m > 0$ such that

$$\tau f_2(t,u,v) + (1-\tau) f_2(t,0,v) \geqslant (\lambda_{2,1} + \varepsilon) v, \tag{3.22}$$

for all $t \in [0,1]$, $(u,v) \in [0,R_1] \times [m,+\infty)$; thus

$$\tau f_2(t,u,v) + (1-\tau) f_2(t,0,v) \geqslant (\lambda_{2,1} + \varepsilon) v - C_{2,1}, \tag{3.23}$$

for all $t \in [0,1]$, $(u,v) \in [0,R_1] \times \mathbb{R}^+$, where $C_{2,1} = (\lambda_{2,1} + \varepsilon)m$.

From (3.23) and the similar argument used in step 2 of proof of Theorem 1.2, it can be proved that if $v_0 = \mu_0 T_{\tau,2}(u_0,v_0)$ for $(u_0,v_0) \in \overline{K_{R_1}} \times K$ and $\mu_0 \geqslant 1$, then $\|v_0\| \leqslant R^*$ (defined in (3.7) with $w_1(t)$, $e_{1,1}(t)$ and $C_{1,1}$ replaced by $w_2(t)$, $e_{2,1}(t)$ and $C_{2,1}$ respectively); in addition, by (3.22) it can also be showed that $\inf_{(u,v)\in \overline{K_{R_1}} \times \partial K_R} \|T_{\tau,2}(u,v)\| > 0$ as $R > m/\sigma$. Hence, choose $R_2 > \max\{r_2, R^*, m/\sigma\}$, then

$$\mu T_{\tau,2}(u,v) \neq v, \quad \inf_{v \in \partial K_{R_2}} \|T_{\tau,2}(u,v)\| > 0, \quad \forall \mu \geqslant 1, (u,v) \in \overline{K_{R_1}} \times \partial K_{R_2}. \tag{3.24}$$

Based on the expressions (3.15), (3.19), (3.21) and (3.24), it is easy to verify that $\{T_\tau\}_{\tau\in[0,1]}$ satisfy the sufficient conditions for the homotopy invariance of fixed point index on ∂D, here $D = (K_{R_1} \backslash \overline{K_{r_1}}) \times (K_{R_2} \backslash \overline{K_{r_2}})$. So, by Lemmas 2.5 and 2.6, we have

$$\begin{aligned} i(T_1, D, K\times K) &= \prod_{j=1}^{2} i(T_{0,j}, K_{R_j} \backslash \overline{K_{r_j}}, K) \\ &= \prod_{j=1}^{2} [i(T_{0,j}, K_{R_j}, K) - i(T_{0,j}, K_{r_j}, K)] = -1. \end{aligned}$$

Therefore, system (1.1) has at least one component-wise positive solution. □

Proof of Theorem 1.10. First, we construct the strict lower solutions to system (1.1). For this purpose, we define the extensions of f_i $(i = 1,2)$ preserving continuity, nonnegativity and quasi-monotone non-decreasing as follows

$$\begin{gathered} f_1(t,u,v) = f_1(t,|u|,v^+), \quad \forall (t,u,v) \in [0,1] \times \mathbb{R} \times \mathbb{R}, \\ f_2(t,u,v) = f_2(t,u^+,|v|), \quad \forall (t,u,v) \in [0,1] \times \mathbb{R} \times \mathbb{R}. \end{gathered}$$

Then, by (H9) and (H10), there exist $\epsilon > 0$ and $r > 0$ such that

$$\begin{gathered} f_1(t,u,v) \geqslant f_1(t,u,0) > (\lambda_{1,1} + \epsilon)|u|, \quad \forall (t,v) \in [0,1] \times \mathbb{R},\ |u| \in (0,r), \\ f_2(t,u,v) \geqslant f_2(t,0,v) > (\lambda_{2,1} + \epsilon)|v|, \quad \forall (t,u) \in [0,1] \times \mathbb{R},\ |v| \in (0,r). \end{gathered} \tag{3.25}$$

Now we give strict lower solutions $(\alpha_u(t), \alpha_v(t))$ of system (1.1) according to the following cases.

Case 1: $a > 0$, $c > 0$. By using (3.25), it is easy to verify that

$$(\alpha_u(t), \alpha_v(t)) = (-r/2,\ -r/2)$$

is a strict lower solution of (1.1).

Case 2: $a > 0$, $c = 0$. Let $e_{i,1}(t)$ be the normalized eigenfunction according to the first eigenvalue $\lambda_{i,1}$ of (1.5) with $a > 0$ and $c = 0$, then by the maximum principles and Hopf's boundary point lemma (see [21]), $e_{i,1}(t) > 0$ for all $t \in (0,1]$ $(i = 1,2)$. In addition, let $\omega_\varepsilon(t)$ be a unique solution to the problem

$$\begin{gathered} -(p(t)u')' + q(t)u = \varepsilon^2, \quad t \in (0,1), \\ au(0) - bp(0)u'(0) = 0, \\ u(1) + u'(1) = 0. \end{gathered} \tag{3.26}$$

Here $\varepsilon > 0$ small enough, so that $\omega_\varepsilon(t) > 0$ for all $t \in (0,1]$ and $\omega_\varepsilon = o(\varepsilon)$ $(\varepsilon \to 0)$.

Now choose $\varepsilon > 0$ small enough such that there hold for all $t \in [0,1]$,

$$\begin{gathered} \varepsilon^2 e_{i,1}(t) + \omega_\varepsilon(t) - \varepsilon \in (-r, 0), \\ w_i(t)[(2\lambda_{i,1} + \epsilon)\varepsilon^2 e_{i,1}(t) + (\lambda_{i,1} + \epsilon)\omega_\varepsilon(t)] + \varepsilon^2 < [q(t) + (\lambda_{i,1} + \epsilon)w_i(t)]\varepsilon, \end{gathered} \tag{3.27}$$

which is valid by the assumptions $w_i \in C([0,1], \mathbb{R}_0^+)$ $(i = 1,2)$. In view of (3.25) and (3.27), it is not difficult to verify that

$$(\alpha_u(t), \alpha_v(t)) = (\varepsilon^2 e_{1,1}(t) + \omega_\varepsilon(t) - \varepsilon,\ \varepsilon^2 e_{2,1}(t) + \omega_\varepsilon(t) - \varepsilon) \tag{3.28}$$

is a strict lower solution of (1.1).

Case 3: $a = 0$, $c > 0$. Let $e_{i,1}(t)$ be the normalized eigenfunction according to the first eigenvalue $\lambda_{i,1}$ of (1.5) with $a = 0$ and $c > 0$, then by the maximum principles and Hopf's boundary point lemma, $e_{i,1}(t) > 0$ for all $t \in [0,1)$ $(i = 1,2)$. In addition, let $\omega_\varepsilon(t)$ be a unique solution to the problem

$$\begin{gathered} -(p(t)u')' + q(t)u = \varepsilon^2, \quad t \in (0,1), \\ u(0) - u'(0) = 0, \\ cu(1) + dp(1)u'(1) = 0. \end{gathered} \tag{3.29}$$

Here $\varepsilon > 0$ small enough, so that $\omega_\varepsilon(t) > 0$ for all $t \in [0,1)$ and $\omega_\varepsilon = o(\varepsilon)$ $(\varepsilon \to 0)$.

Noticing that $w_i \in C([0,1], \mathbb{R}_0^+)$ $(i = 1,2)$, we can choose $\varepsilon > 0$ small enough such that for all $t \in [0,1]$ it holds

$$\begin{gathered} \varepsilon^2 e_{i,1}(t) + \omega_\varepsilon(t) - \varepsilon \in (-r, 0), \\ w_i(t)[(2\lambda_{i,1} + \epsilon)\varepsilon^2 e_{i,1}(t) + (\lambda_{i,1} + \epsilon)\omega_\varepsilon(t)] + \varepsilon^2 < [q(t) + (\lambda_{i,1} + \epsilon)w_i(t)]\varepsilon. \end{gathered} \tag{3.30}$$

In view of (3.25) and (3.30), it is easy to verify that

$$(\alpha_u(t), \alpha_v(t)) = (\varepsilon^2 e_{1,1}(t) + \omega_\varepsilon(t) - \varepsilon,\ \varepsilon^2 e_{2,1}(t) + \omega_\varepsilon(t) - \varepsilon) \tag{3.31}$$

is a strict lower solution of (1.1).

Case 4: $a = 0$, $c = 0$;

Let $e_{i,1}(t)$ be the normalized eigenfunction according to the first eigenvalue $\lambda_{i,1}$ of (1.5) with $a = 0$ and $c = 0$, then by the maximum principles and Hopf's boundary point lemma, $e_{i,1}(t) > 0$ for all $t \in [0,1]$ $(i = 1,2)$. In addition, let $\omega_\varepsilon(t)$ be a unique solution to the problem

$$\begin{gathered} -(p(t)u')' + q(t)u = \varepsilon^2, \quad t \in (0,1), \\ u(0) - u'(0) = 0, \\ u(1) + u'(1) = 0. \end{gathered} \tag{3.32}$$

Here $\varepsilon > 0$ small enough, so that $\omega_\varepsilon(t) > 0$ for all $t \in [0,1]$ and $\omega_\varepsilon = o(\varepsilon)$ $(\varepsilon \to 0)$.

Now choose $\varepsilon > 0$ small enough such that for all $t \in [0,1]$,

$$\begin{gathered} \omega_\varepsilon(t) - \varepsilon e_{i,1}(t) \in (-r, 0), \\ (\lambda_{i,1} + \epsilon) w_i(t) \omega_\varepsilon(t) + \varepsilon^2 < (2\lambda_{i,1} + \epsilon)\varepsilon w_i(t) e_{i,1}(t), \end{gathered} \tag{3.33}$$

which is valid because of the assumptions $w_i \in C([0,1], \mathbb{R}_0^+)$ $(i = 1, 2)$. In view of (3.25) and (3.33), it is not difficult to verify that

$$(\alpha_u(t), \alpha_v(t)) = (\omega_\varepsilon(t) - \varepsilon e_{1,1}(t), \omega_\varepsilon(t) - \varepsilon e_{2,1}(t)) \tag{3.34}$$

is a strict lower solution of (1.1).

Noticing that $f_i \geqslant 0$ and $w_i \geqslant 0$ $(i = 1, 2)$, by the definitions of strict upper solutions, the maximum principles and Hopf's boundary point lemma, we can deduce that $\beta_u(t) > 0$ and $\beta_v(t) > 0$ for all $t \in [0,1]$. Let $\Omega = \{(u, v) \in C[0,1] \times C[0,1] : \alpha_u < u < \beta_u, \text{ and } \alpha_v < v < \beta_v \text{ on } [0,1]\}$, then from Theorem 2.7 we have

$$i(T_1, \Omega \cap (K \times K), K \times K) = \deg(\mathrm{id} - T_1, \Omega, (\theta, \theta)) = 1. \tag{3.35}$$

In the expressions (3.15), (3.9), (3.6) and (3.24), choosing r_1, r_2 small enough and R_1, R_2 large enough such that

$$0 < r_1 < \min_{t \in [0,1]} \beta_u(t) \leqslant \|\beta_u\| < R_1, \quad 0 < r_2 < \min_{t \in [0,1]} \beta_v(t) \leqslant \|\beta_v\| < R_2,$$

In view of Lemmas 2.5 and 2.6, we obtain that

$$i(T_1, K_{r_1} \times K_{r_2}, K \times K) = i(T_1, K_{R_1} \times K_{R_2}, K \times K) = 0. \tag{3.36}$$

Combining with (3.35)-(3.36) and the additivity of fixed point index, we obtain that

$$\begin{gathered} i(T_1, [\Omega \cap (K \times K)] \backslash \overline{K_{r_1} \times K_{r_2}}, K \times K) = 1, \\ i(T_1, (K_{R_1} \times K_{R_2}) \backslash \overline{\Omega \cap (K \times K)}, K \times K) = -1, \end{gathered}$$

which means that system (1.1) has at least two positive solutions. □

4. Application to nonlinear eigenvalue problem (1.2)

As applications, we consider the existence of global component-wise positive solutions for problem (1.2). For Theorem 1.11, our main idea of proof is to analyze the properties and structure of the parameter sets

$$\mathscr{S} = \{(\lambda, \mu) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ : T^1_{\lambda,\mu} \text{ has a fixed point in } [K \backslash \{0\}] \times [K \backslash \{0\}]\}, \tag{4.1}$$

and in the interior of $\mathscr{S}$ denoted by $\operatorname{int} \mathscr{S}$. Then we construct the desired simple arc $\Gamma_0 \subset \mathbb{R}_0^+ \times \mathbb{R}_0^+$ in Theorem 1.11. For this purpose, first we show the existence of component-wise positive solutions to problem (1.2) for small parameters, and then prove the existence of a priori bound of component-wise positive solutions to problem (1.2) for large parameters which implies the nonexistence of component-wise positive solutions to problem (1.2) for sufficiently large parameters. For convenience, we establish several preliminary lemmas.

Lemma 4.1. *Assume that* (H9) *and* (H10*) *hold, for any* $r > 0$ *there exists a* $(\lambda_r, \mu_r) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ *such that* $(0, \lambda_r] \times (0, \mu_r] \subset \mathscr{S}$.

Proof. For any given $r > 0$, denote

$$\alpha = \sup_{(u,v)\in\partial K_r\times K_r} \|\int_0^1 G(t,s)w_1(s)f_1(s,u(s),v(s))\,\mathrm{d}s\|,$$

$$\beta = \sup_{(u,v)\in K_r\times\partial K_r} \|\int_0^1 G(t,s)w_2(s)f_2(s,u(s),v(s))\,\mathrm{d}s\|.$$

It is easy to see that α and β are both positive by $(H10*)$. Let $(\lambda_r,\mu_r) = (\frac{r}{2\alpha},\frac{r}{2\beta})$, then for any $(\lambda,\mu)\in(0,\lambda_r]\times(0,\mu_r]$, from (H9) we have

$$\|A^\tau_\lambda(u,v)\| \leqslant \lambda\alpha < r = \|u\|, \quad \forall(\tau,u,v)\in[0,1]\times\partial K_r\times K_r,$$
$$\|B^\tau_\mu(u,v)\| \leqslant \mu\beta < r = \|v\|, \quad \forall(\tau,u,v)\in[0,1]\times K_r\times\partial K_r,$$

thus $T^\tau_{\lambda,\mu}(u,v) \neq (u,v)$, for all $(\tau,u,v)\in[0,1]\times\partial(K_r\times K_r)$. From Remark 2.4, Lemmas 2.5 and 2.6, it follows that

$$i(T^1_{\lambda,\mu},K_r\times K_r,K\times K) = i(A^0_\lambda,K_r,K)\cdot i(B^0_\mu,K_r,K) = 1.$$

This and (H10*) imply that $T^1_{\lambda,\mu}$ has a fixed point in $[K_r\backslash\{0\}]\times[K_r\backslash\{0\}]$ for all $(\lambda,\mu)\in(0,\lambda_r]\times(0,\mu_r]$. □

Lemma 4.2. *Assume* (H9) *and* (H11*) *hold,* $S_u \equiv \{u : A^\tau_\lambda(u,v) = u,(\lambda,\tau)\in I\times[0,1]$ *and* $(u,v)\in K\times K\}$ *and* $S_v \equiv \{v : B^\tau_\mu(u,v) = v,(\mu,\tau)\in J\times[0,1]$ *and* $(u,v)\in K\times K\}$ *where* $I,J\subset[a,+\infty)$ *for some constant* $a>0$. *Then there exist constants* C_I *and* C_J *such that* $\|u\|\leqslant C_I$ *for all* $u\in S_u$ *and* $\|v\|\leqslant C_J$ *for all* $v\in S_v$.

Remark 4.3. By Lemma 4.2 we know that

$$S_{u,v} \equiv \{(u,v) : T^\tau_{\lambda,\mu}(u,v) = (u,v),\ (\tau,\lambda,\mu)\in[0,1]\times I\times J,\ (u,v)\in K\times K\}$$

is a bounded set. In particular, it implies the nonexistence of positive solutions to (1.2) for sufficiently large parameters (see Lemma 4.5).

Proof. First, we prove that there exists a constant C_I such that $\|u\|\leqslant C_I, \forall u\in S_u$. Suppose, reasoning by the contradiction, that there exist sequences $\{(\lambda_n,\tau_n)\}_{n=1}^\infty\subset I\times[0,1]$ and $\{(u_n,v_n)\}_{n=1}^\infty\subset K\times K$ such that $A^{\tau_n}_{\lambda_n}(u_n,v_n) = u_n$ and $\lim_{n\to\infty}\|u_n\| = +\infty$. By $(H11*)$, for $k > (a\sigma\int_{\delta_0}^{1-\delta_0} G(\frac{1}{2},s)w_1(s)\,\mathrm{d}s)^{-1}$ there exist a $R>0$ such that

$$f_1(t,u,0) \geqslant ku, \quad \forall(t,u)\in[0,1]\times[R,+\infty).$$

Now choose $\|u_N\|\geqslant\sigma^{-1}R$. By $(H9)$ and the inequality above, we obtain that

$$\begin{aligned}
\|u_N\| &= \|A^{\tau_N}_{\lambda_N}(u_N,v_N)\| \\
&\geqslant A^{\tau_N}_{\lambda_N}(u_N,v_N)(\frac{1}{2}) \\
&\geqslant \lambda_N\int_0^1 G(\frac{1}{2},s)w_1(s)f_1(s,u_N(s),0)\,\mathrm{d}s \\
&\geqslant k\lambda_N\int_{\delta_0}^{1-\delta_0} G(\frac{1}{2},s)w_1(s)u_N(s)\,\mathrm{d}s \\
&\geqslant ka\sigma\int_{\delta_0}^{1-\delta_0} G(\frac{1}{2},s)w_1(s)\,\mathrm{d}s\cdot\|u_N\| > \|u_N\|,
\end{aligned}$$

which is a contradiction.

Similarly, we can show that there exists a constant C_J such that $\|v\| \leqslant C_J$ for all $v \in S_v$. □

Lemma 4.4. *Assume* (H9), (H10*), (H11*) *hold. If* $(\overline{\lambda}, \overline{\mu}) \in \mathscr{S}$*, then for any* $(\lambda, \mu) \in (0, \overline{\lambda}) \times (0, \overline{\mu})$*, problem* (1.2) *has at least two component-wise positive solutions, in particular,* $(0, \overline{\lambda}) \times (0, \overline{\mu}) \subset \operatorname{int} \mathscr{S}$*. Moreover,* $(0, \overline{\lambda}] \times (0, \overline{\mu}] \subset \mathscr{S}$*.*

Proof. For any given $(\lambda, \mu) \in (0, \overline{\lambda}) \times (0, \overline{\mu})$, let $(\overline{u}, \overline{v}) \in [K \setminus \{0\}] \times [K \setminus \{0\}]$ is a fixed point of $T^1_{\overline{\lambda}, \overline{\mu}}$. Then there exists an $\epsilon \in (0, 1)$ such that for all $\varepsilon \in (0, \epsilon]$ and $t \in [0, 1]$,

$$
\begin{aligned}
&\lambda[f_1(t, \overline{u}(t) + \varepsilon, \overline{v}(t) + \varepsilon) - f_1(t, \overline{u}(t), \overline{v}(t))] < (\overline{\lambda} - \lambda) q_1/2, \\
&\mu[f_2(t, \overline{u}(t) + \varepsilon, \overline{v}(t) + \varepsilon) - f_2(t, \overline{u}(t), \overline{v}(t))] < (\overline{\mu} - \mu) q_2/2,
\end{aligned} \tag{4.2}
$$

where

$$
\begin{aligned}
q_1 &= \min\{f_1(t, u, 0) : (t, u) \in [0, 1] \times [0, \|\overline{u}\|]\} > 0, \\
q_2 &= \min\{f_2(t, 0, v) : (t, v) \in [0, 1] \times [0, \|\overline{v}\|]\} > 0,
\end{aligned}
$$

and thus for all $\varepsilon \in (0, \epsilon]$ and $t \in [0, 1]$,

$$
\begin{aligned}
&\lambda f_1(t, \overline{u}(t) + \varepsilon, \overline{v}(t) + \varepsilon) - \overline{\lambda} f_1(t, \overline{u}(t), \overline{v}(t)) \leqslant (\lambda - \overline{\lambda}) q_1/2 < 0, \\
&\mu f_2(t, \overline{u}(t) + \varepsilon, \overline{v}(t) + \varepsilon) - \overline{\mu} f_2(t, \overline{u}(t), \overline{v}(t)) \leqslant (\mu - \overline{\mu}) q_2/2 < 0.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\lambda w_1(t) f_1(t, \overline{u}(t) + \varepsilon, \overline{v}(t) + \varepsilon) - \overline{\lambda} w_1(t) f_1(t, \overline{u}(t), \overline{v}(t)) \\
&\leqslant (\lambda - \overline{\lambda}) q_1 \min_{t \in [0,1]} w_1(t)/2 < 0, \\
&\mu w_2(t) f_2(t, \overline{u}(t) + \varepsilon, \overline{v}(t) + \varepsilon) - \overline{\mu} w_2(t) f_2(t, \overline{u}(t), \overline{v}(t)) \\
&\leqslant (\mu - \overline{\mu}) q_2 \min_{t \in [0,1]} w_2(t)/2 < 0,
\end{aligned} \tag{4.3}
$$

which is valid by the assumptions $w_i \in C([0, 1], \mathbb{R}_0^+)$ $(i = 1, 2)$.

Next, we construct strict upper solutions $(\beta_u(t), \beta_v(t))$ of (1.2) according to the following cases.

Case 1: $a > 0$, $c > 0$. Let $(\beta_u(t), \beta_v(t)) = (\overline{u}(t) + \varepsilon, \overline{v}(t) + \varepsilon)$. Then by (4.3), $(\beta_u(t), \beta_v(t))$ on $(0, 1)$ satisfies

$$
\begin{aligned}
-(p(t)\beta_u'(t))' + q(t)\beta_u(t) &\geqslant -(p(t)\overline{u}'(t))' + q(t)\overline{u}(t) \\
&= \overline{\lambda} w_1(t) f_1(t, \overline{u}(t), \overline{v}(t)) \\
&> \lambda w_1(t) f_1(t, \beta_u(t), \beta_v(t)), \\
-(p(t)\beta_v'(t))' + q(t)\beta_v(t) &\geqslant -(p(t)\overline{v}'(t))' + q(t)\overline{v}(t) \\
&= \overline{\mu} w_2(t) f_2(t, \overline{u}(t), \overline{v}(t)) \\
&> \mu w_2(t) f_2(t, \beta_u(t), \beta_v(t)).
\end{aligned} \tag{4.4}
$$

On the other hand,

$$
\begin{aligned}
&a\beta_u(0) - bp(0)\beta_u'(0) = a\beta_v(0) - bp(0)\beta_v'(0) = a\varepsilon > 0, \\
&c\beta_u(1) + dp(1)\beta_u'(1) = c\beta_v(1) + dp(1)\beta_v'(1) = c\varepsilon > 0.
\end{aligned} \tag{4.5}
$$

From (4.4) and (4.5), we obtain that $(\beta_u(t), \beta_v(t))$ is a strict upper solution of (1.2).

Case 2: $a > 0$, $c = 0$. Let

$$(\beta_u(t), \beta_v(t)) = \begin{cases} (\overline{u}(t) + \varepsilon + \frac{a\varepsilon t}{2bp(0)}, \overline{v}(t) + \varepsilon + \frac{a\varepsilon t}{2bp(0)}), & b > 0, \\ (\overline{u}(t) + \varepsilon + \varepsilon t,\ \overline{v}(t) + \varepsilon + \varepsilon t), & b = 0. \end{cases} \tag{4.6}$$

From (4.3), it is not difficult to verity that $(\beta_u(t), \beta_v(t))$ is a strict upper solution of problem (1.2) as $\varepsilon \in (0, \epsilon]$ is sufficiently small.

Case 3: $a = 0$, $c > 0$. Let

$$(\beta_u(t), \beta_v(t)) = \begin{cases} (\overline{u}(t) + \varepsilon + \frac{c\varepsilon(1-t)}{2dp(1)}, \overline{v}(t) + \varepsilon + \frac{c\varepsilon(1-t)}{2dp(1)}, & d > 0, \\ (\overline{u}(t) + \varepsilon + \varepsilon(1-t),\ \overline{v}(t) + \varepsilon + \varepsilon(1-t)), & d = 0. \end{cases} \tag{4.7}$$

Using (4.3), it is easy to verity that $(\beta_u(t), \beta_v(t))$ is a strict upper solution of problem (1.2) as $\varepsilon \in (0, \epsilon]$ is sufficiently small.

Case 4: $a = 0$, $c = 0$. Let

$$(\beta_u(t), \beta_v(t)) = (\overline{u}(t) + \varepsilon(t^2 - t + 1), \overline{v}(t) + \varepsilon(t^2 - t + 1)),$$

where $\varepsilon \in (0, \epsilon]$ is small enough such that

$$\begin{aligned} &\overline{\lambda} w_1(t) f_1(t, \overline{u}(t), \overline{v}(t)) - 2\varepsilon p(t) - \varepsilon(2t-1)p'(t) > \lambda w_1(t) f_1(t, \beta_u(t), \beta_v(t)), \\ &\overline{\mu} w_2(t) f_2(t, \overline{u}(t), \overline{v}(t)) - 2\varepsilon p(t) - \varepsilon(2t-1)p'(t) > \mu w_2(t) f_2(t, \beta_u(t), \beta_v(t)), \end{aligned} \tag{4.8}$$

which is valid in view of (4.3). Then by (4.8), $(\beta_u(t), \beta_v(t))$ satisfies

$$\begin{aligned} -(p(t)\beta_u'(t))' + q(t)\beta_u(t) > \lambda w_1(t) f_1(t, \beta_u(t), \beta_v(t)), \quad t \in (0,1), \\ -(p(t)\beta_v'(t))' + q(t)\beta_v(t) > \mu w_2(t) f_2(t, \beta_u(t), \beta_v(t)), \quad t \in (0,1). \end{aligned} \tag{4.9}$$

On the other hand,

$$\begin{aligned} -bp(0)\beta_u'(0) = -bp(0)\beta_v'(0) = bp(0)\varepsilon > 0, \\ dp(1)\beta_u'(1) = dp(1)\beta_v'(1) = dp(1)\varepsilon > 0. \end{aligned} \tag{4.10}$$

By (4.9) and (4.10), we obtain that $(\beta_u(t), \beta_v(t))$ is a strict upper solution of (1.2).

Hence, problem (1.2) has at least two component-wise positive solutions for $(\lambda, \mu) \in (0, \overline{\lambda}) \times (0, \overline{\mu})$ by Theorem 1.10 and the fact that problem (1.2) has positive solutions in which at least one component is zero if and only if $\lambda\mu = 0$ because of condition $(H10*)$. In particular, $(0, \overline{\lambda}) \times (0, \overline{\mu}) \subset \mathscr{S}$. In addition, if $(\lambda, \mu) \in (0, \overline{\lambda}) \times (0, \overline{\mu})$, then there is an $\varepsilon > 0$ such that $(\lambda - \varepsilon, \lambda + \varepsilon) \times (\mu - \varepsilon, \mu + \varepsilon) \subset (0, \overline{\lambda}) \times (0, \overline{\mu}) \subset \mathscr{S}$, and thus $(\lambda, \mu) \in \operatorname{int} \mathscr{S}$. So, $(0, \overline{\lambda}) \times (0, \overline{\mu}) \subset \operatorname{int} \mathscr{S}$.

Moreover, by Lemma 4.2 and Lebesgue dominated convergence theorem, we know that $(0, \overline{\lambda}] \times (0, \overline{\mu}] \subset \mathscr{S}$. □

In what follows, denote

$$\partial(\operatorname{int} \mathscr{S}) := \text{the boundary of } \operatorname{int}\mathscr{S}, \quad \overline{\operatorname{int}\mathscr{S}} := \text{the closure of } \operatorname{int}\mathscr{S}, \tag{4.11}$$

we will discuss the properties and structures of sets $\mathscr{S}$ and $\operatorname{int} \mathscr{S}$ in the subsequent lemmas.

Lemma 4.5. *Assume that (H9), (H10*), (H11*) hold. Then* $\operatorname{int} \mathscr{S}$ *is nonempty and bounded.*

Proof. By Lemma 4.1 and Lemma 4.4, it is easy to see that $\operatorname{int}\mathscr{S}$ is nonempty.

Next, we prove that $\operatorname{int}\mathscr{S}$ is bounded. In fact, if $\operatorname{int}\mathscr{S}$ is unbounded, then there exist sequences $\{(u_n, v_n)\}_{n=1}^{\infty} \subset [K \setminus \{0\}] \times [K \setminus \{0\}]$ and $\{(\lambda_n, \mu_n)\}_{n=1}^{\infty} \subset \operatorname{int}\mathscr{S}$ such that $T^1_{\lambda_n,\mu_n}(u_n, v_n) = (u_n, v_n)$ and $\lim_{n\to\infty} \lambda_n = +\infty$ or $\lim_{n\to\infty} \mu_n = +\infty$. Without loss of generality, suppose that $\lim_{n\to\infty} \lambda_n = +\infty$. From Lemma 4.2, we know that there exists a constant $M > 0$ such that $\|u_n\| \leqslant M$, $\forall n \in \mathbb{N}$. Let $m = \min\{f_1(t, u, 0) : (t, u) \in [0, 1] \times [0, M]\}$ which is positive, then

$$M \geqslant \|u_n\| = \|A^1_{\lambda_n}(u_n, v_n)\| \geqslant m\lambda_n \|\int_0^1 G(\cdot, s)w_1(s)\,\mathrm{d}s\| > 0,$$

thus $\lambda_n \leqslant (m\|\int_0^1 G(\cdot, s)w_1(s)\,\mathrm{d}s\|)^{-1}M$, a contradiction. Hence, $\operatorname{int}\mathscr{S}$ is bounded. □

Lemma 4.6. *Assume that* (H9), (H10*), (H11*) *hold. Then there is a* $(\lambda_*, \mu_*) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ *such that* $\partial(\operatorname{int}\mathscr{S}) \cap \{(\lambda, \mu) : \lambda\mu = 0\} = \{(\lambda, 0) : \lambda \in [0, \lambda_*]\} \cup \{(0, \mu) : \mu \in [0, \mu_*]\}$ *and* $\overline{\operatorname{int}\mathscr{S}} \subset [0, \lambda_*] \times [0, \mu_*]$.

Proof. By Lemma 4.5, we can define

$$\lambda_* = \sup\{\lambda : (\lambda, \mu) \in \operatorname{int}\mathscr{S}\} \quad \text{and} \quad \mu_* = \sup\{\mu : (\lambda, \mu) \in \operatorname{int}\mathscr{S}\}, \tag{4.12}$$

which are both positive.

Let $(\lambda, \mu) \in \overline{\operatorname{int}\mathscr{S}}$, then there exists a sequence $\{(\lambda_n, \mu_n)\}_{n=1}^{\infty} \subset \operatorname{int}\mathscr{S}$ such that $\lambda = \lim_{n\to\infty} \lambda_n$ and $\mu = \lim_{n\to\infty} \mu_n$. Combining with (4.12), it is easy to see that $(\lambda, \mu) \in [0, \lambda_*] \times [0, \mu_*]$. Thus, $\overline{\operatorname{int}\mathscr{S}} \subset [0, \lambda_*] \times [0, \mu_*]$. In particular, $\partial(\operatorname{int}\mathscr{S}) \cap \{(\lambda, \mu) : \lambda\mu = 0\} \subset \{(\lambda, 0) : \lambda \in [0, \lambda_*]\} \cup \{(0, \mu) : \mu \in [0, \mu_*]\}$. On the other hand, by (4.12) and Lemma 4.4, the reversed inclusion relation is also valid. □

Now define a family of straight lines

$$L(s) = \{(\lambda, \mu) \in \mathbb{R}^2 : \mu = \lambda - s\}, \quad s \in [-\mu_*, \lambda_*]. \tag{4.13}$$

Moreover, by Lemma 4.6, for all $s \in [-\mu_*, \lambda_*]$,

$$\lambda(s) = \sup\{\lambda : (\lambda, \mu) \in L(s) \cap \overline{\operatorname{int}\mathscr{S}}\}, \quad \mu(s) = \lambda(s) - s, \quad \Gamma(s) = (\lambda(s), \mu(s))$$

are well-defined.

Lemma 4.7. *Assume that* (H9), (H10*), (H11*) *hold. Then*

- (i) $\lambda(s)$ *is nondecreasing and* $\mu(s)$ *is nonincreasing, which implies that* $\{\Gamma(s) : s \in [-\mu_*, \lambda_*]\}$ *is a simple arc;*
- (ii) $\{\Gamma(s) : s \in [-\mu_*, \lambda_*]\} \cap \{(\lambda, \mu) : \lambda\mu = 0\} = \{(\lambda_*, 0), (0, \mu_*)\}$*;*
- (iii) $\partial(\operatorname{int}\mathscr{S}) \cap (\mathbb{R}_0^+ \times \mathbb{R}_0^+) = \{\Gamma(s) : s \in (-\mu_*, \lambda_*)\}$*;*
- (iv) $\mathscr{S} = \operatorname{int}\mathscr{S} \cup \{\Gamma(s) : s \in (-\mu_*, \lambda_*)\}$.

Proof. (i) By contradiction, suppose that there exist $s_1, s_2 \in [-\mu_*, \lambda_*]$ with $s_1 < s_2$, such that $\lambda(s_1) > \lambda(s_2)$ or $\mu(s_1) < \mu(s_2)$. Without loss of generality, assume that $\lambda(s_1) > \lambda(s_2)$. Then, by the definition of $\Gamma(s_1)$, there is a $(\lambda', \mu') \in \operatorname{int}\mathscr{S}$ in the neighborhood of $(\lambda(s_1), \mu(s_1))$ with $\lambda' > \lambda(s_2)$ and $\mu' > \mu(s_2)$. On the other hand, by the definition of $\Gamma(s_2)$, there is a $(\lambda'', \mu'') \notin \operatorname{int}\mathscr{S}$ in the neighborhood of $(\lambda(s_2), \mu(s_2))$ with $\lambda(s_2) < \lambda'' < \lambda'$ and $\mu(s_2) < \mu'' < \mu'$, which implies that $(\lambda'', \mu'') \in \operatorname{int}\mathscr{S}$ in view of Lemma 4.4, a contradiction. Hence, $\lambda(s_1) \leqslant \lambda(s_2)$ and $\mu(s_1) \geqslant \mu(s_2)$, for all $s_1, s_2 \in [-\mu_*, \lambda_*]$ with $s_1 < s_2$. Moreover, combining the definition of $\Gamma(s)$ with the nondecreasing of $\lambda(s)$ and the nonincreasing of $\mu(s)$, it is easy to see that

$$|s_1 - s_2|/\sqrt{2} \leqslant |\Gamma(s_1) - \Gamma(s_2)| \leqslant |s_1 - s_2|, \quad \forall s_1, s_2 \in [-\mu_*, \lambda_*].$$

This means that both Γ and Γ^{-1} are continuous, so $\{\Gamma(s) : s \in [-\mu_*, \lambda_*]\}$ and the straight line segment $\{(\lambda, 0) : \lambda \in [-\mu_*, \lambda_*]\}$ are homeomorphic, thus $\{\Gamma(s) : s \in [-\mu_*, \lambda_*]\}$ is a simple arc.

(ii) From Lemma 4.6, it is clear that $\Gamma(-\mu_*) = (0, \mu_*)$ and $\Gamma(\lambda_*) = (\lambda_*, 0)$. Now suppose that there is a $s_0 \in (-\mu_*, \lambda_*)$, such that $\lambda(s_0) = 0$ or $\mu(s_0) = 0$. Without loss of generality, assume that $\lambda(s_0) = 0$, then $s_0 \in (-\mu_*, 0]$. By use of the monotonicity of $\lambda(s)$, we have that $\lambda(s) = 0$, for all $s \in [-\mu_*, s_0]$, which implies that $\{(0, \mu) : \mu \in (-s_0, \mu_*]\} \cap \partial(\operatorname{int} \mathscr{S})$ is an empty set. On the other hand, by Lemma 4.6, $\{(0, \mu) : \mu \in (-s_0, \mu_*]\} \subset \partial(\operatorname{int} \mathscr{S})$, a contradiction. That is, $\{\Gamma(s) : s \in (-\mu_*, \lambda_*)\} \cap \{(\lambda, \mu) : \lambda\mu = 0\}$ is empty. So, item (ii) is valid.

(iii) By the definition of $\Gamma(s)$ and (ii), we obtain that $\{\Gamma(s) : s \in (-\mu_*, \lambda_*)\} \subset \partial(\operatorname{int} \mathscr{S}) \cap (\mathbb{R}_0^+ \times \mathbb{R}_0^+)$. Contrarily, for any given $(\lambda, \mu) \in \partial(\operatorname{int} \mathscr{S}) \cap (\mathbb{R}_0^+ \times \mathbb{R}_0^+)$, we will show that $(\lambda, \mu) \in \{\Gamma(s) : s \in (-\mu_*, \lambda_*)\}$. Let $s_0 = \lambda - \mu$, then $s_0 \in (-\mu_*, \lambda_*)$. We claim that $(\lambda, \mu) = \Gamma(s_0)$. In fact, from the definition of $\Gamma(s_0)$, we have that $\lambda(s_0) \geqslant \lambda$; in addition, if $\lambda(s_0) > \lambda$, then there is a $(\overline{\lambda}, \overline{\mu}) \in \operatorname{int} \mathscr{S}$ in the neighborhood of $(\lambda(s_0), \mu(s_0))$ with $\overline{\lambda} > \lambda$ and $\overline{\mu} > \mu$, which implies that $(\lambda, \mu) \in \operatorname{int} \mathscr{S}$ by Lemma 4.4, a contradiction.

(iv) For any given $s \in (-\mu_*, \lambda_*)$, there exists a sequence $\{(\lambda_n, \mu_n)\}_{n=1}^{\infty} \subset \operatorname{int} \mathscr{S}$ such that $\lim_{n\to\infty}(\lambda_n, \mu_n) = (\lambda(s), \mu(s))$. Combining with Lemma 4.2 and Lebesgue dominated convergence theorem, $(\lambda(s), \mu(s)) \in \mathscr{S}$. In other words, $\{\Gamma(s) : s \in (-\mu_*, \lambda_*)\} \subset \mathscr{S}$. So, $\operatorname{int} \mathscr{S} \cup \{\Gamma(s) : s \in (-\mu_*, \lambda_*)\} \subset \mathscr{S}$.

On the other hand, $\mathscr{S} \subset \operatorname{int} \mathscr{S} \cup \{\Gamma(s) : s \in (-\mu_*, \lambda_*)\}$. In fact, if there is $(\lambda, \mu) \in \mathscr{S}$ and $(\lambda, \mu) \notin \operatorname{int} \mathscr{S} \cup \{\Gamma(s) : s \in (-\mu_*, \lambda_*)\}$, then by Lemma 4.4, $\{\Gamma(s) : s \in (-\mu_*, \lambda_*)\} \cap \operatorname{int} \mathscr{S}$ is nonempty, a contradiction. □

Proof of Theorem 1.11. By Lemma 4.7, $\{\Gamma(s) : s \in [-\mu_*, \lambda_*]\}$ ia a simple arc with both end $(\lambda_*, 0)$ and $(0, \mu_*)$, and separates $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ into two disjoint subsets $\operatorname{int} \mathscr{S}$ and $\mathbb{R}_0^+ \times \mathbb{R}_0^+ \backslash \mathscr{S}$. Denote

$$\mathscr{O}_1 = \operatorname{int} \mathscr{S}, \quad \Gamma_0 = \{\Gamma(s) : s \in (-\mu_*, \lambda_*)\}, \quad \mathscr{O}_2 = \mathbb{R}_0^+ \times \mathbb{R}_0^+ \setminus \mathscr{S}. \tag{4.14}$$

It is clear that $\mathscr{O}_2 \cap \mathscr{S} = \phi$, and that $\mathscr{O}_1 \cup \Gamma_0 = \mathscr{S}$ from item (iv) of Lemma 4.7. By the definition of $\mathscr{S}$ and the fact that problem (1.2) has positive solutions in which at least one component is zero if and only if $\lambda\mu = 0$ due to the condition $(H10*)$, we obtain that problem (1.2) has at least one component-wise positive solution for $(\lambda, \mu) \in \Gamma_0 \cup \mathscr{O}_1$ and no solutions for $(\lambda, \mu) \in \mathscr{O}_2$. Next, it is sufficient to show that problem (1.2) has at least two component-wise positive solutions for $(\lambda, \mu) \in \mathscr{O}_1$. In fact, by $(\lambda, \mu) \in \mathscr{O}_1$, there is a $(\overline{\lambda}, \overline{\mu}) \in \mathscr{O}_1$ such that $(\lambda, \mu) \in (0, \overline{\lambda}) \times (0, \overline{\mu})$, then the desired conclusion derives from Lemma 4.4. □

Acknowledgments. This work was partially supported by the Fundamental Research Funds for the Central Universities Grant (lzujbky-2018-112).

References

[1] R. P. Agarwal, D. O'Regan, P. J. Y. Wong; *Positive Solutions of Differential, Difference, and Integral Equations*, Kluwer Academic Publishers, Boston, 1999.

[2] H. Amann; *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Review, 18 (1976), 620-709.
[3] A. Benmezaï; *On the number of solutions of two classes of Sturm-Liouville boundary value problems*, Nonlinear Anal., 70 (2009), 1504-1519.
[4] S. R. Bernfeld, V. Lakshmikantham; *An Introduction to Nonlinear Boundary Value Problems*, Academic Press, New York, 1974.
[5] G. Bonanno, G. Riccobono; *Multiplicity results for Sturm-Liouville boundary value problems*, Appl. Math. and Comput., 210 (2009), 294-297.
[6] X. Cheng; *Existence of positive solutions for a class of second-order ordinary differential systems*, Nonlinear Anal., 69 (2008), 3042-3049.
[7] X. Cheng, Z. Feng; *Existence and multiplicity of positive solutions to systems of nonlinear Hammerstein integral equations*, Electron. J. Differential Equations, 52 (2019), 1-16.
[8] X. Cheng, G. Dai; *Positive solutions of sub-superlinear Sturm-Liouville problems*, Appl. Math. Comput., 261 (2015), 351-359.
[9] X. Cheng, H. Lü; *Multiplicity of positive solutions for a (p_1, p_2)-Laplacian system and its applications*, Nonlinear Anal. RWA, 13 (2012), 2375-2390.
[10] X. Cheng, Z. Zhang; *Existence of positive solutions to systems of nonlinear integral or differential equations*, Topol. Methods Nonlinear Anal., 34 (2009), 267-277.
[11] X. Cheng, Z. Zhang; *Positive solutions for a class of multi-parameter elliptic sytsems*, Nonlinear Anal. RWA, 14 (2013), 1551-1562.
[12] X. Cheng, Z. Feng, Z. Zhang; *Multiplicity of positive solutions to nonlinear systems of hammerstein integral equations with weighted functions*, Commun. Pure Appl. Anal., 19 (2020), 221-240.
[13] X. Cheng, C. Zhong; *Existence of positive solutions for a second-order ordinary differential system*, J. Math. Anal. Appl., 312 (2005), 14-23.
[14] K. Deimling; *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.
[15] D. R. Dunninger, H. Y. Wang; *Existence and multiplicity of positive solutions for elliptic systems*, Nonlinear Anal., 29 (1997), 1051-1060.
[16] D. R. Dunninger, H. Y. Wang; *Multiplicity of positive radial solutions for an elliptic system on an annulus*, Nonlinear Anal., 42 (2000), 803-811.
[17] L. H. Erbe, S. Hu, H.Y. Wang; *Multiple positive solutions of some boundary value problems*, J. Math. Anal. Appl., 184 (1994), 640-648.
[18] L. H. Erbe, H.Y. Wang; *On the existence of the positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc., 120 (1994), 743-748.
[19] D. G. de Figueiredo; *Positive solutions of semilinear elliptic problems*, in Proc. of the First Latin American School of Differential Equations, Springer Verlag Lecture Notes in Mathematics, Vol. 957 (1982), 34-87.
[20] D. G. de Figueiredo, P. L. Lions; *On pairs of positive solutions for a class of semilinear elliptic problems*, Indiana Univ. Math. J., 34 (1985), 591-606.
[21] D. Gilbarg, N. Trudinger; *Elliptic Partial Differential Equations of Second Order* (2nd ed.), Springer, New York, Berlin, 1983.
[22] D. Guo, V. Lakshmikantham; *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.
[23] J. Henderson, H. Y. Wang; *Positive solutions for nonlinear eigenvalue problems*, J. Math. Anal. Appl., 208 (1997) 252-259.
[24] K. Lan, R. L. Webb; *Positive solutions of semilinear equations with singularities*, J. Diff. Equations 148 (1998), 407-421.
[25] Y. H. Lee; *A multiplicity result of positive radial solutions for a multiparameter elliptic system on an exterior domain*, Nonlinear Anal., 45 (2001), 597-611.
[26] Y. H. Lee; *Multiplicity of positive radial solutions for multiparameter semilinear elliptic systems on an annulus*, J. Differential Equations, 174 (2001), 420-441.
[27] H. Li, J. Sun; *Positive solutions of sublinear Sturm-Liouville problems with changing sign nonlinearity*, Computers & Mathematics with Applications, 58 (2009), 1808-1815.
[28] Y. Li; *On the existence and nonexistence of positive solutions for nonlinear Sturm-Liouville boundary value problems*, J. Math. Anal. Appl., 304 (2005), 74-86.
[29] P. L. Lions; *On the existence of positive solutions of semilinear elliptic equations*, SIAM Review, 24 (1982), 441-467.

[30] R. Ma, G. Dai; *Global bifurcation and nodal solutions for a Sturm-Liouville problem with a nonsmooth nonlinearity*, J. Funct. Anal., 265 (2013), 1443-1459.
[31] J. M. do Ó, S. Lorca, P. Ubilla; *Local superlinearity for elliptic systems involving parameters*, J. Differential Equations, 211 (2005), 1-19.
[32] J. M. do Ó, S. Lorca, J. Sánchez, P. Ubilla; *Positive solutions for a class of multiparameter ordinary elliptic systems*, J. Math. Anal. Appl., 332 (2007), 1249-1266.
[33] J. Sun, G. Zhang; *Nontrivial solutions of singular superlinear Sturm-Liouville problems*, J. Math. Anal. Appl., 313 (2006), 518-536.
[34] J. Sun, G. Zhang; *Nontrivial solutions of singular sublinear Sturm-Liouville problems*, J. Math. Anal. Appl., 326 (2007), 242-251.
[35] H. Y. Wang; *On the number of positive solutions of nonlinear systems*, J. Math. Anal. Appl., 281 (2003), 287-306.
[36] G. T. Whyburn; *Topological Analysis*, Princeton University Press, Princeton, 1958.
[37] C. Zhong, X. Fan, W. Chen; *An Introduction to Nonlinear Functional Analysis*, Lanzhou University Press, Lanzhou, 1998.

Jinfang He
School of Mathematics and Statistics, Lanzhou University, Lanzhou, 730000, China
Email address: hejf14@lzu.edu.cn

Lu Yang (corresponding author)
School of Mathematics and Statistics, Lanzhou University, Lanzhou, 730000, China
Email address: yanglu@lzu.edu.cn