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MARKOV SEMIGROUP APPROACH TO THE ANALYSIS OF A NONLINEAR STOCHASTIC PLANT DISEASE MODEL

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Abstract. In this article, we consider a stochastic plant disease model with logistic growth and saturated incidence rate. We analyze long-term behaviors of densities of the distributions of the solution. On the basis of the theory of Markov semigroup, we obtain the existence of asymptotically stable stationary distribution density of the stochastic system. We demonstrate that the densities can converge in L^1 to an invariant density under appropriate conditions. Moreover, we obtain the sufficient conditions for extinction of the disease. Also, we present a series of numerical simulations to illustrate our theoretical results.

1. Introduction

Plants not only provide necessary basic living materials for humans, but also provide food and shelter for other species on Earth. However, plants are invaded by various diseases during their growth and development, causing huge crop losses and global threats to food security during the spread of plant diseases [30]. Plant viruses or pathogens are an important constraint to crop production worldwide and cause serious losses in agricultural production and economic efficiency. For example, the cassava plant, which is a staple in many lesser-developed African countries, is vulnerable to the cassava mosaic virus. This virus has ravaged plants in Kenya, Uganda and Tanzania[9]. In India, tomato leaf curl disease (TLCD) causes the leaves of tomato plants to curl and may become sterile[9]. In the United States, the annual loss caused by plant diseases accounts for about 15% of the total agricultural production, that is, more than $15 billion[16]. At present, plant diseases are still important diseases that endanger human health and have not been effectively controlled. In order to prevent plant disease disasters, people are constantly studying the disease mechanism, transmission laws and prevention and control strategies of plant diseases. The dynamics of infectious diseases is an important method for the theoretical quantitative research of the epidemic of infectious diseases. Meanwhile, mathematics plays a large role in studying the dynamic behaviors of infectious diseases. Since the pioneer work of Kermack and McKendrick[11], mathematical models have been contributing to improve our understanding of the dynamics of infectious diseases and helping us develop preventive measures to control infection spread qualitatively and quantitatively. Therefore, plant diseases have attracted

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the interest of many mathematical modeling researchers and epidemiologists (see [3, 6, 7, 13, 18, 19, 20, 21, 29, 33]). For example, Meng and Li have investigated the dynamic behaviors of the vegetatively propagated plant disease models with continuous and impulsive cultural control strategies (see[18]). The scholars in [7] studied the invasion and persistence of plant pathogens. Here, a plant disease model with logistic growth and saturated incidence rate is described by

$$\begin{aligned} \mathrm{d}S_t &= \Big[rS_t\big(1-\frac{S_t}{K}\big)-\frac{\beta S_t I_t}{1+\alpha S_t}+\gamma I_t\Big]\mathrm{d}t, \\ \mathrm{d}I_t &= \Big[\frac{\beta S_t I_t}{1+\alpha S_t}-(\mu+\gamma)I_t\Big]\mathrm{d}t, \end{aligned} \tag{1.1}$$

where the parameters r, K, α, β, γ, μ are positive constants. In model (1.1), $S(t)$ and $I(t)$ denote the number of susceptible and infected plants, respectively, r is an intrinsic growth rate of susceptible plants, K represents the carrying capacity of susceptible plants, β is the transmission coefficient, α denotes potentially density dependent, γ is the recovery rate of the cured diseased plants, and μ is the disease-related death rate of the infected plants. The basic reproduction number $R_0 = \frac{\beta K}{(1+\alpha K)(\mu+\gamma)}$ is the threshold of system (1.1) for an epidemic to occur. If $0 < R_0 < 1$, model (1.1) has a unique disease-free equilibrium $E_0 = (K, 0)$ which is globally asymptotically stable. This means the plants disease will disappear and the entire plant population will become susceptible. If $R_0 > 1$, E_0 becomes unstable and system (1.1) has a unique positive equilibrium

$$E^* = (S^*, I^*) = \Big(\frac{\mu+\gamma}{\beta-\alpha(\mu+\gamma)}, \frac{rS^*}{\mu}\big(1-\frac{S^*}{K}\big)\Big)$$

which is globally asymptotically stable. This means the plants disease always remains.

However, any system is always subject to environmental noise in nature. Plant disease models that have a significant impact on human survival are inevitably affected by environmental fluctuations in the ecosystem. Therefore, it is more realistic to study the stochastic model than to study the deterministic model. Consequently, many researchers have investigated the effect of environmental noise on stochastic models (see[5, 10, 14, 15, 22, 25, 32]). There are several approaches to study the effect of environmental noises on the dynamic behaviors of stochastic models. For example, Pasquali [22] discussed the stability in distributions of solutions of stochastic logistic equations by solving the explicit solution of the corresponding Fokker-Planck equations. Cai et al. [5] obtained thresholds of the stochastic SIRS model which determine the extinction and persistence by using the theory of Markov semigroups. Based on the discussion above, in this paper, we consider a stochastic plant disease model

$$\begin{aligned} \mathrm{d}S_t &= \Big[rS_t\big(1-\frac{S_t}{K}\big)-\frac{\beta S_t I_t}{1+\alpha S_t}+\gamma I_t\Big]\mathrm{d}t-\frac{\sigma S_t I_t}{1+\alpha S_t}\mathrm{d}B_t, \\ \mathrm{d}I_t &= \Big[\frac{\beta S_t I_t}{1+\alpha S_t}-(\mu+\gamma)I_t\Big]\mathrm{d}t+\frac{\sigma S_t I_t}{1+\alpha S_t}\mathrm{d}B_t, \end{aligned} \tag{1.2}$$

where $B(t)$ is independent standard Brownian motion with $B(0) = 0$ and $\sigma^2(t)$ is the intensities of Wiener processes $B(t)$.

In this article, we discuss the long-time dynamical behaviors of system (1.2). Particularly, as the main purpose, we will investigate the extinction and the existence of stable stationary distribution density by establishing the corresponding sufficient conditions. Furthermore, we will validate the main conclusions obtained in this paper by the numerical simulations.

This article is organized as follows. In Section 2, we present some auxiliary definitions and results concerning Markov semigroups. In Section 3, we prove that there exists a unique global positive solution of system (1.2). In section 4, we obtain the sufficient conditions for extinction of model (1.2). In Section 5, we investigate the existence of an invariant and asymptotically stable density of system (1.2). In section 6, we give the main conclusions and make numerical simulations to illustrate our conclusions.

2. Preliminaries

In this section, we provide some auxiliary definitions and results about Markov semigroups and asymptotic properties (see [26, 27, 28, 23, 12]) to prove our main results.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0, 1 \leq i \leq n\}$.

Markov semigroups. Let $\mathbb{X} = \mathbb{R}^2_+$, $\Sigma = \mathfrak{B}$ be the σ-algebra of Borel subset of $\mathbb{X}$ and m be the Lebesgue measure on $(\mathbb{X}, \Sigma)$. Let the triple $(\mathbb{X}, \Sigma, m)$ be a σ-finite measure space. Denote $\mathbb{D}$ be the subset of the space $L^1 = L^1(\mathbb{X}, \Sigma, m)$ which contains all densities, i.e.

$$\mathbb{D} = \{f \in L^1 : f \geq 0, \|f\| = 1\}.$$

A linear mapping $P : L^1 \to L^1$ is called a Markov operator if $P(\mathbb{D}) \subset \mathbb{D}$.

The Markov operator P is called an integral or kernel operator if there exists a measurable function $\mathcal{K} : \mathbb{X} \times \mathbb{X} \to [0, \infty)$ such that

$$Pf(x) = \int_{\mathbb{X}} \mathcal{K}(x, y) f(y) m(dy)$$

for every density f. One can check that from the condition $P(\mathbb{D}) \subset \mathbb{D}$ it follows that

$$\int_{\mathbb{X}} \mathcal{K}(x, y) m(dx) = 1 \tag{2.1}$$

for all $y \in \mathbb{X}$.

A a Markov semigroup is a family $\{P(t)\}_{t\geq 0}$ of Markov operators that satisfies the following conditions:

(a) $P(0) = \mathrm{I}d$,
(b) $P(t + s) = P(t)P(s)$ for s, $t \geq 0$,
(c) The function $t \to P(t)f$ is continuous with respect to the L^1 norm for each $f \in L^1$.

A Markov semigroup $\{P(t)\}_{t\geq 0}$ is called integral, if for each $t > 0$, the operator $P(t)$ is an integral Markov operator. That is, there exists a measurable function $\mathcal{K} : (0, \infty) \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$, called a kernel, such that

$$Pf(x) = \int_{\mathbb{X}} \mathcal{K}(x, y) f(y) m(dy)$$

for every density f.

We also need two definitions concerning the asymptotic behavior of a Markov semigroup. A density f_* is called invariant if $P(t)f_* = f_*$ for each $t > 0$. The Markov semigroup $\{P(t)\}_{t\geq 0}$ is called asymptotically stable if there is an invariant density f_* such that

$$\lim_{t\to\infty} \|P(t)f - f_*\| = 0 \quad \text{for } f \in \mathbb{D}.$$

A Markov semigroup $\{P(t)\}_{t\geq 0}$ is called sweeping with respect to a set $A \in \Sigma$ if for every $f \in \mathbb{D}$

$$\lim_{t\to\infty} \int_A P(t)f(x)m(dx) = 0.$$

The following lemma summarizes some result concerning asymptotic stability and sweeping.

Lemma 2.1 ([26, 27])**.** *Let X be a metric space and Σ be the σ-algebra of Borel sets. Let $\{P(t)\}_{t\geq 0}$ be an integral Markov semigroup with a continuous kernel $\mathcal{K}(t;x;y)$ for $t > 0$, which satisfies* (2.1) *for all $y \in X$. We assume that for every $f \in D$ we have*

$$\int_0^\infty P(t)f dt > 0 \quad a.e.$$

Then this semigroup is asymptotically stable or is sweeping with respect to compact sets.

The property that a Markov semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable or sweeping from a sufficiently large family of sets is called the Foguel alternative [12].

Fokker-Planck equation. For $A \in \Sigma$, we denote the transition probability function by $\mathcal{P}(t, x, y, A)$ for the diffusion process (S_t, I_t), i.e.

$$\mathcal{P}(t, x, y, A) = \text{prob}\{(S_t, I_t) \in A\}$$

with the initial condition $(S_0, I_0) = (x, y)$. Assume that (S_t, I_t) is a solution of system (1.2) such that the distribution of (S_0, I_0) is absolutely continuous and has the density $U(t, x, y)$. Then (S_t, I_t) also has the density $U(t, x, y)$ and U satisfies the Fokker-Planck equation [28],

$$\frac{\partial U}{\partial t} = \frac{1}{2}\sigma^2\Big(\frac{\partial^2(\varphi U)}{\partial x^2} - 2\frac{\partial^2(\varphi U)}{\partial x \partial y} + \frac{\partial^2(\varphi U)}{\partial y^2}\Big) - \frac{\partial(f_1 U)}{\partial x} - \frac{\partial(f_2 U)}{\partial y}, \tag{2.2}$$

where $\varphi(x, y) = x^2y^2/(1 + \alpha x)^2$ and

$$\begin{gathered} f_1(x, y) = rx\big(1 - \frac{x}{K}\big) - \frac{\beta xy}{1 + \alpha x} + \gamma y, \\ f_2(x, y) = \frac{\beta xy}{1 + \alpha x} - (\mu + \gamma)y. \end{gathered} \tag{2.3}$$

Now we introduce a Markov semigroup associated with (2.2). Let $P(t)V(x, y) = U(x, y, t)$ for $V \in \mathbb{D}$. Since the operator $P(t)$ is a contraction on $\mathbb{D}$, it can be extended to a contraction on L^1. Thus the operators $\{P(t)\}_{t\geq 0}$ form a Markov semigroup. Let $\mathcal{A}$ be the infinitesimal generator of semigroup $\{P(t)\}_{t\geq 0}$, i.e.

$$\mathcal{A}V = \frac{1}{2}\sigma^2\Big(\frac{\partial^2(\varphi U)}{\partial x^2} - 2\frac{\partial^2(\varphi U)}{\partial x \partial y} + \frac{\partial^2(\varphi U)}{\partial y^2}\Big) - \frac{\partial(f_1 U)}{\partial x} - \frac{\partial(f_2 U)}{\partial y}.$$

The adjoint operator of $\mathcal{A}$ is of the form

$$\mathcal{A}^*V = \frac{1}{2}\sigma^2\varphi\Big(\frac{\partial^2(U)}{\partial x^2} - 2\frac{\partial^2(U)}{\partial x\partial y} + \frac{\partial^2(U)}{\partial y^2}\Big) - \frac{\partial(f_1U)}{\partial x} - \frac{\partial(f_2U)}{\partial y}. \tag{2.4}$$

3. Global positive solution

We first prove the existence and uniqueness of positive solution of system (1.2).

Theorem 3.1. *For each initial value $(S_0, I_0) \in \mathbb{R}^2_+$, there is a unique positive solution (S_t, I_t) of system (1.2) on $t \geq 0$ and the solution will remain in $\mathbb{R}^2_+$ with probability one, namely, $(S_t, I_t) \in \mathbb{R}^2_+$ for all $t \geq 0$ almost surely.*

Proof. Note that the coefficients of system (1.2) are locally Lipschitz conditions, then for any given initial value $(S_0, I_0) \in \mathbb{R}^2_+$, there is a unique positive local solution (S_t, I_t) on $t \in [0, \tau_e)$, where τ_e is the explosion time. To demonstrate that this solution is global, we only need to prove that $\tau_e = \infty$ a.s.

Let $k_0 > 0$ be sufficiently large for any initial value S_0 and I_0 lying within the interval $[1/k_0, k]$. For each integer $k \geq k_0$, define the following stopping time

$$\tau_k = \inf\big\{t \in [-\omega, \tau_e) : S_t \notin \big(\frac{1}{k}, k\big),\ \text{or}\ I_t \notin \big(\frac{1}{k}, k\big)\big\},$$

where we set $\inf\emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_k is increasing as $k \to \infty$. Let $\tau_\infty = \lim_{k\to\infty}\tau_k$, hence $\tau_\infty \leq \tau_k$ a.s. Next, we only need to verify $\tau_\infty = \infty$ a.s. If this statement is false, then there exist two constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \epsilon.$$

Thus there is an integer $k_1 \geq k_0$ such that

$$\mathbb{P}\{\tau_k \leq T\} \geq \epsilon, \quad k \geq k_1.$$

Define a C^2-function $V : \mathbb{R}^2_+ \to \mathbb{R}_+$ as follows

$$V(S_t, I_t) = S_t - \frac{\mu}{\beta} - \frac{\mu}{\beta}\ln\frac{\beta S_t}{\mu} + I_t - 1 - \ln I_t,$$

the nonnegativity of this function can be obtained from $u - 1 - \ln u \geq 0$, $u > 0$. Applying Itô's formula yields

$$\mathrm{d}V(S_t, I_t) = LV(S_t, I_t)\mathrm{d}t + \frac{\sigma\big(2S_tI_t - S_t - \frac{\mu}{\beta}I_t\big)}{1 + \alpha S_t}dB_t,$$

where

$$\begin{aligned}
LV(S_t, I_t) =&\big(1 - \frac{\mu}{\beta S_t}\big)\Big[rS_t\big(1 - \frac{S_t}{K}\big) - \frac{\beta S_tI_t}{1 + \alpha S_t} + \gamma I_t\Big] + \frac{\sigma^2 I_t^2}{2(1 + \alpha S_t)^2} \\
&+ \big(1 - \frac{1}{I_t}\big)\Big[\frac{\beta S_tI_t}{1 + \alpha S_t} - (\mu + \gamma)I_t\Big] + \frac{\sigma^2 S_t^2}{2(1 + \alpha S_t)^2} \\
=&rS_t - \frac{r}{K}S_t^2 - \mu I_t - \frac{r\mu}{\beta} + \frac{r\mu}{K\beta}S_t + \frac{\mu I_t}{1 + \alpha S_t} - \frac{\mu\gamma}{\beta}\frac{I_t}{S_t} - \frac{\beta S_t}{1 + \alpha S_t} \\
&+ \mu + \gamma + \frac{\sigma^2 I_t^2}{2(1 + \alpha S_t)^2} + \frac{\sigma^2 S_t^2}{2(1 + \alpha S_t)^2} \\
\leq& -\Big[\frac{r}{K}S_t^2 - \big(\frac{r\mu}{K\beta} + r\big)S_t\Big] + \mu + \gamma + \frac{1}{2}\sigma^2(S_t^2 + I_t^2).
\end{aligned}$$

From system (1.2), we have

$$\frac{\mathrm{d}\,(S_t+I_t)}{\mathrm{d}t}+\mu(S_t+I_t)=rS_t\big(1-\frac{S_t}{K}\big)+\mu S_t\leq\frac{K(r+\mu)^2}{4r}=P.$$

Then

$$S_t+I_t\leq\frac{P}{\mu}+e^{-\mu t}\Big(S_0+I_0-\frac{K(r+\mu)^2}{4r\mu}\Big).$$

For each $t\leq\tau_k$ and k, we have

$$S_t+I_t\leq\begin{cases}\frac{P}{\mu}, & \text{if } S_0+I_0\leq\frac{K(r+\mu)^2}{4r\mu},\\ S_0+I_0, & \text{if } S_0+I_0>\frac{K(r+\mu)^2}{4r\mu}.\end{cases}$$

Therefore,

$$LV(S_t,I_t)\leq-\big[\frac{r}{K}S_t^2-(\frac{r\mu}{K\beta}+r)S_t\big]+\mu+\gamma+\frac{\sigma^2P^2}{\mu^2}:=M_0,$$

where M_0 is a positive constant.

The rest of the proof follows that in [8], se we omit it here. $\square$

4. Extinction

The aim of this section is to investigate the stochastic extinction of the plant disease in system (1.2). We denote

$$R_0^s=R_0-\frac{\sigma^2K^2}{2(1+\alpha K)^2(\mu+\gamma)}.$$

Theorem 4.1. *Let (S_t,I_t) be a solution of system (1.2) with any given initial value $(S_0,I_0)\in\mathbb{R}_+^2$. If*

$$\sigma^2>\max\big\{\frac{\beta(1+\alpha K)}{K},\frac{\beta^2}{2(\mu+\gamma)}\big\},\tag{4.1}$$

or

$$R_0^s<1\quad\textit{and}\quad\sigma^2\leq\frac{\beta(1+\alpha K)}{K},\tag{4.2}$$

then

$$\limsup_{t\to\infty}\frac{\ln I_t}{t}\leq-a<0\quad a.s.,$$

where $a=(\mu+\gamma)-\frac{\beta^2}{2\sigma^2}$ under condition (4.1) and $a=(\mu+\gamma)(1-R_0^s)$ corresponding to conditions (4.2). In other words, the disease I_t dies out with probability one.

Proof. By Itô's formula, we have

$$d\ln I_t=\Big[\frac{\beta S_t}{1+\alpha S_t}-(\mu+\gamma)-\frac{\sigma^2S_t^2}{2(1+\alpha S_t)^2}\Big]dt+\frac{\sigma S_t}{1+\alpha S_t}dB_t.\tag{4.3}$$

Integrating both sides of (4.3) from 0 to t gives

$$\ln I_t=\ln I_0+\int_0^t\Big[\frac{\beta S_s}{1+\alpha S_s}-(\mu+\gamma)-\frac{\sigma^2S_s^2}{2(1+\alpha S_s)^2}\Big]ds+\int_0^t\frac{\sigma S_s}{1+\alpha S_s}dB_s.$$

Note that $M(t)=\int_0^t\frac{\sigma S_s}{1+\alpha S_s}dB_s$ implies

$$\frac{\langle M,M\rangle_t}{t}=\frac{1}{t}\int_0^t\frac{\sigma^2S_s^2}{(1+\alpha S_s)^2}ds\leq\frac{\sigma^2K^2}{(1+\alpha K)^2}<+\infty.$$

By the strong law of large numbers for martingales [17], we have $\limsup_{t\to\infty}\frac{M(t)}{t}=0$ a.s. Under conditions (4.1), we obtain

$$
\begin{aligned}
\ln I_t &= \ln I_0 + \int_0^t \Big[\frac{\beta S_s}{1+\alpha S_s} - (\mu+\gamma) - \frac{\sigma^2 S_s^2}{2(1+\alpha S_s)^2}\Big] ds + \int_0^t \frac{\sigma S_s}{1+\alpha S_s} dB_s \\
&= \ln I_0 + \int_0^t \Big[-\frac{\sigma^2}{2}\Big(\frac{S_s}{1+\alpha S_s} - \frac{\beta}{\sigma^2}\Big)^2 + \frac{\beta^2}{2\sigma^2} - (\mu+\gamma)\Big] ds \\
&\quad + \int_0^t \frac{\sigma S_s}{1+\alpha S_s} dB_s \\
&\le \ln I_0 + \int_0^t \Big[\frac{\beta^2}{2\sigma^2} - (\mu+\gamma)\Big] ds + M(t).
\end{aligned}
\tag{4.4}
$$

Taking the limit superior of both sides of (4.4), we obtain

$$
\limsup_{t\to\infty} \frac{\ln I_t}{t} \le \frac{\beta^2}{2\sigma^2} - (\mu+\gamma) < 0 \ a.s.
$$

Using the condition (4.2), one obtains

$$
\begin{aligned}
\ln I_t &= \ln I_0 + \int_0^t \Big[\frac{\beta S_s}{1+\alpha S_s} - (\mu+\gamma) - \frac{\sigma^2 S_s^2}{2(1+\alpha S_s)^2}\Big] ds + \int_0^t \frac{\sigma S_s}{1+\alpha S_s} dB_s \\
&\le \ln I_0 + \int_0^t \Big[\frac{\beta K}{1+\alpha K} - (\mu+\gamma) - \frac{\sigma^2 K^2}{2(1+\alpha K)^2}\Big] ds + \int_0^t \frac{\sigma S_s}{1+\alpha S_s} dB_s \\
&= \ln I_0 + \int_0^t [(\mu+\gamma)(R_0^s - 1)] ds + M(t).
\end{aligned}
$$

Hence,

$$
\limsup_{t\to\infty} \frac{\ln I_t}{t} \le (\mu+\gamma)(R_0^s - 1) < 0 \ a.s.
$$

Therefore, $\lim_{t\to\infty} I_t = 0$,a.s. The proof is complete. ☐

In system (1.2), $R_0^s = R_0 - \frac{\sigma^2 K^2}{2(1+\alpha K)^2(\mu+\gamma)} \le R_0$. From the condition (4.2) of Theorem 4.1, we can easily conclude that the disease I_t will disappear and the disease I_t will also become extinct in the corresponding deterministic model. Therefore, if the noises are not considered, then R_0^s is coincide with R_0 of the deterministic system (1.1).

5. Asymptotic stability of stationary distribution density

In this section, our aim is to investigate the existence of an invariant and asymptotically stable density of model (1.2).

Theorem 5.1. *Let (S_t, I_t) be a solution of system (1.2) with any given initial value $(S_0, I_0) \in \mathbb{R}_+^2$. For every $t > 0$, the distribution of (S_t, I_t) has a density $U(t, x, y)$. If*

$$
R_0^s > 1, \quad \frac{\alpha\beta S^*}{\mu(1+\alpha S^*)} + a < \frac{r}{K} + \frac{1}{K - S^*},
$$

and

$$
p_3 < \min\Big\{\Big[\frac{r}{K} + r\big(1 - \frac{S^*}{K}\big)\Big(\frac{\gamma}{K} - \frac{\alpha\beta S^*}{\mu(1+\alpha S^*)} - a\Big)\Big](S^*)^2, a\mu(I^*)^2\Big\},
$$

where

$$p_3=\gamma\big(1-\frac{S^*}{K}\big)\big(\frac{P}{\mu}-I^*\big)+\frac{\sigma^2 S^* P^2}{2\mu^2},\quad a=\frac{\beta K}{r(1+\alpha K)(K-S^*)},$$
$$S^*=\frac{\mu+\gamma}{\beta-\alpha(\mu+\gamma)},\quad I^*=\frac{rS^*}{\mu}\big(1-\frac{S^*}{K}\big).$$

Then there exists a unique density $U_*(t,x,y)$ which is a stationary solution of system (1.2) *and*

$$\lim_{t\to\infty}\iint_{\mathbb{R}^2_+}|U(t,x,y)-U_*(t,x,y)|\,dx\,dy=0.$$

In addition, we have

$$\Xi:=\operatorname{supp}U_*=\big\{(x,y)\in\mathbb{R}^2_+:0<x+y<\frac{P}{\mu}\big\}.$$

The strategy of the proof of theorem 5.1 is as follows. First, using the Hörmander condition [4] we show that the transition function of the process (S_t,I_t) is absolutely continuous. Then, we prove that the density of the transition function is positive on $\mathbb{R}^2_+$ via using support theorems [1, 31, 2]. Next, we verify that the Markov semigroup satisfies the "Foguel alternative". Finally, we exclude sweeping by showing that there exists a Khasminskiǐ function. We realize this strategy by lemma 5.2–5.6.

Lemma 5.2. *The transition probability function $\mathcal{P}(t,x_0,y_0,A)$ has a continuous density $\mathcal{K}(t,x,y;x_0,y_0)$ with respect to the Lebesgue measure.*

Proof. In the proof of this lemma, we use the Hörmander theorem (see [4]) on the existence of smooth densities of the transition probability function for degenerate diffusion processes. If $a(x)$ and $b(x)$ are vector fields on $\mathbb{R}^d$, then the Lie bracket $[a,b]$ is a vector field given by

$$[a,b]_j(x)=\sum_{k=1}^{d}\Big(a_k\frac{\partial b_j}{\partial x_k}(x)-b_k\frac{\partial a_j}{\partial x_k}(x)\Big),\quad j=1,2,\dots,d.$$

Let

$$a_1(x,y)=\begin{pmatrix} rx(1-\frac{x}{K})-\frac{\beta xy}{1+\alpha x}+\gamma y\\ \frac{\beta xy}{1+\alpha x}-(\mu+\gamma)y\end{pmatrix},\quad a_2(x,y)=\begin{pmatrix}-\frac{\sigma xy}{1+\alpha x}\\ \frac{\sigma xy}{1+\alpha x}\end{pmatrix}.$$

Then, by direct calculations, the Lie bracket $[a_1,a_2]$ is a vector field given by

$$[a_1,a_2]=\begin{pmatrix}-\frac{\sigma y}{(1+\alpha x)^2}(rx-\frac{r}{K}x^2+\gamma y)+\frac{\sigma xy}{1+\alpha x}\big(r+\mu-\frac{2r}{K}x\big)\\ \frac{\sigma y}{(1+\alpha x)^2}(rx-\frac{r}{K}x^2+\gamma y)\end{pmatrix}.$$

Consequently

$$\begin{aligned}|a_2[a_1,a_2]|&=\begin{vmatrix}-\frac{\sigma xy}{1+\alpha x} & -\frac{\sigma y}{(1+\alpha x)^2}\big(rx-\frac{r}{K}x^2+\gamma y\big)+\frac{\sigma xy}{1+\alpha x}(r+\mu-\frac{2r}{K}x)\\ \frac{\sigma xy}{1+\alpha x} & \frac{\sigma y}{(1+\alpha x)^2}\big(rx-\frac{r}{K}x^2+\gamma y\big)\end{vmatrix}\\ &=-\frac{\sigma^2x^2y^2}{(1+\alpha x)^2}\big(r+\mu-\frac{2r}{K}x\big).\end{aligned}$$

For each $(x,y)\in\mathbb{R}_+/\frac{K(\mu+r)}{2r}\times\mathbb{R}_+$, vectors $a_2(x,y)$, $[a_1,a_2](x,y)$ span the space $\mathbb{R}^2_+$. In view of Hörmander Theorem, the transition probability function $\mathcal{P}(t,x_0,y_0,A)$

has a continuous density $\mathcal{K}(t, x, y; x_0, y_0)$ and

$$\mathcal{K} \in C^\infty\Big((0,\infty) \times \big(\mathbb{R}_+/\frac{K(\mu+r)}{2r} \times \mathbb{R}_+\big) \times \big(\mathbb{R}_+/\frac{K(\mu+r)}{2r} \times \mathbb{R}_+\big)\Big).$$

This completes the proof. $\square$

Lemma 5.3. *Let $\mathbb{E} = \big(\mathbb{R}_+/\frac{K(\mu+r)}{2r} \times \mathbb{R}_+\big)$. For each $(x_0, y_0) \in \mathbb{E}$ and $(x, y) \in \mathbb{E}$, there exists $T > 0$ such that $\mathcal{K}(T, x, y; x_0, y_0) > 0$.*

Proof. We briefly describe the method based on support theorems (see [1, 31, 2]) for checking positivity of $\mathcal{K}$. Itô's SDEs in system (1.2) need to be rewritten in the Stratonovitch's form

$$\begin{aligned} \mathrm{d}S_t &= f_1(S_t, I_t)\mathrm{d}t - \frac{\sigma S_t I_t}{1+\alpha S_t} \circ \mathrm{d}B_t, \\ \mathrm{d}I_t &= f_2(S_t, I_t)\mathrm{d}t + \frac{\sigma S_t I_t}{1+\alpha S_t} \circ \mathrm{d}B_t, \end{aligned}$$

where

$$\begin{aligned} f_1(x,y) &= rx\big(1-\frac{x}{K}\big) - \frac{\beta xy}{1+\alpha x} + \gamma y + \frac{1}{2}\Big[\frac{\sigma^2 x^2 y}{(1+\alpha x)^2} - \frac{\sigma^2 x y^2}{(1+\alpha x)^3}\Big], \\ f_2(x,y) &= \frac{\beta xy}{1+\alpha x} - (\mu+\gamma)y - \frac{1}{2}\Big[\frac{\sigma^2 x^2 y}{(1+\alpha x)^2} - \frac{\sigma^2 x y^2}{(1+\alpha x)^3}\Big]. \end{aligned} \tag{5.1}$$

Fix a point $(x_0, y_0) \in \mathbb{E}$ and a continuous function $\phi \in L^2([0,T];\mathbb{R})$, consider the following system of integral equations

$$\begin{aligned} x_\phi(t) &= x_0 + \int_0^t \Big(f_1(x_\phi(s), y_\phi(s)) - \sigma\phi\frac{x_\phi(s)y_\phi(s)}{1+\alpha x_\phi(s)}\Big)\mathrm{d}s, \\ y_\phi(t) &= y_0 + \int_0^t \Big(f_2(x_\phi(s), y_\phi(s)) + \sigma\phi\frac{x_\phi(s)y_\phi(s)}{1+\alpha x_\phi(s)}\Big)\mathrm{d}s, \end{aligned} \tag{5.2}$$

Denote $D_{x_0,y_0;\phi}$ be the Frechét derivative of the function $h \mapsto X_{\phi+h}(T)$ from $L^2([0,T];\mathbb{R})$ to $\mathbb{R}^2_+$, where $X_{\phi+h} = [x_{\phi+h}, y_{\phi+h}]^T$. If for some $\phi \in L^2([0,T];\mathbb{R})$ the derivative $D_{x_0,y_0;\phi}$ has rank 2, then $\mathcal{K}(T, x, y; x_0, y_0) > 0$ for $(x, y) = (x_\phi(T), y_\phi(T))$. The derivative $D_{x_0,y_0;\phi}$ can be found by means of the perturbation method for ODEs. In other words, let

$$\Psi = \mathbf{f}'(x_\phi, y_\phi) + \mathbf{g}'(x_\phi, y_0)\phi,$$

where $\mathbf{f}'$ and $\mathbf{g}'$ are the Jacobians of

$$\mathbf{f} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} -\frac{\sigma xy}{1+\alpha x} \\ \frac{\sigma xy}{1+\alpha x} \end{pmatrix},$$

respectively. For $0 \le t_0 \le t \le T$, let $Q(t, t_0)$ be a matrix function such that $Q(t_0, t_0) = I$, $\frac{\partial Q(t,t_0)}{\partial t} = \Psi(t)Q(t, t_0)$. Then

$$D_{x_0,y_0;\phi}h = \int_0^T Q(T,s)\mathbf{g}(s)h(s)ds.$$

Since we consider a continuous control function ϕ, the system (5.2) can be replaced by the following system of differential equations

$$\begin{aligned} x_\phi' &= f_1(x_\phi, y_\phi) - \sigma\phi\frac{x_\phi y_\phi}{1+\alpha x_\phi},\\ y_\phi' &= f_2(x_\phi, y_\phi) + \sigma\phi\frac{x_\phi y_\phi}{1+\alpha x_\phi}. \end{aligned} \tag{5.3}$$

Step 1. We claim that the rank of $D_{x_0,y_0;\phi}$ is 2. Let $\epsilon \in (0,T)$ and $h(t) = \frac{1+\alpha x_\phi(t)}{x_\phi(t)y_\phi(t)}\mathbf{1}_{[T-\epsilon,T]}$ for $t \in [0,T]$, where $\mathbf{1}_{[T-\epsilon,T]}$ is the characteristic function of interval $[T-\epsilon,T]$. By Taylor expansion, we obtain

$$Q(T,s) = I + \Psi(T)(s-T) + o((T-s)).$$

Then

$$\begin{aligned} D_{x_0,y_0;\phi}h &= \int_0^T [I + \Psi(T)(s-T) + o((T-s))]g(s)h(s)ds\\ &= \epsilon\mathbf{v} + \frac{\epsilon^2}{2}\Psi(T)\mathbf{v} + o(\epsilon^2), \end{aligned}$$

where $\mathbf{v} = (-\sigma,\sigma)^T$, and

$$\begin{aligned} \Psi(T)\mathbf{v} = \sigma\big(&-r + \frac{2r}{K}x + \gamma + \beta\frac{\partial a}{\partial x} - \frac{1}{2}\frac{\partial b}{\partial x} - \beta\frac{\partial a}{\partial y} + \frac{1}{2}\frac{\partial b}{\partial y} + \sigma\frac{\partial a}{\partial x}\phi - \sigma\frac{\partial a}{\partial y}\phi\\ &- \mu - \gamma - \beta\frac{\partial a}{\partial x} + \frac{1}{2}\frac{\partial b}{\partial x} + \beta\frac{\partial a}{\partial y} - \frac{1}{2}\frac{\partial b}{\partial y} - \sigma\frac{\partial a}{\partial x}\phi + \sigma\frac{\partial a}{\partial y}\phi\big), \end{aligned}$$

where $a(x,y) = \frac{xy}{1+\alpha x}$, $b(x,y) = \frac{\sigma^2x^2y}{(1+\alpha x)^2} - \frac{\sigma^2xy^2}{(1+\alpha x)^3}$. Since

$$\begin{aligned} |\mathbf{v}\ \Psi(T)\mathbf{v}| &= \begin{vmatrix} -\sigma & -\sigma[-r + \frac{2r}{K}x + \gamma + \beta\frac{\partial a}{\partial x} - \frac{1}{2}\frac{\partial b}{\partial x} - \beta\frac{\partial a}{\partial y} + \frac{1}{2}\frac{\partial b}{\partial y} + \sigma\frac{\partial a}{\partial x}\phi - \sigma\frac{\partial a}{\partial y}\phi]\\ \sigma & \sigma\big[-\mu - \gamma - \beta\frac{\partial a}{\partial x} + \frac{1}{2}\frac{\partial b}{\partial x} + \beta\frac{\partial a}{\partial y} - \frac{1}{2}\frac{\partial b}{\partial y} - \sigma\frac{\partial a}{\partial x}\phi + \sigma\frac{\partial a}{\partial y}\phi\big] \end{vmatrix}\\ &= -\frac{\sigma^2x^2y^2}{(1+\alpha x)^2}\Big(r + \mu - \frac{2r}{K}x\Big) \neq 0. \end{aligned}$$

Then $\mathbf{v}$, $\Psi(T)\mathbf{v}$ are linearly independent and the derivative $D_{x_0,y_0;\phi}$ is rank 2.

Step 2. We prove that for any two points $(x_0,y_0) \in \mathbb{E}$ and $(x_1,y_1) \in \mathbb{E}$, there exist a control function ϕ and $T > 0$ such that $(x_\phi(0), y_\phi(0)) = (x_0,y_0)$ and $(x_\phi(T), y_\phi(T)) = (x,y)$. Let $z_\phi = x_\phi + y_\phi$. then system (5.2) becomes

$$\begin{aligned} x_\phi'(t) &= g_1(x_\phi(t), z_\phi(t)) - \sigma\phi\frac{x_\phi(t)(z_\phi(t) - x_\phi(t))}{1+\alpha x_\phi(t)},\\ z_\phi'(t) &= g_2(x_\phi(t), z_\phi(t)), \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} g_1(x,z) &= f_1(x, z-x),\\ g_2(x,z) &= (r+\mu)x - \frac{r}{K}x^2 - \mu z. \end{aligned}$$

Let

$$\Xi_0 = \big\{(x,z) \in \mathbb{E} : 0 < x < K, 0 < z < \frac{P}{\mu}, \text{ and } x < z\big\}.$$

Now we prove for any $(x_0,z_0) \in \Xi_0$ and $(x_1,z_1) \in \Xi_0$, there exist a control function ϕ and $T > 0$ such that $(x_\phi(0), z_\phi(0)) = (x_0,z_0)$ and $(x_\phi(T), z_\phi(T)) = (x_1,z_1)$.

We construct the control function ϕ in the following way. First, we find a positive constant T and a differentiable function

$$z_\phi : [0,T] \to \big(0, \frac{P}{\mu}\big),$$

such that $z_\phi(0) = z_0$, $z_\phi(T) = z_1$, $z'_\phi(0) = g_2(x_0, z_0) = z_0^d$, $z'_\phi(T) = g_2(x_1, z_1) = z_T^d$ and

$$0 < (r+\mu)x - \frac{r}{K}x^2 - \mu z = z'_\phi(t) = rx - \frac{r}{K}x^2 - \mu z_\phi(t) < P - \mu z_\phi(t), \tag{5.5}$$

for $t \in [0,T]$. To do this, we split the construction of the function z_ϕ on three intervals $[0,\epsilon]$, $[\epsilon, T-\epsilon]$ and $[T-\epsilon, T]$, where $0 < \epsilon < \frac{T}{2}$. Let

$$\xi = \frac{1}{2}\min\big\{z_0 - 0, z_1 - 0, \frac{P}{\mu} - z_0, \frac{P}{\mu} - z_1\big\}.$$

If $z_\phi \in \big(0+\xi, \frac{P}{\mu} - \xi\big)$, we have

$$(r+\mu)x - \frac{r}{K}x^2 - \mu z_\phi(t) < -\mu\xi < 0, \quad P - \mu z_\phi(t) > \mu\xi > 0 \quad \text{for } t \in [0,T]. \tag{5.6}$$

Hence, it follows from (5.6) that we can construct a C^2-function $z_\phi : [0,\epsilon] \to (0+\xi, \frac{P}{\mu} - \xi)$ such that

$$z_\phi(0) = z_0, z'_\phi(0) = z_0^d, z'_\phi(\epsilon) = 0$$

and z_ϕ satisfies (5.5) for $t \in [0,\epsilon]$. Analogously, we construct a C^2-function z_ϕ : $[T-\epsilon, T] \to \Big(0+\xi, \frac{P}{\mu} - \xi\Big)$ such that

$$z_\phi(T) = z_1, z'_\phi(T) = z_T^d, z'_\phi(T-\epsilon) = 0$$

and z_ϕ satisfies (5.5) for $t \in [T-\epsilon, T]$.

Taking T sufficiently large, we can extend the function

$$z_\phi : [0,\epsilon] \cap [T-\epsilon, T] \to \big(0+\xi, \frac{P}{\mu} - \xi\big)$$

to a C^2-function z_ϕ defined on the whole interval $[0,T]$ such that

$$(r+\mu)x - \frac{r}{K}x^2 - \mu z_\phi(t) < -\mu\xi < z'_\phi(t) < \mu\xi < P - \mu z_\phi(t) \quad \text{for } [\epsilon, T-\epsilon]$$

and therefore, the function z_ϕ satisfies (5.5) on $[0,T]$. It follows that we can find a C^1-function x_ϕ which satisfies the second equation of (5.3) and finally we can determine a continuous function ϕ from the first equation of (5.3). This completes the proof. $\square$

Lemma 5.4. *Assume that $R_0^s > 1$. For the semigroup $\{P(t)\}_{t\geq 0}$ and every density f, we have*

$$\lim_{t\to\infty} \iint_{\Xi} P(t)f(x,y)\,dx\,dy = 1.$$

Proof. We substitute $J_t = S_t + I_t$. Then system (1.2) can be replaced by

$$\begin{aligned} \mathrm{d}S_t &= \Big[rS_t\big(1 - \frac{S_t}{K}\big) - \frac{\beta S_t(J_t - S_t)}{1+\alpha S_t} + \gamma(J_t - S_t)\Big]\mathrm{d}t - \frac{\sigma S_t(J_t - S_t)}{1+\alpha S_t}\mathrm{d}B_t, \\ \mathrm{d}J_t &= \Big[(r+\mu)S_t - \frac{r}{K}S_t^2 - \mu J_t\Big]\mathrm{d}t. \end{aligned} \tag{5.7}$$

Since (S_t, I_t) is a positive solution of system (1.2) with probability one, from the second equation of (5.7), we obtain

$$0 < \frac{dJ_t}{dt} < P - \mu J_t, \quad t \in (0, \infty) \text{ a.s.} \tag{5.8}$$

Now we claim that for almost every $\omega \in \Omega$, there exists $t_0 = t_0(\omega)$ such that

$$0 < J_t(\omega) < \frac{P}{\mu}, \quad t > t_0.$$

In fact, there are two possible cases.

(1) $J_0 \in (0, \frac{P}{\mu})$. In this case, our claim is evident from (5.8).

(2) $J_0 \in (\frac{P}{\mu}, +\infty)$. Suppose that our assertion does not hold. Then there exists $\Omega' \subset \Omega$ with $\text{prob}(\Omega') > 0$ such that

$$\lim_{t\to\infty} J_t(\omega) = \frac{P}{\mu}, \quad \omega \in \Omega'.$$

From the second equation of (5.7), we obtain, for any $\omega \in \Omega'$,

$$J_t(\omega) = e^{-\mu t}\Big(J_0 + \int_0^t e^{\mu s}[(r+\mu)S_s(\omega) - \frac{r}{K}S_s^2(\omega)]ds\Big),$$

hence, for any $\omega \in \Omega'$,

$$\lim_{t\to\infty} S_t(\omega) = K, \quad \lim_{t\to\infty} I_t(\omega) = \frac{P}{\mu} - K.$$

namely,

$$\lim_{t\to\infty} \frac{\ln I_t - \ln I_0}{t} = 0, \quad \omega \in \Omega'.$$

By Itô's formula, we have

$$d\ln I_t = \Big[\frac{\beta S_t}{1+\alpha S_t} - (\mu+\gamma) - \frac{\sigma^2 S_t^2}{2(1+\alpha S_t)^2}\Big]dt + \frac{\sigma S_t}{1+\alpha S_t}dB_t.$$

Hence,

$$\begin{aligned}
\lim_{t\to\infty} \frac{\ln I_t - \ln I_0}{t} &= \lim_{t\to\infty}\Big(\frac{1}{t}\int_0^t \Big(\frac{\beta K}{1+\alpha K} - (\mu+\gamma) - \frac{\sigma^2 K^2}{2(1+\alpha K)^2}\Big)ds \\
&\quad + \frac{1}{t}\int_0^t \frac{\sigma S_s}{1+\alpha S_t}dB_s\Big) \\
&= \frac{\beta K}{1+\alpha K} - (\mu+\gamma) - \frac{\sigma^2 K^2}{2(1+\alpha K)^2} \\
&= (\mu+\gamma)(R_0^s - 1) \\
&> 0, \quad \text{a.s. on } \Omega',
\end{aligned}$$

which is contradiction. Thus our claim holds for $J_0 \in (\frac{P}{\mu}, +\infty)$. $\square$

From Lemmas 5.3 and 5.4, we know that if the Fokker-Planck equation (2.2) has a stationary solution U_*, then $\operatorname{supp} U_* = \Xi$.

Lemma 5.5. *Assume that $R_0^s > 1$. The semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable or sweeping with respect to compact sets.*

Proof. By Lemma 5.2, it follows that $\{P(t)\}_{t\geq 0}$ is an integral Markov semigroup with a continuous kernel $\mathcal{K}(t,x,y,x_0,y_0)$ for $t>0$. According to Lemma 5.3, for every $f\in\mathbb{D}$ we have

$$\int_0^\infty P(t)f\,dt>0,\quad \text{a.s. on } \Xi.$$

From Lemma 5.4, we know that it is sufficient to demonstrate the restriction of the semigroup $\{P(t)\}_{t\geq 0}$ to the space $L^1(\Xi)$. From Lemma 2.1 it follows immediately that the semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable or is sweeping with respect to compact sets. $\square$

Lemma 5.6. *If* $R_0^s>1$, $\frac{\alpha\beta S^*}{\mu(1+\alpha S^*)}+a<\frac{r}{K}+\frac{1}{K-S^*}$, *and*

$$p_3<\min\Big\{\Big[\frac{r}{K}+r\Big(1-\frac{S^*}{K}\Big)\Big(\frac{\gamma}{K}-\frac{\alpha\beta S^*}{\mu(1+\alpha S^*)}-a\Big)\Big](S^*)^2, a\mu(I^*)^2\Big\},$$

where

$$p_3=\gamma\big(1-\frac{S^*}{K}\big)\big(\frac{P}{\mu}-I^*\big)+\frac{\sigma^2S^*P^2}{2\mu^2},\quad a=\frac{\beta K}{r(1+\alpha K)(K-S^*)},$$
$$S^*=\frac{\mu+\gamma}{\beta-\alpha(\mu+\gamma)},\quad I^*=\frac{rS^*}{\mu}\big(1-\frac{S^*}{K}\big).$$

Then the semigroup $\{P(t)\}_{t\geq 0}$ *is asymptotically stable.*

Proof. According to Lemma 5.5, the semigroup $\{P(t)\}_{t\geq 0}$ satisfies the Foguel alternative. To exclude sweeping it is sufficient to construct a non-negative C^2-function V and a closed set $O\in\Sigma$ such that

$$\sup_{(S,I)\in\mathbb{X}\setminus O}\mathcal{A}^*V<0.$$

The function V is called a Khasminskiǐ function [24]. Using similar arguments to those in [24], the existence of a Khasminskiǐ function implies that the semigroup is not sweeping from the set O, which will complete the proof. In fact, when $R_0>1$, there is an endemic equilibrium E^* of system (1.1). Then we have

$$\begin{gathered}rS^*\big(1-\frac{S^*}{K}\big)-\frac{\beta S^*I^*}{1+\alpha S^*}+\gamma I^*=0,\\ \frac{\beta S^*I^*}{1+\alpha S^*}-(\mu+\gamma)I^*=0.\end{gathered}\tag{5.9}$$

Let

$$V(S,I)=\Big(S-S^*-S^*\ln\frac{S}{S^*}\Big)+a\frac{(S-S^*+I-I^*)^2}{2}:=V_1+aV_2.$$

Then V is a nonnegative C^2-function. It follows from (2.4) and (5.9) that

$$\begin{aligned}\mathcal{A}^*V_1=&\frac{S-S^*}{S}\big[rS\big(1-\frac{S}{K}\big)-\frac{\beta SI}{1+\alpha S}+\gamma I\big]+\frac{\sigma^2S^*I^2}{2(1+\alpha S)^2}\\ =&(S-S^*)\big[r-\frac{r}{K}S-\frac{\beta I}{1+\alpha S}+\gamma\frac{I}{S}\big]+\frac{\sigma^2S^*I^2}{2(1+\alpha S)^2}\\ =&(S-S^*)\Big[-\frac{r}{K}(S-S^*)-\frac{\beta(1+\alpha S^*)(I-I^*)-\alpha\beta I^*(S-S^*)}{(1+\alpha S^*)(1+\alpha S)}\\ &+\gamma\frac{S^*(I-I^*)-I^*(S-S^*)}{S^*S}\Big]+\frac{\sigma^2S^*I^2}{2(1+\alpha S)^2}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{r}{K}(S-S^*)^2 + \frac{\alpha\beta I^*}{(1+\alpha S^*)(1+\alpha S)}(S-S^*)^2 - \frac{\beta}{1+\alpha S}(S-S^*)(I-I^*) \\
&\quad - \frac{\gamma I^*}{S^* S}(S-S^*)^2 + \frac{\gamma}{S}(S-S^*)(I-I^*) + \frac{\sigma^2 S^* I^2}{2(1+\alpha S)^2} \\
&\le -\Big[\frac{r}{K}\big[1+\gamma\big(1-\frac{S^*}{K}\big)\big] - \frac{\alpha\beta I^*}{(1+\alpha S^*)}\Big](S-S^*)^2 - \frac{\beta}{1+\alpha K}(S-S^*)(I-I^*) \\
&\quad + \gamma\big(1-\frac{S^*}{K}\big)\big(\frac{P}{\mu}-I^*\big) + \frac{\sigma^2 S^* P^2}{2\mu^2},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{A}^* V_2 =&(S-S^*+I-I^*)[rS - \frac{r}{K}S^2 - \mu I] \\
=&(S-S^*+I-I^*)[r(S-S^*) - \frac{r}{K}(S^*+S)(S-S^*) - \mu(I-I^*)] \\
\le& r\big(1-\frac{S^*}{K}\big)(S-S^*)^2 + r\big(1-\frac{S^*}{K}\big)(S-S^*)(I-I^*) - \mu(I-I^*)^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathcal{A}^* V =&\mathcal{A}^* V_1 + a\mathcal{A}^* V_2 \\
\le& -\Big[\frac{r}{K}\big[1+\gamma\big(1-\frac{S^*}{K}\big)\big] - \frac{\alpha\beta I^*}{(1+\alpha S^*)} - ar\big(1-\frac{S^*}{K}\big)\Big](S-S^*)^2 \\
& - a\mu(I-I^*)^2 + \gamma\big(1-\frac{\mu S^*}{P}\big)\big(\frac{P}{\mu}-I^*\big) + \frac{\sigma^2 S^* P^2}{2\mu^2} \\
:=& -p_1(S-S^*)^2 - p_2(I-I^*)^2 + p_3.
\end{aligned}
$$

Conditions $R_0^s > 1$, $\frac{\alpha\beta S^*}{\mu(1+\alpha S^*)} + a < \frac{r}{K} + \frac{1}{K-S^*}$, and

$$
p_3 < \min\Big\{\big[\frac{r}{K} + r\big(1-\frac{S^*}{K}\big)\big(\frac{\gamma}{K} - \frac{\alpha\beta S^*}{\mu(1+\alpha S^*)} - a\big)\big](S^*)^2, a\mu(I^*)^2\Big\}.
$$

It then follows that the ellipsoid

$$
-p_1(S-S^*)^2 - p_2(I-I^*)^2 + p_3 = 0
$$

lies entirely in $\mathbb{X}$. Therefore there exist a closed set $O \in \Sigma$ which contains this ellipsoid and $c > 0$ such that

$$
\sup_{(S,I)\in\mathbb{X}\setminus O} \mathcal{A}V \le -c < 0.
$$

The proof is complete. □

6. Numerical simulations and conclusions

We present some numerical examples that illustrate our main results. We employ the following discrete equations

$$
\begin{aligned}
S_{k+1} =&S_k + \Big[rS_k\big(1-\frac{S_k}{K}\big) - \frac{\beta S_k I_k}{1+\alpha S_k} + \gamma I_k\Big]\Delta t - \frac{\sigma S_k I_k}{1+\alpha S_k}\sqrt{\Delta t}\xi_k \\
&- \frac{\sigma^2 S_k I_k}{2(1+\alpha S_k)}(\xi_k^2 - 1)\Delta t, \\
I_{k+1} =& I_k + \Big[\frac{\beta S_k I_k}{1+\alpha S_k} - (\mu+\gamma)I_k\Big]\Delta t + \frac{\sigma S_k I_k}{1+\alpha S_k}\sqrt{\Delta t}\xi_k + \frac{\sigma^2 S_k I_k}{2(1+\alpha S_k)}(\xi_k^2 - 1)\Delta t,
\end{aligned}
$$

where $\xi_k, k = 1, 2, \dots, n$, are independent Gaussian random variables $N(0, 1)$.

Example 6.1. For the deterministic system (1.1) and its stochastic system (1.2), the parameters are taken as follows: $r = 0.4$, $K = 1.2$, $\beta = 0.4$, $\mu = 0.2$, $\alpha = 0.4$, $\gamma = 0.2$.

Case 1. Let $\sigma = 0.8$. Then

$$\sigma^2 = 0.64 > \max\{0.4933, 0.2\},$$

as a consequence result of conditions (4.1) in Theorem 4.1, the disease I_t dies out with probability one. Figure 1(a) shows the paths of S_t, I_t in the deterministic system (1.1) and Figure 1(b) shows the paths of S_t, I_t in the stochastic system corresponding to a deterministic system (1.1). Figure 1(b') is the phase portrait of Figure 1(a) and (b).

Case 2. Let $\sigma = 0.3$. Then

$$R_0^s = 0.7369 < 1, \quad \text{and} \quad \sigma^2 = 0.09 \le \frac{\beta(1+\alpha K)}{K} = 0.4933.$$

Then from conditions (4.2) in Theorem 4.1, the disease I_t also dies out with probability one. Figure 1(a) and Figure 1(c) show the paths of S_t, I_t in the deterministic system (1.1) and the stochastic system (1.2), respectively. Figure 1(c') is the phase portrait of Figure 1(a) and (c).

Simulations in Figure 1 show that σ can affect the persistent and extinction of the disease I_t. When σ is large, the disease I_t must be extinct. But the σ is not large, the disease will also die out under certain conditions. Consequence, we can control the persistent and extinction of the disease by controlling the size of σ.

Example 6.2. For the deterministic model (1.1) and its stochastic model (1.2), the parameters are taken as follows $r = 1.2$, $K = 2.5$, $\beta = 0.4$, $\mu = 0.2$, $\alpha = 0.35$, $\gamma = 0.3$, $\sigma = 0.046$. Then

$$R_0^s = 1.063 > 1 \quad \text{and} \quad p_3 = 0.121 < \min\{0.819, 0.702\},$$

according to Theorem 5.1, we can conclude that the density functions of S_t and I_t will convergent. By Figure 2(a) and (c), we have that the disease I_t will persistent in long time. Figure 2(b) and (d) describe the density function images of the stationary distribution of S_t and I_t, respectively. Simulations in Figure 2 show that σ keep the processes S_t, I_t for stochastic system (1.2) moving around the orbits for the deterministic system (1.1) in a confined region. This indicates that the semigroup of system (1.2) is asymptotically stable. Hence, Figure 2 approve the conclusion of Theorem 5.1.

This article explores a stochastic plant disease model with logistic growth and saturated incidence rate. The diffusion matrix of the stochastic system is the degenerate form. Thus, the theory on Markov semigroup is used to analyze the asymptotic behaviors of the distributions of the solutions. The densities of the distributions of the solutions are absolutely continuous, and the densities will converge in L^1 to an invariant density under appropriate conditions. Moreover, the sufficient conditions is provided for the extinction of the disease under the different white noise intensity. Our results are given as follows

(I) If one of the following two conditions holds

$$\sigma^2 > \max\Big\{\frac{\beta(1+\alpha K)}{K}, \frac{\beta^2}{2(\mu+\gamma)}\Big\};$$

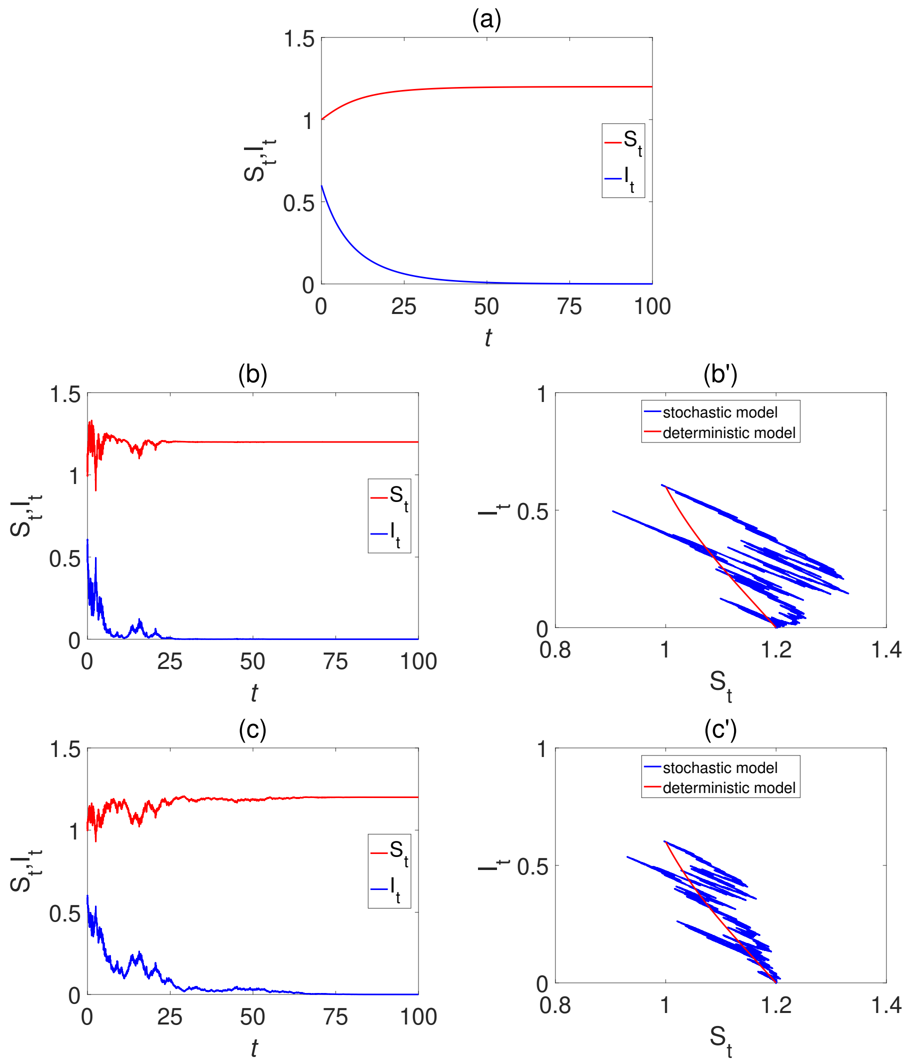

FIGURE 1. (a) is the deterministic system, (b) and (c) are the corresponding stochastic systems of (a), (b') and (c') are phase portrait of (b) and (c), respectively. The initial values $(S_0, I_0) = (1, 0.6)$, (a) $\sigma = 0$, (b) $\sigma = 0.3$, (c) $\sigma = 0.8$, (b') $\sigma = 0.3$, (c') $\sigma = 0.8$.

$$R_0^s < 1 \quad \text{and} \quad \sigma^2 \leq \frac{\beta(1 + \alpha K)}{K},$$

then the diseases I of the system (1.2) will be extinct with probability one.

(II) If

$$R_0^s > 1, \ \frac{\alpha\beta S^*}{\mu(1 + \alpha S^*)} + a < \frac{r}{K} + \frac{1}{K - S^*},$$

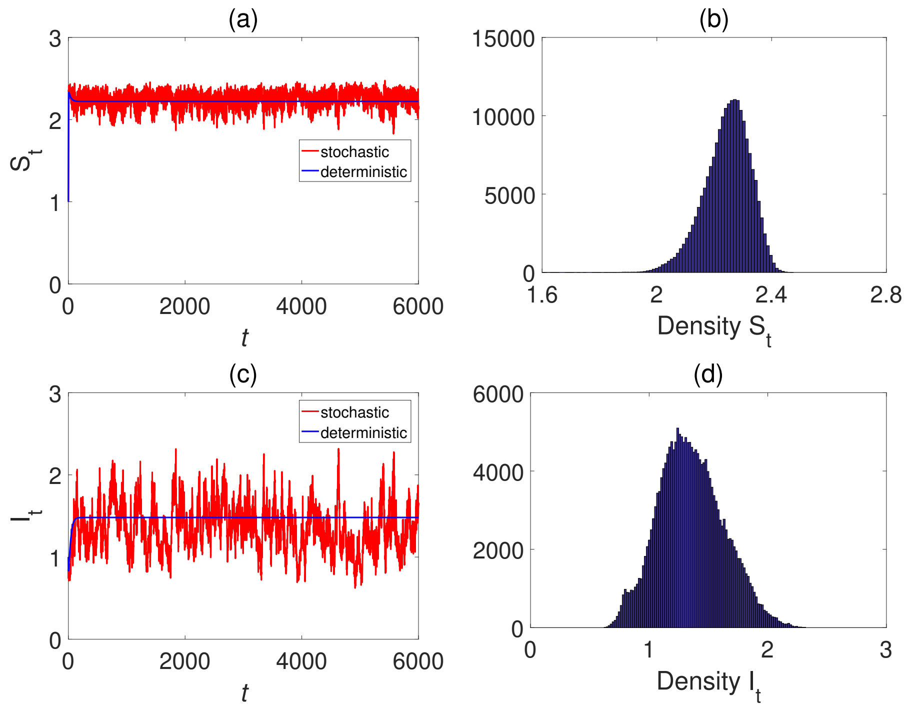

FIGURE 2. (a) and (c) are the sample paths for the solutions of system (1.2) and its corresponding deterministic system (1.1). (b) and (d) are the density functions of S_t and I_t in stochastic systems (1.2). $(S_0, I_0) = (1, 1)$, $\sigma = 0.046$.

and

$$p_3 < \min\Big\{\big[\frac{r}{K} + r\big(1 - \frac{S^*}{K}\big)\big(\frac{\gamma}{K} - \frac{\alpha\beta S^*}{\mu(1+\alpha S^*)} - a\big)\big](S^*)^2, a\mu(I^*)^2\Big\},$$

then system (1.2) exists a unique density $U_*(t, x, y)$ of a stationary solution.

Through theoretical proof and numerical simulation, we have obtained that environmental noises have a critical influence on the development of plant infectious diseases. Compared to the deterministic model, the stochastic model is more realistic for the research of plant disease models. For the stochastic system (1.2), we can control the persistence and extinction of the plant disease by the size of σ. When σ is very large, the disease I_t will disappear (see Figure 1(b) and (b')); when σ is not large, the disease I_t may disappear or it may exist (see Figure 1(c), (c') and Figure 2), which requires to predict the development of the disease by R_0^s. When $R_0^s > 1$, the disease will persist; when $R_0^s < 1$, the disease will become extinct. Therefore, farmers can make the disease extinct by controlling the size of σ, so that farmers can get more and better food, which is conducive to the country's stability and economic development.

Some interesting questions deserve further investigation. On the one hand, we may explore some realistic but complex models, considering the effect of predators hunting on diseased plants. On the other hand, we can use the methods to research epidemic models, chemostat models or other population dynamics models. We will leave these cases as our future work.

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