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OPTIMAL BILINEAR CONTROL FOR GROSS-PITAEVSKII EQUATIONS WITH SINGULAR POTENTIALS

KAI WANG, DUN ZHAO

Abstract. We study the optimal bilinear control problem of the generalized Gross-Pitaevskii equation

$$i\partial_t u = -\Delta u + U(x)u + \phi(t)\frac{1}{|x|^\alpha}u + \lambda|u|^{2\sigma}u, \quad x \in \mathbb{R}^3,$$

where $U(x)$ is the given external potential, $\phi(t)$ is the control function. The existence of an optimal control and the optimality condition are presented for suitable α and σ. In particular, when $1 \le \alpha < 3/2$, the Fréchet-differentiability of the objective functional is proved for two cases: (i) $\lambda < 0$, $0 < \sigma < 2/3$; (ii) $\lambda > 0$, $0 < \sigma < 2$. Comparing with the previous studies in [6], the results fill the gap for $\sigma \in (0, 1/2)$.

1. Introduction

In the study of optimal control of partial differential equations [10], optimal control of Gross-Pitaevskii (GP) equations is a new topic [5, 6, 7, 8, 9, 11, 12] which was originated from the experiments of quantum control for Bose-Einstein condensates. In this article, we consider the optimal bilinear control problem governed by the generalized GP equation

$$\begin{gathered} i\partial_t u = -\Delta u + U(x)u + \phi(t)\frac{1}{|x|^\alpha}u + \lambda|u|^{2\sigma}u, \quad (t,x) \in [0,\infty) \times \mathbb{R}^3, \\ u(0,x) = u_0(x). \end{gathered} \tag{1.1}$$

where $U(x)$ is the given external potential to confine the atoms in the experiment, $\lambda \in \mathbb{R}$, $\phi : [0, +\infty) \to \mathbb{R}$ is the control function to manipulate the the control potential $1/|x|^\alpha$. In the whole text, we assume $U \in C^\infty(\mathbb{R}^3; \mathbb{R})$ and U is subquadratic, i.e.,

$$\partial^k U \in L^\infty(\mathbb{R}^3), \quad \text{for all } |k| \ge 2.$$

In [7], the mathematical frame for the study on the optimal control of GP equation is established for the first time, the existence of an optimal control with bounded controlled potential is obtained, and under the assumption $\lambda > 0$, $\sigma \in \mathbb{N}$ with $\sigma < 2/(d-2)$, the first-order optimality conditions is derived by virtue of the Gâteaux-differentiability of the objective functional. After that, in [6], similar results are extended to Coulombian potential ($\alpha = 1$) with $1/2 \le \sigma < 2/3$ for

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$\lambda < 0$ or $1/2 \leq \sigma < 2$ for $\lambda > 0$. Furthermore, the Fréchet-differentiability with respect to the control of the objective functional is proved. However, both the Gâteaux-differentiability in [7] and the Fréchet-differentiability in [6] of the objective functionals are dependent on the local Lipschitz continuity of the solution $u(\phi)$ with respect to the control ϕ. However, when $0 < \sigma < 1/2$, the local Lipschitz continuity no longer holds, the method becomes invalid, and the corresponding results are absent.

In this note, we consider a more general unbounded control potential $|x|^{-\alpha}$ with $1 \leq \alpha < 3/2$ rather than $\alpha = 1$ in [6]. With the aid of Σ^2 regularity of the solution, through a rather elaborate analysis, we obtain a new kind of continuity estimate for the state u with respect to the control ϕ when $0 < \sigma < 1/2$ (see equation (3.3) below). Based on this estimate, we prove that the Fréchet-differentiability of the objective functional is still true for $0 < \sigma < 1/2$. This fills the gap in the results of [6].

This article is organized as follows: in section 2, some estimates and inequalities are given. In addition, we show the global existence and Σ^2 regularity of the solution; in section 3, the property of continuity of the Σ solution is discussed; and in section 4, the first-order Fréchet-differentiability of the objective functional is obtained. Besides, the rigorous characterization of the optimal control is derived.

Notation and conventions. Throughout this article, we use the abbreviations $L^r = L^r(\mathbb{R}^3)$, $W^{m,r} = W^{m,r}(\mathbb{R}^3)$, and L^2 which is equipped with the scalar product

$$\langle \zeta, \xi \rangle_{L^2} = \Re \int_{\mathbb{R}^3} \bar{\zeta}(x)\xi(x)dx,$$

where $\Re z$ denotes the real part of a complex number z. We define

$$\Sigma^m := \{u \in L^2 : x^j \nabla^k u \in L^2 \text{ for all multi-indices } j \text{ and } k \text{ with } |j| + |k| \leq m\},$$

with the norm

$$\|u\|_{\Sigma^m} = \sum_{|j|+|k| \leq m} \|x^j \nabla^k u\|_{L^2},$$

we will write Σ in stead of Σ^1, and set

$$\Sigma^{1,r} := \{u \in L^r : xu, \nabla u \in L^r\}.$$

Recall that [2] a pair of exponents (q, r) is admissible on $\mathbb{R}^N$ if $2/q = N(1/2 - 1/r)$ with $q \geq 2$. In what follows, $C > 0$ will stand for a constant that may be different from line to line when it does not cause any confusion.

2. Preliminaries

In this section, we firstly recall a Gronwall-type estimate (see [4]), which would be invoked throughout the paper. Thereafter, we study the existence and regularity of the solution of system (1.1).

Lemma 2.1 ([4]). *Assume that $B = (B_1, B_2, \dots, B_n)$, $p = (p_1, p_2, \dots, p_n)$ and $q = (q_1, q_2, \dots, q_n)$ satisfy $B_j > 0$, $1 \leq p_j < q_j \leq \infty$, for $j = 1, 2, \dots, n$. Then fore ach A, $T > 0$, there exists $\Gamma = \Gamma(T, B, p, q)$, such that if $f_j \in L^{q_j}(0, T)$ satisfy*

$$\sum_{j=1}^{n} \|f_j\|_{L^{q_j}(0,t)} \leq A + \sum_{j=1}^{n} B_j \|f_j\|_{L^{p_j}(0,t)} \quad \textit{for all } 0 < t < T,$$

then

$$\sum_{j=1}^{n} \|f_j\|_{L^{q_j}(0,T)} \leq A\Gamma\,.$$

Next, we establish the existence and regularity of the solution for system (1.1).

Lemma 2.2. *Assume that $1 \leq \alpha < 2$, $0 < \sigma < 2/3$ if $\lambda < 0$, or $0 < \sigma < 2$ if $\lambda > 0$. Let $\phi \in H^1_{\rm loc}(0,\infty)$ be a real-valued function, $U \in C^\infty(\mathbb{R}^3)$ be subquadratic. Then for every $u_0 \in \Sigma$, system (1.1) admits an unique mild solution $u \in C([0,\infty),\Sigma) \cap L^\gamma_{\rm loc}((0,\infty),\Sigma^{1,\rho})$ for all admissible pair (γ,ρ). Moreover, for every $T > 0$, we have*

$$\|\nabla u(t)\|^2_{L^2} + \|xu(t)\|^2_{L^2} \leq C_0 \exp\big\{C\big(T + T^{\frac12}\|\phi'\|_{L^2(0,T)}\big)\big\} \tag{2.1}$$

for all $t \in [0,T]$, where C_0 depends continuously on $E(0)$, $\|u_0\|_\Sigma$, $\|\phi\|_{H^1(0,T)}$ and T.

Furthermore, if $u_0 \in \Sigma^2$, then for every admissible pair (q,r), the solution of (1.1) satisfies $u \in C([0,\infty),\Sigma^2) \cap C^1([0,\infty),L^2) \cap W^{1,q}_{\rm loc}((0,\infty),L^r)$.

Proof. Firstly, we prove the local well-posedness for (1.1). The Duhamel's formulation for (1.1) reads

$$u(t) = S(t)u_0 - i\int_0^t S(t-s)\phi(s)\frac{u(s)}{|x|^\alpha}ds - i\lambda\int_0^t S(t-s)|u|^{2\sigma}u(s)ds, \tag{2.2}$$

where $S(t) = e^{-itH}$ with $H = -\Delta + U$. Since $[\nabla, H] = \nabla U$ and $[x,H] = \nabla$, then we have

$$[\nabla, S(t)] = -i\int_0^t S(t-s)\nabla U S(s)ds, \quad [x, S(t)] = -i\int_0^t S(t-s)\nabla S(s)ds. \tag{2.3}$$

We denote $\Phi(u)$ the right hand side of (2.2). It follows from (2.3) that

$$\begin{aligned}\nabla\Phi(u)(t) = S(t)\nabla u_0 &- i\int_0^t S(t-s)\phi(s)\frac{1}{|x|^\alpha}\big(\nabla u - \frac{\alpha x}{|x|^2}u\big)(s)ds\\ &- i\int_0^t S(t-s)\nabla(|u|^{2\sigma}u)(s)ds - i\int_0^t S(t-s)\nabla U\Phi(u)(s)ds,\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}x\Phi(u)(t) = S(t)xu_0 &- i\int_0^t S(t-s)\big(\phi(s)\frac{x}{|x|^\alpha}u + |u|^{2\sigma}xu\big)(s)ds\\ &- i\int_0^t S(t-s)\nabla\Phi(u)(s)ds.\end{aligned} \tag{2.5}$$

Let $1/|x|^\alpha = V_1(x) + V_2(x)$, where

$$V_1(x) = \begin{cases} 1/|x|^\alpha & |x| \leq 1\\ 0 & |x| \geq 2,\end{cases} \quad\text{and}\quad V_2(x) = \begin{cases} 0 & |x| \leq 1\\ 1/|x|^\alpha & |x| \geq 2\end{cases}$$

are nonnegative. Apparently, $V_2 \in L^\infty$ and $V_1 \in L^{\frac{3}{2-\epsilon}}$ for any $0 < \epsilon \ll \min\{1/2, 2-\alpha\}$.

We denote

$$G(u) := -i\int_0^t S(t-s)\phi(s)\frac{u(s)}{|x|^\alpha}\,ds,$$

$$G_\nabla(u) := -i\int_0^t S(t-s)\phi(s)\frac{1}{|x|^\alpha}\big(\nabla u - \frac{\alpha x}{|x|^2}u\big)(s)\,ds,$$

$$G_x(u) := -i\int_0^t S(t-s)\phi(s)\frac{xu(s)}{|x|^\alpha}\,ds,$$

and let $(q_0, r_0) = (4(\sigma+1)/3\sigma, 2\sigma+2)$, by Strichartz's estimates (see [1, Proposition 2.2]) and Hölder's inequality, there exists $l > 0$, such that

$$\begin{aligned}\|G(u)\|_{L_t^\gamma L_x^\rho(0,l)} \le& Cl^{\epsilon/2}\|\phi\|_{L^\infty(0,l)}\|v_1\|_{L^{\frac{3}{2-\epsilon}}}\|u\|_{L_t^{\frac{2}{1-\epsilon}}L_x^{\frac{6}{1+2\epsilon}}(0,l)}\\ &+ Cl\|\phi\|_{L^\infty(0,l)}\|V_2\|_{L^\infty}\|u\|_{L_t^\infty L_x^2(0,l)}.\end{aligned} \tag{2.6}$$

Similarly, we have

$$\begin{aligned}\|G_x(u)\|_{L_t^\gamma L_x^\rho(0,l)} \le& Cl^{\epsilon/2}\|\phi\|_{L^\infty(0,l)}\|v_1\|_{L^{\frac{3}{2-\epsilon}}}\|xu\|_{L_t^{\frac{2}{1-\epsilon}}L_x^{\frac{6}{1+2\epsilon}}(0,l)}\\ &+ Cl\|\phi\|_{L^\infty(0,l)}\|V_2\|_{L^\infty}\|xu\|_{L_t^\infty L_x^2(0,l)}.\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\|G_\nabla(u)\|_{L_t^\gamma L_x^\rho(0,l)} &\le C\|\phi V_1\nabla u\|_{L_t^2 L_x^{\frac{6}{5}}(0,l)} + C\|\phi_2\nabla u\|_{L_t^1 L_x^2(0,l)}\\ &\quad + C\big\|\phi V_1\frac{u}{|\cdot|}\big\|_{L_t^2 L_x^{\frac{6}{5}}(0,l)} + C\|\phi V_2\frac{u}{|\cdot|}\|_{L_t^1 L_x^2(0,l)}\\ &\le Cl^{\epsilon/2}\|\phi\|_{L^\infty(0,l)}\|v_1\|_{L^{\frac{3}{2-\epsilon}}}\|\nabla u\|_{L_t^{\frac{2}{1-\epsilon}}L_x^{\frac{6}{1+2\epsilon}}(0,l)}\\ &\quad + Cl\|\phi\|_{L^\infty(0,l)}\|V_2\|_{L^\infty}\|\nabla u\|_{L_t^\infty L_x^2(0,l)},\end{aligned} \tag{2.8}$$

where the second inequality in (2.8) holds by using Hardy's inequality (see [2, Lemma 7.6.1]).

Set $X_1 := C((0,l),\Sigma) \cap L^{q_0}((0,l),\Sigma^{1,r_0}) \cap L^{\frac{2}{1-\epsilon}}((0,l),\Sigma^{1,\frac{6}{1+2\epsilon}})$. It is easily to deduce that Φ defines a contraction mapping from a suitable ball in X_1 into itself by Choosing l sufficiently small. Then, the local-existence holds by a standard contraction mapping argument. One can find more details in [2, 6].

Now, we show the global existence. For every $T > 0$, the only obstruction to well-posedness on $[0,T]$ is the existence of a time $0 < T_0 < T$ such that $\|\nabla u(t)\|_{L^2}^2 + \|xu(t)\|_{L^2}^2 \to +\infty$ as $t \to T_0$. So the key is to prove (2.1).

It is easily to check that system (1.1) enjoys mass conservation, i.e., $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$ for all $t \in \mathbb{R}$. However, the energy is not conserved. Indeed, the energy corresponding to (1.1) may be written as:

$$\begin{aligned}E(t) =& \int_{\mathbb{R}^3}\Big(|\nabla u(t,x)|^2 + \big(U(x) + \phi(t)\frac{1}{|x|^\alpha}\big)|u(t,x)|^2\Big)dx\\ &+ \frac{\lambda}{\sigma+1}\int_{\mathbb{R}^3}|u(t,x)|^{2\sigma+2}dx,\end{aligned} \tag{2.9}$$

and its evolution reads

$$E'(t) = \phi'(t)\int_{\mathbb{R}^3}\frac{1}{|x|^\alpha}|u(t,x)|^2dx. \tag{2.10}$$

Since $U(x)$ is subquadratic, there exists a constant $C_U > 0$, such that

$$\Big|\int_{\mathbb{R}^3}U(x)|u(t,x)|^2dx\Big| \le C_U\big(\|xu(t)\|_{L^2}^2 + \|u_0\|_{L^2}^2\big).$$

When $0 < \sigma < 2/3$ and $\lambda \in \mathbb{R}$, it follows from (2.9) that

$$\begin{aligned} \|\nabla u(t)\|_{L^2}^2 \le E(0) + \int_0^t E'(s)ds + C_U\big(\|xu(t)\|_{L^2}^2 + \|u_0\|_{L^2}^2\big) \\ + |\phi(t)| \int_{\mathbb{R}^3} \frac{1}{|x|^\alpha}|u(t,x)|^2dx + \frac{|\lambda|}{\sigma+1}\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2}. \end{aligned} \tag{2.11}$$

By Hardy's inequality, we have

$$\int_{\mathbb{R}^3} \frac{1}{|x|^\alpha}|u(t,x)|^2dx \le C(\alpha)\|\nabla u\|_{L^2}^\alpha \|u_0\|_{L^2}^{2-\alpha}. \tag{2.12}$$

Gagliardo-Nirenberg's inequality implies that

$$\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2} \le C_{GN}\|\nabla u(t)\|_{L^2}^{3\sigma}\|u_0\|_{L^2}^{2-\sigma} \quad \text{for all } t \in [0,T]. \tag{2.13}$$

Substituting (2.12)-(2.13) into (2.11), and using Young's inequality, we infer that

$$\|\nabla u(t)\|_{L^2}^2 \le C_1 + 2C_U\|xu(t)\|_{L^2}^2 + \int_0^t |\phi'(s)|\|\nabla u(s)\|_{L^2}^2 ds, \tag{2.14}$$

with

$$\begin{aligned} C_1 \le |E(0)| + C_U\|u_0\|_{L^2}^2 + CT^{\frac12}\|u_0\|_{L^2}^2\|\phi'\|_{L^2(0,T)} \\ + C\big(\|\phi\|_{L^\infty(0,T)} + 1\big)^{\frac{2}{2-\alpha}} + C\|u_0\|_{L^2}^{\frac{2(2-\sigma)}{2-3\sigma}}\big(\|\phi\|_{L^\infty(0,T)} + 1\big)^{\frac{3\sigma}{2-3\sigma}}. \end{aligned} \tag{2.15}$$

where C in (2.15) depend on α, σ. Hence C_1 depends continuously on $E(0)$, $\|u_0\|_{L^2}$, $\|\phi\|_{H^1(0,T)}$ and T.

On the other hand

$$\Big|\frac{d}{dt}\|xu(t)\|_{L^2}^2\Big| = 4\Big|\Im \int_{\mathbb{R}^3} x\bar{u}(t)\nabla u(t)dx\Big| \le 8\|xu(t)\|_{L^2}^2 + \frac12\|\nabla u(t)\|_{L^2}^2, \tag{2.16}$$

where $\Im z$ denotes the imaginary part of a complex number z.

Combining (2.14) and (2.16), we have

$$\begin{aligned} &\Big|\frac{d}{dt}\|xu(t)\|_{L^2}^2\Big| + \frac12\|\nabla u(t)\|_{L^2}^2 \\ &\le C_1 + C\|xu_0\|_{L^2}^2 + \int_0^t 2\big(C + |\phi'(s)|\big)\Big(\Big|\frac{d}{dt}\|xu(s)\|_{L^2}^2\Big| + \frac12\|\nabla u(s)\|_{L^2}^2\Big)ds. \end{aligned}$$

Then, using Gronwall's inequality twice, we obtain (2.1).

When $\lambda > 0$ and $0 < \sigma < 2$, it follows from (2.9) that

$$\begin{aligned} \|\nabla u(t)\|_{L^2}^2 \le E(0) + \int_0^t E'(s)ds + C_U\big(\|xu(t)\|_{L^2}^2 + \|u_0\|_{L^2}^2\big) \\ + |\phi(t)| \int_{\mathbb{R}^3} \frac{1}{|x|^\alpha}|u(t,x)|^2dx. \end{aligned}$$

Then, by a similar but slightly simpler argument as above, we obtain (2.1).

Finally, combining [6, Proposition 2.5], [2, Theorems 4.8.1 and 5.3.1], we can obtain the Σ^2 regularity of the mild solution, we omit the details here. This completes the proof. □

3. Continuity with respect to the control

In [7], the Lipschitz property of the mild solution u with respect to the control ϕ, which is heavily depended on the nonlinearity, was used in the heart of the argument. In order to get the same property of $u(\phi)$, the authors in [6] considered the problem under the assumption $\sigma \geq 1/2$. When $0 < \sigma < 1/2$, the Lipschitz estimate is failed by following the methods of [6, 7]. In this section, we will establish a new kind of continuity estimate for the mild solution with respect to the control ϕ. Our result reads

Theorem 3.1. *Assume that $1 \leq \alpha < 2$, $0 < \sigma < 2/3$ if $\lambda \in \mathbb{R}$, or $0 < \sigma < 2$ if $\lambda > 0$. Let $u_0 \in \Sigma$, $U \in C^\infty(\mathbb{R}^3)$ be subquadratic, and u_1, u_2 be two mild solutions of (1.1) corresponding to control parameters ϕ_1, $\phi_2 \in H^1(0,T)$, respectively. Then there exists $\delta > 0$, such that when $\|\phi_1 - \phi_2\|_{H^1(0,T)} < \delta$, we have*

$$\|u_1 - u_2\|_{L_t^\gamma L_x^\rho(0,T)} \leq C\|\phi_1 - \phi_2\|_{H^1(0,T)}. \tag{3.1}$$

Furthermore, when $\sigma \geq 1/2$, we have

$$\begin{aligned} &\|\nabla u_1 - \nabla u_2\|_{L_t^\gamma L_x^\rho(0,T)} + \|xu_1 - xu_2\|_{L_t^\gamma L_x^\rho(0,T)} \\ &\leq C\|\phi_1 - \phi_2\|_{H^1(0,T)} \end{aligned} \tag{3.2}$$

for any admissible pair (γ, ρ), where $C = C(T, \gamma, \|u_0\|_\Sigma, \|\phi_1\|_{H^1(0,T)})$.

Let $0 < \sigma < 1/2$, if we assume further that $u_0 \in \Sigma^2$, then

$$\begin{aligned} &\|\nabla u_1 - \nabla u_2\|_{L_t^\gamma L_x^\rho(0,T)} + \|xu_1 - xu_2\|_{L_t^\gamma L_x^\rho(0,T)} \\ &\leq C\|\phi_1 - \phi_2\|_{H^1(0,T)} + C\|\phi_1 - \phi_2\|_{H^1(0,T)}^{2\sigma}, \end{aligned} \tag{3.3}$$

where $C = C(T, \gamma, \|u_0\|_\Sigma, \|\phi_1\|_{H^1(0,T)}, \|u_1\|_{L^\infty((0,T),\Sigma^2)})$.

Proof. Since $u_1 = u(\phi_1)$ and $u_2 = u(\phi_2)$ are two mild solutions of system (1.1), we have

$$\begin{aligned} &u_1(t) - u_2(t) \\ &= -i\int_0^t S(t-s)\big(\frac{1}{|x|^\alpha}(\phi_1 u_1 - \phi_2 u_2) + \lambda(|u_1|^{2\sigma}u_1 - |u_2|^{2\sigma}u_2)\big)(s)ds. \end{aligned} \tag{3.4}$$

Let $(q_0, r_0) = (4(\sigma+1)/3\sigma, 2\sigma+2)$ and $(q_1, r_1) = (2/(1-\epsilon), 6/(1+2\epsilon))$ be admissible pairs. For every $t \in [0,T]$, applying Strichartz's estimate to (3.4), we have

$$\begin{aligned} \|u_1 - u_2\|_{L_t^\gamma L_x^\rho(0,t)} &\leq C\|V_1(\phi_1 u_1 - \phi_2 u_2)\|_{L_t^2 L_x^{\frac{6}{5}}(0,t)} + C\|V_2(\phi_1 u_1 - \phi_2 u_2)\|_{L_t^1 L_x^2(0,t)} \\ &\quad + C\||u_1|^{2\sigma}u_1 - |u_2|^{2\sigma}u_2\|_{L_t^{q_0'} L_x^{r_0'}(0,t)}. \end{aligned}$$

It then follows from Hölder's inequality that

$$\begin{aligned} \|u_1 - u_2\|_{L_t^\gamma L_x^\rho(0,t)} &\leq C\|u_1 - u_2\|_{L_t^{q_1} L_x^{r_1}(0,t)} + C\|u_1 - u_2\|_{L_t^1 L_x^2(0,t)} \\ &\quad + C\|u_1 - u_2\|_{L_t^{q_0'} L_x^{r_0}(0,t)} + C\|\phi_1 - \phi_2\|_{H^1(0,t)}. \end{aligned} \tag{3.5}$$

This and Lemma 2.1 imply

$$\|u_1 - u_2\|_{L_t^\gamma L_x^\rho(0,T)} \leq C\|\phi_1 - \phi_2\|_{H^1(0,T)}, \tag{3.6}$$

where C depends on T, γ, $\|u_1\|_{L^\infty((0,T),\Sigma)}$, $\|u_2\|_{L^\infty((0,T),\Sigma)}$, $\|\phi_1\|_{H^1(0,T)}$.

On the other hand, combining (2.4) and (2.5), we obtain

$$\nabla u_1(t) - \nabla u_2(t)$$

$$
\begin{aligned}
&= -i\int_0^t S(t-s)\frac{1}{|x|^\alpha}\Big(\phi_1\big(\nabla u_1 - \frac{\alpha x}{|x|^2}u_1\big) - \phi_2\big(\nabla u_2 - \frac{\alpha x}{|x|^2}u_2\big)\Big)(s)ds \\
&\quad - i\int_0^t S(t-s)\big(\nabla(|u_1|^{2\sigma}u_1) - \nabla(|u_2|^{2\sigma}u_2)\big)(s)ds \\
&\quad - i\int_0^t S(t-s)\nabla U(u_1-u_2)(s)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&xu_1(t) - xu_2(t) \\
&= -i\int_0^t S(t-s)\frac{x}{|x|^\alpha}(\phi_1 u_1 - \phi_2 u_2)(s)ds - i\int_0^t S(t-s)(\nabla u_1 - \nabla u_2)(s)ds \\
&\quad - i\int_0^t S(t-s)(|u_1|^{2\sigma}xu_1 - |u_2|^{2\sigma}xu_2)(s)ds.
\end{aligned}
$$

Consider the complex-valued function $g(\xi) = |\xi|^\alpha\xi$ with $\alpha > 0$, the first-order Wirtinger derivatives $\partial_z g(\xi)$ and $\partial_{\bar z} g(\xi)$ satisfy the follow properties [3]:

$$
|\partial_z g(\xi)| \le C|\xi|^\alpha, \quad |\partial_{\bar z} g(\xi)| \le C|\xi|^\alpha,
$$

$$
|\partial_z g(\xi_1) - \partial_z g(\xi_2)| \le \begin{cases} C|\xi_1-\xi_2|^\alpha & \text{if } 0<\alpha<1, \\ C(|\xi_1|^{\alpha-1}+|\xi_2|^{\alpha-1})|\xi_1-\xi_2| & \text{if } \alpha \ge 1. \end{cases}
$$

This estimate also holds for $\partial_{\bar z} g(\xi)$. Thus if $0<\alpha<1$, we have

$$
|\nabla g(\xi_1(x)) - \nabla g(\xi_2(x))| \le C|\xi_1-\xi_2|^\alpha|\nabla\xi_1| + C|\xi_2|^\alpha\big|\nabla\xi_1 - \nabla\xi_2\big|.
$$

And if $\alpha \ge 1$, then

$$
\begin{aligned}
&|\nabla g(\xi_1(x)) - \nabla g(\xi_2(x))| \\
&\le C(|\xi_1|^{\alpha-1}+|\xi_2|^{\alpha-1})|\xi_1-\xi_2||\nabla\xi_1| + C|\xi_2|^\alpha\big|\nabla\xi_1-\nabla\xi_2\big|.
\end{aligned}
$$

Therefore, when $0<\sigma<1/2$, applying Strichartz's estimates and Hardy's inequality, we obtain

$$
\begin{aligned}
\|\nabla u_1 - \nabla u_2\|_{L_t^\gamma L_x^\rho(0,t)} &\le C\|\phi_1-\phi_2\|_{H^1(0,T)} + C\|\nabla u_1 - \nabla u_2\|_{L_t^{q_1}L_x^{r_1}(0,t)} \\
&\quad + C\|u_1-u_2\|_{L^1((0,t),H^1)} + C\|xu_1 - xu_2\|_{L_t^1 L_x^2(0,t)} \\
&\quad + C\|u_1-u_2\|^{2\sigma}_{L_t^{q_0}L_x^{r_0}(0,t)}\|\nabla u_1\|_{L_t^{\frac{4(\sigma+1)}{4+\sigma-6\sigma^2}}L_x^{r_0}(0,t)} \\
&\quad + C\|u_2\|^{2\sigma}_{L_t^\infty L_x^{r_0}(0,t)}\|\nabla u_1 - \nabla u_2\|_{L_t^{q_0'}L_x^{r_0}(0,t)}.
\end{aligned}
$$

Since $\sigma+1 < 4(\sigma+1)/(4+\sigma-6\sigma^2) < +\infty$, it follows that

$$
\|\nabla u_1\|_{L_t^{\frac{4(\sigma+1)}{4+\sigma-6\sigma^2}}L_x^{r_0}(0,T)} \le C\|u_1\|_{L^\infty((0,T),\Sigma^2)}.
$$

It thus follows from (3.6) that

$$
\begin{aligned}
&\|\nabla u_1 - \nabla u_2\|_{L_t^\gamma L_x^\rho(0,t)} \\
&\le C\|\phi_1-\phi_2\|_{H^1(0,T)} + C\|\phi_1-\phi_2\|^{2\sigma}_{H^1(0,T)} \\
&\quad + C\|\nabla u_1-\nabla u_2\|_{L_t^{q_1}L_x^{r_1}(0,t)} + C\|\nabla u_1 - \nabla u_2\|_{L_t^1 L_x^2(0,t)} \\
&\quad + C\|xu_1 - xu_2\|_{L_t^1L_x^2(0,t)} + C\|\nabla u_1 - \nabla u_2\|_{L_t^{q_0'}L_x^{r_0}(0,t)},
\end{aligned} \tag{3.7}
$$

where C depends on T, γ, $\|u_1\|_{L^\infty((0,T),\Sigma^2)}$, $\|u_2\|_{L^\infty((0,T),\Sigma)}$, $\|\phi_1\|_{H^1(0,T)}$.

On the other hand, we have

$$\begin{aligned} &\|xu_1 - xu_2\|_{L_t^\gamma L_x^\rho(0,t)} \\ &\le C\|\phi_1 - \phi_2\|_{H^1(0,T)} + C\|\nabla u_1 - \nabla u_2\|_{L_t^{q_1} L_x^{r_1}(0,t)} \\ &\quad + C\|xu_1 - xu_2\|_{L_t^{q_0'} L^{r_0}(0,t)} + C\|\nabla u_1 - \nabla u_2\|_{L_t^1 L^2(0,t)}, \end{aligned} \tag{3.8}$$

where C depends on γ, $\|u_1\|_{L^\infty((0,T),\Sigma)}$, $\|u_2\|_{L^\infty((0,T),\Sigma)}$.

Collecting (3.7) and (3.8), using Lemma 2.1, we deduce that

$$\begin{aligned} &\|\nabla u_1 - \nabla u_2\|_{L_t^\gamma L_x^\rho(0,T)} + \|xu_1 - xu_2\|_{L_t^\gamma L_x^\rho(0,T)} \\ &\le C\|\phi_1 - \phi_2\|_{H^1(0,T)} + C\|\phi_1 - \phi_2\|_{H^1(0,T)}^{2\sigma}, \end{aligned} \tag{3.9}$$

where C depends on T, γ, $\|u_1\|_{L^\infty((0,T),\Sigma^2)}$, $\|u_2\|_{L^\infty((0,T),\Sigma)}$, $\|\phi_1\|_{H^1(0,T)}$.

To prove (3.1) and (3.3), we firstly notice that $E(0) = E_\phi(0)$ depends on $\|u_0\|_\Sigma$ and $|\phi(0)|$, thus it depends on T, $\|u_0\|_\Sigma$ and $\|\phi\|_{H^1(0,T)}$. So it remains to show that there exists $\delta > 0$ such that if $\|\phi_1 - \phi_2\|_{H^1(0,T)} < \delta$, then $\|u_2\|_{L^\infty((0,T),\Sigma)} \le C\big(T, E_{\phi_1}(0), \|u_0\|_\Sigma, \|\phi_1\|_{H^1(0,T)}\big)$. Indeed, by (2.1), we know that u depends continuously on ϕ, so it can be obtained by choosing δ sufficiently small.

When $\sigma \ge 1/2$, it holds that

$$\begin{aligned} \|\nabla u_1 - \nabla u_2\|_{L_t^\gamma L_x^\rho(0,t)} &\le C\|\phi_1 - \phi_2\|_{H^1(0,T)} + C\|\nabla u_1 - \nabla u_2\|_{L_t^{q_1} L_x^{r_1}(0,t)} \\ &\quad + C\|\nabla u_1 - \nabla u_2\|_{L_t^1 L_x^2(0,t)} + C\|xu_1 - xu_2\|_{L_t^1 L_x^2(0,t)} \\ &\quad + C\|\nabla u_1 - \nabla u_2\|_{L_t^{q_0'} L_x^{r_0}(0,t)}. \end{aligned}$$

Together with (3.8), we can obtain (3.2). This completes the proof. □

4. Minimizers of the optimal control problem

For $T > 0$, we consider $H^1(0,T)$ as the real vector space of control parameter ϕ. We denote by Σ^* the dual of the energy space Σ. Let $X(0,T) := L^2((0,T),\Sigma) \cap W^{1,2}((0,T),\Sigma^*)$. then we set

$$\begin{aligned} \Lambda(0,T) := \big\{&(u,\phi) \in X(0,T) \times H^1(0,T) : \ u \text{ is a mild solution of (1.1)} \\ &\text{with } \phi(0) \in B_R\big\}, \end{aligned}$$

where $R > 0$ is a given constant and $B_R := \{\phi(0) \in \mathbb{R} : |\phi(0)| \le R\}$.

Following [7], we define the objective functional as

$$F(u,\phi) := \langle u(T), Au(T)\rangle_{L^2}^2 + \gamma_1 \int_0^T |E'(t)|^2 dt + \gamma_2 \int_0^T |\phi'(t)|^2 dt. \tag{4.1}$$

Then our optimal problem is to study the following minimizing problem

$$F^* = \min_{\Lambda(0,T)} F(u,\phi) \tag{4.2}$$

With the same argument as in [6], we can obtain the existence of a minimizer for the optimal control problem (4.2) as follows.

Lemma 4.1. *Let $1 \le \alpha < 2$, and $U \in C^\infty(\mathbb{R}^3)$ be subquadratic. Assume that $0 < \sigma < 2/3$ if $\lambda \in \mathbb{R}$, or $0 < \sigma < 2$ if $\lambda > 0$. Then, for any $T > 0$, $R > 0$ and $u_0 \in \Sigma$, the optimal control problem (4.2) has a minimizer $(u_*, \phi_*) \in \Lambda(0,T)$.*

Using the Lagrange methods in [10], we define the Lagrangian of the optimal control problem (4.2) as

$$L(u,v,\phi)=F(u,\phi)-\langle v,P(u,\phi)\rangle_{L^2_tL^2_x(0,T)},$$

where

$$P(u,\phi):=i\partial_t u+\Delta u-U(x)u-\phi(t)\frac{1}{|x|^\alpha}u-\lambda|u|^{2\sigma}u.$$

Then we can derive the adjoint equation

$$\begin{aligned}
&iv_t+\Delta v-U(x)v-\phi(t)\frac{1}{|x|^\alpha}v-\lambda(\sigma+1)|u|^{2\sigma}v-\lambda\sigma|u|^{2\sigma-2}u^2\bar v\\
&=\frac{\delta F(u,\phi)}{\delta u(t)}, \qquad (4.3)\\
&v(T)=i\frac{\delta F(u,\phi)}{\delta u(T)},
\end{aligned}$$

where $\frac{\delta F(u,\phi)}{\delta u(t)}$ and $\frac{\delta F(u,\phi)}{\delta u(T)}$ denote the first variation of $F(u,\phi)$ with respect to $u(t)$ and $u(T)$ respectively. Easily, we have

$$\begin{gathered}
\frac{\delta F(u,\phi)}{\delta u(t)}=4\gamma_1(\phi'(t))^2\Big(\int_{\mathbb{R}^3}\frac{1}{|x|^\alpha}|u(t,x)|^2dx\Big)\frac{1}{|x|^\alpha}u,\\
\frac{\delta F(u,\phi)}{\delta u(T)}=4\langle u(T),Au(T)\rangle_{L^2}Au(T).
\end{gathered}$$

Thus, (4.3) defines a cauchy problem for v with the initial data $v(T)\in L^2$. And we have the following existence results for the adjoint system.

Lemma 4.2. *Assume that $1\le\alpha<3/2$, $0<\sigma<2/3$ if $\lambda\in\mathbb{R}$, or $0<\sigma<2$ if $\lambda>0$. Let $U\in C^\infty(\mathbb{R}^3)$ be subquadratic. Then, for every $T>0$, $\phi\in H^1(0,T)$ and $u_0\in\Sigma^2$, the cauchy problem (4.3) admits a unique mild solution $v\in C([0,T];L^2)\cap L^\gamma((0,T),L^\rho)$ for all admissible pair (γ,ρ).*

Proof. It follows from Lemma 2.2 that $u\in C([0,T],\Sigma^2)$ is a mild solution for (1.1). And then, when $\alpha=1$, by the Hardy inequality, it is easily to check that $\frac{\delta F(u,\phi)}{\delta u(t)}\in L^1((0,T),L^2)$. When $1<\alpha<3/2$, there exists $\epsilon_0>0$, such that for every ball $B_0(r)\subset\mathbb{R}^3$, it holds $1/|x|^{2\alpha-2}\in L^{\frac{3}{1-2\epsilon_0}}(B_0(r))$. Combining Hölder's inequality and the Strichartz's estimates, we have

$$\int_{\mathbb{R}^3}\frac{|u(t,x)|^2}{|x|^{2\alpha}}dx\le C\Big\|\frac{1}{|\cdot|^{2\alpha-2}}\Big\|_{L^{\frac{3}{1-2\epsilon_0}}(B_0(r))}\|\nabla u\|^2_{L^{\frac{3}{1+\epsilon_0}}}+C\|\nabla u\|^2_{L^2}. \qquad (4.4)$$

Hence we deduce that $\frac{\delta F(u,\phi)}{\delta u(t)}\in L^1((0,T),L^2)$ for $1\le\alpha<3/2$. Then, we can get the local existence in time by a standard contraction mapping argument.

Multiplying (4.3) by $\bar v$, integrating over $\mathbb{R}^3$ and taking the imaginary part, we obtain

$$\frac{d}{dt}\|v(t)\|^2_{L^2}=\lambda\sigma\Im\int_{\mathbb{R}^3}|u|^{2\sigma-2}u^2\bar v^2dx+\Im\int_{\mathbb{R}^3}\frac{\delta F(u,\phi)}{\delta u(t)}\bar vdx.$$

Noting that $u\in L^\infty([0,T]\times\mathbb{R}^3)$, it then holds that

$$\frac{d}{dt}\|v(t)\|^2_{L^2}\le C+C\|v(t)\|^2_{L^2}.$$

By Gronwall's inequality, we infer that $v \in C([0,T],L^2)$. And $v \in L^\gamma((0,T),L^\rho)$ can be concluded by Strichartz's estimates. This proves the existence of a global solution. □

According to the argument of well-posedness for equation (1.1) and Theorem 3.1, for any given initial data $u_0 \in \Sigma$, u behaves as a continuous function of ϕ. Then the objective functional can be treated as a functional of ϕ, i.e., $\mathcal{F}(\phi) = F(u(\phi),\phi)$. In the following theorem, we consider the Fréchet differentiability of the objective functional $\mathcal{F}$.

Theorem 4.3. *Assume that $1 \le \alpha < 3/2$, $0 < \sigma < 2/3$ if $\lambda \in \mathbb{R}$, or $0 < \sigma < 2$ if $\lambda > 0$. Let $u_0 \in \Sigma^2$, $\phi \in H^1(0,T)$ and $U \in C^\infty(\mathbb{R}^3)$ be subquadratic, then the objective functional $\mathcal{F}(\phi)$ is Fréchet differentiable, and for any direction $h \in H^1(0,T)$,*

$$\begin{aligned} \mathcal{F}'(\phi)h = \Re \int_0^T h(t)\int_{\mathbb{R}^3} \frac{1}{|x|^\alpha}\bar{u}(t,x)v(t,x)\,dx\,dt \\ + 2\int_0^T \phi'(t)h'(t)(\gamma_2 + \gamma_1\omega^2(t))dt, \end{aligned} \tag{4.5}$$

where $v \in C([0,T];L^2(\mathbb{R}^3))$ is the solution of the adjoint equation (4.3) and $\omega(t)$ is a weight factor defined as

$$\omega(t) = \int_{\mathbb{R}^3} \frac{1}{|x|^\alpha}|u(t,x)|^2 dx. \tag{4.6}$$

Proof. Recalling the definition of Fréchet differentiability, we need to verify that

$$\mathcal{F}(\phi_1) - \mathcal{F}(\phi) = \text{linear terms in } (\phi_1 - \phi) + o(\|\phi_1 - \phi\|_{H^1(0,T)}),$$

then as $\|\phi_1 - \phi\|_{H^1(0,T)} \to 0$, the desired result can be obtained.

Consider the difference of $\mathcal{F}(\phi_1)$ and $\mathcal{F}(\phi)$, which can be written as a sum of three terms

$$\mathcal{F}(\phi_1) - \mathcal{F}(\phi) = \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3,$$

where

$$\begin{gathered} \mathcal{F}_1 := \langle u_1(T), Au_1(T)\rangle_{L^2}^2 - \langle u(T), Au(T)\rangle_{L^2}^2, \\ \mathcal{F}_2 := \gamma_2 \int_0^T \big[\big(\phi_1'(t)\big)^2 - \big(\phi'(t)\big)^2\big]dt, \\ \mathcal{F}_3 := \gamma_1 \int_0^T \big((\phi_1'(t))^2\omega_1(t)^2 - (\phi'(t))^2\omega(t)^2\big)dt, \end{gathered}$$

where $\omega_1(t)$ defined as (4.6) with u replaced by u_1.

Firstly, we consider the case $0 < \sigma < 1/2$, we start from the term $\mathcal{F}_1$, which can be written in the form

$$\begin{aligned} \mathcal{F}_1 =& 2\langle u(T), Au(T)\rangle_{L^2}\big(\langle u_1(T), Au_1(T)\rangle_{L^2} - \langle u(T), Au(T)\rangle_{L^2}\big) \\ &+ \big(\langle u_1(T), Au_1(T)\rangle_{L^2} - \langle u(T), Au(T)\rangle_{L^2}\big)^2. \end{aligned} \tag{4.7}$$

By the essential self-adjointness of A, we have

$$\begin{aligned} &\langle u_1(T), Au_1(T)\rangle_{L^2} - \langle u(T), Au(T)\rangle_{L^2} \\ &= 2\langle u_1(T) - u(T), Au(T)\rangle_{L^2} + \langle u_1(T) - u(T), A\big(u_1(T) - u(T)\big)\rangle_{L^2}. \end{aligned}$$

It follows from Theorem 3.1 that

$$\begin{aligned}\big|\langle u_1(T)-u(T),A\big(u_1(T)-u(T)\big)\rangle_{L^2}\big| &\le\|u_1(T)-u(T)\|_{L^2}\|A\|_{\mathfrak{L}}\|u_1(T)-u(T)\|_\Sigma\\ &\le C\|\phi_1-\phi\|^2_{H^1(0,T)}+C\|\phi_1-\phi\|^{2\sigma+1}_{H^1(0,T)}.\end{aligned}$$

Substituting this into (4.7), we obtain

$$\mathcal{F}_1=4\langle u(T),Au(T)\rangle_{L^2}\langle u_1(T)-u(T),Au(T)\rangle_{L^2}+o(\|\phi_1-\phi_2\|_{H^1(0,T)}). \tag{4.8}$$

The quadratic expansion of ϕ_1 is given by

$$(\phi_1'(t))^2=(\phi'(t))^2+2\phi'(t)\big(\phi_1'(t)-\phi'(t)\big)+\big(\phi_1'(t)-\phi'(t)\big)^2.$$

It then holds that

$$\mathcal{F}_2=2\gamma_2\int_0^T\phi'(t)\big(\phi_1'(t)-\phi'(t)\big)dt+O(\|\phi_1-\phi\|^2_{H^1(0,T)}). \tag{4.9}$$

Finally, we consider $\mathcal{F}_3$. Note that

$$\omega_1(t)=\omega(t)+2\Re\int_{\mathbb{R}^3}\frac{1}{|x|^\alpha}\big(\bar u(u_1-u)\big)(t,x)dx+\int_{\mathbb{R}^3}\frac{1}{|x|^\alpha}|u_1-u|^2(t,x)dx.$$

Using Hölder's inequality, Hardy's inequality and Theorem 3.1, we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^3}\frac{1}{|x|^\alpha}|u_1-u|^2(t,x)dx\\ &\le C\|\nabla u_1-\nabla u\|^\alpha_{L^2}\|u_1-u\|^{2-\alpha}_{L^2}\\ &\le C\|\phi_1-\phi\|^{2-\alpha}_{H^1(0,T)}(\|\phi_1-\phi\|_{H^1(0,T)}+\|\phi_1-\phi\|^{2\sigma}_{H^1(0,T)})^\alpha\\ &=o(\|\phi_1-\phi\|_{H^1(0,T)}).\end{aligned}$$

Hence

$$\omega_1(t)^2=\omega(t)^2+4\omega(t)\Re\int_{\mathbb{R}^3}\frac{1}{|x|^\alpha}\big(\bar u(u_1-u)\big)(t,x)dx+o(\|\phi_1-\phi\|_{H^1(0,T)}).$$

Therefore,

$$\begin{aligned}\mathcal{F}_3&=\gamma_1\int_0^T(\phi_1'(t))^2\big(\omega_1^2(t)-\omega^2(t)\big)dt+\gamma_1\int_0^T\big((\phi_1'(t))^2-(\phi'(t))^2\big)\omega^2(t)dt\\ &=4\gamma_1\int_0^T(\phi'(t))^2\omega(t)\Re\int_{\mathbb{R}^3}\frac{1}{|x|^\alpha}\big(\bar u(u_1-u)\big)(t,x)\,dx\,dt\\ &\quad+2\gamma_1\int_0^T\phi'(t)\big(\phi_1'(t)-\phi'(t)\big)\omega^2(t)dt+o(\|\phi_1-\phi\|_{H^1(0,T)}).\end{aligned}$$

The adjoint equation (4.3) yields

$$\begin{aligned}&4\gamma_1\int_0^T(\phi'(t))^2\omega(t)\Re\int_{\mathbb{R}^3}\frac{1}{|x|^\alpha}\big(\bar u(u_1-u)\big)(t,x)\,dx\,dt\\ &=\Re\int_0^T\int_{\mathbb{R}^3}\bar vQ(u_1-u)(t,x)\,dx\,dt-\Re\int_{\mathbb{R}^3}i\bar v(T,x)(u_1-u)(T,x)dx,\end{aligned} \tag{4.10}$$

where v is the solution of (4.3) and

$$\begin{aligned}Q(u_1-u)=&i\partial_t(u_1-u)+\Delta(u_1-u)-U(x)(u_1-u)-\phi(t)\frac{1}{|x|^\alpha}(u_1-u)\\ &-\lambda(\sigma+1)|u|^{2\sigma}(u_1-u)-\lambda\sigma|u|^{2\sigma-2}u^2(\overline{u_1-u}).\end{aligned}$$

Sine $u,\, u_1 \in C([0,T];\Sigma^2) \cap W^{1,\infty}((0,T);L^2)$, the right hand side of (4.10) is well defined. Moreover, it is easily to check that the last term of the right hand side of (4.10) is equal to $-\mathcal{F}_1 + o(\|\phi_1 - \phi_2\|_{H^1(0,T)})$.

By the fact that u and u_1 are solutions of (1.1), we infer that

$$Q(u_1 - u) = (\phi_1(t) - \phi(t))\frac{1}{|x|^\alpha}u_1 + \mathcal{R}(u_1, u), \tag{4.11}$$

where the remainder is given by

$$\frac{1}{\lambda}\mathcal{R}(u_1,u) = |u_1|^{2\sigma}u_1 - |u|^{2\sigma}u - (\sigma+1)|u|^{2\sigma}(u_1 - u) - \sigma|u|^{2\sigma-2}u^2(\overline{u_1 - u}).$$

Since $0 < \sigma < 1/2$, the remainder $\mathcal{R}(u_1,u)$ can be bounded by

$$|\mathcal{R}(u_1,u)| \le C|u_1 - u|^{2\sigma+1}.$$

Let $(q_0, r_0) = (4(\sigma+1)/3\sigma, 2\sigma+2)$, and in view of Theorem 3.1, we obtain

$$\begin{aligned}
&\Big|\Re\int_0^T\int_{\mathbb{R}^3}\bar{v}\mathcal{R}(u_1,u)(t,x)\,dx\,dt\Big| \\
&\le C\|v\|_{L_t^{q_0'}L_x^{r_0}(0,T)}\|u_1 - u\|_{L_t^{q_0}L_x^{r_0}(0,T)}\|u_1 - u\|^{2\sigma}_{L_t^{\infty}L_x^{r_0}(0,T)} \\
&\le C\|\phi_1 - \phi\|_{H^1(0,T)}\big(\|\phi_1 - \phi\|_{H^1(0,T)} + \|\phi_1 - \phi\|^{2\sigma}_{H^1(0,T)}\big)^{2\sigma} \\
&= o(\|\phi_1 - \phi\|_{H^1(0,T)}).
\end{aligned} \tag{4.12}$$

On the other hand,

$$(\phi_1(t) - \phi(t))\frac{1}{|x|^\alpha}u_1 = (\phi_1(t) - \phi(t))\frac{1}{|x|^\alpha}u - (\phi_1(t) - \phi(t))\frac{1}{|x|^\alpha}(u_1 - u). \tag{4.13}$$

Thus, using (4.4) for $1 < \alpha < 3/2$ and Hardy's inequality for $\alpha = 1$, from Theorem 3.1 it follows that

$$\begin{aligned}
&\int_0^T\big(\phi_1(t) - \phi(t)\big)\Re\int_{\mathbb{R}^3}\frac{1}{|x|^\alpha}\bar{v}(u_1 - u)(t,x)\,dx\,dt \\
&\le C\int_0^T\big(\phi_1(t) - \phi(t)\big)\|v(t)\|_{L^2}\Big(\int_{\mathbb{R}^3}\frac{|u_1 - u|^2}{|x|^{2\alpha}}(t,x)dx\Big)^{\frac{1}{2}}dt \\
&= o(\|\phi_1 - \phi\|_{H^1(0,T)}).
\end{aligned} \tag{4.14}$$

Combining (4.10)-(4.14), we deduce that

$$\begin{aligned}
\mathcal{F}_3 = &-\mathcal{F}_1 + \int_0^T\big(\phi_1(t) - \phi(t)\big)\Re\int_{\mathbb{R}^3}\frac{1}{|x|^\alpha}\bar{v}u(t,x)\,dx\,dt \\
&+ 2\gamma_1\int_0^T\phi'(t)\big(\phi_1'(t) - \phi'(t)\big)\omega^2(t)dt + o(\|\phi_1 - \phi\|_{H^1(0,T)}).
\end{aligned} \tag{4.15}$$

Collecting (4.8), (4.9) and (4.15), we obtain (4.5) by taking $\|\phi_1 - \phi\|_{H^1(0,T)} \to 0$.

When $\sigma \ge 1/2$, the argument is slightly simpler. Indeed, by Theorem 3.1, $u_1 - u$ has the Lipschitz property as (3.1) and (3.2). Therefore, any higher order (at least quadratic) error of $\|u_1 - u\|$ is bounded by $O\big(\|\phi_1 - \phi\|^2_{H^1(0,T)}\big)$. Then (4.5) can be derived by the same argument as above. This completes the proof. □

Assume that (u_*, ϕ_*) is a minimizer of the optimal control problem (4.2), and if $\phi \in H^1(0,T)$ satisfies $(\phi - \phi_*)(0) = (\phi - \phi_*)(T) = 0$, it holds that $\mathcal{F}'(\phi_*)(\phi - \phi_*) = 0$, see [7]. Furthermore, if we assume that the ϕ_* is sufficiently smooth, we have the following characterization, which is the control equation corresponding to our optimal control problem.

Corollary 4.4. *Assume that* $(u_*, \phi_*) \in \Lambda(0,T)$ *be a minimizer of the control problem* (4.2)*. Let* v_* *be the corresponding solution of the adjoint equation* (4.3)*. Also, denote by* ω_* *the function defined in* (4.6) *with* u *replaced by* u_**. Then* $\phi_* \in C^2[0,T]$ *is a classical solution of the following ordinary differential equation*

$$\frac{d}{dt}\big(\phi_*'(t)(\gamma_2 + \gamma_1 \omega_*^2(t)\big) = \frac{1}{2}\Re \int_{\mathbb{R}^3} \frac{1}{|x|^\alpha} \bar{v}_*(t,x) u_*(t,x)\, dx\,,$$

subject to the initial data $\phi_*(0) = \phi_0$ *and* $\phi_*'(T) = 0$*.*

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References

[1] R. Carles; *Nonlinear Schrödinger equation with time dependent potential*, Commun. Math. Sci., 9 (2011), 937-964.

[2] T. Cazenave; *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics, vol.10, New York University, Courant Institute of Mathematical Sciences, American Math. Society, Providence, 2003.

[3] T. Cazenave, D. Fang, Z. Han; *Continuous dependence for NLS in fractional order spaces*, Ann. I. H. Poincaré Anal. Non Linéaire, 28 (2011), 135-147.

[4] T. Cazenave, M. Scialom; *A Schrödinger equation with time-oscillating nonlinearity*, Rev. Mat. Complut., 23 (2010), 321-339.

[5] B. Feng, K. Wang; *Optimal bilinear control of nonlinear Hartree equations with singular potentials*. J. Optim. Theory Appl., 170 (2016) no. 3, 756-771.

[6] B. Feng, D. Zhao; *Optimal bilinear control of Gross-Pitaevskii equations with Coulombian potentials*, J. Differential Equations, 260 (2016), 2973-2993.

[7] M. Hintermüller, D. Marahrens, P. A. Markowich, C. Sparber; *Optimal bilinear control of Gross-Pitaevskii equations*, SIAM J. Control Optim., 51 (2013), 2509-2543.

[8] U. Hohenester, P. K. Rekdel, A. Borzi, J. Schmiedmayer; *Optimal quantum control of Bose-Einstein condensates in magnetic microtraps*, Phys. Rev. A, 75 (2007), 023602.

[9] J. F. Mennemann, D. Matthes, R. M. Weishaup, T. Langen; *Optimal control of Bose-Einstein condensates in three dimensions*, New J. Phys., 17 (2015), 113027.

[10] F. Tröltzsch; *Optimal control of partial differential equations: theory, methods and applications*, vol. 112. American Math. Society, Providence, 2010.

[11] K. Wang, D. Zhao, B. Feng; *Optimal nonlinearity control of Schrödinger equation*, Evol. Equ. Control Theory, 7 (2018), no. 2, 317-334.

[12] K. Wang, D. Zhao, B. Feng; *Optimal bilinear control of the coupled nonlinear Schrödinger system*. Nonlinear Anal. Real World Appl., 47 (2019), 142-167.

Kai Wang
School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China.
School of Physical Science and Technology, Lanzhou University, Lanzhou 730000, China
Email address: `wkai558@163.com`

Dun Zhao (corresponding author)
School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China
Email address: `zhaod@lzu.edu.cn`